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# STOCHASTIC FLOWS AND AN INTERFACE SDE ON METRIC GRAPHS

HATEM HAJRI<sup>(1)</sup> AND OLIVIER RAIMOND<sup>(2)</sup>

## ABSTRACT

We study a stochastic differential equation (SDE) driven by a finite family of independent white noises on a star graph, each of these white noises driving the SDE on a ray of the graph. This equation extends the perturbed Tanaka's equation recently studied by Prokaj [14] and Le Jan-Raimond [9] among others. We prove that there exists a coalescing stochastic flow of mappings solution of this equation. This flow is unique in law and is coalescing. Our proofs involve the study of a reflected Brownian motion in the two dimensional quadrant obliquely reflected at the boundary, with time dependent reflections. Filtering this flow solution of the SDE with respect to the family of white noises yields a Wiener stochastic flow of kernels also solution of this SDE. This Wiener solution is also unique. Moreover, if  $N$  denotes the number of rays in the star graph, the Wiener solution and the coalescing solution coincide if and only if  $N = 2$ . When  $N \geq 3$ , the problem of classifying all solutions is left open. Finally, we define an extension of this equation on a general metric graph with finite sets of vertices and edges to which we apply our previous results [7]. As a consequence, we get a flow of mappings and a Wiener flow solutions for this SDE.

## 1. INTRODUCTION AND MAIN RESULTS

In [14], Prokaj proved that pathwise uniqueness holds for the perturbed Tanaka's equation

$$(1) \quad dX_t = \text{sgn}(X_t)dW_t^1 + \lambda dW_t^2$$

for all  $\lambda \neq 0$  where  $W^1$  and  $W^2$  are two independent Brownian motions. When  $\lambda = 1$ , after rescaling, setting  $W^+ = \frac{W^1+W^2}{\sqrt{2}}$  and  $W^- = \frac{W^2-W^1}{\sqrt{2}}$ ,

(1) rewrites

$$(2) \quad dX_t = 1_{\{X_t>0\}}dW_t^+ + 1_{\{X_t\leq 0\}}dW_t^-.$$

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Using different techniques, the same result in the case of (2) has been obtained also by Le Jan and Raimond [9] (see also [4, 12]) and they proved in addition that (2) generates a stochastic coalescing flow. Intuitively, a solution to (2) is a Brownian motion that follows  $W^+$  on its positive excursions and that follows  $W^-$  on its negative excursions. In this paper, we consider first the analogous problem on a star graph (by a star graph, we mean a metric space consisting of a finite number of pieces of  $\mathbb{R}_+$  in which all origins are identified) and then extend this framework to more general metric graphs.

### 1.1. Notations.

- In all this paper, we fix  $N \geq 2$  and  $p_1, \dots, p_N > 0$  such that  $\sum_{i=1}^N p_i = 1$  and let  $G$  be a star graph with origin denoted by 0 and  $N$  edges  $(E_i)_{1 \leq i \leq N}$ . Then  $G$  is such that  $E_i \cap E_j = \{0\}$  if  $i \neq j$  and such that for each  $i$ ,  $E_i$  is isometric to  $[0, \infty[$  via a mapping  $e_i : [0, \infty[ \rightarrow E_i$ . Define  $\sim$  the equivalence relation on  $G$  by  $x \sim y$  if there exists  $i$  such that  $x$  and  $y$  both belong to  $E_i$ , and when it is not the case, we use the notation  $x \not\sim y$ . Let  $d$  be the metric on  $G$  such that if  $x = e_i(r)$  then  $|x| := d(x, 0) = r$ , if  $x \sim y$  then  $d(x, y) = ||y| - |x||$  and if  $x \not\sim y$ ,  $d(x, y) = |x| + |y|$ . We equip  $G$  with its Borel  $\sigma$ -field  $\mathcal{B}(G)$  and set  $G^* = G \setminus \{0\}$ . For each  $i$ , set  $E_i^* = E_i \setminus \{0\}$ .

We denote by  $C_b^2(G^*)$  the set of all continuous functions  $f : G \rightarrow \mathbb{R}$  such that for all  $i \in [1, n]$ ,  $f \circ e_i$  is  $C^2$  on  $]0, \infty[$  with bounded first and second derivatives both with finite limits at 0. For  $f \in C_b^2(G^*)$  and  $x = e_i(r) \in G^*$ , set  $f'(x) = (f \circ e_i)'(r)$ ,  $f''(x) = (f \circ e_i)''(r)$ . When  $x = 0$  define  $f'(0) = \sum_{i=1}^N p_i (f \circ e_i)'(0+)$  and  $f''(0) = \sum_{i=1}^N p_i (f \circ e_i)''(0+)$ . Set

$$\mathcal{D} = \{f \in C_b^2(G^*) : f'(0) = 0\}.$$

- The two-dimensional quadrant is the set  $\mathcal{Q} := [0, \infty[^2$ . Its boundary is denoted by  $\partial\mathcal{Q} := \partial_1\mathcal{Q} \cup \partial_2\mathcal{Q}$ , where  $\partial_1\mathcal{Q} = [0, \infty[ \times \{0\}$  and  $\partial_2\mathcal{Q} = \{0\} \times [0, \infty[$ . We also set  $\mathcal{Q}^* = \mathcal{Q} \setminus \{(0, 0)\}$ .
- For  $X$  a continuous semimartingale, we will denote by  $L_t(X)$  its symmetric local time process at 0, i.e.

$$L_t(X) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s| \leq \epsilon\}} d\langle X \rangle_s.$$

- For a family of random variables  $Z = (Z_{s,t})_{s \leq t}$  and a process  $X = (X_t)_{t \geq 0}$ , we will use the usual notations

$$\mathcal{F}_{s,t}^Z = \sigma(Z_{u,v}, s \leq u \leq v \leq t), \quad \mathcal{F}_t^X = \sigma(X_u, 0 \leq u \leq t).$$

- A filtration generated by a finite or infinite family of independent Brownian motions will be called a Brownian filtration.
- The Walsh's Brownian motion on  $G$  is the Feller diffusion defined via its Feller semigroup  $(P_t, t \geq 0)$  as in [1]: Let  $(T_t^+, t \geq 0)$  be the semigroup of reflecting Brownian motion on  $\mathbb{R}_+$  and let  $(T_t^0, t \geq 0)$  be the semigroup of Brownian motion on  $\mathbb{R}_+$  killed at 0, then for  $f \in C_0(G)$  and  $x \in E_i$ , denoting  $f_j(r) = f \circ e_j(r)$  for  $1 \leq j \leq N$  and  $\bar{f}(r) = \sum_{j=1}^N p_j f_j$ ,

$$P_t f(x) = T_t^+ \bar{f}(|x|) + T_t^0 (f_i - \bar{f})(|x|).$$

- For a filtration  $(\mathcal{G}_t)_t$ ,  $X$  is a  $(\mathcal{G}_t)$ -Walsh's Brownian motion if it is adapted to  $(\mathcal{G}_t)$  and if given  $\mathcal{G}_t$ ,  $(X_{t+s}, s \geq 0)$  is a Walsh's Brownian motion started at  $X_t$ .

**1.2. The interface SDE on a star graph.** Our main interest in this paper is the following SDE, we call the interface SDE, which is the natural extension of (2) to star graphs.

**Definition 1.1.** *A solution of the interface SDE (E) on a star graph  $G$  is a pair  $(X, W)$  of processes defined on a filtered probability space  $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$  such that*

- $W = (W^1, \dots, W^N)$  is a standard  $(\mathcal{F}_t)$ -Brownian motion in  $\mathbb{R}^N$ ;
- $X$  is a  $(\mathcal{F}_t)$ -adapted continuous process in  $G$ ;
- For all  $f \in \mathcal{D}$ ,

$$(3) \quad f(X_t) = f(X_0) + \sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i + \frac{1}{2} \int_0^t f''(X_s) ds.$$

We will say it is a strong solution if  $X$  is adapted to the filtration  $(\mathcal{F}_t^W)_t$ .

Note that it can easily be seen (by choosing for each  $i$  a function  $f \in \mathcal{D}$  such that  $f(x) = |x|$  if  $x \in E_i$ ) that on  $E_i$ , away from 0,  $X$  follows the Brownian motion  $W^i$ . Our first result is the following

**Theorem 1.2.** *For all  $x \in G$ ,*

- There exists a solution  $(X, W)$  with  $X_0 = x$ , unique in law, of the SDE (E). Moreover  $X$  is a Walsh's Brownian motion.*
- The solution of the SDE (E) is a strong solution, if and only if  $N = 2$ .*

To prove (ii), when  $N = 2$ , we will prove that pathwise uniqueness holds for (E). Then, this implies that the solution  $(X, W)$  is a strong one. The fact that when  $N \geq 3$ ,  $(X, W)$  is not a strong solution is a

consequence of a result of Tsirelson [15] (see Theorem 3.6 below) which states that if  $N \geq 3$ , there does not exist any  $(\mathcal{F}_t)$ -Walsh's Brownian motion on  $G$  with  $(\mathcal{F}_t)$  a Brownian filtration (see also [3]).

When  $N = 2$ , one can assume  $G = \mathbb{R}$ ,  $E_1 = ]-\infty, 0]$  and  $E_2 = [0, \infty[$ . Applying Itô-Tanaka's formula (or Theorem 3.1 below), we see that (E) is equivalent to the skew Brownian motion version of (2):

$$(4) \quad dX_t = 1_{\{X_t > 0\}} dW_t^+ + 1_{\{X_t \leq 0\}} dW_t^- + (2p - 1) dL_t(X)$$

where  $p = p_1$  (note that when  $p = 1/2$ , (2) and (4) coincide).

In this paper a stochastic flow of mappings as defined by Le Jan and Raimond [10] will be called a SFM. We will be interested in SFM's solving (E) in the following sense.

**Definition 1.3.** *On a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , let  $\mathcal{W} = (W^i, 1 \leq i \leq N)$  be a family of independent real white noises (see Definition 1.10 in [10]) and  $\varphi$  be a SFM on  $G$ . We say that  $(\varphi, \mathcal{W})$  solves (E) if for all  $s \leq t$ ,  $f \in \mathcal{D}$  and  $x \in G$ , a.s.*

$$f(\varphi_{s,t}(x)) = f(x) + \sum_{i=1}^N \int_s^t (1_{E_i} f')(\varphi_{s,u}(x)) dW_u^i + \frac{1}{2} \int_s^t f''(\varphi_{s,u}(x)) du.$$

We will say it is a Wiener solution if for all  $s \leq t$ ,  $\mathcal{F}_{s,t}^\varphi \subset \mathcal{F}_{s,t}^\mathcal{W}$ .

It will be shown that as soon as  $(\varphi, \mathcal{W})$  solves (E), we have  $\mathcal{F}_{s,t}^\mathcal{W} \subset \mathcal{F}_{s,t}^\varphi$  for all  $s \leq t$  and thus we may just say  $\varphi$  solves (E). Note that when  $\varphi$  is a Wiener solution, then  $\mathcal{F}_{s,t}^\varphi = \mathcal{F}_{s,t}^\mathcal{W}$  for all  $s \leq t$ .

Our main result is the following

**Theorem 1.4.** (i) *There exists a SFM  $\varphi$  solution of (E). This solution is unique in law.*

(ii) *The SFM  $\varphi$  is coalescing in the sense that for all  $s \in \mathbb{R}$  and  $(x, y) \in G^2$ , a.s.,*

$$T_s(x, y) = \inf\{t \geq s : \varphi_{s,t}(x) = \varphi_{s,t}(y)\} < \infty$$

*and  $\varphi_{s,t}(x) = \varphi_{s,t}(y)$  for all  $t \geq T_s(x, y)$ .*

(iii) *The SFM  $\varphi$  is a Wiener solution if and only if  $N = 2$ .*

Note that (iii) in Theorem 1.4 is a consequence of (ii) in Theorem 1.2. Let  $\varphi$  be a SFM on  $G$  and  $\mathcal{W}$  be a family of independent white noises such that  $(\varphi, \mathcal{W})$  is a solution to (E). As  $\mathcal{F}_{s,t}^\mathcal{W} \subset \mathcal{F}_{s,t}^\varphi$ , Lemma 3.2 in [10] ensures that there exists a stochastic flow of kernels  $K^\mathcal{W}$  (see [10] for the definition) such that : for all  $s \leq t$ ,  $x \in G$ , a.s.

$$K_{s,t}^\mathcal{W}(x) = E[\delta_{\varphi_{s,t}(x)} | \mathcal{F}_{s,t}^\mathcal{W}].$$

A stochastic flow of kernels will be denoted later on simply by SFK. We will also be interested on SFK's solving (E) in the following sense.

**Definition 1.5.** *Let  $K$  be a SFK on  $G$  and  $\mathcal{W} = (W^i, 1 \leq i \leq N)$  be a family of independent real white noises. We say that  $(K, \mathcal{W})$  solves (E) if for all  $s \leq t$ ,  $f \in \mathcal{D}$  and  $x \in G$ , a.s.*

$$(5) \quad K_{s,t}f(x) = f(x) + \sum_{i=1}^N \int_s^t K_{s,u}(1_{E_i}f')(x)dW_u^i + \frac{1}{2} \int_s^t K_{s,u}f''(x)du.$$

We will say it is a Wiener solution if for all  $s \leq t$ ,  $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^{\mathcal{W}}$ .

Since we also have that  $\mathcal{F}_{s,t}^{\mathcal{W}} \subset \mathcal{F}_{s,t}^K$ , we simply say that  $K$  solves (E). Note that when  $K = \delta_\varphi$ , then  $K$  solves (E) if and only if  $\varphi$  also solves (E). In this case, the SFK  $K$  will be called a SFM. We have the following

**Proposition 1.6.**  *$K^{\mathcal{W}}$  is the unique (up to modification) Wiener solution of (E).*

We do not give a proof of this proposition here. This can be done following Proposition 8 in [6] where this result is proved when all the  $W^i$  are equal, or following the proof of Proposition 3.1 in [9] where this result is proved in the case of (2).

A consequence of Proposition 1.6 and Theorem 1.4 (ii) is

**Corollary 1.7.**  *$K^{\mathcal{W}}$  is the only SFK solution of (E) if and only if  $N = 2$ .*

*Proof.* When  $N = 2$ ,  $\varphi$  is a Wiener solution of (E). Suppose  $(K, \mathcal{W})$  is another solution of (E), then  $\mathbb{E}[K|\mathcal{W}]$  is a Wiener solution of (E). Since the Wiener solution is unique, for all  $s \leq t$  and  $x \in G$  a.s.

$$\delta_{\varphi_{s,t}(x)} = E[K_{s,t}(x)|\mathcal{F}_{s,t}^{\mathcal{W}}].$$

This yields that  $\delta_{\varphi_{s,t}(x)} = K_{s,t}(x)$  a.s. □

For  $N \geq 3$ , the SDE (E) may have other SFK's solutions different from  $\varphi$  and  $K^{\mathcal{W}}$ . The problem of a complete classification of the laws of all these flows is left open.

**1.3. Brownian motion with oblique reflection.** To prove Theorems 1.2 and 1.4 we shall study a Brownian motion in the two dimensional quadrant, obliquely reflected at the boundary and with time dependent angles of reflections. We now give an application of our methods to the obliquely reflected Brownian motion defined by Varadhan and Williams in [16].

Fix  $\theta_1, \theta_2 \in ]0, \frac{\pi}{2}[$  and  $x > 0$ . Let  $(B^1, B^2)$  be a two dimensional Brownian motion and  $(X, Y)$  be the reflected Brownian motion in  $\mathcal{Q}$  started from  $(x, 0)$  with angles of reflections on  $\partial_1 \mathcal{Q}$  and on  $\partial_2 \mathcal{Q}$  respectively given by  $\theta_1$  and  $\theta_2$ , and killed at time  $\sigma_0$ , the hitting time of  $(0, 0)$ . More precisely, for  $t < \sigma_0$ ,

$$\begin{aligned} dX_t &= dB_t^1 - \tan(\theta_1)dL_t(X) + dL_t(Y), \quad X_0 = x; \\ dY_t &= dB_t^2 + dL_t(X) - \tan(\theta_2)dL_t(Y), \quad Y_0 = 0. \end{aligned}$$

Denote by  $L_t = L_t(X) + L_t(Y)$  the local time accumulated at  $\partial \mathcal{Q}$ . Then it is known that  $\sigma_0$  and  $L_{\sigma_0}$  are finite (see [16] and [17]). Our next result gives a necessary and sufficient condition for  $L_{\sigma_0}$  to be integrable with an explicit expression of its expectation.

**Proposition 1.8.** *We have that*

$$\mathbb{E}[L_{\sigma_0}] < \infty \text{ if and only if } \tan(\theta_1) \tan(\theta_2) > 1.$$

*In this case*

$$\mathbb{E}[L_{\sigma_0}] = \frac{x(\tan(\theta_2) + 1)}{\tan(\theta_1) \tan(\theta_2) - 1}.$$

The assumptions on the wedge and the angles considered here are more suitable to our framework but our techniques may be applied to give an expression of  $\mathbb{E}[L_{\sigma_0}]$  in other situations.

**1.4. Extension to oriented metric graphs.** Instead of considering only star graphs, it is natural to define an analogous version of  $(E)$  on more general metric graphs. Let  $G$  be a metric graph (see section 2.1 [7] for the definition) and denote by  $V$ , the set of its vertices, and by  $\{E_i; i \in I\}$  the set of its edges. We suppose that  $I$  and  $V$  are finite. To each edge  $E_i$ , we associate an isometry  $e_i : J_i \rightarrow \bar{E}_i$ , with  $J_i = [0, L_i]$  when  $L_i < \infty$  and  $J_i = [0, \infty)$  or  $J_i = (-\infty, 0]$  when  $L_i = \infty$ . When  $L_i < \infty$ , denote  $\{g_i, d_i\} = \{e_i(0), e_i(L_i)\}$ . When  $L_i = \infty$ , denote  $\{g_i, d_i\} = \{e_i(0), \infty\}$  when  $J_i = [0, \infty)$  and  $\{g_i, d_i\} = \{\infty, e_i(0)\}$  when  $J_i = (-\infty, 0]$ . For all  $v \in V$ , denote  $I_v^+ = \{i \in I; g_i = v\}$ ,  $I_v^- = \{i \in I; d_i = v\}$  and  $I_v = I_v^+ \cup I_v^-$ . To each  $v \in V$  and  $i \in I_v$ , we associate a transmission parameter  $p_v^i$  such that  $\sum_{i \in I_v} p_v^i = 1$ . Let  $G^* = G \setminus V$ . We denote by  $C_b^2(G^*)$  the set of all continuous functions  $f : G \rightarrow \mathbb{R}$  such that for all  $i \in I$ ,  $f \circ e_i$  is  $C^2$  on the interior of  $J_i$  with bounded first and second derivatives both extendable by continuity to  $J_i$  and such that for all  $v \in V$

$$\sum_{i \in I_v^+} p_v^i \lim_{r \rightarrow 0^+} (f \circ e_i)'(r)$$

$$= \sum_{i \in I_v^-} p_v^i \left( \lim_{r \rightarrow L_i^-} (f \circ e_i)'(r) 1_{\{L_i < \infty\}} + \lim_{r \rightarrow 0^-} (f \circ e_i)'(r) 1_{\{L_i = \infty\}} \right).$$

For  $f \in C_b^2(G^*)$  and  $x = e_i(r) \in G \setminus V$ , set  $f'(x) = (f \circ e_i)'(r)$ ,  $f''(x) = (f \circ e_i)''(r)$  and take the following conventions for all  $v \in V$ ,

$$\begin{aligned} f'(v) &= \sum_{i \in I_v^+} p_v^i (f \circ e_i)'(0+) \\ &\quad - \sum_{i \in I_v^-} p_v^i \left( \lim_{r \rightarrow L_i^-} (f \circ e_i)'(r) 1_{\{L_i < \infty\}} + \lim_{r \rightarrow 0^-} (f \circ e_i)'(r) 1_{\{L_i = \infty\}} \right). \end{aligned}$$

Define the analogous convention for  $f''(v)$  by only replacing the first derivatives with the second ones. Finally set

$$\mathcal{D} = \{f \in C_b^2(G^*) : f'(v) = 0 \text{ for all } v \in V\}.$$

**Definition 1.9.** (Equation (E)) On a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , let  $\mathcal{W} = (W^i)_{i \in I}$  be a real white noise and  $K$  be a stochastic flow of kernels on  $G$ . We say that  $(K, \mathcal{W})$  solves (E) if for all  $s \leq t$ ,  $f \in \mathcal{D}$  and  $x \in G$ , a.s.

$$K_{s,t}f(x) = f(x) + \sum_{i \in I} \int_s^t K_{s,u}(1_{E_i}f')(x)W^i(du) + \frac{1}{2} \int_s^t K_{s,u}f''(x)du.$$

Suppose there exists a SFM solution of (E) on  $G$ , then before hitting two distinct vertices, the motion of any point under  $\varphi$  is well described by Theorem 1.4 : it is governed by a SFM on a star graph. Then the problem reduces to well concatenate flows on star graphs to get a flow on  $G$ . Technically, this is not so easy and has been the subject of our previous work [7]. There, a general result is established and applies well here. We state now

**Theorem 1.10.** (1) *There exists a SFM on  $G$  solution of (E).*  
 (2) *There exists a unique Wiener SFK  $K^{\mathcal{W}}$  solution of (E).*

*Proof.* The proof of the existence of these flows is immediate from Theorem 3.2 in [7]. The unicity of the Wiener solution is a consequence of Theorem 4.1 in [7].  $\square$

From Theorem 4.1 in [7], we can also deduce

**Corollary 1.11.**  $K^{\mathcal{W}} = \delta_\varphi$  if and only if each vertex point on  $G$  has at most two adjacent edges. This is also the only case when (E) has a unique solution.



**1.5. Outline of contents.** Let us describe the content of this paper. In section 2, we study a variation of the obliquely reflected Brownian motion in  $\mathcal{Q}$ , where the angles of reflections depend on time and which is absorbed when it hits the corner. In section 3, we prove Theorem 1.2. In section 4, we prove Theorem 1.4 (i) and (ii), using in particular the results of section 2.

## 2. BROWNIAN MOTION IN THE QUADRANT WITH TIME DEPENDENT ANGLES OF REFLECTION

In this section, we study a variation of the obliquely reflected Brownian motion in  $\mathcal{Q}$  where the angles of reflections depend on time and which is absorbed when it hits the corner. This process is defined in section 2.2. We will be interested in the following two questions:

- (I) Is the hitting time  $\sigma_0$  of  $(0, 0)$  finite a.s.?
- (II) Is  $L_{\sigma_0}$ , the local time accumulated at  $\partial\mathcal{Q}$  at time  $\sigma_0$ , finite a.s.?

In sections 2.3 and 2.4, we prove that, under some assumptions on the sequence of the angles of reflections, the answer to these two questions is positive. The tools used are a scaling property and a precise study, done in section 2.1, of an obliquely reflected Brownian motion on the quadrant started at  $(x, 0)$ , with  $x > 0$ , and stopped when it hits  $\{y = 0\}$ . Finally in section 2.5, we calculate  $\mathbb{E}[L_{\sigma_0}]$ .

**2.1. Brownian motion on the half-plane with oblique reflection.** We fix  $\theta \in ]0, \pi/2[$ . Let  $Z = (X, Y)$  be the process started from  $(x, y)$  in  $\mathbb{R} \times \mathbb{R}_+$  obliquely reflected at  $\{y = 0\}$ , with angle of reflection given by  $\theta$ . More precisely,

$$\begin{aligned} dX_t &= dB_t^1 - \tan(\theta)dL_t, & X_0 &= x \\ dY_t &= dB_t^2 + dL_t, & Y_0 &= y \end{aligned}$$

where  $B^1$  and  $B^2$  are two independent Brownian motions and  $L_t$  is the local time at 0 of  $Y$ . Set  $S = \inf\{s : X_s = 0\}$ . Denote by  $\mathbb{P}_x^\theta$  the law of  $(Z_t; t \leq S)$  when  $y = 0$  and  $x > 0$ . Note that for all  $t \leq S$ ,  $Z_t \in \mathcal{Q}$ . Observe that we have the following scaling property:

**Proposition 2.1.** *For all  $x > 0$ , if the law of  $(Z_t; t \leq S)$  is  $\mathbb{P}_1^\theta$ , then the law of  $(xZ_{x^{-2}t}; t \leq x^2S)$  is  $\mathbb{P}_x^\theta$ .*

For  $z \in \mathbb{C}$ ,  $\arg(z)$ ,  $\mathcal{R}(z)$  and  $\mathcal{I}(z)$  will denote respectively the argument, the real part and the imaginary part of  $z$ . Following [16], if  $f$  is holomorphic on an open set  $U$  containing  $\mathcal{Q}^*$  such that  $f(z) \in \mathbb{R}$  for all  $z \in ]0, \infty[$ , then  $\phi(x, y) := \mathcal{R}(f(x + iy)e^{-i\theta})$  is harmonic on  $U$ .

Moreover,

$$(6) \quad v_1(\theta) \cdot \nabla \phi(x, 0) = 0 \text{ for } x > 0, \text{ where } v_1(\theta) = (-\tan(\theta), 1).$$

Indeed, the fact that  $f$  is holomorphic with the condition  $f(z) \in \mathbb{R}$  for all  $z \in ]0, \infty[$  implies that  $f'(z) \in \mathbb{R}$  for all  $z \in ]0, \infty[$ . Thus

$$\nabla \phi(x, 0) = (\mathcal{R}(f'(x)e^{-i\theta}), \mathcal{R}(if'(x)e^{-i\theta})) = f'(x)(\cos(\theta), \sin(\theta))$$

and (6) follows. These properties imply in particular that  $(\phi(Z_{t \wedge S}))_t$  is a local martingale. For  $b \in \mathbb{R}$  and  $f(z) = z^b$  the function  $\phi$  defined above will be denoted  $\phi_b$ .

**Lemma 2.2.** *Let  $(Z_t; t \leq S)$  be a process of law  $\mathbb{P}_x^\theta$ .*

(i) *If  $0 < b < 1 + 2\theta/\pi$ , then for all  $a > x$ ,*

$$\mathbb{P} \left( \sup_{s \leq S} |Z_s| > a \right) \leq c_b \left( \frac{x}{a} \right)^b,$$

*where  $c_b = 1$  if  $b\pi/2 \leq \theta$  and  $c_b = \cos(\theta)/\cos(b\pi/2 - \theta)$  otherwise.*

(ii) *If  $0 < b < 1 - 2\theta/\pi$ , then for all  $a < x$ ,*

$$\mathbb{P} \left( \inf_{s \leq S} |Z_s| < a \right) \leq c_b \left( \frac{a}{x} \right)^b,$$

*where  $c_b = \cos(\theta)/\cos(b\pi/2 + \theta)$ .*

*Proof.* Using the scaling property we may take  $x = 1$ . For  $a \geq 0$ , set  $\sigma_a = \inf\{t : |Z_t| = a\}$ . Recall that for all  $b \in \mathbb{R}$ ,  $(\phi_b(Z_{t \wedge S}))_t$  is a local martingale.

Proof of (i): Fix  $a > 1$  and  $0 < b < 1 + 2\theta/\pi$ . For  $c_b^0 = \inf\{\cos(\theta), \cos(b\pi/2 - \theta)\}$  and  $t \leq S$ , we have

$$c_b^0 |Z_t|^b \leq \phi_b(Z_t) \leq |Z_t|^b.$$

Moreover

$$\mathbb{P}(\sup_{s \leq S} |Z_s| > a) = \mathbb{P}(\sigma_a < S).$$

By the martingale property, for all  $t \geq 0$ ,

$$\cos(\theta) = \phi_b(1) = \mathbb{E}[\phi_b(Z_{t \wedge \sigma_a \wedge S})]$$

which is larger than

$$\mathbb{E}[\phi_b(Z_{t \wedge \sigma_a}) 1_{\{\sigma_a < S\}}].$$

As  $t \rightarrow \infty$ , this last term converges using dominated convergence to

$$\mathbb{E}[\phi_b(Z_{\sigma_a}) 1_{\{\sigma_a < S\}}] \geq c_b^0 a^b \mathbb{P}(\sigma_a < S).$$

This easily implies (i).

The proof of (ii) is similar: Fix  $a < 1$  and  $0 < b < 1 - 2\theta/\pi$ . For  $c_b^1 = \cos(b\pi/2 + \theta)$  and  $t \leq S$ ,

$$c_b^1 |Z_t|^{-b} \leq \phi_{-b}(Z_t) \leq |Z_t|^{-b}.$$

We also have that

$$\mathbb{P}(\inf_{s \leq S} |Z_s| < a) = \mathbb{P}(\sigma_a < S).$$

By the martingale property, for all  $t \geq 0$ ,

$$\cos(\theta) = \phi_{-b}(1) = \mathbb{E}[\phi_{-b}(Z_{t \wedge \sigma_a \wedge S})]$$

which is larger than

$$\mathbb{E}[\phi_{-b}(Z_{t \wedge \sigma_a}) 1_{\{\sigma_a < S\}}]$$

and this converges as  $t \rightarrow \infty$  to

$$\mathbb{E}[\phi_{-b}(Z_{\sigma_a}) 1_{\{\sigma_a < S\}}] \geq c_b^1 a^{-b} \mathbb{P}(\sigma_a < S).$$

This easily implies (ii).  $\square$

**Corollary 2.3.** *Let  $(Z_s; s \leq S)$  be distributed as  $\mathbb{P}_x^\theta$ . If  $-1 + 2\theta/\pi < b < 1 + 2\theta/\pi$ , then*

$$\mathbb{E}(\sup_{s \leq S} |Z_s|^b) < \infty.$$

*Proof.* To simplify, assume  $x = 1$ . For  $b \in ]0, 1 + 2\theta/\pi[$ , let  $b' \in ]b, 1 + 2\theta/\pi[$ . Then

$$\begin{aligned} \mathbb{E}(\sup_{s \leq S} |Z_s|^b) &= \int_0^\infty \mathbb{P}[\sup_{s \leq S} |Z_s| > a^{1/b}] da \\ &\leq 1 + c_b \int_1^\infty a^{-b'/b} da < \infty. \end{aligned}$$

For  $b \in ]-1 + 2\theta/\pi, 0[$ , let  $b' \in ]-1 + 2\theta/\pi, b[$ . Then

$$\begin{aligned} \mathbb{E}(\sup_{s \leq S} |Z_s|^b) &= \int_0^\infty \mathbb{P}[\inf_{s \leq S} |Z_s| < a^{1/b}] da \\ &\leq 1 + c_b \int_1^\infty a^{-b'/b} da < \infty. \end{aligned}$$

$\square$

**Corollary 2.4.** *Let  $(Z_s; s \leq S)$  be distributed as  $\mathbb{P}_x^\theta$ . Let  $f$  be an holomorphic function on an open set containing  $\mathcal{Q}^*$  such that  $f(z) \in \mathbb{R}$  for all  $z \in ]0, \infty[$ . Assume there exists  $C > 0$ ,  $b_+ \in ]0, 1 + 2\theta/\pi[$  and  $b_- \in ]0, 1 - 2\theta/\pi[$  with*

$$|f(z)| \leq C(|z|^{-b_-} + |z|^{b_+}) \text{ for all } z \in \mathcal{Q}^*.$$

then setting  $\phi(x, y) = \mathcal{R}(f(x + iy)e^{-i\theta})$ , we have

$$E[\phi(iY_S)] = \cos(\theta)f(x).$$

*Proof.* Recall that  $(\phi(Z_{t \wedge S}))_t$  is a local martingale (stopped at time  $S$ ). Using Corollary 2.3, it is a uniformly integrable martingale. And we conclude using the martingale property.  $\square$

Note that the functions  $f(z) = z^b$ , for  $b \in ]-1 + 2\theta/\pi, 1 + 2\theta/\pi[$ ,  $f(z) = \log(z)^\ell$  for  $\ell > 0$  satisfy the assumptions of Corollary 2.4.

**Corollary 2.5.** *Let  $(Z_s; s \leq S)$  be distributed as  $\mathbb{P}_x^\theta$ . Then*

- $E[Y_S^b] = x^b \frac{\cos(\theta)}{\cos(\theta - b\pi/2)}$  for  $b \in ]-1 + 2\theta/\pi, 1 + 2\theta/\pi[$ ,
- $\mathbb{E}[\log(Y_S)] = \log(x) - \frac{\pi}{2} \tan(\theta)$ ,
- $\mathbb{E}[(\log(x^{-1}Y_S))^2] = \frac{\pi^2}{4}(1 + 2 \tan^2(\theta))$ .

*Proof.* The calculation of  $\mathbb{E}[Y_S^b]$  is immediate. Using the scaling property one only needs to do the next calculations when  $x = 1$ . Now, for all  $\ell > 0$  and  $x = 1$ ,

$$\mathbb{E}[\mathcal{R}((\log(Y_S) + i\pi/2)^\ell e^{-i\theta})] = 0.$$

Applying this identity for  $\ell = 1$ , we get the value of  $\mathbb{E}[\log(Y_S)]$ . For  $\ell = 2$ , we get

$$\mathbb{E}[(\log(Y_S))^2 - (\pi/2)^2] \cos(\theta) + \pi \log(Y_S) \sin(\theta) = 0.$$

Thus

$$\begin{aligned} \mathbb{E}[(\log(Y_S))^2] &= (\pi/2)^2 - \pi \mathbb{E}[\log(Y_S)] \tan(\theta) \\ &= (\pi/2)^2 + 2(\pi/2)^2 (\tan(\theta))^2. \end{aligned}$$

$\square$

**2.2. Brownian motion on the quadrant with time dependent reflections.** In all this section, we fix  $z = (x, 0)$  with  $x > 0$ , and  $\theta_{min} \in ]0, \frac{\pi}{2}[$ . Suppose we are given on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  a sequence of random variables  $(\Theta_n)_{n \geq 0}$  and a sequence of processes  $(Z^n)_{n \geq 1}$ , with  $Z^n = (Z_t^n = (X_t^n, Y_t^n); t \leq S_n)$ , such that :

- (i) With probability 1, for all  $n \geq 0$ ,  $\Theta_n \in ]\theta_{min}, \frac{\pi}{2}[$ .
- (ii) Set  $U_0 = x$  and for  $n \geq 1$ ,  $U_n = Y_{S_n}^n$ . Set also for  $n \geq 0$ ,

$$\mathcal{G}_n = \sigma((\Theta_k, Z^k); 1 \leq k \leq n) \vee \sigma(\Theta_0).$$

Then given  $\mathcal{G}_n$ ,  $Z^{n+1}$  is distributed as  $\mathbb{P}_{U_n}^{\Theta_n}$ .

Define for  $\theta \in ]0, \pi/2[$ ,

$$v_1(\theta) = (-\tan(\theta), 1) \text{ and } v_2(\theta) = (1, -\tan(\theta)).$$

Our purpose in this section and in section 2.3 is to construct a process  $Z = (X, Y)$ , a reflected Brownian motion in  $\mathcal{Q}$  stopped at time  $\sigma_0$ , the first hitting time of  $(0, 0)$  by  $Z$ .

Set  $T_0 = 0$  and  $T_n = \sum_{k=1}^n S_k$  for  $n \geq 1$ . For  $n \geq 0$ , set

$$\begin{aligned} Z_t &= (X_{t-T_{2n}}^{2n+1}, Y_{t-T_{2n}}^{2n+1}) \quad \text{for all } t \in [T_{2n}, T_{2n+1}[ , \\ Z_t &= (Y_{t-T_{2n+1}}^{2n+2}, X_{t-T_{2n+1}}^{2n+2}) \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[ . \end{aligned}$$

Using this procedure, we have defined a process  $(Z_t; t < T_\infty)$ , where  $T_\infty = \lim_{n \rightarrow \infty} T_n$ . Set for  $t \geq T_\infty$ ,  $Z_t = (0, 0)$ . Then, by construction,  $T_\infty = \sigma_0$ . It will be checked in section 2.3 (see Corollary 2.7) that  $Z$  is a continuous process.

Note that there exists  $B$  a two-dimensional Brownian motion such that for  $n \geq 0$ ,

$$\begin{aligned} dZ_t &= dB_t + v_1(\Theta_{2n})dL_t^1 \quad \text{for all } t \in [T_{2n}, T_{2n+1}[ , \\ dZ_t &= dB_t + v_2(\Theta_{2n+1})dL_t^2 \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[ , \end{aligned}$$

with  $L^1$  and  $L^2$  being the local times processes of  $X$  and  $Y$ . Define  $(v_t; t < \sigma_0)$  by: for  $n \geq 0$

$$\begin{aligned} v_t &= v_1(\Theta_{2n}) \quad \text{for all } t \in [T_{2n}, T_{2n+1}[ , \\ v_t &= v_2(\Theta_{2n+1}) \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[ . \end{aligned}$$

Then for all  $t < \sigma_0$ ,

$$(7) \quad Z_t = Z_0 + B_t + \int_0^t v_s dL_s$$

where  $Z_0 = (x, 0)$  and  $L = L^1 + L^2$  is the accumulated local time at  $\partial\mathcal{Q}$  until  $t$ .

The purpose of the following sections is to answer the questions (I) and (II) addressed in the beginning of section 2.

**2.3. The corner is reached.** For  $a \geq 0$ , denote  $\sigma_a := \inf\{t; |Z_t| = a\}$ . Following [16], we will first prove that  $\mathbb{P}(\sigma_0 \wedge \sigma_K < \infty) = 1$  for all  $K > x$ . This is the major difficulty we encountered here although the proof when the angles of reflections remain constant on each boundary is quite easy (Lemma 2.1 [16]). The main idea is inspired from [2]. Define for  $n \geq 1$ ,  $V_n = \frac{U_n}{U_{n-1}}$ . Then using the scaling property (Proposition 2.1) and the strong Markov property, we have that for all  $n \geq 0$ , given  $\mathcal{G}_n$ ,  $V_{n+1}$  is distributed as  $\tilde{Y}_{\tilde{S}}$ , where  $((\tilde{X}_t, \tilde{Y}_t); t \leq \tilde{S})$  has law  $\mathbb{P}_1^{\Theta_n}$ .

**Lemma 2.6.** *With probability 1,  $\sum_{n \geq 0} U_n$  is finite.*

*Proof.* For all  $n \geq 1$ , we have that

$$U_n = x \exp \left( \sum_{k=1}^n \log(V_k) \right).$$

We denote by  $\mathbb{E}_{\mathcal{G}_k}$  the conditional expectation with respect to  $\mathcal{G}_k$ . By Corollary 2.5, for all  $k \geq 1$ ,  $\mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)] = -\frac{\pi}{2} \tan(\Theta_{k-1})$  and  $\mathbb{E}_{\mathcal{G}_{k-1}}[(\log(V_k))^2] = \frac{\pi^2}{4}(1 + 2 \tan^2(\Theta_{k-1}))$ . Note now that

$$\sum_{k=1}^n \log(V_k) = M_n + \sum_{k=1}^n \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)]$$

where  $M_n := \sum_{k=1}^n (\log(V_k) - \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)])$  is a martingale. Denote by  $\langle M \rangle_n$  its quadratic variation given by

$$\sum_{k=1}^n \mathbb{E}_{\mathcal{G}_{k-1}}[(\log(V_k) - \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)])^2] = \sum_{k=1}^n \frac{\pi^2}{4}(1 + \tan^2(\Theta_{k-1})).$$

Thus  $\langle M \rangle_\infty = \infty$  and a.s.  $\lim_{n \rightarrow \infty} M_n/n = 0$ . Since  $\inf_{k \geq 0} \Theta_k \geq \theta_{\min} > 0$ , this easily implies the lemma.  $\square$

A first consequence of Lemma 2.6 is

**Corollary 2.7.** *With probability 1,  $\lim_{t \uparrow \sigma_0} Z_t = 0$ .*

*Proof.* For  $\epsilon > 0$  and  $n \geq 0$ , set

$$A_n^\epsilon = \left\{ \sup_{s \in [T_n, T_{n+1}]} |Z_s| > \epsilon \right\}.$$

By Lemma 2.2 (i), with  $b = 1$ , for all  $n \geq 0$ ,

$$\mathbb{P}(A_n^\epsilon | \mathcal{G}_n) \leq \sup_{\theta \in [\theta_{\min}, \frac{\pi}{2}[} \cotan(\theta) U_n = \cotan(\theta_{\min}) U_n.$$

Thus by Lemma 2.6  $\sum_n \mathbb{P}(A_n^\epsilon | \mathcal{G}_n) < \infty$  and the corollary follows by applying the conditional Borel-Cantelli lemma.  $\square$

Lemma 2.6 will be also used to prove

**Lemma 2.8.** *For all  $K > x$ ,  $\mathbb{P}(\sigma_0 \wedge \sigma_K < \infty) = 1$ .*

*Proof.* For all  $n \geq 0$  and  $t \in [0, S_{n+1}]$ , set

$$W_t^{n+1} = \cos(\Theta_n)(X_t^{n+1} - U_n) + \sin(\Theta_n)Y_t^{n+1}$$

Recall  $\sigma_0 = \lim_{n \rightarrow \infty} T_n$ . Define the continuous process  $(W_t; t \leq \sigma_0)$  such that  $W_0 = 0$  and for  $n \geq 0$  and  $t \in [T_n, T_{n+1}]$ ,  $W_t = W_{t-T_n}^{n+1} + W_{T_n}$ . Then, it is straightforward to check that  $(W_t; t \leq \sigma_0)$  is a Brownian

motion stopped at  $\sigma_0$ . Since for all  $n \geq 0$ ,  $U_n \geq 0$  and  $\Theta_n \in ]0, \pi/2[$ , we get that on the event  $\{\sigma_K \geq T_{n+1}\}$ ,

$$\sup_{t \in [T_n, T_{n+1}]} W_t \leq 2K + W_{T_n}.$$

Thus, on  $\{\sigma_K = \infty\}$ ,  $\sup_{t \leq \sigma_0} W_t \leq 2K + \sup_{n \geq 0} W_{T_n}$ . Now for all  $n \geq 0$ ,  $W_{S_{n+1}}^{n+1} = \sin(\Theta_n)U_{n+1} - \cos(\Theta_n)U_n \leq U_{n+1}$ . Note that for all  $n \geq 0$ ,

$$W_{T_{n+1}} - W_{T_n} = W_{S_{n+1}}^{n+1}.$$

This implies that on the event  $\{\sigma_K = \infty\}$ ,  $\sup_{t \leq \sigma_0} W_t \leq 2K + \sum_{n \geq 0} U_n$ , which is a.s. finite using Lemma 2.6. This shows that a.s.  $\{\sigma_K = \infty\} \subset \{\sigma_0 < \infty\}$  and finishes the proof.  $\square$

And following [16], we can prove

**Theorem 2.9.** *With probability 1, we have  $\sigma_0 < \infty$ .*

*Proof.* Set  $b = \frac{4\theta_{min}}{\pi}$ . Let  $\phi(x, y) = \mathcal{R}((x + iy)^b e^{-i\theta_{min}})$ , then  $\phi$  is harmonic on some open set  $U$  containing  $\mathcal{Q}^*$ . Using  $b = \frac{4\theta_{min}}{\pi}$ , we have that

$$\begin{aligned} \nabla\phi(x, 0) &= bx^{b-1}(\cos(\theta_{min}), \sin(\theta_{min})), \\ \nabla\phi(0, y) &= by^{b-1}(\sin(\theta_{min}), \cos(\theta_{min})). \end{aligned}$$

Thus for all  $t < \sigma_0$  with  $Z_t \in \partial\mathcal{Q}$ , we have  $v_t \cdot \nabla\phi(Z_t) \leq 0$ . It follows by (7) and Itô's formula that for all  $0 < \epsilon < x < K$  and  $t \geq 0$ ,

$$\mathbb{E}[\phi(Z_{t \wedge \sigma_\epsilon \wedge \sigma_K})] \leq \phi(x, 0).$$

Letting  $t \rightarrow \infty$  and using dominated convergence, we deduce

$$\mathbb{E}[\phi(Z_{\sigma_\epsilon \wedge \sigma_K})] \leq \phi(x, 0).$$

Obviously  $\phi(z) \geq \cos(\theta_{min})|z|^b$  for all  $z \in \mathcal{Q}$ . Setting  $p_{\epsilon, K} = \mathbb{P}(\sigma_\epsilon < \sigma_K)$ , we get

$$\cos(\theta_{min})(\epsilon^b p_{\epsilon, K} + K^b(1 - p_{\epsilon, K})) \leq x^b.$$

From this, we deduce

$$p_{\epsilon, K} \geq \frac{(K^b - x^b / \cos(\theta_{min}))}{K^b - \epsilon^b}.$$

As in [16], since  $\sigma_0 \wedge \sigma_K < \infty$ ,  $\lim_{\epsilon \rightarrow 0} p_{\epsilon, K} = \mathbb{P}(\sigma_0 < \sigma_K)$ , this yields

$$(8) \quad \mathbb{P}(\sigma_0 < \sigma_K) \geq 1 - \frac{x^b}{K^b \cos(\theta_{min})}.$$

Letting  $K \rightarrow \infty$ , it comes that  $\mathbb{P}(\sigma_0 < \infty) = 1$ .  $\square$

**Remark 2.10.** Using the inclusion  $\{\sup_{t < \sigma_0} |Z_t| > \epsilon\} \subset \{\sigma_\epsilon < \sigma_0\}$  and (8), we deduce that for all  $\epsilon > 0$ ,

$$(9) \quad \lim_{x \rightarrow 0^+} \mathbb{P}(\sup_{t < \sigma_0} |Z_t| > \epsilon) = 0$$

This fact will be used in section 3.

**2.4. The local time process.** Following Williams [17], we prove in this section that

**Theorem 2.11.** *With probability 1,  $L_{\sigma_0} := \lim_{t \uparrow \sigma_0} L_t$  is finite.*

*Proof.* In what follows, we refer to the proof of Theorem 1 in [17] for more details. Let  $\tilde{\theta} \in ]0, \theta_{min} \wedge \pi/4[$  and set  $\tilde{b} = \frac{4\tilde{\theta}}{\pi}$ . Let  $\tilde{\phi}$  be defined as the function  $\phi$  in the proof of Theorem 2.9, with the parameters  $(b, \theta_{min})$  replaced by  $(\tilde{b}, \tilde{\theta})$ . Then there exists  $c > 0$  such that for all  $t$  for which  $Z_t \in \partial\mathcal{Q}$ , we have  $v_t \cdot \nabla \phi(Z_t) \leq -c|Z_t|^{\tilde{b}-1}$ . For each  $\gamma > 0$ , define  $f_\gamma = e^{-\gamma\phi}$ . Then  $f_\gamma$  is twice continuously differentiable in  $\mathcal{Q}^*$  and

$$\Delta f_\gamma(z) = \gamma^2 f_\gamma(z) (\tilde{b}|z|^{\tilde{b}-1})^2 \text{ for } z \in \mathcal{Q}^*.$$

Moreover for all  $t$  such that  $Z_t \in \partial\mathcal{Q}$ ,

$$v_t \cdot \nabla f_\gamma(Z_t) = -\gamma f_\gamma(Z_t) (v_t \cdot \nabla \phi(Z_t)).$$

For  $t < \sigma_0$ , set

$$A_t = -\gamma \int_0^t (v_s \cdot \nabla \phi(Z_s)) dL_s + \frac{\gamma^2}{2} \int_0^t (\tilde{b}|Z_s|^{\tilde{b}-1})^2 ds.$$

and  $A_{\sigma_0} = \lim_{t \uparrow \sigma_0} A_t$ . Then

$$A_{\sigma_0} \geq c\gamma \int_0^{\sigma_0} |Z_s|^{\tilde{b}-1} dL_s + \frac{\gamma^2}{2} \int_0^{\sigma_0} (\tilde{b}|Z_s|^{\tilde{b}-1})^2 ds.$$

Itô's formula implies that for  $t < \sigma_0$ ,

$$f_\gamma(Z_t) e^{-A_t} = f_\gamma(Z_0) + \int_0^t e^{-A_s} (\nabla f_\gamma(Z_s) \cdot dB_s).$$

Taking the expectation, we get

$$\mathbb{E} \left[ \exp \left( -c\gamma \int_0^{\sigma_0} |Z_s|^{\tilde{b}-1} dL_s \right) \right] \geq f_\gamma(Z_0).$$

This easily implies that for all  $r > 0$ ,

$$\mathbb{E} \left[ \exp \left( -\gamma cr^{\tilde{b}-1} \int_0^{\sigma_0} 1_{\{|Z_s| \leq r\}} dL_s \right) \right] \geq f_\gamma(Z_0).$$



Letting  $\gamma \downarrow 0$ , we get a.s.

$$(10) \quad \int_0^{\sigma_0} 1_{\{|Z_s| \leq r\}} dL_s < \infty.$$

Let  $S_r = \sup\{t \geq 0 : |Z_t| > r\}$ , then by the continuity of  $Z$ ,  $S_r < \sigma_0$  and thus  $L_{S_r} < \infty$ . By combining this with (10), we get  $L_{\sigma_0} < \infty$ .  $\square$

**2.5. On the integrability of  $L_{\sigma_0}$ .** In this section, Proposition 1.8 is proved. We use the notation of section 2.2 in which the process  $Z$  is constructed. Note that  $L_{\sigma_0} = \sum_{n=1}^{\infty} L_{S_n}^n$ , where  $L^n$  is the local time at 0 of  $Y^n$  and where  $Z^n = (X^n, Y^n)$ . Recall that for  $n \geq 0$ , given  $\mathcal{G}_n$ , the law of  $Z^{n+1}$  is  $\mathbb{P}_{U_n}^{\Theta_n}$ , where  $U_0 = x$  and  $U_n = Y_{S_n}^n$  for  $n \geq 1$ .

Let  $Z^0 = (X_t^0, Y_t^0)_{t \leq S^0}$  be a process of law  $\mathbb{P}_x^\theta$ . Then, if  $L_t^0 = L_t(X^0)$ , for all  $t \geq 0$ ,

$$Y_{t \wedge S^0}^0 = B_{t \wedge S^0}^2 + L_{t \wedge S^0}^0$$

where  $(B_{t \wedge S^0}^2)_t$  is a Brownian motion stopped at time  $S^0$ . Thus

$$\mathbb{E}[Y_{t \wedge S^0}^0] = \mathbb{E}[L_{t \wedge S^0}^0].$$

Taking the limit as  $t \rightarrow \infty$  and using Corollary 2.3 leads to  $\mathbb{E}[L_{S^0}^0] = \mathbb{E}[Y_{S^0}^0]$ . But  $\mathbb{E}[Y_{S^0}^0] = x \cotan(\theta)$  by Corollary 2.5. This implies that

$$\mathbb{E}[L_{S_{n+1}}^{n+1} | \mathcal{G}_n] = U_n \cotan(\Theta_n).$$

Consequently

$$\mathbb{E}[L_{\sigma_0}] = \sum_{n \geq 0} \mathbb{E}[U_n \cotan(\Theta_n)].$$

Assume that for all  $n$ ,  $U_n$  and  $\Theta_n$  are independent, then

$$\mathbb{E}[U_n \cotan(\Theta_n)] = \mathbb{E}[\cotan(\Theta_n)] \mathbb{E}[U_n] = \cdots = x \prod_{k=0}^n \mathbb{E}[\cotan(\Theta_k)].$$

Therefore

$$\mathbb{E}[L_{\sigma_0}] = x \sum_{n \geq 0} \prod_{k=0}^n \mathbb{E}[\cotan(\Theta_k)].$$

This gives a necessary and sufficient condition to get  $\mathbb{E}[L_{\sigma_0}] < \infty$ .

Assume that  $\Theta_n = \theta \in ]0, \pi/2[$  for all  $n$ , we get

$$\mathbb{E}[L_{\sigma_0}] = x \sum_{n \geq 0} (\cotan(\theta))^{n+1}$$

which is finite if and only if  $\theta \in ]\pi/4, \pi/2[$ . In this case

$$\mathbb{E}[L_{\sigma_0}] = \frac{x \cotan(\theta)}{1 - \cotan(\theta)} = \frac{x}{\tan(\theta) - 1}.$$

Assume that  $\Theta_{2n} = \theta_1$  and  $\Theta_{2n+1} = \theta_2$ . Set  $c_1 = \cotan(\theta_1)$  and  $c_2 = \cotan(\theta_2)$ . Then

$$\begin{aligned}\mathbb{E}[L_{\sigma_0}] &= x(c_1 + c_1c_2 + c_1^2c_2 + c_1^2c_2^2 + \cdots) \\ &= c_1(1 + c_2 + c_1c_2 + c_1c_2^2 + \cdots) \\ &= c_1((1 + c_2) + (1 + c_2)c_1c_2 + \cdots)\end{aligned}$$

which is finite if and only if  $c_1c_2 < 1$ . In this case, we have

$$\mathbb{E}[L_{\sigma_0}] = \frac{xc_1(1 + c_2)}{1 - c_1c_2}.$$

Proposition 1.8 is proved.

### 3. PROOF OF THEOREM 1.2

Theorem 1.2 (i) is proved in section 3.1. For the construction of a solution, we will use Freidlin-Sheu formula for the Walsh's Brownian motion (see Theorem 3.1 below). The uniqueness in law of the solution of the SDE  $(E)$  will follow from the fact that the Walsh's Brownian motion is the unique solution of a martingale problem.

Theorem 1.2 (ii) is proved in section 3.2. To prove pathwise uniqueness for  $(E)$  when  $N = 2$ , we proceed as in [4] using the local times techniques introduced in [8, 13]. The fact that the solution of  $(E)$  is not a strong solution when  $N \geq 3$  is a consequence of a Theorem by Tsirelson (see Theorem 3.6 below).

We prove Theorem 1.2 only for  $x = 0$ , the case  $x \neq 0$  following easily.

**3.1. Proof of Theorem 1.2 (i).** Let us recall Freidlin-Sheu formula (see [5] and also Theorem 3 in [6]).

**Theorem 3.1.** [5] *Let  $(X_t)_{t \geq 0}$  be a Walsh's Brownian motion on  $G$  started from  $X_0$  and  $B_t^X = |X|_t - |X_0| - L_t(|X|)$ . Then  $B^X$  is a Brownian motion and for all  $f \in C_b^2(G^*)$ , we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dB_s^X + \frac{1}{2} \int_0^t f''(X_s)ds + f'(0)L_t(|X|).$$

We call  $B^X$  the Brownian motion associated to  $X$ .

Remark that in this formula the local martingale part of  $f(X_t)$  is always a stochastic integral with respect to  $B^X$ . This is an expected fact since  $B^X$  has the martingale representation property for  $(\mathcal{F}_t^X)_t$  (Theorem 4.1 [1]). This martingale representation property will be used to prove the uniqueness in law of the solutions to  $(E)$ . Theorem 1.2 (i) is proved in sections 3.1.1 and 3.1.2. Theorem 1.2 (ii) is proved in sections 3.2.1 and 3.2.2.

3.1.1. *Construction of a solution to (E).* Let  $X$  be a Walsh's Brownian motion with  $X_0 = 0$  and let  $B^X$  be the Brownian motion associated to  $X$ . Take a  $N$ -dimensional Brownian motion  $V = (V^1, \dots, V^N)$  independent of  $X$ . Let  $(\mathcal{F}_t)$  denote the filtration generated by  $X$  and  $V$ . For  $i \in [1, N]$ , define

$$W_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \int_0^t 1_{\{X_s \notin E_i\}} dV_s^i.$$

Then  $W := (W^1, \dots, W^N)$  is a  $N$ -dimensional  $(\mathcal{F}_t)$ -Brownian motion by Lévy's theorem and

$$B_t^X = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}} dW_s^i.$$

Then, using Theorem 3.1,  $(X, W)$  solves (E). Denote by  $\mu$  the law of  $(X, W)$ .

3.1.2. *Uniqueness in law.* To prove the uniqueness in law, we will apply the two following lemmas. The first Lemma states that the Walsh's Brownian motion is the unique solution of a martingale problem. The second Lemma gives conditions that ensure that a Walsh's Brownian motion is independent of a given family of Brownian motions.

**Lemma 3.2.** *Let  $(\mathcal{F}_t)$  be a filtration and let  $X$  be a  $G$ -valued  $(\mathcal{F}_t)$ -adapted and continuous process such that for all  $f \in \mathcal{D}$ ,*

$$(11) \quad M_t^f := f(X_t) - f(x) - \frac{1}{2} \int_0^t f''(X_s) ds.$$

*is a martingale with respect to  $(\mathcal{F}_t)$ , then  $X$  is a  $(\mathcal{F}_t)$ -Walsh's Brownian motion.*

*Proof.* We exactly follow the proof of Theorem 3.2 of [1] and only check that with our conventions for  $f'(0)$  and  $f''(0)$  when  $f \in \mathcal{D}$ , we avoid all trivial solutions to the previous martingale problem (with the hypothesis of Theorem 3.2 of [1], the trivial process  $X_t = 0$  is a possible solution of the martingale problem (3.3) in [1]). For  $i \in [1, N]$ , set  $q_i = 1 - p_i$  and let  $f_i$  and  $g_i$  be defined by

$$\begin{aligned} f_i(x) &= q_i |x| 1_{\{x \in E_i\}} - p_i |x| 1_{\{x \notin E_i\}} \\ g_i(x) &= (f_i(x))^2 = q_i^2 |x|^2 1_{\{x \in E_i\}} + p_i^2 |x|^2 1_{\{x \notin E_i\}}. \end{aligned}$$

Then  $f_i$  and  $g_i$  are  $C^2$  on  $G^*$ . We have  $f_i'(x) = q_i$  for  $x \in E_i^*$ ,  $f_i'(x) = -p_i$  for  $x \notin E_i$  and  $f_i'(0) = 0$ . Moreover, for all  $x \in G$ ,  $f_i''(x) = 0$ . We also have  $g_i'(x) = 2q_i^2 |x|$  for  $x \in E_i^*$ ,  $g_i'(x) = 2p_i^2 |x|$  for  $x \notin E_i$  and  $g_i'(0) = 0$ . Moreover,  $g_i''(x) = 2q_i^2$  for  $x \in E_i^*$ ,  $g_i''(x) = 2p_i^2$  for  $x \notin E_i$

and  $g_i''(0) = 2p_iq_i$ . Set  $Y_t^i := f_i(Z_t)$ . Although  $f_i$  is not bounded, by a localization argument, we have that  $Y_t^i$  is a local martingale. Using the function  $g_i$ , we also have that  $(Y_t^i)^2 - \frac{1}{2} \int_0^t g_i''(Z_s) ds$  is a local martingale. Thus

$$\langle Y^i \rangle_t = \int_0^t (q_i^2 1_{\{Z_s \in E_i^*\}} + p_i^2 1_{\{Z_s \notin E_i\}} + p_i q_i 1_{\{Z_s = 0\}}) ds.$$

Set

$$U_t^i = \int_0^t (q_i^{-1} 1_{\{Y_s^i > 0\}} + p_i^{-1} 1_{\{Y_s^i < 0\}} + (p_i q_i)^{-1/2} 1_{\{Y_s^i = 0\}}) dY_s^i.$$

Then  $U_t^i$  is a local martingale with  $\langle U^i \rangle_t = t$ ; that is  $U_t^i$  is a Brownian motion. Let  $\phi(y) = q_i^{-1} 1_{\{y > 0\}} + p_i^{-1} 1_{\{y < 0\}} + \sqrt{p_i q_i} 1_{\{y = 0\}}$ . Then  $Y_t^i$  is a solution of the stochastic differential equation

$$Y_t^i = Y_0^i + \int_0^t \phi(Y_s^i) dU_s^i.$$

As in [1], the solution of this SDE is pathwise unique and following the end of the proof of Theorem 3.2 of [1], we arrive at

$$\mathbb{E}[f(Z_t) | \mathcal{F}_s] = P_{t-s} f(Z_s)$$

for all  $s \leq t$  and  $f : G \rightarrow \mathbb{R}$  a bounded measurable where  $P_t$  is the semigroup of Walsh's Brownian motion.  $\square$

**Lemma 3.3.** *Let  $(\mathcal{G}_t)$  be a filtration. Let  $X$  be a  $(\mathcal{G}_t)$ -Walsh's Brownian motion,  $B^X$  its associated Brownian motion and  $B = (B^1, \dots, B^d)$  be a  $(\mathcal{G}_t)$ -Brownian motion in  $\mathbb{R}^d$ , with  $d \geq 1$ . If  $B$  and  $B^X$  are independent, then  $X$  and  $B$  are independent.*

*Proof.* Let  $U$  be a bounded  $\sigma(B)$ -measurable random variable. Then

$$U = \mathbb{E}[U] + \sum_{i=1}^d \int_0^\infty H_s^i dB_s^i$$

where  $H^i$  predictable for the filtration  $\mathcal{F}^B$  and  $E[\int_0^\infty (H_s^i)^2 ds] < \infty$ . Let  $U'$  be a bounded  $\sigma(X)$ -measurable random variable. Since  $B^X$  has the martingale representation property for  $\mathcal{F}^X$  (Theorem 4.1 [1]), we deduce that

$$U' = \mathbb{E}[U'] + \int_0^\infty H_s dB_s^X$$

with  $H$  predictable for  $\mathcal{F}^X$  and  $E[\int_0^\infty (H_s)^2 ds] < \infty$ . Then  $H$  and  $(H^i)_{1 \leq i \leq d}$  are also predictable for  $(\mathcal{G}_t)$ . It is also easy to check that  $B^X$

is a  $(\mathcal{G}_t)$ -Brownian motion. Now

$$\begin{aligned}\mathbb{E}[UU'] &= \mathbb{E}[U]\mathbb{E}[U'] + \mathbb{E}\left[\sum_{i=1}^d \int_0^\infty H_s^i dB_s^i \int_0^\infty H_s dB_s^X\right] \\ &= \mathbb{E}[U]\mathbb{E}[U'] + \sum_{i=1}^d \mathbb{E}\left[\int_0^\infty H_s^i H_s d\langle B^i, B^X \rangle_s\right] \\ &= \mathbb{E}[U]\mathbb{E}[U'].\end{aligned}$$

□

Let  $(X, W)$  be a solution of  $(E)$ , defined on a filtered probability space  $(\Omega, (\mathcal{F}_t), \mathbb{P})$ , and such that  $X_0 = 0$ . Without loss of generality, we can assume that  $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$ . For all  $f \in \mathcal{D}$ ,  $\sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i$  is a martingale, and therefore  $X$  is a solution to the martingale problem of Lemma 3.2. Thus  $X$  is a Walsh's Brownian motion. Let  $B$  be a Brownian motion independent of  $(X, W)$ , denote by  $B^X$  the Brownian motion associated to  $X$  and set  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^B$ . Note that  $B^X$  is a  $(\mathcal{G}_t)$ -Brownian motion. For  $i \in [1, N]$ , define

$$V_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s + \int_0^t 1_{\{X_s \notin E_i\}} dW_s^i.$$

Then  $V := (V^1, \dots, V^N)$  is a  $N$ -dimensional  $(\mathcal{G}_t)$ -Brownian motion independent of  $B^X$ . By the previous Lemma  $V$  is also independent of  $X$ . It is easy to check that for all  $i \in [1, N]$ ,

$$W_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \int_0^t 1_{\{X_s \notin E_i\}} dV_s^i.$$

This proves that the law of  $(X, W)$  is  $\mu$ .

### 3.2. Proof of Theorem 1.2 (ii).

3.2.1. *The case  $N = 2$ .* To prove that the solution is a strong one, it suffices to prove that pathwise uniqueness holds for  $(E)$ . Fix  $p \in ]0, 1[$ , and set  $\beta = \frac{1-p}{p}$ .

**Lemma 3.4.** *Let  $B^+$  and  $B^-$  be two independent Brownian motions. Let also  $X$  and  $Y$  be two continuous processes, with  $Y_t = \beta X_t 1_{\{X_t \geq 0\}} + X_t 1_{\{X_t < 0\}}$ . Then  $(X, B^+, B^-)$  is a solution to  $(E)$  or equivalently of*

$$(12) \quad dX_t = 1_{\{X_t > 0\}} dB_t^+ + 1_{\{X_t \leq 0\}} dB_t^- + (2p - 1)dL_t(X)$$

*if and only if  $(Y, B^+, B^-)$  is a solution of the following SDE*

$$(13) \quad dY_t = \beta 1_{\{Y_t > 0\}} dB_t^+ + 1_{\{Y_t \leq 0\}} dB_t^-.$$

*Proof.* Suppose  $(X, B^+, B^-)$  solves (12). Set  $B_t = \int_0^t 1_{\{X_s > 0\}} dB_s^+ + 1_{\{X_s \leq 0\}} dB_s^-$ . Then  $B_t$  is a Brownian motion, and  $(X, B)$  is a solution of the SDE  $X_t = B_t + (2p - 1)L_t(X)$ . It well known (see for example section 5.2 in the survey [11]) that  $(Y, B)$  solves

$$dY_t = \beta 1_{\{Y_t > 0\}} dB_t + 1_{\{Y_t \leq 0\}} dB_t$$

and thus that  $(Y, B^+, B^-)$  solves (13). The converse can be proved in the same way.  $\square$

**Proposition 3.5.** *Pathwise uniqueness holds for (E).*

*Proof.* Lemma 3.4 implies that the proposition holds if pathwise uniqueness holds for (13). Let  $(X, B^+, B^-)$  and  $(Y, B^+, B^-)$  be two solutions of (13) with  $X_0 = Y_0 = 0$ . Set  $\text{sgn}(x) = 1_{\{x > 0\}} - 1_{\{x < 0\}}$ . We shall use the same techniques as in [4] (see also [8] and [13]) and first prove that a.s.

$$(14) \quad \int_{]0, +\infty[} \frac{L_t^a(X - Y)}{a} da < \infty.$$

By the occupation times formula

$$\int_{]0, +\infty[} \frac{L_t^a(X - Y)}{a} da = \int_0^t 1_{\{X_s - Y_s > 0\}} \frac{d\langle X - Y \rangle_s}{X_s - Y_s}.$$

It is easily verified that

$$d\langle X - Y \rangle_s \leq C |\text{sgn}(X_s) - \text{sgn}(Y_s)| ds$$

where  $C = (1 + \beta^2)/2$ . Let  $(f_n)_n \subset C^1(\mathbb{R})$  such that  $f_n \rightarrow \text{sgn}$  pointwise and  $(f_n)_n$  is uniformly bounded in total variation. By Fatou's Lemma, we get

$$\begin{aligned} \int_{]0, +\infty[} \frac{L_t^a(X - Y)}{a} da &\leq C \liminf_n \int_0^t 1_{\{X_s - Y_s > 0\}} \frac{|f_n(X_s) - f_n(Y_s)|}{X_s - Y_s} ds \\ &\leq C \liminf_n \int_0^t 1_{\{X_s - Y_s > 0\}} \left| \int_0^1 f_n'(Z_s^u) du \right| ds \end{aligned}$$

where

$$Z_s^u = (1 - u)X_s + uY_s.$$

It is easy to check the existence of a constant  $A > 0$  such that for all  $s \geq 0$  and  $u \in [0, 1]$ ,  $\frac{d}{du} \langle Z^u \rangle_s \geq A^{-1}$ . Hence, setting  $C' = A \times C$ , we have

$$\begin{aligned} \int_{]0, +\infty[} \frac{L_t^a(X - Y)}{a} da &\leq C' \liminf_n \int_0^1 \int_0^t |f_n'(Z_s^u)| d\langle Z^u \rangle_s du \\ &\leq C' \liminf_n \int_0^1 \int_{\mathbb{R}} |f_n'(a)| L_t^a(Z^u) da du. \end{aligned}$$

Now taking the expectation and using Fatou's Lemma, we get

$$\mathbb{E} \left[ \int_{]0,+\infty[} \frac{L_t^a(X-Y)}{a} da \right] \leq C' \liminf_n \int_{\mathbb{R}} |f'_n(a)| da \sup_{a \in \mathbb{R}, u \in [0,1]} \mathbb{E}[L_t^a(Z^u)].$$

It remains to prove that  $\sup_{a \in \mathbb{R}, u \in [0,1]} \mathbb{E}[L_t^a(Z^u)] < \infty$ . By Tanaka's formula, we have

$$\begin{aligned} \mathbb{E}[L_t^a(Z^u)] &= \mathbb{E}[|Z_t^u - a|] - \mathbb{E}[|Z_0^u - a|] - \mathbb{E} \left[ \int_0^t \text{sgn}(Z_s^u - a) dZ_s^u \right] \\ &\leq E[|Z_t^u - Z_0^u|] \end{aligned}$$

It is easy to check that the right-hand side is uniformly bounded with respect to  $(a, u)$  which permits to deduce (14). Consequently  $L_t^a(X - Y) = 0$  and thus by Tanaka's formula,  $|X - Y|$  is a local martingale which is also a nonnegative supermartingale, with  $|X_0 - Y_0| = 0$  and finally  $X$  and  $Y$  are indistinguishable.  $\square$

3.2.2. *The case  $N \geq 3$ .* Let  $(X, W)$  be a solution to  $(E)$ . Then  $X$  is a  $(\mathcal{F}_t)$ -Walsh's Brownian motion, where  $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$ . If  $(X, W)$  is a strong solution we thus have that  $X$  is a  $(\mathcal{F}_t^W)$ -Walsh's Brownian motion, which is impossible when  $N \geq 3$  because of the following Tsirelson's theorem :

**Theorem 3.6.** [6] *There does not exist any  $(\mathcal{G}_t)_t$ -Walsh's Brownian motion on a star graph with three or more rays with  $(\mathcal{G}_t)_t$  a Brownian filtration.*

#### 4. PROOF OF THEOREM 1.4

In this section, we prove assertions (i) and (ii) of Theorem 1.4. We first construct a coalescing SFM solution of  $(E)$ . To construct this SFM, we will use the following

**Theorem 4.1.** [10] *Let  $(P^{(n)}, n \geq 1)$  be a consistent family of Feller semigroups acting respectively on  $C_0(M^n)$  where  $M$  is a locally compact metric space such that*

$$(15) \quad P_t^{(2)} f^{\otimes 2}(x, x) = P_t^{(1)} f^2(x) \text{ for all } f \in C_0(M), x \in M, t \geq 0.$$

*Then there exists a (unique in law) SFM  $\varphi = (\varphi_{s,t})_{s \leq t}$  defined on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that*

$$P_t^{(n)} f(x) = \mathbb{E}[f(\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n))]$$

*for all  $n \geq 1, t \geq 0, f \in C_0(M^n)$  and  $x \in M^n$ .*

To apply this Theorem, we construct a consistent family of  $n$ -point motions (i.e. the Markov process associated to  $P^{(n)}$ ) up to their first coalescing times in section 4.1. After associating to the two-point motion an obliquely reflected Brownian motion in  $\mathcal{Q}$  in section 4.2, we prove the coalescing property in section 4.3 and the Feller property in section 4.4. It is then possible to apply Theorem 4.1 and as a result we get a flow  $\varphi$ . In section 4.4, we also show that  $\varphi$  solves (E). Finally, we prove in section 4.5 that  $\varphi$  is the unique SFM solving (E).

Note finally that in the case of Le Jan and Raimond [9], all the angles of reflection of the obliquely reflected Brownian motion associated to the two-point motion are equal to  $\pi/4$ . This simplifies greatly the study of section 2.

**4.1. Construction of the  $n$ -point motion up to the first coalescing time.** Fix  $x_1, \dots, x_n \in G$  such that  $|x_1| < \dots < |x_n|$  and let  $(X, W)$  be a solution of the SDE (E), with  $X_0 = x_1$ .

Set, for  $t \geq 0$ ,  $X_t^{1,0} = X_t$  and for all  $j \in [2, n]$ , if  $x_j \in E_i$ , define

$$X_t^{j,0} = e_i(|x_j| + W_t^i).$$

Set

$$\tau_1 = \inf\{t \geq 0 : \exists j \neq 1 : X_t^{j,0} = 0\}.$$

For  $t \leq \tau_1$ , set  $X_t^{(n)} = (X_t^{1,0}, \dots, X_t^{n,0})$ .

Assume now that  $(\tau_k)_{k \leq \ell}$  and  $(X_t^{(n)})_{t \leq \tau_\ell}$  have been defined such that a.s.

- $(\tau_k)_{1 \leq k \leq \ell}$  is an increasing sequence of stopping times with respect to the filtration associated to  $(X_t^{(n)})_{t \leq \tau_\ell}$ ;
- for all  $k$ , there exists an integer  $j_k$  such that  $X_{\tau_k}^{j_k} = 0$ .

Now introduce an independent solution  $(X, W)$  of the SDE (E), with  $X_0 = 0$ . Define  $(X_t^{(n)})_{t \in [\tau_\ell, \tau_{\ell+1}]}$  by analogy with the construction of  $(X_t^{(n)})_{t \in [0, \tau_1]}$  by replacing  $(x_1, \dots, x_n)$  with  $(X_{\tau_\ell}^{j_1^\ell}, \dots, X_{\tau_\ell}^{j_n^\ell})$ , where  $(j_1^\ell, \dots, j_n^\ell)$  are such that

$$0 = |X_{\tau_\ell}^{j_1^\ell}| < \dots < |X_{\tau_\ell}^{j_n^\ell}|.$$

Thus, we have defined  $X_t^{(n)}$  for all  $t < \tau_\infty$ , where  $\tau_\infty := \lim_{l \rightarrow \infty} \tau_l$ .

We denote by  $\mathbb{P}_x^{(n),0}$  the law of  $(X_t^{(n)})_{t < \tau_\infty}$ . Notice that if we denote  $X^{(n)} = (X^1, \dots, X^n)$ , then for all  $i$ ,  $(X_t^i, t \leq \tau_\infty)$  is a Walsh's Brownian motion stopped at time  $\tau_\infty$ . Note also that a.s. on the event  $\{\tau_\infty < \infty\}$ , there exist  $i \neq j$  such that  $X_{\tau_n}^i = X_{\tau_{n+1}}^j = 0$  for infinitely many  $n$ 's. This implies that a.s. on the event  $\{\tau_\infty < \infty\}$ , there exist  $i \neq j$  such that  $\lim_{t \uparrow \tau_\infty} X_t^i = \lim_{t \uparrow \tau_\infty} X_t^j = 0$ , and thus that  $\lim_{t \uparrow \tau_\infty} X_t^{(n)} \in \Delta_n$ ,



with  $\Delta_n := \{(x_1, \dots, x_n) \in G^n : \exists i \neq j, x_i = x_j\}$ . Therefore, if we denote by  $X_{\tau_\infty}^{(n)}$  this limit, then, by construction,  $\tau_\infty$  coincides with

$$(16) \quad T_{\Delta_n} = \inf\{t \geq 0 : X_t^{(n)} \in \Delta_n\}.$$

Note that in the particular case  $n = 2$ , on the event  $\{\tau_\infty < \infty\}$ , a.s.  $X_{\tau_\infty}^{(2)} = (0, 0)$ . We will prove in section 4.3 that  $\tau_\infty < \infty$  a.s.

**4.2. An obliquely reflected Brownian motion associated to the 2-point motion.** Fix  $x \in G$ , and let  $i$  such that  $x \in E_i$ . Recall the construction of  $(X, Y)$  of law  $\mathbb{P}_{(x,0)}^{(2)}$ . We have  $\tau_0 = 0$  and for  $k \geq 0$ ,

$$\begin{aligned} \tau_{2k+1} &= \inf\{t \geq \tau_{2k} : X_t = 0\}, \\ \tau_{2k+2} &= \inf\{t \geq \tau_{2k+1} : Y_t = 0\}. \end{aligned}$$

For  $n \geq 0$ , let  $i_{2n}$  and  $i_{2n+1}$  be in  $\{1, \dots, N\}$  such that  $X_{\tau_{2n}} \in E_{i_{2n}}$  and  $Y_{\tau_{2n+1}} \in E_{i_{2n+1}}$ . Then for  $n \geq 0$ ,

$$\begin{aligned} X_t &= e_{i_{2n}} \left( |X_{\tau_{2n}}| + W_t^{i_{2n}} - W_{\tau_{2n}}^{i_{2n}} \right) && \text{for } t \in [\tau_{2n}, \tau_{2n+1}[ , \\ Y_t &= e_{i_{2n+1}} \left( |Y_{\tau_{2n+1}}| + W_t^{i_{2n+1}} - W_{\tau_{2n+1}}^{i_{2n+1}} \right) && \text{for } t \in [\tau_{2n+1}, \tau_{2n+2}[ . \end{aligned}$$

Define, for  $i \in [1, N]$ ,  $f^i : G \rightarrow \mathbb{R}$  by

$$f^i(x) = -|x| \text{ if } x \in E_i \quad \text{and} \quad f^i(x) = |x| \text{ if not.}$$

Define now  $(U_t, V_t)_{t < \tau_\infty}$  such that for  $n \geq 0$

$$(U_t, V_t) = \begin{cases} (|X_t|, f^{i_{2n}}(Y_t)) & \text{for } t \in [\tau_{2n}, \tau_{2n+1}[ \\ (f^{i_{2n+1}}(X_t), |Y_t|) & \text{for } t \in [\tau_{2n+1}, \tau_{2n+2}[ \end{cases}$$

Remark that  $(U_t, V_t)_{t < \tau_\infty}$  is a continuous process with values in  $\{(u, v) \in \mathbb{R}^2 : u + v > 0\}$  and such that for all  $n \geq 0$ ,  $U_{\tau_{2n}} > 0$ ,  $V_{\tau_{2n}} = 0$ ,  $U_{\tau_{2n+1}} = 0$  and  $V_{\tau_{2n+1}} > 0$ . Note that the excursions of this process outside of  $\mathcal{Q}$  occur on straight lines parallel to  $\{y = -x\}$ .

Let, for  $n \geq 0$ ,

$$\Theta_n = \arctan \left( \frac{p_{i_n}}{1 - p_{i_n}} \right).$$

Define for  $t < \tau_\infty$ ,

$$A(t) = \int_0^t 1_{\{(U_s, V_s) \in \mathcal{Q}\}} ds = \int_0^t 1_{\{X_s \neq Y_s\}} ds.$$

Set  $\gamma(t) = \inf\{s \geq 0 : A(s) > t\}$ . Set for  $n \geq 0$ ,  $T_n = A(\tau_n)$  and  $S_{n+1} = T_{n+1} - T_n$ . Define for  $t < T_\infty := \lim_{n \rightarrow \infty} T_n$ ,

$$(U_t^r, V_t^r) = (U_{\gamma(t)}, V_{\gamma(t)})$$

and for  $t \geq T_\infty$ ,  $(U_t^r, V_t^r) = (0, 0)$ . Note that  $T_{2n+1} = \inf\{t \geq T_{2n} : V_t^r = 0\}$  and  $T_{2n+2} = \inf\{t \geq T_{2n+1} : U_t^r = 0\}$  and that  $\gamma(T_n) = \tau_n$ .

**Lemma 4.2.** *Given  $\Theta_0$ , the law of  $(U_t^r, V_t^r)_{t \leq S_1}$  is  $\mathbb{P}_{|x|}^{\Theta_0}$ .*

The proof of this lemma is given at the end of this section.

Notice that since a.s.  $|Y_{\tau_1}| = V_{T_1}^r \neq 0$ , then the sequence  $(\tau_k)_k$  defined above is a.s. strictly increasing. It is also a sequence of stopping times with respect to the filtration  $\mathcal{F}_t = \sigma((X_s, Y_s); s \leq t)$ ,  $t \geq 0$ .

Define the sequence of processes  $(Z^n)_{n \geq 1}$  such that for  $n \geq 0$ ,

$$\begin{aligned} Z^{2n+1} &= (U_{t+T_{2n}}^r, V_{t+T_{2n}}^r)_{t \leq S_{2n+1}}, \\ Z^{2n+2} &= (V_{t+T_{2n+1}}^r, U_{t+T_{2n+1}}^r)_{t \leq S_{2n+2}}. \end{aligned}$$

Set also for  $n \geq 0$ ,  $U_{2n} = U_{T_{2n}}^r$  and  $U_{2n+1} = V_{T_{2n+1}}^r$ .

Applying Lemma 4.2 and using the strong Markov property at the stopping times  $\tau_n$ , with the fact that if  $(X, Y)$  is distributed as  $\mathbb{P}_{(x,y)}^{(2),0}$ , then  $(Y, X)$  is distributed as  $\mathbb{P}_{(y,x)}^{(2),0}$ , one has the following

**Lemma 4.3.** *For all  $n \geq 0$ , given  $\mathcal{F}_{\tau_n}$ , the law of  $Z^{n+1}$  is  $\mathbb{P}_{U_n}^{\Theta_n}$ .*

This lemma shows that the sequences  $(\Theta_n)_{n \geq 0}$  and  $(Z^n)_{n \geq 1}$  satisfy (i) and (ii) in the beginning of Section 2.2 since for all  $n \geq 0$ ,

$$\mathcal{G}_n = \sigma((\Theta_k, Z^k); 1 \leq k \leq n) \vee \sigma(\Theta_0) \subset \mathcal{F}_{\tau_n}.$$

Thus  $(U_t^r, V_t^r)_{t < T_\infty}$  is a Brownian motion in  $\mathcal{Q}^*$  started from  $(|x|, 0)$ , with time dependent angle of reflections at the boundaries given by  $(\Theta_n)_{n \geq 0}$  and stopped when it hits  $(0, 0)$ , as defined in section 2. In particular,  $(U^r, V^r)$  is a continuous process and  $\lim_{t \uparrow T_\infty} (U_t^r, V_t^r) = (0, 0)$ . We will now denote the process  $(U^r, V^r)$  by  $Z$ .

**Remark 4.4.** *Note that  $(i_n)_{n \geq 0}$  is an homogeneous Markov chain started from  $i_0 = 1$  with transition matrix  $(P_{i,j})$  given by : for  $(i, j) \in [1, N]^2$ ,  $P_{i,j} = \frac{p_j}{\sum_{k \neq i} p_k}$ . Remark also that given  $\mathcal{G}_n$ ,  $Z^{n+1}$  and  $i_{n+1}$  are independent and a fortiori  $Z^{n+1}$  and  $\Theta_{n+1}$  are also independent.*

**Proof of Lemma 4.2.** Let  $i$  be such that  $x \in E_i$ . Let  $(Y, W)$  be a solution of  $(E)$  with  $Y_0 = 0$  and define  $X_t = e_i(|x| + W_t^i)$  for  $0 \leq t \leq \tau_1$  where  $\tau_1 = \inf\{s \geq 0 : |x| + W_s^i = 0\}$ . Set for  $t \geq 0$ ,  $(U_t, V_t) := (|x| + W_t^i, f^i(Y_t))$  where  $f^i(y) = |y| \mathbf{1}_{y \notin E_i} - |y| \mathbf{1}_{y \in E_i}$ . Note that for  $t \leq \tau_1$ ,  $U_t = |X_t|$ .

Since  $Y$  is a Walsh's Brownian motion started at 0, it is well known that  $V$  is a skew Brownian motion with parameter  $1 - p_i$ . This can be

seen using Freidlin-Sheu formula, which shows that

$$(17) \quad V_t = \int_0^t (\mathbf{1}_{\{V_s > 0\}} - \mathbf{1}_{\{V_s \leq 0\}}) dB_s^Y + (1 - 2p_i)L_t(V).$$

Define  $A(t) = \int_0^t \mathbf{1}_{\{V_s \geq 0\}} ds = \int_0^t \mathbf{1}_{\{Y_s \notin E_i\}} ds$  and  $\gamma(t) = \inf\{s \geq 0 : A(s) > t\}$ . It is also well known that  $V_t^r := V_{\gamma(t)}$  is a reflecting Brownian motion on  $\mathbb{R}_+$ . Set  $M_t = \int_0^t \mathbf{1}_{\{V_s > 0\}} dV_s = \int_0^t \mathbf{1}_{\{Y_s \notin E_i\}} dB_s^Y$ . Then  $B_t^2 := M_{\gamma(t)}$  is a Brownian motion. We also have that  $V_t \vee 0 = M_t + (1 - p_i)L_t(V)$ , which implies that  $V_t^r = B_t^2 + (1 - p_i)L_{\gamma(t)}(V)$  and therefore that  $L_t(V^r) = (1 - p_i)L_{\gamma(t)}(V)$ . Note finally that  $L(V) = L(|Y|)$ .

Set for  $t \geq 0$ ,  $B_t^1 = \int_0^{\gamma(t)} \mathbf{1}_{\{V_s > 0\}} dW_s^i$ . By Lévy's theorem  $B^1$  and  $B^2$  are two independent Brownian motions. Finally, set  $U_t^r = U_{\gamma(t)}$ . Then  $(U_t^r, V_t^r)_{t \leq \gamma(\tau_1)}$  is equal in law to the process  $(U_t^r, V_t^r)_{t \leq S_1}$  given in the statement of Lemma 4.2.

Lemma 4.2 is a direct consequence of the following.

**Lemma 4.5.** *For all  $t \geq 0$ ,*

$$\begin{aligned} U_t^r &= |x| + B_t^1 - \frac{p_i}{1 - p_i} L_t(V^r) \\ V_t^r &= B_t^2 + L_t(V^r). \end{aligned}$$

*Proof.* We closely follow the proof of Lemma 4.3 in [9]. Let  $\epsilon > 0$  and define the sequences of stopping times  $\sigma_k^\epsilon$  and  $\tau_k^\epsilon$  such that  $\tau_0^\epsilon = 0$  and for  $k \geq 0$ ,

$$\begin{aligned} \sigma_k^\epsilon &= \inf\{t \geq \tau_k^\epsilon; V_t = -\epsilon\}, \\ \tau_{k+1}^\epsilon &= \inf\{t \geq \sigma_k^\epsilon; V_t = 0\}. \end{aligned}$$

Note first that (17) implies that

$$\sum_{k \geq 0} (V_{\sigma_k^\epsilon \wedge \gamma(t)} - V_{\tau_k^\epsilon \wedge \gamma(t)})$$

converges in probability as  $\epsilon \rightarrow 0$  towards  $B_t^2 + (1 - 2p_i)L_{\gamma(t)}(V)$ . Since  $V_t^r = B_t^2 + (1 - p_i)L_{\gamma(t)}(V)$ , if we set

$$L_t^{\epsilon, r} = \sum_{k \geq 0} (V_{\tau_{k+1}^\epsilon \wedge \gamma(t)} - V_{\sigma_k^\epsilon \wedge \gamma(t)}),$$

as  $\epsilon \rightarrow 0$ ,  $L_t^{\epsilon, r}$  converges towards  $p_i L_{\gamma(t)}(V)$  in probability. Now for  $t > 0$ ,

$$U_t^r = |x| + \sum_{k \geq 0} (U_{\tau_{k+1}^\epsilon \wedge \gamma(t)} - U_{\sigma_k^\epsilon \wedge \gamma(t)}).$$

Set for  $t \geq 0$ ,

$$B_t^{\epsilon,1} = \sum_{k \geq 0} (W_{\sigma_k^\epsilon \wedge \gamma(t)}^i - W_{\tau_k^\epsilon \wedge \gamma(t)}^i).$$

Note that  $d(U_s + V_s) = \sum_{j \neq i} \mathbf{1}_{\{Y_s \in E_j\}} dW_s^j$  and thus when  $Y_s \in E_i^*$  (i.e. when  $V_s$  is negative),  $U_s + V_s$  remains constant, and we have

$$\begin{aligned} U_t^r &= |x| + \sum_{k \geq 0} (U_{\tau_{k+1}^\epsilon \wedge \gamma(t)} - U_{\sigma_k^\epsilon \wedge \gamma(t)}) + \sum_{k \geq 0} (U_{\sigma_k^\epsilon \wedge \gamma(t)} - U_{\tau_k^\epsilon \wedge \gamma(t)}) \\ &= |x| - L_t^{\epsilon,r} + B_t^{\epsilon,1}. \end{aligned}$$

Since  $B_t^{\epsilon,1}$  converges in probability towards  $B_t^1$ , we get

$$U_t^r = |x| + B_t^1 - p_i L_{\gamma(t)}(V).$$

And we conclude using that  $L_t(V^r) = (1 - p_i)L_{\gamma(t)}(V)$ .  $\square$

**4.3. Coalescing property.** Our purpose in this section is to prove that  $\tau_\infty$  defined above is finite a.s. By symmetry and the strong Markov property, it suffices to prove this for  $n = 2$  and  $(X_0, Y_0) = (x, 0)$  for some  $x \in G^*$ . We use the notations of section 4.2.

**Proposition 4.6.** *With probability 1,  $\tau_\infty < \infty$ .*

*Proof.* In order to show that  $\tau_\infty < \infty$  a.s., we prove that a.s.  $L_{\tau_\infty}(|X|) < \infty$ . Since  $(|X_t|, t < \tau_\infty)$  is a reflected Brownian motion stopped at time  $\tau_\infty$ , this implies that  $\tau_\infty < \infty$  a.s.

Denote by  $L_t^1$  and  $L_t^2$  the local times accumulated by  $Z$  respectively on  $\{u = 0\}$  and  $\{v = 0\}$  up to  $t$  and  $L_t = L_t^1 + L_t^2$ . First, note that for  $t \leq S_1$ ,  $L_t(V^r) = (1 - p_i)L_{\gamma(t)}(V) = (1 - p_i)L_{\gamma(t)}(|Y|)$ . Thus  $L_{\tau_1}(|Y|) = \frac{L_{S_1}(V^r)}{1 - p_i}$ . Note also that  $L_{\tau_1}(|X|) = 0$ . Thus

$$L_{\tau_1}(|X|) + L_{\tau_1}(|Y|) = \frac{L_{S_1}}{1 - p_i}.$$

By induction, we get that

$$L_{\tau_\infty}(|X|) + L_{\tau_\infty}(|Y|) = \sum_{n \geq 0} \frac{L_{T_{n+1}} - L_{T_n}}{1 - p_{i_n}} \leq C L_{T_\infty}$$

with  $C = \sup_{\{1 \leq i \leq N\}} (1 - p_i)^{-1}$ . By Theorem 2.11 a.s.  $L_{T_\infty} < \infty$ , and so  $L_{\tau_\infty}(|X|) + L_{\tau_\infty}(|Y|) < \infty$ .  $\square$

The fact that when  $n \geq 3$ ,  $\tau_\infty < \infty$  a.s., with  $\tau_\infty$  defined in section 4.1, is an immediate consequence of Proposition 4.6.

4.4. **Construction of  $\varphi$ .** Let  $(P^{(n)}, n \geq 1)$  be the unique consistent family of Markovian semigroups such that

- (i)  $P^{(1)}$  is the semigroup of the Walsh's Brownian motion on  $G$ .
- (ii) The  $n$ -point motion of  $P^{(n)}$  started from  $x \in G^n$  up to its entrance time in  $\Delta_n$  is distributed as  $\mathbb{P}_x^{(n),0}$ .
- (iii) The  $n$ -point motion  $(X^1, \dots, X^n)$  of  $P^{(n)}$  is such that if  $X_s^i = X_s^j$  then  $X_t^i = X_t^j$  for all  $t \geq s$ .

We will prove that all  $P^{(n)}$  are Feller and that (15) holds. By Lemma 1.11 [10], this amounts to check the following condition.

**Lemma 4.7.** *Let  $(X, Y)$  be the two point motion associated to  $P^{(2)}$ , then for all positive  $\epsilon > 0$*

$$\lim_{d(x,y) \rightarrow 0} \mathbb{P}_{(x,y)}^{(2),0}[d(X_t, Y_t) > \epsilon] = 0.$$

*Proof.* As in the proof of Proposition 4.6, we take  $y = 0$ . Then using the same notations, for all positive  $\epsilon$ ,  $\{d(X_t, Y_t) > \epsilon\} \subset \{\sup_{t < \sigma_0} |Z_t| > \epsilon\}$ . Now the result of the lemma follows from Remark 2.10.  $\square$

By Theorem 4.1, a SFM  $\varphi$  can be associated to  $(P^{(n)})_n$ .

**Proposition 4.8.** *Let  $\varphi$  be a SFM associated to  $(P^{(n)})_n$ . Then there exists a family of independent white noises  $\mathcal{W} = (W^i, 1 \leq i \leq N)$  such that*

- (i)  $\mathcal{F}_{s,t}^{\mathcal{W}} \subset \mathcal{F}_{s,t}^{\varphi}$  for all  $s \leq t$  and
- (ii)  $(\varphi, \mathcal{W})$  solves (E).

*Proof.* Let  $V_{s,\cdot}(x)$  be the Brownian motion associated to  $\varphi_{s,\cdot}(x)$ . For all  $i \in [1, N]$  and  $s \leq t$ , set

$$W_{s,t}^i = \lim_{|x| \rightarrow \infty, x \in E_i, |x| \in \mathbb{Q}} V_{s,t}(x).$$

For all  $i \in [1, N]$  and  $s \leq t$ , with probability 1, this limit exists. Indeed if  $x, y \in E_i$  are such that  $|x| \leq |y|$ , then a.s.  $V_{s,t}(x) = V_{s,t}(y)$  for all  $s \leq t \leq \tau_s^x = \inf\{u \geq s; \varphi_{s,u}(x) = 0\}$ . Moreover  $W^i = (W_{s,t}^i, s < t)$  is a real white noise. Indeed,  $W^i$  is centered and Gaussian, and by the flow property of  $\varphi$  and using  $\varphi_{s,u}(x) = e_i(|x| + W_{s,u}^i)$  if  $s \leq u \leq \tau_s^x$  and  $x \in E_i$ , we have  $W_{s,u}^i = W_{s,t}^i + W_{t,u}^i$ . It is also clear that  $W^i$  has independent increments with respect to  $(s, t)$ . Thus,  $W^i$  is a real white noise. The fact that  $\mathcal{W} = (W^i, 1 \leq i \leq N)$  is a family of independent real white noises easily holds.

For  $x \in G$  and  $t \geq 0$ ,

$$\langle W_{s,\cdot}^i, V_{s,\cdot}(x) \rangle_t = \lim_{|y| \rightarrow \infty, y \in E_i, |y| \in \mathbb{Q}} \langle V_{s,\cdot}(y), V_{s,\cdot}(x) \rangle_t = \int_s^t 1_{\{\varphi_{s,u}(x) \in E_i\}} du.$$

This yields

$$V_{s,t}(x) = \sum_{i=1}^N \int_s^t 1_{\{\varphi_{s,u}(x) \in E_i\}} dW_u^i.$$

By Theorem 3.1, we deduce that  $(\varphi, \mathcal{W})$  solves (E).  $\square$

Denote by  $\mathbb{P}_E$  the law of  $(\varphi, \mathcal{W})$ .

**4.5. Uniqueness in law of a SFM solution of (E).** In this section, we show that the SFM  $\varphi$  constructed in section 4.4 is the only SFM solution of (E). More precisely, we show

**Proposition 4.9.** *Let  $(\varphi, \mathcal{W})$  be a solution of (E), with  $\varphi$  a SFM. Then the law of  $(\varphi, \mathcal{W})$  is  $\mathbb{P}_E$ .*

*Proof.* We start by showing

**Lemma 4.10.** *For all  $x = e_i(r) \in G$ , we have  $\varphi_{s,t}(x) = e_i(r + W_{s,t}^i)$  for all  $s \leq t \leq \tau_s^x = \inf\{t \geq s : \varphi_{s,t}(x) = 0\}$ . In particular for all  $1 \leq i \leq N$ ,  $s \leq t$ , we have  $\mathcal{F}_{s,t}^{W^i} \subset \mathcal{F}_{s,t}^\varphi$ .*

*Proof.* Let  $f \in \mathcal{D}$  such that  $f(x) = |x|$  for all  $x \in E_i$ . By applying  $f$  in (E), we deduce the first claim. The second claim is then an immediate consequence by taking a sequence  $(x_k)_k \subset E_i$  converging to  $\infty$ .  $\square$

With this Lemma and Theorem 1.2 we prove the following

**Lemma 4.11.** *Let  $x = (x_1, \dots, x_n) \in G^n$ . Let  $S = \inf\{t \geq 0 : (\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n)) \in \Delta_n\}$ . Then  $(\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n))_{t \leq S}$  is distributed like  $\mathbb{P}_x^{(n),0}$ .*

*Proof.* Suppose  $|x_1| < \dots < |x_n|$ . For  $k \in [1, n]$ , set  $Y_t^k = \varphi_{0,t}(x_k)$  and  $Y_t^{(n)} = (Y_t^1, \dots, Y_t^n)$ . Set for  $i \in [1, N]$ ,  $W_t^i = W_{0,t}^i$  and  $W_t = (W_t^1, \dots, W_t^n)$ . Note that for all  $k \in [1, n]$ ,  $(Y^k, W)$  is a solution of (E). Set

$$\sigma_1 = \inf\{t \geq 0 : \exists k \neq 1 : Y_t^k = 0\}$$

and for  $\ell \geq 1$ , set

$$\sigma_{\ell+1} = \inf\{t \geq \sigma_\ell : \exists k \in [1, n] : Y_t^k = 0, Y_{\sigma_\ell}^k \neq 0\}.$$

Let  $S^n = \lim_{\ell \rightarrow \infty} \sigma_\ell$ , then  $S^n = S = \inf\{t : Y_t^{(n)} \in \Delta_n\}$ . From Theorem 1.2, the law of  $(Y^1, W)$  is uniquely determined. Now, for  $k \in [2, n]$  with  $x_k \in E_i$ , we have that for  $t \leq \sigma_1$ ,  $Y_t^k = e_i(|x_k| + W_t^i)$ . This shows that  $(Y_t^{(n)})_{t \leq \sigma_1}$  is distributed as  $(X_t^{(n)})_{t \leq \tau_1}$ , constructed in Subsection 4.1. Adapting the previous argument on the time interval  $[\sigma_\ell, \sigma_{\ell+1}]$ , we show that for all  $\ell \geq 1$ ,  $(Y_t^{(n)})_{t \leq \sigma_\ell}$  is distributed as  $(X_t^{(n)})_{t \leq \tau_\ell}$ . This thus shows the Lemma.  $\square$

Lemma 4.11 permits to conclude the proof of Proposition 4.9. Indeed, the law of a SFM is uniquely determined by its family of  $n$ -point motions  $X^{(n)}$ . Using the fact that  $\Delta_n$  is an absorbing set for  $X^{(n)}$ , the strong Markov property at time  $T^n = \inf\{t; X_t^{(n)} \in \Delta_n\}$  and the consistency of the family of  $n$ -point motions, we see that the law of a SFM is uniquely determined by its family of  $n$ -point motions stopped at its first entrance time in  $\Delta_n$ .  $\square$

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#### REFERENCES

- [1] M. Barlow, J. Pitman, and M. Yor. On Walsh's Brownian motions. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 275–293. Springer, Berlin, 1989.
- [2] Martin Barlow, Krzysztof Burdzy, Haya Kaspri, and Avi Mandelbaum. Coalescence of skew Brownian motions. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 202–205. Springer, Berlin, 2001.
- [3] M.T. Barlow, M. Émery, F.B. Knight, S. Song, and M. Yor. Autour d'un théorème de Tsirelson sur des filtrations browniennes et non browniennes. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 264–305. Springer, Berlin, 1998.
- [4] E. Robert Fernholz, Tomoyuki Ichiba, Ioannis Karatzas, and Vilmos Prokaj. Planar diffusions with rank-based characteristics and perturbed Tanaka equations. *Probab. Theory Related Fields*, 156(1-2):343–374, 2013.
- [5] M. Freidlin and S. Sheu. Diffusion processes on graphs: stochastic differential equations, large deviation principle. *Probab. Theory Related Fields*, 116(2):181–220, 2000.
- [6] H. Hajri. Stochastic flows related to Walsh Brownian motion. *Electron. J. Probab.*, 16:no. 58, 1563–1599, 2011.
- [7] H. Hajri and O. Raimond. Stochastic flows on metric graphs. Available via <http://arxiv.org/abs/1305.0839>, 2013.
- [8] J.-F. Le Gall. Applications du temps local aux équations différentielles stochastiques unidimensionnelles. In *Seminar on probability, XVII*, volume 986 of *Lecture Notes in Math.*, pages 15–31. Springer, Berlin, 1983.
- [9] Y. Le Jan and O. Raimond. Three examples of Brownian flows on  $\mathbb{R}$ . To appear in *AIHP. Probab. Stat.*
- [10] Y. Le Jan and O. Raimond. Flows, coalescence and noise. *Ann. Probab.*, 32(2):1247–1315, 2004.
- [11] Antoine Lejay. On the constructions of the skew Brownian motion. *Probab. Surv.*, 3:413–466 (electronic), 2006.
- [12] S. Bouhadou M. Benabdallah and Y. Ouknine. On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations with jumps. *Preprint, arXiv:1108.4016.*, 2011.

- [13] Edwin Perkins. Local time and pathwise uniqueness for stochastic differential equations. In *Seminar on Probability, XVI*, volume 920 of *Lecture Notes in Math.*, pages 201–208. Springer, Berlin, 1982.
- [14] V. Prokaj. The solution of the perturbed tanaka-equation is pathwise unique. *Ann. Probab.*, 41(3B):2376–2400, 2013.
- [15] B. Tsirelson. Triple points: from non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal.*, 7(6):1096–1142, 1997.
- [16] S. R. S. Varadhan and R. J. Williams. Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.*, 38(4):405–443, 1985.
- [17] R. J. Williams. Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahrsch. Verw. Gebiete*, 69(2):161–176, 1985.