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Abstract. We study the structure of subgroups of minimal connected simple groups of finite Morley rank. We first establish a Jordan decomposition for a large family of minimal connected simple groups including those with a non-trivial Weyl group. We then show that definable, connected, solvable subgroups of such a simple group are the semi-direct product of their unipotent part extended by a maximal torus. This is an essential step in the proof of the main theorem which provides a precise structural description of Borel subgroups.

1. Introduction

This article aspires to contribute to the progress towards the resolution of the Cherlin-Zil'ber conjecture, which states that an infinite simple group of finite Morley rank, seen as a pure group structure, is a linear algebraic group over an algebraically closed field. This conjecture is in fact a natural question in the context of the model theory of algebraic structures. It has served as a reference point in that any work on simple groups of finite Morley rank is an attempt to measure how far one is from a family of algebraic groups. These attempts have had recourse to two main sources in addition to model-theoretic foundations: the structure of linear algebraic groups, and finite group theory, especially the classification of the finite simple groups.

This paper touches upon all these resources. Its main theme is the description of the solvable subgroup structure of a large class of connected minimal simple groups. Connected minimal simple groups of finite Morley rank are those whose proper definable connected subgroups are solvable, and their analysis has well-known analogues in the classification of the finite simple groups. The main theorem of the present paper (Theorem 6.16) yields a precise structural description in our context, of the *Borel subgroups* (maximal definable connected solvable subgroups) as a semi-direct product of two definable, connected subgroups. In order to achieve this objective, a Jordan decomposition is established by introducing notions of *semisimple* and *unipotent* elements.

Our main theorems were initially proven for connected minimal simple groups having a non-trivial *Weyl group* (see Section 3 for the definition). In fact, our methods cover a more general class of groups. They will be proven for a connected minimal simple group G that satisfies the *negation* of the following assumption:

$(*)$ the *Borel subgroups* of G are all non-nilpotent and there exists one that is generically disjoint from its conjugates.

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The groups not covered by our theorems form a subclass of groups of *type (1)*, one of the four mutually exclusive families that cover all connected minimal simple groups and were introduced in [ABF12] (Fact 3.9). The groups satisfying the (*) hypothesis are groups whose structure is close to that of a *bad group* (see Fact 2.12), but having no nilpotent Borel subgroup. They form a class of groups for which known group theoretical methods are ineffective. However, as it will be discussed in the final section of this article, our results have potential extensions that are both natural and relevant.

Our main result is Theorem 6.16 that gives a precise structural description of the Borel subgroups of a connected minimal simple group satisfying the negation of the $(*)$ hypothesis. This theorem involves the use of *Carter subgroups*, definable, connected, nilpotent subgroups of finite index in their normalizers.

Main Theorem (special case of Theorem 6.16, Corollary 6.17) – *Let G be a connected minimal simple group of finite Morley rank which satisfies the negation of $(*)$. Then any Borel subgroup B of G satisfies*

$$B \;=\; U \rtimes D\ .$$

where D is any Carter subgroup of B and U is a normal nilpotent connected definable subgroup of B.

Furthermore, if D is abelian, then $B' = U$ and $Z(B) = F(B) \cap D$.

A noticeable consequence of this result is a positive answer to a question posed by Deloro in the end of his Ph. D. Thesis [Del07, p. 93] about the splitting of Borel subgroups in a specific class of connected minimal simple groups of *odd type* (see Section 2 after Fact 2.14 for the definition).

The main ingredient for our analysis is the introduction in Section 5 of notions of *semisimple* and *unipotent* elements in the context of groups of finite Morley rank. The *semisimple elements* are those belonging to a Carter subgroup of the ambient group, and an element u is *unipotent* if its definable envelope $d(u)$ contains no nontrivial semisimple element. In the context of reductive algebraic groups over an algebraically closed field, these definitions corresponds exactly to the algebraic ones. We succeed in obtaining the following Jordan Decomposition in the context of connected minimal simple groups.

Theorem 5.12 – *Let G be a minimal, connected, simple group of finite Morley rank which satisfies the negation of the $(*)$ hypothesis. Then for each $x \in G$, there exists a unique semisimple element x_s and a unique unipotent element x_u satisfying the following conclusions:*

(1) *$x = x_s x_u = x_u x_s$;*
(2) *for each $x \in G$, we have $d(x) = d(x_s) \times d(x_u)$;*
(3) *for each $(x, y) \in G \times G$ such that $xy = yx$, we have $(xy)_u = x_u y_u$ and $(xy)_s = x_s y_s$.*

Using our Jordan Decomposition, we obtain the following decomposition of solvable subgroups:

Theorem 6.12 – *Let G be a connected minimal simple group of finite Morley rank which satisfies the negation of $(*)$. In each connected solvable definable subgroup H of G, the set H_u of the unipotent elements of H is a definable connected subgroup such that $H = H_u \rtimes T$ for any maximal torus T of H.*

This is an analogue in our context of the well-known decomposition of closed, connected, solvable subgroups of algebraic groups. The proof of Theorem 6.16 builds on this essential step.

It should be noted that the Jordan Decomposition in groups of finite Morley rank has been studied independently by Poizat in [Poiz12]. The approaches are different because Poizat aims to obtain a description of the centralizer a generic element of a group of finite Morley rank.

2. Background on groups of finite Morley rank

This is a long section that covers the entire bacground on groups of finite Morley rank needed in this paper. Keeping in mind specialists not familiar with groups of finite Morley rank, in order to be as self-consistant as possible, we will start from the most fundamental results. For further details, one can consult [BN94], [ABC08]. [PoizGrSt] contains a more model-theoretic approach to some of the themes of this paper. [WagStGr] offers a solid introduction to probable extensions of our results to higher levels of generality. Some recent results that do not appear in these books will be exposed in detail. Readers familiar with groups of finite Morley rank can skip this section, but have a look at the next one devoted to very recent progress.

Morley rank is one of the many dimension notions in model theory. It generalizes the notion of *Zariski dimension* of closed sets in algebraic geometry over algebraically closed fields. In the case of a structure that admits Morley rank, *definable* sets are those that yield themselves to the measurement by the Morley rank. We will note the Morley rank of a definable set X by $rk(X)$.

The ordinal character of the Morley rank imposes strong finiteness conditions, the most fundamental being the *descending chain condition on definable subgroups*: in a group of finite Morley rank, there is no infinite descending chain of definable subgroups. This property allows one to introduce various notions in the abstract context of groups of finite Morley rank, analogous to geometric aspects of algebraic groups. Thus, the *connected component* of a group G of finite Morley rank, noted G° and defined as the smallest definable subgroup of finite index, does exist and is the intersection of all definable subgroups of finite index in G. A group of finite Morley rank is said to be *connected* if it is equal to its connected component.

The connected component of a group is an example of a "large" definable set in that it is of the same rank as the ambient group. In general, a definable subset X of G is said to be *generic* if $rk(X) = rk(G)$.

In a dual vein, if X is an arbitrary subset of a group G of finite Morley rank, then one defines its *definable hull*, noted $d(X)$ as the intersection of all definable subgroups of G containing X. Thanks to the descending chain condition, the definable hull of a set is well-defined and yields an analogue of the Zariski closure in algebraic geometry. The existence of a definable hull allows to introduce the connected component of an arbitrary subgroup of the ambient group G: if X is subgroup, then X° is defined as $X \cap d(X)^\circ$, and X is said to be connected if $X = X^\circ$. It is worth noting that the notion of definable hull has proven to be very effective in illuminating the algebraic structure of groups of finite Morley rank since many algebraically interesting subgroups such as Sylow subgroups, are not definable. Various algebraic properties are preserved as one passes to the definable hull:

Fact 2.1. – (*Zil'ber*) [BN94, Corollary 5.38] *Let G be a group of finite Morley rank and H be a solvable (resp. nilpotent) subgroup of class n. Then $d(H)$ has the same properties.*

Another fundamental notion that also has connections with definability and connectedness is that of an *indecomposable set*. A definable set in a group G of finite Morley rank is said to be indecomposable if for any definable subgroup $H \leq G$ whenever cosets of H decompose X into more than one subset, then they decompose into infinitely many. In particular, an indecomposable subgroup is a connected subgroup.

The notion of indecomposable set, that has analogues well-known to algebraic group theorists, is of fundamental importance in that it helps clarify the definable structure of a group of finite Morley rank. This is mostly due to the *Zil'ber's indecomposability theorem* which states that indecomposable sets which contain the identity element of the group generate definable connected subgroups. We will use its following corollaries frequently, mostly without mention:

Fact 2.2. – [BN94, Corollary 5.28] *Let G be a group of finite Morley rank. Then the subgroup generated by a family of definable connected subgroups of G is definable and the setwise product of finitely many of them.*

Fact 2.3. – [BN94, Corollaries 5.29 and 5.32] *Let G be a group of finite Morley rank.*

1. *Let $H \leq G$ be a definable connected subgroup of G and X an arbitrary subset of G. Then the subgroup $[H, X]$ is definable and connected.*
2. *Let H be a definable subgroup of G. Then the members of the derived ($H^{(n)}$) and lower central series (H^n) of H are definable. If H is connected, then so are these subgroups of H.*

As a linear algebraic group, a group of finite Morley rank is built up from definable, minimal subgroups that are abelian:

Fact 2.4. – [Rei] [BN94, Theorem 6.4] *In a group of finite Morley rank, a minimal, infinite, definable subgroup A is abelian. Furthermore, either A is divisible or is an elementary abelian p-group for some prime p.*

This simple and historically old fact is what permits many inductive arguments using Morley rank. The additional structural conclusions in Fact 2.4 are related to the following general structural description of abelian groups of finite Morley rank.

Fact 2.5. – [Mac70Gr, Theorems 1 and 2] [BN94, Theorem 6.7] *Let G be an abelian group of finite Morley rank. Then the following hold:*

1. *$G = D \oplus C$ where D is a divisible subgroup and C is a subgroup of bounded exponent;*
2. *$D \cong \oplus_{p\ prime}(\oplus_{I_p}\mathbb{Z}_{p^\infty}) \oplus \oplus_I \mathbb{Q}$ where the index sets I_p are finite;*
3. *$G = DB$ where D and B are definable characteristic subgroups, D is divisible, B has bounded exponent and $D \cap B$ is finite. The subgroup D is connected. If G is connected, then B can be taken to be connected.*

It easily follows from this detailed description of abelian groups of finite Morley rank that, in general, groups of finite Morley rank enjoy the property of *lifting torsion from definable quotients*. More precisely, if G is a group of finite Morley rank, $H \leq G$ a definable subgroup of G and $g \in G$ such that $g^n \in H$ for some $n \in \mathbb{N}^*$, where n is assumed to be the order of g in $d(g)/d(g) \cap H$ and is a π-number with π a set of prime numbers, then there exists $g' \in gH \cap d(g)$ such that g' is again a π-element. Here, a *π-number* is a natural number whose prime divisors belong to π, and a *π-element* is an element whose order is a π-number. Weyl groups (see Section 3) provide a good example of the importance of this property. The torsion-lifting property will be used without mention.

Fact 2.5 was later generalized to the context of nilpotent groups of finite Morley rank using techniques of algebraic character:

Fact 2.6. – [Nes91, Theorem 2] [BN94, Theorem 6.8 and Corollary 6.12] *Let G be a nilpotent group of finite Morley rank. Then G is the central product $B * D$ where D and B are definable characteristic subgroups of G, D is divisible, B has bounded exponent. The torsion elements of D are central in G.*

The structural description provided by Facts 2.5 and 2.6 can be regarded as a weak "Jordan decomposition" in groups of finite Morley rank since, using the notation of the fact, B and D are respectively abstract analogues of unipotent and semisimple parts of a nilpotent algebraic group. This viewpoint is indeed weak in that when $B = 1$ and D is a torsion-free group, it is not possible to decide whether D is semisimple or unipotent (characteristic 0).

The description of the divisible nilpotent groups can be refined further:

Fact 2.7. – [Nes91, Theorem 3] [BN94, Theorem 6.9] *Let G be a divisible nilpotent group of finite Morley rank. Let T be the torsion part of G. Then T is central in G and $G = T \oplus N$ for some torsion-free divisible nilpotent subgroup N.*

This description has been extensively exploited in most works on groups of finite Morley rank and this paper is no exception to this. Remarkably, as will be explained later in this section, and used later in this paper, a finer analysis of nilpotent groups of finite Morley rank, even when torsion elements are absent, is possible using a suitable notion of unipotence.

We also include the following two elementary properties of nilpotent groups of finite Morley rank that generalize similar well-known properties of algebraic groups. Other similarities involving normalizer conditions will be mentioned later in this section in the context of the finer unipotent analysis.

Fact 2.8. –

(1) [BN94, Lemma 6.3] *Let G be a nilpotent group of finite Morley rank and H a definable subgroup of infinite index in G. Then $N_G(H)/H$ is infinite.*
(2) [BN94, Exercice 6.1.5] *Let G be a nilpotent group of finite Morley rank. Any infinite normal subgroup has infinite intersection with $Z(G)$.*

As in many other classes of groups, there is a long way between nilpotent and solvable groups of finite Morley rank. The differences are best measured by field structures that are definable in solvable non-nilpotent groups of finite Morley rank. All the solvable results used in this paper illustrate this "definably linear" aspect of solvable groups of finite Morley rank. The most fundamental one is the following:

Fact 2.9. – (*Zil'ber*) [BN94, Theorem 9.1] *Let G be a connected, solvable, non-nilpotent group of finite Morley rank. Then there exist a field K and definable connected sections U and T of G' and G/G' respectively such that $U \cong (K, +)$,* *and T embeds in $(K^\times, .)$. Moreover, these mappings are definable in the pure group G, and each element of K is the sum of a bounded number of elements of T. In particular, K is definable in G and hence of finite Morley rank.*

The ability to define an algebraically closed field in a connected solvable eventually culminates in the following result that generalizes a well-known property of connected solvable algebraic groups.

Fact 2.10. – [BN94, Corollary 9.9] *Let G be a connected solvable group of finite Morley rank. Then G' is nilpotent.*

The *Fitting subgroup* of a group of finite Morley rank G, noted $F(G)$, is defined to be the maximal, definable, normal, nilpotent subgroup of G. By the works of Belegradek and Nesin, this definition is equivalent to the one used in finite group theory: the subgroup generated by all normal, nilpotent subgroups. The following result of Nesin shows that the Fitting subgroup shares properties of its unipotent analogues in algebraic groups. This is another consequence of the linear behaviour of solvable groups of finite Morley rank of which various refinements have been obtained first in the works of Altseimer and Berkman, later of the third author.

Fact 2.11. – [BN94, Theorem 9.21] *Let G be a connected solvable group of finite Morley rank. Then $G/F(G)^\circ$, thus $G/F(G)$ are divisible abelian groups.*

Beyond solvable?... Since this paper is about minimal connected simple groups of finite Morley rank and we already mentioned examples that motivate the Algebraicity Conjecture, at this point we will be content with the most extreme minimal counterexample whose existence is a major open problem, namely *bad groups*. By definition a bad group is a connected, non-solvable, group of finite Morley rank whose proper definable connected subgroups are nilpotent. One easily shows that if a bad group exists, then there exists a simple one. In particular, such a group is minimal, connected and simple. The following make up for most of the few but striking known properties of simple bad groups.

Fact 2.12. – [BN94, Theorem 13.3] *The following hold in a simple bad group G:*

1. *The Borel subgroups of G are conjugate.*
2. *Distinct Borel subgroups of G are intersect trivially.*
3. *G is covered by its Borel subgroups.*
4. *G has no involutions.*
5. *$N_G(B) = B$ for any Borel subgroup B of G.*

A *Borel subgroup* of a group of finite Morley rank is a maximal, definable, connected, solvable subgroup.

Clearly, the stated properties are far from those of simple algebraic groups. Except for the primes 2 and 3, it is not even known whether a simple bad group can be of prime exponent. This is the main reason why below we will be careful while treating p-subgroups of groups of finite Morley rank.

In this paper, for each prime p, a *Sylow p-subgroup* of any group G is defined to be a maximal *locally finite* p-subgroup. By Fact 2.13 (1), such a subgroup of a group of finite Morley rank is nilpotent-by-finite.

Fact 2.13. –

1. [BN94, Theorem 6.19] *For any prime number p, a locally finite p-subgroup of a group of finite Morley rank is nilpotent-by-finite.*

(2) [BN94, Proposition 6.18 and Corollary 6.20] *If P is a nilpotent-by-finite p-subgroup of a group of finite Morley rank, then $P^\circ = B * T$ is the central product of a definable, connected, subgroup B of bounded exponent and a divisible abelian p-group. In particular, P° is nilpotent.*

The assumption of local finiteness for p-subgroups is rather restrictive but unavoidable as was implied by the remarks after Fact 2.12. The only prime for which the mere assumption of being a p-group is equivalent to being a nilpotent-by-finite in groups of finite Morley rank is 2. The prime 2 is also the only one for which a general Sylow theorem is known for groups of finite Morley rank:

Fact 2.14. – [BN94, Theorem 10.11] *In a group of finite Morley rank the maximal 2-subgroups are conjugate.*

Before reviewing the Sylow theory in the context of solvable groups where it is better understood, we introduce some terminology related to the unipotent/semisimple decomposition, as well as some of its implications for the analysis of simple groups of finite Morley rank. For each prime p, a nilpotent definable connected p-group of finite Morley rank is said to be *p-unipotent* if it has bounded exponent while a *p-torus* is a divisible abelian p-group.

In general, a p-torus is not definable but enjoys a useful finiteness property in a group of finite Morley rank. It is the direct sum of finitely many copies of $\mathbb{Z}_{p^\infty}$, the Sylow p-subgroup of the multiplicative group of complex numbers. In particular, the p-elements of order at most p form an finite elementary abelian p-group of which the rank is called the *Prüfer p-rank* of the torus in question. Thus, in any group of finite Morley rank where maximal p-tori are conjugate, the Prüfer p-rank of the ambient group is defined as the Prüfer p-rank of a maximal p-torus.

The choice of terminology, "unipotent" and "torus", is not coincidental. Fact 2.13 (2) shows that the Sylow p-subgroups of a group of finite Morley rank have similarities with those of algebraic groups. These are of bounded exponent when the characteristic of the underlying field is p, and divisible abelian when this characteristic is different from p. In the notation of Fact 2.13 (2), this case division corresponds to $T = 1$ or $B = 1$ respectively when the Sylow p-subgroup in question is non-trivial.

A similar case division for the prime 2 has played a major role in developing a strategy to attack parts of the Cherlin-Zil'ber conjecture. In this vein, a group of finite Morley rank is said to be of *even type* if its Sylow 2-subgroups are infinite of bounded exponent ($B \neq 1,\ T = 1$), of *odd type* if its Sylow 2-subgroups are infinite and their connected components are divisible ($B = 1,\ T \neq 1$), of *mixed type* if $B \neq 1$ and $T \neq 1$ and of *degenerate type* if they are finite.

The main result of [ABC08] states that a simple group of finite Morley rank that contains a non-trivial unipotent 2-subgroup is an algebraic group over an algebraically closed field of characteristic 2. In particular, there exists no simple group of finite Morley rank of mixed type. In this article, we will use this result and refer to it as the *classification of simple groups of even type*. Despite spectacular advances for groups of odd type, no such extensive conclusion has been achieved. In the degenerate type, it has been shown in [BBC07] that a connected group of finite Morley rank of degenerate type has no involutions:

Fact 2.15. – [BBC07, Theorems 1 and 3] *Let G be a connected group of finite Morley rank whose maximal p-subgroups are finite. Then G contains no elements of order p.*

As was mentioned above, the Sylow theory is much better understood in solvable groups of finite Morley rank. This is one reason why one expects to improve the understanding of the structure of minimal connected simple groups of finite Morley rank although even in the minimal context additional tools are indispensable. We first review the parts of what can now be called the classical Hall theory for solvable groups of finite Morley rank that are relevant for this paper. Then we will go over more recent notions of tori, unipotence and Carter theory as was developed in the works Cherlin, Deloro, Jaligot, the second and third authors.

One now classical result on maximal π-subgroups of solvable groups of finite Morley rank is the Hall theorem for this class of groups:

Fact 2.16. – [BN94, Theorem 9.35] *In a solvable group of finite Morley rank, any two Hall π-subgroups are conjugate.*

Hall π-subgroups are by definition maximal π-subgroups. The Hall theorem was motivated by finite group theory while the next two facts have their roots in the structure of connected solvable algebraic groups:

Fact 2.17. –

(1) [BN94, Corollary 6.14] *In a connected nilpotent group of finite Morley rank, the Hall π-subgroups are connected.*
(2) [BN94, Theorem 9.29] [Fré00a, Corollaire 7.15] *In a connected solvable group of finite Morley rank, the Hall π-subgroups are connected.*

We also recall the following easy but useful consequence of Fact 2.11.

Fact 2.18. – *A solvable group of finite Morley rank G has a unique maximal p-unipotent subgroup.*

On the toral side, we will need the following analogue of well-known properties of solvable algebraic groups:

Fact 2.19. – [Fré00b, Lemma 4.20] *Let G be a connected solvable group of finite Morley rank, p a prime number and T a p-torus. Then $T \cap F(G) \leq Z(G)$.*

Attempts to understand the nature of a generic element of a group of finite Morley rank have given rise to two important notions of tori. A divisible abelian group G of finite Morley rank is said to be: a *decent torus* if $G = d(T)$ for T its (divisible) torsion subgroup; a *pseudo-torus* if no definable quotient of G is definably isomorphic to K_+ for an interpretable field K.

The following remark based on important work of Wagner on bad fields of non zero characteristic was the first evidence of the relevance of these notions of tori.

Fact 2.20. – [AC04, Lemma 3.11] *Let F be a field of finite Morley rank and nonzero characteristic. Then $F^\times$ is a* good torus*.*

A good torus is a stronger version of a decent torus in that the defining property of a decent torus is assumed to be hereditary.

Using the geometry of groups of finite Morley rank provided by genericity arguments, Cherlin and later the third author obtained the following conjugacy results.

It is worth mentioning that such results were attained to the extent that one can describe the generic element of a group of finite Morley rank. This is the case when a group of finite Morley rank has non-trivial decent or pseudo-tori.

Fact 2.21. –

(1) [Che05, Extended nongenericity] *In a group of finite Morley rank, maximal decent tori are conjugate.*
(2) [Fré09, Theorem 1.7] *In a group of finite Morley rank, maximal pseudo-tori are conjugate.*

Below, we include several facts about decent and pseudo-tori mostly for the practical reason that we will need them. They illustrate that these more general notions of tori, introduced to investigate more efficiently the structure of groups of finite Morley rank, share crucial properties of tori in algebraic groups, and thus clarify which aspects of algebraic tori influence the structure of algebraic groups.

The last point below mentions the *generosity* of a set. A definable subset X of a group G of finite Morley rank is said to be *generous in G* (or shortly, "generous" in case the ambient group is clear) if the union of its conjugates is generic in G. This notion was introduced and studied in [Jal06].

Fact 2.22. –

(1) [Fré06b, Lemma 3.1] *Let G be a group of finite Morley rank, N be a normal definable subgroup of G, and T be a maximal decent torus of G. Then TN/N is a maximal decent torus of G/N and every maximal decent torus of G/N has this form.*
(2) [Fré09, Corollary 2.9] *Let G be a connected group of finite Morley rank. Then the maximal pseudo-torus of $F(G)$ is central in G.*
(3) [AB09, Theorem 1] *Let T be a decent torus of a connected group G of finite Morley rank. Then $C_G(T)$ is connected.*
(4) [Fré09, Corollary 2.12] *Let T be a pseudo-torus of a connected group G of finite Morley rank. Then $C_G(T)$ is connected and generous in G, and $N_G(C_G(T))^\circ = C_G(T)$.*

So far, we have emphasized notions of tori and their generalizations in groups of finite Morley rank. Before moving to the unipotent side, it is necessary to go over a notion that is related to both sides and thus fundamental to the understanding of groups of finite Morley rank: *Carter subgroups*. In groups of finite Morley rank, Carter subgroups are defined as being the definable connected nilpotent subgroups of finite index in their normalizers. We summarize the main results concerning these subgroups in Fact 2.23.

In a reductive algebraic group, Carter subgroups are the maximal tori. Hence, the notion of Carter subgroup yields a group-theoretical tool to analyze properties of algebraic tori. Carter subgroups have strong ties with the geometry of groups of finite Morley rank stemming from genericity arguments.

Fact 2.23. – *Let G be a group of finite Morley rank.*

(1) [FJ05], [FJ08, Theorem 3.11] *G has a Carter subgroup.*
(2) [Fré09, Corollary 2.10] *Each pseudo-torus is contained in a Carter subgroup of G.*
(3) [Wag94, Theorem 29] *If G is solvable, its Carter subgroups are conjugate.*

(4) [Fré08, Theorem 1.2] *If G is a minimal connected simple group, its Carter subgroups are conjugate.*
(5) [Fré00a, Théorèmes 1.1 and 1.2] *If G is connected and solvable, any subgroup of G containing a Carter subgroup of G is definable, connected and self-normalizing.*
(6) [Fré00a, Corollaire 5.20], [FJ08, Corollary 3.13] *If G is connected and solvable, for each normal subgroup N, Carter subgroups of G/N are exactly of the form CN/N, with C a Carter subgroup of G.*
(7) [Fré00a, Corollaire 7.7] *Let G be a connected solvable group of class* 2 *and C be a Carter subgroup of G. Then there exists $k \in \mathbb{N}$ such that $G = G^k \rtimes C$.*
(8) [Fré00a, Théorème 1.1] *If G is connected and solvable, and if C is a definable nilpotent subgroup of finite index in its normalizer in G, then C is connected,* i.e. *a Carter subgroup of G.*

The following observation illustrates the connection between genericity, Carter subgroups and torsion elements in connected minimal simple groups.

Fact 2.24. – [AB09, Proposition 3.6] *Let G be minimal connected simple group. Then*
(1) *either G does not have torsion,*
(2) *or G has a generous Carter subgroup.*

The notion of *abnormality* is tightly connected to that of a Carter subgroup in solvable groups. In the context of solvable groups of finite Morley rank, abnormal subgroups of solvable groups were analyzed in detail in [Fré00a]. A subgroup H of any group G is said to be *abnormal* if $g \in \langle H, H^g \rangle$ for every $g \in G$. In a connected solvable group of finite Morley rank, abnormal subgroups are definable and connected. Their relation to Carter subgroups is as follows:

Fact 2.25. –
(1) [Fré00a, Théorème 1.1] *In a connected solvable group of finite Morley rank, a definable subgroup is a Carter subgroup if and only if it is a minimal abnormal subgroup.*
(2) [Fré00a, Théorème 1.2] *Let G be a connected solvable group of finite Morley rank, and H be a subgroup of G. Then the following are equivalent:*
  (i) *H is abnormal;*
  (ii) *H contains a Carter subgroup of G.*

An important class of abnormal subgroup is formed by *generalized centralizers.* If G is an arbitrary group, A a subgroup and $g \in N_G(A)$, then the generalized centralizer of g in A is $E_A(g) = \{x \in A |\text{ il existe } n \in \mathbb{N} \text{ tel que } [x,_n g] = 1\}$. Let us remind that $[x,_0 g] = x$ and $[x,_{n+1} g] = [[x,_n g], g]$ for every $n \in \mathbb{N}$. More generally, if $Y \subseteq N_G(A)$ then $E_A(Y) = \cap_{y \in Y} E_A(y)$.

In general, a generalized centralizer need not even be a subgroup. In a connected solvable group of finite Morley rank, it turns out to be a definable, connected subgroup that sheds considerable light on the structure of the ambient group:

Fact 2.26. – [Fré00a, Corollaire 7.4] *Let G be a connected solvable group of finite Morley rank and H be a nilpotent subgroup of G. Then $E_G(H)$ is abnormal in G.*

In addition to the information they provide, the generalized centralizers are in a sense more practical tools than the centralizers of sets. This is mainly because a generalized centralizer contains the elements that they "centralize", and this containment is rather special:

Fact 2.27. – [Fré00a, Corollaire 5.17] *Let G be a connected solvable group of finite Morley rank and H a subset of G that generates a locally nilpotent subgroup. Then $E_G(H) = E_G(d(H))$, is definable, connected, and H is contained in $F(E_G(H))$. In particular, $d(H)$ is nilpotent and the set of nilpotent subgroups of G is inductive.*

Thus generalized centralizers provide definable connected enveloping subgroups for arbitrary subsets of connnected solvable groups of finite Morley rank.

The notion of a p-unipotent group gives a robust analogue of a unipotent element in an algebraic group over an algebraically closed field of characteristic p. As was mentioned after Fact 2.6 however, there is no such analogue for unipotent elements in characteristic 0, and this has been a major question to which answers of increasing levels of efficiency have been given. The first step in this direction can be traced back to the notion of *quasiunipotent radical* introduced in unpublished work by Altseimer and Berkman. This notion is still of relevance, and yields a refinement of Fact 2.11, proven by the third author.

A definable, connected, nilpotent subgroup of group G of finite Morley rank is said to be *quasi-unipotent* if it does not contain any non-trivial p-torus. The quasi-unipotent radical of a group of finite Morley rank G, noted $Q(G)$, is the subgroup generated by its quasi-unipotent subgroups. By Fact 2.2, $Q(G)$ is a definable, connected subgroup. Clearly, $Q(G) \lhd G$. Less clearly, though naturally, the following is true:

Fact 2.28. – [Fré00b, Proposition 3.26] *Let G be connected solvable group of finite Morley rank. Then $G/Q(G)$ is abelian and divisible.*

The notions of *reduced rank* and *$U_{0,r}$-groups* were introduced by the second author in order to carry out an analogue of local analysis in the theory of the finite simple groups. In a similar vein, a theory of *Sylow $U_{0,r}$-subgroups* was developed. The notion of *homogeneity* was introduced by the third author in his refinement of the unipotence analysis. We summarize these in the following definition:

Definition 2.29. – [Bur04], [Fré06a], [Bur06]

- *An abelian connected group A of finite Morley rank is* indecomposable *if it is not the sum of two proper definable subgroups. If $A \neq 1$, then A has a unique maximal proper definable connected subgroup $J(A)$, and if $A = 1$, let $J(1) = 1$.*
- *The* reduced rank *of any abelian indecomposable group A of finite Morley rank is $\overline{r}(A) = rk(A/J(A))$.*
- *For any group G of finite Morley rank and any positive integer r, we define*

$$U_{0,r}(G) = \langle A \leq G \mid A \textit{ is indecomposable definable abelian, } \overline{r}(A) = r,\ A/J(A) \textit{ is torsion-free}\rangle.$$

- *A group G of finite Morley rank is said to be a $U_{0,r}$-*group *whenever $G = U_{0,r}(G)$, and to be* homogeneous *if each definable connected subgroup of G is a $U_{0,r}$-subgroup.*
- *The* radical *$U_0(G)$ is defined as follows. Set $\overline{r}_0(G) = \max\{r \mid U_{0,r}(G) \neq 1\}$ and set $U_0(G) = U_{0,\overline{r}_0(G)}(G)$.*

- *In any group G of finite Morley rank, a* Sylow $U_{0,r}$-subgroup *is a maximal, definable, nilpotent $U_{0,r}$-subgroup.*
- *In a group G of finite Morley rank, $U(G)$ is defined as the subgroup of G generated by its normal homogeneous $U_{0,s}$-subgroups where s covers $\mathbb{N}^*$ and by its normal definable connected subgroups of bounded exponent. A U-group is a group G of finite Morley rank such that $G = U(G)$.*

The notion of reduced rank and the resulting unipotence theory, allowed a finer analysis of connected solvable groups in a way reminiscent of what torsion elements had allowed to achieve in such results as Facts 2.6, 2.7, 2.10, 2.18. Indeed, the first point of Fact 2.30 can be regarded as an analogue of Fact 2.18 while the points (6) and (7) refine Facts 2.6 and 2.7. The points (3), (4) and (5) are clear examples of nilpotent behaviour. It should also be emphasized that the "raison d'être" of the first two points is nothing but Fact 2.9.

Fact 2.30. –

(1) [Bur04, Theorem 2.16] *Let H be a connected solvable group of finite Morley rank. Then $U_0(H) \leq F(H)$.*
(2) [FJ05, Proposition 3.7] *Let $G = NC$ be a group of finite Morley rank where N and C are nilpotent definable connected subgroups and N is normal in G. Assume that there is an integer $n \geq 1$ such that $N = \langle U_{0,s}(N) | 1 \leq s \leq n \rangle$ and $C = \langle U_{0,s}(C) | s \geq n \rangle$. Then G is nilpotent.*
(3) [Bur06, Lemma 2.3] *Let G be a nilpotent group satisfying $U_{0,r}(G) \neq 1$. Then $U_{0,r}(Z(G)) \neq 1$.*
(4) [Bur06, Lemma 2.4] *Let G be a nilpotent $U_{0,r}$-group. If H is a definable proper subgroup of G then $U_{0,r}(N_G(H)/H) > 1$.*
(5) [Bur06, Theorem 2.9] *Let G be a nilpotent $U_{0,r}$-group. Let $\{H_i | 1 \leq i \leq n\}$ be a family of definable subgroups such that $G = \langle \cup_i H_i \rangle$. Then $G = \langle U_{0,r}(H_i) | 1 \leq i \leq n \rangle$.*
(6) [Bur06, Theorem 3.4] *Let G be a divisible nilpotent group of finite Morley rand, and let T be the torsion subgroup G. Then*
$$G \;=\; d(T) * U_{0,1}(G) * U_{0,2}(G) * \ldots * U_{0,\mathrm{rk}(G)}(G) \;.$$
(7) [Bur06, Corollary 3.5] *Let G be a nilpotent group of finite Morley rank. Then $G = D * B$ is a central product of definable characteristic subgroups D, B where D is divisible and B has bounded exponent. The latter group is connected if and only if G is connected.*
Let T be the torsion part of D. Then we have decompositions of D and B as follows.
$$D \;=\; d(T) * U_{0,1}(G) * U_{0,2}(G) \ldots$$
$$B \;=\; U_2(G) \oplus U_3(G) \oplus \ldots$$

For a prime p, $U_p(G)$ is the largest normal p-unipotent subgroup of G.

The new notion of unipotence behaves well under homomorphisms:

Fact 2.31. – **(Burdges** [Bur04, Lemma 2.11]**)** *Let $f : G \longrightarrow H$ be a definable homomorphism between two groups of finite Morley rank. Then*

- *(Push-forward) $f(U_{0,r}(G)) \leq U_{0,r}(H)$ is a $U_{0,r}$-subgroup.*
- *(Pull-back) If $U_{0,r}(H) \leq f(G)$ then $f(U_{0,r}(G)) = U_{0,r}(H)$.*

In particular, an extension of a $U_{0,r}$-subgroup by a $U_{0,r}$-subgroup is a $U_{0,r}$-subgroup.

The work of the third author showed that the theory of unipotence is much better behaved when the unipotent groups in question are homogeneous in the sense of Definition 2.29. Remarkably, as points (1), (3) and (4) of Fact 2.32 illustrate, in order to find homogeneous groups it suffices to avoid central elements.

Fact 2.32. –

(1) [Fré06a, Theorem 4.11] *Let G be a connected group of finite Morley rank. Assume that G acts definably by conjugation on H, a nilpotent $U_{0,r}$-group. Then $[G,H]$ is a homogeneous $U_{0,r}$-group.*

(2) [Fré06a, Theorem 5.4] *Let G be a U-group. Then G has the following decomposition:*

$$G \;=\; B * U_{0,1}(G) * U_{0,2}(G) * \ldots * U_{0,\overline{r}(G)}(G),$$

where

(i) *B is definable, connected, definably characteristic and of bounded exponent;*

(ii) *$U_{0,s}(G)$ is a homogeneous $U_{0,s}$-subgroup for each $s \in \{1,2,\ldots,\overline{r}(G)\}$;*

(iii) *the intersections of the form $U_{0,s}(G) \cap U_{0,t}(G)$ are finite. In particular, if G does not contain a bad group, then*

$$G \;=\; B \times U_{0,1}(G) \times U_{0,2}(G) \times \ldots \times U_{0,\overline{r}(G)}(G).$$

(3) [Fré06a, Corollary 6.8] *Let G be a solvable connected group of finite Morley rank. Then G' is a U-group.*

(4) [Fré06a, Lemma 4.3] *Let G be a nilpotent $U_{0,r}$-group. Then $G/Z(G)^\circ$ is a homogeneous $U_{0,r}$-group.*

A natural question was whether it was possible to develop a Sylow theory using the notions introduced in Definition 2.29. The second author answered this affirmatively in the context of connected solvable groups of finite Morley rank.

Fact 2.33. –

(1) [Bur06, Lemma 6.2] *In a group G of finite Morley rank, the Sylow $U_{0,r}$-subgroups are exactly those nilpotent $U_{0,r}$-subgroups S such that $U_{0,r}(N_G(S)) = S$.*

(2) [Bur06, Theorem 6.5] *Let H be a connected solvable group of finite Morley rank. Then the Sylow $U_{0,r}$-subgroups of H are conjugate in H.*

(3) [Bur06, Theorem 6.7] *Let H be a connected solvable group of finite Morley rank and let Q be a Carter subgroup of H. Then $U_{0,r}(H')U_{0,r}(Q)$ is a Sylow $U_{0,r}$-subgroup of H, and every Sylow $U_{0,r}$-subgroup has this form for some Carter subgroup of H.*

(4) [Bur06, Corollary 6.9] *Let H be a connected solvable group of finite Morley rank and let S be a Sylow $U_{0,r}$-subgroup of H. Then $N_H(S)$ contains a Carter subgroup of H.*

(5) [FJ05] *Let G be a group of finite Morley rank. If r is an integer and if S is a Sylow $U_{0,r}$-subgroup of G such that $N_G(S)$ has no nontrivial decent torus and such that $U_{0,s}(N_G(S)) = 1$ for each $s < r$, then S is contained in a Carter subgroup of G as a normal subgroup.*

These results that we will use intensively in this paper have been key to the progress in local analysis in connected minimal simple groups of finite Morley rank. The facts below summarize the major ingredients of local analysis.

Fact 2.34. –

(1) [Bur07, Lemma 2.1] *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups satisfying $U_{p_1}(B_1) \neq 1$ and $U_{p_2}(B_2) \neq 1$. Then $F(B_1) \cap F(B_2) = 1$.*

(2) [Bur07, Corollary 2.2] *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups of G. Then $F(B_1) \cap F(B_2)$ is torsion-free.*

Fact 2.35. – [Bur07, Proposition 4.1] *Let G be a minimal connected simple group. Let B_1, B_2 be two distinct Borel subgroups of G. Let H be a definable connected subgroup of the intersection $B_1 \cap B_2$. Then the following hold:*

(1) *H' is homogeneous or trivial.*

(2) *Any definable connected nilpotent subgroup of $B_1 \cap B_2$ is abelian.*

Fact 2.36. – [Bur07, Theorem 4.3]

(1) *Let G be a minimal connected simple, and let B_1, B_2 be two distinct Borel subgroups of G. Suppose that $H = (B_1 \cap B_2)^\circ$ is non-abelian. Then the following are equivalent:*
   (a) *B_1 and B_2 are the only Borel subgroups G containing H.*
   (i) *If B_3 and B_4 are distinct Borel subgroups of G containing H, then $(B_3 \cap B_4)^\circ = H$.*
   (ii) *$C_G^\circ(H')$ is contained in B_1 or B_2.*
   (iii) *$\overline{r}_0(B_1) \neq \overline{r}_0(B_2)$.*

(2) [Bur07, Lemma 3.28] *If one of the equivalent conditions of* (1) *holds and $\overline{r}_0(B_1) > \overline{r}_0(B_2)$, then B_1 is the only Borel subgroup containing $N_G(H')^\circ$.*

(3) [Bur07, Consequence of Theorem 4.5 (4)] *If one of the equivalent conditions of* (1) *holds and $\overline{r}_0(B_1) > \overline{r}_0(B_2)$ and $r = \overline{r}_0(H')$, then $F_r(B_2)$ is non-abelian, where $F_r(X)$ denotes $U_{0,r}(F(X))$ with X a solvable connected group of finite Morley rank.*

3. Recent progress around the Weyl group

In this section, we will go over recent results that will play a major role in this article. The three main references are [Del08] that generalizes the fundamental [CJ04] to a non-tame context, [BD09] and [ABF12]. The main theme is the notion of the Weyl group of a group G of finite Morley rank, denoted $W(G)$, and defined as $N_G(T)/C_G(T)$ where T is any maximal decent torus of G. By Fact 2.21, the Weyl group of a group of finite Morley rank G is well-defined. By Facts 2.22 (3) and (4), $W(G)$ is finite, and by the torsion-lifting properties of groups of finite Morley rank, the non-triviality of the Weyl group implies the non-triviality of torsion in the ambient group. It thus follows from Fact 2.24 that, in a connected minimal simple group of finite Morley rank, there is a strong connection between the generic element of the group and the Weyl group. This connection is also illustrated by Fact 3.9.

In a simple algebraic group G, one could alternately define the Weyl group as $N_G(C)/C$ where C is any Carter subgroup of G. The following fact shows that the same equivalence holds in a connected minimal simple group of finite Morley rank:

Fact 3.1. – [ABF12, Proposition 3.2] *Let G be a minimal connected simple group, and let C be a Carter subgroup of G. Then the Weyl group $W(G)$ of G is isomorphic to $N_G(C)/C$.*

The following statements are rapid corollaries:

Fact 3.2. –

(1) *If C is a Carter subgroup of a minimal connected simple group G, then C is a maximal nilpotent subgroup.*
(2) *Let G be a minimal connected simple group, and let S be a non-trivial p-torus for a prime p. Then $N_G(S)/C_G(S)$ is isomorphic to a subgroup of $W(G)$. Moreover, if S is maximal, then we have $N_G(S)/C_G(S) \simeq W(G)$.*
(3) *Let G be a minimal connected simple group, and let T be a maximal pseudo-torus of G. Then $W(G)$ is isomorphic to $N_G(T)/C_G(T)$.*

Next, we revise a variety of results on the structure of $W(G)$ when G is minimal connected simple.

Fact 3.3. – (Particular case of [Del08, Théorème-Synthèse]) *Let G be a minimal connected simple group of odd type. Then G satisfies one of the following four conditions:*

- *$G \simeq \mathrm{PSL}_2(K)$ for an algebraically closed field K of characteristic $p \neq 2$;*
- *$|W(G)| = 1$, and the Prüfer 2-rank of G is one;*
- *$|W(G)| = 2$, the Prüfer 2-rank of G is one, and G has an abelian Borel subgroup C;*
- *$|W(G)| = 3$, the Prüfer 2-rank of G is two, and the Carter subgroups of G are not Borel subgroups.*

Fact 3.4. – [BD09, Theorem 4.1] *Let G be a minimal connected simple group, T a maximal decent torus of G, and τ the set of primes p such that $\mathbb{Z}_{p^\infty}$ embeds into T. Then $W(G)$ is cyclic, and has an isomorphic lifting to G. Moreover, no element of τ divides $|W(G)|$, except possibly 2.*

The results of [BD09, §3] do not need that the group G be degenerate, but just that $|W(G)|$ be odd. This increases their relevance for us in conjonction with results from [BC08b]. In particular, the following fact holds.

Fact 3.5. – [BD09, §3][BC08b, §5] *Let G be a minimal connected simple group, T a maximal decent torus of G, τ the set of primes p such that $\mathbb{Z}_{p^\infty}$ embeds into T, and τ' its complement. If $W(G)$ is non-trivial and of odd order, then the following conditions hold:*

(1) [BC08b, Corollary 5.3] *the minimal prime divisor of $|W(G)|$ does not belong to τ;*
(2) *if a is a τ'-element of $N_G(T)$, then $C_{C_G(T)}(a)$ is trivial;*
(3) [BD09, Corollary 3.8] *$C_G(T)$ is a Carter subgroup of G;*

In the rest of this section, we will recall various facts from [ABF12].

Fact 3.6. – [ABF12, Lemma 3.9] *Let B_1 and B_2 be two generous Borel subgroups of a minimal connected simple group G. Then there exists $g \in G$ such that $B_1 \cap B_2^g$ contains a generous Carter subgroup of G.*

Fact 3.7. – [ABF12, Lemma 3.10] *Let G be a minimal connected simple group with a nilpotent Borel subgroup B. Then B is a Carter subgroup of G, and the generous Borel subgroups of G are conjugate with B, and they are generically disjoint.*

The following theorem, that will not be directly used in this article, is involved in the proofs of many other crucial facts such as 3.9, 3.12, 3.13.

Fact 3.8. – [ABF12, Theorem 3.12] *Any non-nilpotent generous Borel subgroup B of a minimal connected simple group G is self-normalizing.*

In [ABF12], a uniform approach to the analysis of connected minimal simple groups of finite Morley rank was introduced through a case division that consists of four mutually exclusive classes of groups. This case division follows two criteria: generic disjointness of Borel subgroups from their conjugates and (non-)triviality of the Weyl group. The following table introduces the four types of groups that emerge from these two criteria:

		A Borel subgroup generically disjoint from its conjugates	
		exists	does not exist
Weyl group	trivial	(1)	(2)
	non-trivial	(3)	(4)

Fact 3.9. – **(Tetrachotomy theorem)** [ABF12, Theorem 4.1] *Any minimal connected simple group G satisfies exactly one of the following four conditions:*

- *G is of type (1), its Carter subgroups are generous and any generous Borel subgroup is generically disjoint from its conjugates;*
- *G is of type (2), it is torsion-free and it has neither a generous Carter subgroup, nor a generous Borel subgroup;*
- *G is of type (3), its generous Borel subgroups are nilpotent: they are the Carter subgroups;*
- *G is of type (4), its Carter subgroups are generous, and there is no nilpotent Borel subgroup.*

In the sequel, by "type (i)" we will mean one of the four types caracterized in Theorem 3.9. Although we will try to obtain results as general as possible, the terminology and conclusions of the Tetrachotomy theorem will be essential in the development of this article. It should be emphasized the (*) condition in the introduction is also best appreciated in the light of this four-way case division and its consequences. Indeed, the condition (*) describes a strict subclass of groups of type (1), and it in particular *excludes* the bad groups. We recall some remarks from [ABF12] concerning the four types of groups.

Remark 3.10. –

- *Bad groups* [BN94, Chapter 13], and more generally *full Frobenius groups* [Jal01], are examples of groups of type (1). The existence of any of these groups is a well-known open problem.
- The minimal connected simple groups with a nongenerous Carter subgroup are of type (2) and are analyzed in [Fré08].
- The group $\mathrm{PSL}_2(K)$ for an algebraically closed field K, is of type (4).
- By Fact 2.15, the classification of simple groups of even type, and Theorem 3.9, a non-algebraic minimal connected simple group with involutions is of odd type and not of type (2). But the existence of a minimal connected simple group with involutions and either of type (1), or of type (3), or not algebraic and of type (4), is an open problem. A comparison of Fact 3.3

and Theorem 3.9 show that the three pathological configurations by Deloro in [Del08] corresponds to the groups of type (1), (3) and (4) respectively.

As Remark 3.10 suggests, minimal connected simple algebraic groups over algebraically closed fields are of type (4). Thus, one expects simple groups of type (4) to have properties close to those of algebraic groups. Fact 3.11 provides evidence in this direction.

Fact 3.11. – [ABF12, Theorem 5.1] *Let G be a minimal connected simple group of type (4). Then there is an interpretable field K such that each Carter subgroup definably embeds in $K^* \times K^*$.*

The following fact is of fundamental importance for the analysis of groups of type (4). Many proofs in this article depend very much on the non-triviality of the Weyl group when *all* the Borel subgroups of the ambient connected minimal simple group are non-nilpotent.

Fact 3.12. – [ABF12, Proposition 3.13] *Let H be a subgroup of a minimal connected simple group G. If H contains a Carter subgroup C of G, then H is definable, and either it is contained in $N_G(C)$, or it is connected and self-normalizing.*

The following is a corollary proven in [ABF12].

Fact 3.13. – [ABF12, Corollary 3.14] *Let G be a minimal connected simple group with a non-trivial Weyl group, and let T be a non-trivial maximal p-torus of G for a prime p. Then $C_G(T)$ is a Carter subgroup of G.*

The following corollary has a similar proof.

Corollary 3.14. – *Let G be a minimal connected simple group, and T a maximal decent torus of G. If $W(G)$ is non-trivial, then $C_G(T)$ is a Carter subgroup of G and any Carter subgroup of G has this form.*

4. Major Borel subgroups

In this section, we will introduce and analyze the structure of a special class of Borel subgroups that we will call *major*.

Definition 4.1. – *Let G be a group of finite Morley rank. A Borel subgroup B of G is said to be a* major Borel *subgroup if it satisfies the following conditions:*

(1) *every Carter subgroup of B is contained in a Carter subgroup of G;*
(2) *for every non-nilpotent Borel subgroup A and Carter subgroup C of G such that $A \cap C$ contains a Carter subgroup of B, $\mathrm{rk}(A \cap C) = \mathrm{rk}(B \cap C)$.*

Except for pathological cases, in a connected minimal simple group G that is *not* a bad group, that thus possesses a non-nilpotent Borel subgroup, a Carter subgroup C of G group and a non-nilpotent Borel that contains C, present a picture closer to that of a connected minimal simple algebraic group over an algebraically closed field. In presence of a Carter subgroup that is itself a Borel subgroup, this picture is lost. The notion of a major Borel subgroup tries to remedy this deficiency and yields a picture sufficiently close to the natural one. Indeeed, the main result of this section, Theorem 4.9, proves the existence of a factorization of major Borel subgroups in minimal simple groups with a non-trivial Weyl group in a way very reminiscent of the decomposition of connected solvable algebraic groups as semidirect product of their unipotent part by their maximal tori [Hum81, Theorem 19.3].

Theorem 4.9 is proven under a technical assumption that may look exotic: Conditions (N) and (W) in its statement. The assumption describes in fact the groups covered by the negation of the (*) condition in the introduction, except a subclass of groups of type (2). The groups of type (2) will be treated separately in Subsection 5.2. As a result, the structural description provided by Theorem 4.9 will involve all connected minimal simple groups except a rather pathological subclass of groups of type (1), namely those covered by the (*) assumption.

Remark 4.2. – 1. Every nilpotent Borel subgroup of a group of finite Morley rank is a major Borel subgroup. Indeed, if B is such a Borel subgroup, then by Fact 2.4, $N_G(B)/B$ is finite, and as a result, B is a Carter subgroup of G and satisfies the two conditions defining a major Borel subgroup.

2. In a group of finite Morley rank whose Borel subgroups are not nilpotent, the major Borel subgroups are those containing a Carter subgroup of G.

Lemma 4.3. – *Let G be a minimal connected simple group that has a nilpotent and a non-nilpotent Borel subgroup. Let C be a Carter subgroup of G. Then there exists a non-nilpotent Borel subgroup A of G such that $A \cap C \neq 1$.*

Proof – We assume toward a contradiction that $A \cap C$ is trivial for each non-nilpotent Borel subgroup A. By Fact 2.23 (2) and (4), each pseudo-torus of G is contained in a conjugate of C.

Suppose toward a contradiction that G has a non-trivial p-unipotent subgroup U_0 not contained in a conjugate of C for a prime integer p. Let U be a maximal p-unipotent subgroup containing U_0. Then $N_G(U)^\circ$ centralizes all the $N_G(U)^\circ$-minimal sections of U, otherwise it would contain a nontrivial pseudo-torus by Facts 2.9, 2.20 and 2.22 (1), so $N_G(U)^\circ$ would have a non-trivial intersection with a conjugate of C, and it would be contained in C^g for a $g \in G$ by our contradictory assumption, contradicting our choice of U. Let D be a Carter subgroup of $N_G(U)^\circ$. Since D centralizes all the $N_G(U)^\circ$-minimal sections of U, it contains U. Moreover, since U is a maximal p-unipotent subgroup of G, it is definably characteristic in D. So we obtain $N_G(D) \leq N_G(U)$, and D is a Carter subgroup of G. Thus D is conjugate to C by Fact 2.23 (4), contradicting our choice of U. This proves that, for each prime p, any p-unipotent subgroup of G is contained in a conjugate of C.

Let B be a non-nilpotent Borel subgroup of G. By our contradictory assumption, $B \cap C^g$ is trivial for each $g \in G$. Then the pseudo-tori of B and its p-unipotent subgroups are trivial for each prime p, so B is torsion-free by Fact 2.15. In particular, B contains a non-trivial nilpotent $U_{0,r}$-group for a positive integer r. Let r be the smallest positive integer such that there is a non-trivial nilpotent $U_{0,r}$-group intersecting trivially any conjugate of C.

Let T be a Sylow $U_{0,r}$-subgroup of G contained in no conjugate of C, B a Borel subgroup of G containing $N_G(T)^\circ$, and D a Carter subgroup of $N_G(T)^\circ$. By our choice of T and by our contradictory assumption, $B \cap C^g$ is trivial for each $g \in G$. This implies that $N_G(T)^\circ$ is torsion-free, and by minimality of r, we have $U_{0,s}(D) = 1$ for each $s < r$. Thus, by Fact 2.30 (2) and (6), the group TD is nilpotent, and since D is a Carter subgroup of $N_G(T)^\circ$, we have $T \leq D$. Since T is a Sylow $U_{0,r}$-subgroup of G, it is normal in $N_G(D)$. Therefore we have $N_G(D)^\circ \leq N_G(T)^\circ$ and D is a Carter subgroup of G. By Fact 2.23 (4), D

is conjugate to C, contradicting that T is contained in no conjugate of C. Hence there exists a non-nilpotent Borel subgroup A of G such that $A \neq C$ and $A \cap C \neq 1$. □

Lemma 4.4. – *Let G be a minimal connected simple group that has a nilpotent Borel subgroup C. Then, for each Borel subgroup B of G, either $B \cap C$ is a Carter subgroup of B, or $B \cap C$ is torsion-free.*

Proof – If $B \cap C$ is of finite index in $N_B(B \cap C)$, then Fact 2.23 (8) shows that $B \cap C$ is a Carter subgroup of B. So we may assume that $B \cap C$ is of infinite index in $N_B(B \cap C)$.

We assume toward a contradiction that the torsion part R of $B \cap C$ is non-trivial. If $U_p(C)$ is trivial for each prime p, then R is central in C by Fact 2.6, and $N_G(R)$ contains $N_G(B \cap C)$ and C. Since C is a Borel subgroup of G, this implies that $C = N_G(R)^\circ$ and that $B \cap C$ is of finite index in $N_B(B \cap C)$, contradicting that $B \cap C$ is of infinite index in $N_B(B \cap C)$. Therefore $U_p(C)$ is non-trivial for a prime p. As a result $U_p(C_C(R))$ is non-trivial by Fact 2.6, and C is the only Borel subgroup of G containing $N_G(R)^\circ$ by Fact 2.34 (1). Thus, once again we conclude that C contains $N_G(B \cap C)^\circ$ and thus $B \cap C$ is of finite index in $N_B(B \cap C)$, contradicting that $B \cap C$ is of infinite index in $N_B(B \cap C)$. Hence $B \cap C$ is torsion-free, as desired. □

Corollary 4.5. – *Let G be a minimal connected simple group that has a nilpotent Borel subgroup. Then, for each Carter subgroup C of G and each Borel subgroup $B \neq C$, the subgroup $B \cap C$ is abelian and divisible.*

Proof – First, we note that by the hypothesis on the Borel subgroups of G and Fact 2.23 (4), C is a Borel subgroup of G. Lemma 4.4 and Fact 2.35 (2) imply that the subgroup $B \cap C$ is connected and abelian. On the other hand, since $F(B) \cap C$ is torsion-free by Fact 2.34 (2), we have $U_p(B \cap C) = 1$ for each prime p. Thus, $B \cap C$ is divisible by Fact 2.5. □

Lemma 4.6. – *Let G be a minimal connected simple group that has a nilpotent and a non-nilpotent Borel subgroup. Then, for each Carter subgroup C of G and each Borel subgroup $B \neq C$, there is a Borel subgroup $A \neq C$ such that $A \cap C$ contains $B \cap C$ and is a Carter subgroup of A.*

Moreover, if $B \cap C$ has torsion or if $rk(B_0 \cap C) = rk(B \cap C)$ for each Borel subgroup $B_0 \neq C$ containing $B \cap C$, then $B \cap C$ is a Carter subgroup of B.

Proof – First, we note that by the hypothesis on the Borel subgroups of G and Fact 2.23 (4), C is a Borel subgroup of G. By Lemma 4.4, we may assume that $B \cap C$ is torsion-free and is of infinite index in $N_B(B \cap C)$. Moreover, we may assume that, for each Borel subgroup $A \neq C$ containing $B \cap C$ we have $rk(A \cap C) = rk(B \cap C)$. By Lemma 4.3, this implies that $B \cap C$ is non-trivial. We consider a Borel subgroup A containing $N_G(B \cap C)^\circ$. In particular, A contains $B \cap C$ since $B \cap C$ is torsion-free. Since $B \cap C$ is of infinite index in $N_B(B \cap C)$, we have $A \neq C$ and it follows that $rk(A \cap C) = rk(B \cap C)$. On the other hand, since $C > B \cap C$ is nilpotent, $B \cap C$ is of infinite index in $N_C(B \cap C)^\circ \leq A \cap C$. This contradiction finishes the proof. □

Proposition 4.7. – *Let G be a minimal connected simple group that has a nilpotent and a non-nilpotent Borel subgroup. Then the following two conditions are equivalent for any Borel subgroup B of G:*

(1) *B is a non-nilpotent major Borel subgroup;*
(2) *there is a Carter subgroup $C \neq B$ of G such that, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $rk(A \cap C) = rk(B \cap C)$;*

In this case, $B \cap C$ is an abelian divisible Carter subgroup of B, and each Carter subgroup of B has the form $B \cap C^b$ for $b \in B$.

Moreover, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $A \cap C = B \cap C$.

Proof – First we assume that B is a non-nilpotent major Borel subgroup of B. Let D be a Carter subgroup of B. Then D is contained in a Carter subgroup C of G, and we have $C \neq B$ since B is non-nilpotent. Moreover, for each Borel subgroup $A \neq C$ containing $B \cap C$, either A is nilpotent or $rk(A \cap C) = rk(B \cap C)$. But Lemma 4.6 applied to A shows that $A \cap C$ is a Carter subgroup of A, so A is non-nilpotent, and we have $rk(A \cap C) = rk(B \cap C)$. Hence, since $A \cap C$ is connected by Corollary 4.5, we obtain $A \cap C = B \cap C$.

Now we assume that there is a Carter subgroup $C \neq B$ of G such that, for each Borel subgroup $A \neq C$ containing $B \cap C$, we have $rk(A \cap C) = rk(B \cap C)$. Then $B \cap C$ is a Carter subgroup of B by Lemma 4.6. In particular, B is non-nilpotent, and $B \cap C$ is abelian and divisible by Corollary 4.5. Moreover, Fact 2.23 (3) shows that any Carter subgroup of B has the form $B \cap C^b$ for $b \in B$. This implies the result. The final conclusion follows from Fact 2.23 (5). □

Corollary 4.8. – *Let G be a minimal connected simple group. Either G is a bad group or it has a non-nilpotent major Borel subgroup.*

Proof – Since every Borel subgroup of a bad group is nilpotent, we may assume G is not bad using Remark 4.2 (1). If all Borel subgroups of G are non-nilpotent, then any Borel containing a Carter subgroup of G is major. Indeed, it can be easily checked that a non-nilpotent Borel subgroup containing a Carter subgroup of G is major. Thus, we may assume G has a nilpotent Borel subgroup, say C, which is evidently a Carter subgroup of G.

By Lemma 4.6, there exists a Borel subgroup B of G such that $B \neq C$ and $B \cap C$ is a Carter subgroup of B. The same lemma allows us to assume that the intersection $B \cap C$ is of maximal Morley rank with respect to these properties. To such a pair and their intersection applies clause (2) of Proposition 4.7. □

Now, we can prove the main theorem of this section.

Theorem 4.9. – *Let G be a minimal connected simple group that is not bad and that satisfies one of the following conditions:*

(N) *G has a nilpotent Borel subgroup;*
(W) *no Borel subgroup of G is nilpotent and $W(G) \neq 1$.*

Let B be a non-nilpotent major Borel subgroup G and C be a Carter subgroup of G containing a Carter subgroup D of B. Then the following conclusions follow

$$D = B \cap C, \quad B = B' \rtimes D \quad \text{and} \quad Z(B) = F(B) \cap D \ ;$$

moreover, C is the only Carter subgroup of G such that $D = B \cap C$, and for each Borel subgroup $A \neq C$ containing $B \cap C$, the equality $rk(A \cap C) = rk(B \cap C)$ holds.

Furthermore, B has the following properties:

(1) *for each prime p, either $U_p(B')$ is the unique Sylow p-subgroup of B, or each Sylow p-subgroup of B is a p-torus contained in a conjugate of D;*

(2) *for each positive integer* $r \leq \overline{r}_0(D)$*, each Sylow* $U_{0,r}$*-subgroup of* B *has the form* $U_{0,r}(D^b)$ *for* $b \in B$*.*

Proof – Before going any further, we emphasize that condition (W) in the statement of the theorem describes exactly the minimal connected simple groups of type (4), as is justified by Theorem 3.9. We will stick to this latter terminology during the proof. As for condition (N), it covers completely groups of type (3), but is more general.

We note that if G is of type (4), then B contains a Carter subgroup of G (Remark 4.2 (2) and Fact 3.9 (4)). Thus, $D = C$ by Fact 2.23 (3), so $D = B \cap C$. If G has a nilpotent Borel subgroup, then C is a Borel subgroup of G (Fact 3.7) and B, despite being major, is relatively small. Nevertheless, as we will now show, it still controls the conjugacy of the Carter subgroups of G that it intersects non-trivially. By Proposition 4.7 there exists a Carter subgroup C_0 of G such that, for each Borel subgroup $A \neq C_0$ containing $B \cap C_0$, we have $rk(A \cap C_0) = rk(B \cap C_0)$, and that $D = B \cap C_0^b$ for some $b \in B$. We thus conclude that $rk(B_0 \cap C_0^b) = rk(B \cap C_0^b)$ for each Borel subgroup $B_0 \neq C_0^b$ containing $B \cap C_0^b$. Since C is a Borel subgroup of G that contains D, if $C \neq C_0^b$, then $rk(B_0 \cap C_0^b) = rk(C \cap C_0^b)$ for each Borel subgroup $B_0 \neq C_0^b$ containing $C \cap C_0^b$. Thus by Lemma 4.6 $C \cap C_0^b$ is a proper Carter subgroup of C, a contradiction to the nilpotence of C. Hence we have $C = C_0^b$ and $D = B \cap C$. This argument also shows the following two conclusions:

- for each Borel subgroup $A \neq C$ containing $B \cap C$, $rk(A \cap C) = rk(B \cap C)$;
- C is the only Carter subgroup of G such that $D = B \cap C$.

Now, by Fact 2.23 (6), we have $B = B'D$ and, by Fact 2.23 (5), we obtain $Z(B) \leq N_B(D) = D$, so $Z(B) \leq F(B) \cap D$. On the other hand, using Fact 3.11 when G is of type (4) and Corollary 4.5 in presence of a nilpotent Borel subgroup, one concludes that D is divisible and abelian. We also remind that B/B' is divisible by Facts 2.10 and 2.11.

We verify assertion (1). Let p be a prime integer. We will show that, either $U_p(B')$ is the unique Sylow p-subgroup of B, or each Sylow p-subgroup of B is a p-torus contained in a conjugate of D. We may assume that $U_p(B')$ is not a Sylow p-subgroup of B. By Fact 2.28, there is no non-trivial p-torus in B'. It then follows from Facts 2.6 and 2.17 (1) that $U_p(B')$ is the Sylow p-subgroup of B'. Since $B = B'D$ and since D is abelian and divisible, the Sylow p-subgroup T of D is a non-trivial p-torus. Then, Facts 2.17 (2) and 2.13 (2) imply that there is a Sylow p-subgroup of B in $C_B(T)$. On the other hand, $C = C_G(T)$. Indeed, if G has a nilpotent Borel subgroup then C is one such, and since $C_G(T) \geq C$, we have equality using Fact 2.22 (3); if, on the other hand, G is of type (4), then $D = C$ and Fact 3.13 implies that $C = C_G(T)$. It follows from the preceding conclusions that T is a Sylow p-subgroup of B, and (1) is then a consequence of Fact 2.16.

We note that, since D is abelian and divisible, assertion (1) implies that $B' \cap D$ is torsion-free.

Now we assume that $s = \overline{r}_0(D)$ is positive, and we consider a Sylow $U_{0,s}$-subgroup S of G containing $U_{0,s}(D) = U_0(D)$. We suppose toward a contradiction that C does not contain S. We note that the hypothesis $s > 0$ implies that $U_0(D) \neq 1$. Let $R = U_{0,s}(S \cap C)$. If G is of type (4), then we have $D = C$, so $R = U_0(D)$, and R is normal in $N_G(D)$, and D is not self-normalizing in $N_G(R)$ as we have $N_G(D)/D \simeq W(G) \neq 1$ by Fact 3.1. On the other hand, Fact 2.33 (1) gives $R < U_{0,s}(N_S(R))$, and we obtain $D < N_G(R)^\circ$. Therefore Fact 3.12 shows that

$N_G(R)$ is a solvable connected subgroup of G. In particular D is self-normalizing in $N_G(R)$ (Fact 2.23 (5)), contradicting that D is not self-normalizing in $N_G(R)$.

If G has a nilpotent Borel subgroup, then C is one such. Fact 2.30 (7) implies that $D = U_0(D)C_D(U_{0,s}(C))$, so D normalizes R. Thus $N_G(R)^\circ$ contains D, and the maximality of the intersection $D = B \cap C$ implies either $N_C(R)^\circ = D$ and $R = U_0(D)$, or $N_G(R)^\circ \leq C$. But, as S is not contained in C, Fact 2.33 (1) implies $R < U_{0,s}(N_S(R))$, and we obtain $N_G(R)^\circ \not\leq C$, so we have $N_C(R)^\circ = D$ and $R = U_0(D)$. Consequently, we obtain $N_C(D)^\circ \leq N_C(U_0(D))^\circ = N_C(R)^\circ = D$, contradicting $D < N_C(D)^\circ$. Thus, in all the cases, $U_{0,s}(C)$ is the only $U_{0,s}$-subgroup of G containing $U_0(D)$.

We assume toward a contradiction that there exists a positive integer $r \leq \overline{r}_0(D)$ such that $U_{0,r}(B')$ is non-trivial. Then, by Fact 2.30 (2), the subgroup $U_{0,r}(B')U_0(D)$ is nilpotent. On the other hand, by Facts 2.32 (2) and (3), there is a definable connected definably characteristic subgroup A of B' such that $B' = A \times U_{0,r}(B')$. But, since $U_{0,r}(B')$ is non-trivial, B/A is not abelian. Hence, since D is abelian and satisfies $B = B'D$, the group $U_{0,r}(B')$ is not contained in D. Now, in the case $r = \overline{r}_0(D)$, the group $U_{0,r}(B')U_0(D)$ is a nilpotent $U_{0,r}$-subgroup of B containing the $U_{0,r}(D) = U_0(D)$ and not contained in C. Since this contradicts the previous paragraph, we obtain $r < \overline{r}_0(D)$, and by Fact 2.30 (7) $U_{0,r}(B')$ centralizes $U_0(D)$. In particular, this gives $U_{0,r}(B') \leq N_G(U_0(D))^\circ$. If G is of type (4), this yields $C < N_G(U_0(C))^\circ$, and Fact 3.12 shows that $N_G(U_0(C))$ is a definable connected solvable subgroup of G. Since it contains $N_G(C)$, we have a contradiction with Facts 2.23 (5) and 3.1. If G has a nilpotent Borel subgroup, then $D < N_C(D)^\circ \leq N_C(U_0(D))^\circ$, and the maximality of $D = B \cap C$ yields $N_G(U_0(D))^\circ \leq C$ and $U_{0,r}(B') \leq C$, contradicting $U_{0,r}(B') \not\leq D = B \cap C$. Consequently, in all the cases, $U_{0,r}(B')$ is trivial for each positive integer $r \leq \overline{r}_0(D)$.

We note that, since $B' \cap D$ is torsion-free, the last paragraph yields $B' \cap D = 1$ and $B = B' \rtimes D$. On the other hand, for each positive integer $r \leq \overline{r}_0(D)$, the group $[B, U_{0,r}(F(B))]$ is a homogeneous $U_{0,r}$-group by Fact 2.32 (1), so $U_{0,r}(F(B))$ is central in B. Since the torsion part of $F(B) \cap D$ is central in B by Fact 2.19, we obtain $F(B) \cap D \leq Z(B)$ by (Fact 2.30 (7)). Thus $Z(B) = F(B) \cap D$, and the same holds for every Carter subgroup of B by Fact 2.23 (5).

Now we prove assertion (2). Let $r \leq \overline{r}_0(D)$ be a positive integer, and let U be a Sylow $U_{0,r}$-subgroup of B. Since $U_{0,r}(B')$ is trivial, by Fact 2.33 (3) there exists a Carter subgroup Q of B such that $U = U_{0,r}(Q)$. Hence assertion (2) follows from Fact 2.23 (3). □

Corollary 4.10. – *Let G be a minimal connected simple group that is not bad and that satisfies one of the following conditions:*

(N) *G has a nilpotent Borel subgroup;*
(W) *no Borel subgroup of G is nilpotent and $W(G) \neq 1$.*

Let B be a non-nilpotent major Borel subgroup G and C be a Carter subgroup of G containing a Carter subgroup D of B. If H is a subgroup of B containing a Carter subgroup D of B, then the following conditions are satisfied:

$$H = H' \rtimes D \quad \text{and} \quad Z(H) = F(H) \cap D.$$

Furthermore, H has the following properties:

(1) *for each prime p, either $U_p(H')$ is the unique Sylow p-subgroup of H, or each Sylow p-subgroup of H is a p-torus contained in a conjugate of D;*

(2) *for each positive integer* $r \leq \overline{r}_0(D)$*, each Sylow* $U_{0,r}$*-subgroup of* H *has the form* $U_{0,r}(D^h)$ *for* $h \in H$*.*

PROOF – By Fact 2.23 (3) and Theorem 4.9, we may assume $D = B \cap C$. By Fact 2.23 (6), we have $H = H'D$, and Theorem 4.9 gives $H' \cap D \leq B' \cap D = 1$, so $H = H' \rtimes D$. In particular, we have $H' = B' \cap H$.

Now we prove assertion (1). Let p be a prime, and let S be a Sylow p-subgroup of H. By Facts 2.17 (2) and 2.13 (2), we have $S = U_p(H) * T$ for a p-torus T. Then Theorem 4.9 (1) says that we have either $S = U_p(H) \leq U_p(B')$, or $S = T$. In the first case, we have $S \leq B' \cap H = H'$ and $S = U_p(H')$. In the second case, S is contained in a conjugate of D by Fact 2.23 (2) and (3). Now the conjugacy of Sylow p-subgroups in H yields (1).

We prove the second assertion. Let $r \leq \overline{r}_0(D)$ be a positive integer, and let S be a Sylow $U_{0,r}$-subgroup of H. By Theorem 4.9 (2), we have $S \cap H' \leq S \cap B' = 1$, so Fact 2.33 (3) provides a Carter subgroup Q of H such that $U = U_{0,r}(Q)$. Hence assertion (2) follows from Fact 2.23 (3).

From now on, we have just to prove the equality $Z(H) = F(H) \cap D$. By Fact 2.23 (5), we have $Z(H) \leq N_H(D) = D$, so $Z(H)$ is contained in $F(H) \cap D$. On the other hand, since $H = H' \rtimes D$, we have $F(H) = H' \times (F(H) \cap D)$, so the Sylow structure description of H obtained in assertions (1) and (2), together with Fact 2.30 (7), yields the conclusion. □

5. Jordan Decomposition

In the following definition, we propose our *Jordan decomposition.* This section is devoted to proving that for a large subclass of connected minimal simple groups, this decomposition has the same fundamental properties as the one in linear algebraic groups. This subclass is identified by the negation of the (*) hypothesis in the introduction.

Definition 5.1. – *Let* G *be a group of finite Morley rank.*

(1) *We denote by* $\mathcal{S}$ *the union of its Carter subgroups and by* $\mathcal{U}$ *its elements* x *satisfying* $d(x) \cap \mathcal{S} = \{1\}$*.*
(2) *The elements of* $\mathcal{S}$ *are called* semisimple *and the ones of* $\mathcal{U}$ unipotent*.*
(3) *For each subgroup* H *of* G*, we denote by* H_u *the set* $H \cap \mathcal{U}$ *of its unipotent elements, and by* H_s *the set* $H \cap \mathcal{S}$ *of its semisimple elements. A definable connected subgroup* H *is said to be a* semisimple torus *if* $H = H_s$*.*

Remark 5.2. –

(1) If G is to $\mathrm{PSL}_2(K)$ for an algebraically closed field K, then in the language of pure fields its Carter subgroups are the maximal tori and its non-trivial unipotent subgroups have the form B' for B a Borel subgroup. Moreover, each element belongs to a maximal torus or a unipotent subgroup, hence our definitions of semisimple and unipotent elements coincide with the classical definitions in *simple* algebraic groups.
(2) For each definable automorphism α of the pure group G, we have $\alpha(\mathcal{S}) = \mathcal{S}$ and $\alpha(\mathcal{U}) = \mathcal{U}$.
(3) The notion of semisimple torus should be handled with care. Under favorable hypotheses, it describes groups that are similar to algebraic tori, a phenomenon illustrated in Subsection 5.2 as well as in Section 6. Nevertheless, a simple bad group is semisimple torus as Fact 2.12 (3) shows.

Although Definition 5.1 is for an arbitrary group of finite Morley rank, in the rest of this section, G will denote a connected minimal simple group that satisfies the negation of the $(*)$ hypothesis. In Subsection 5.1, preparatory lemmas will be proven under additional hypotheses that exclude a subclass of groups of type (2). These excluded ones will be recovered in Subsection 5.2; tools from [Fré08] permit a uniform treatment of that type. In Subsection 5.3, the properties of the Jordan decomposition will be verified at the level of generality described by the negation of the $(*)$ hypothesis.

5.1. **Preparatory lemmas.** In this subsection, unless otherwise stated, G will denote a conected minimal simple group that is not bad and that is subject to one of the following conditions:

(N) G has a nilpotent Borel subgroup;

(W) no Borel subgroup of G is nilpotent and $W(G) \neq 1$.

We find it useful to remind that the groups satisfying the condition (W) are exactly those of type (4).

Lemma 5.3. – *Let x be an element of a Carter subgroup C of G. Then one of the following three conditions is satisfied:*

(A) *either $C_G(x)$ is connected;*

(B) *or $C_G(x)$ is not connected, $C_G(x) \subseteq \mathcal{S}$ and one of the following holds:*
  (1) *$|W(G)|$ is odd, G has a nilpotent and a non-nilpotent Borel subgroup, $C_G(x) \leq C$ and C is the only Borel subgroup of G that contains $C_G(x)$;*
  (2) *$|W(G)| = 2$, $I(G) \neq \emptyset$, G is of odd type of Prüfer 2-rank 1, x is an involution and belongs to C, $C = C_G(x)^\circ$, $C_G(x) = C_G(x)^\circ \rtimes \langle i \rangle$ where $i \in I(G)$ and inverts $C_G(x)^\circ$.*

Proof – We may assume that $C_G(x)$ is not connected. First we assume that $|W(G)|$ is even. By Corollary 3.14, we have $C = C_G(T)$ for a maximal decent torus T of G. We may assume that G is not isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K. Then Fact 2.15, the classification of simple groups of even type, and Fact 3.3 imply that G is of odd type and of Prüfer 2-rank one. It follows from Fact 3.3 that $|W(G)| = 2$, that involutions of G are conjugate, and that G has an abelian Borel subgroup D such that $N_G(D) = D \rtimes \langle i \rangle$ for an involution i inverting D. By the conjugacy of C and D (Fact 2.23 (4)), we obtain $C_G(x) = N_G(C) = C \rtimes \langle j \rangle$ for an involution j inverting C. In particular, x is an involution, and the elements of jC are involutions, which are semisimple by conjugacy. Hence we may assume that $|W(G)|$ is odd.

We first assume that G has a nilpotent Borel subgroup. In particular, C is a nilpotent Borel subgroup by Theorem 3.9. For each prime p and each p-element $a \in N_G(C) \backslash C$, the prime p divides $|W(G)|$, and by Fact 3.4 there is no non-trivial p-torus in T. Then Fact 3.5 (2) implies $a \notin C_G(x)$, and we conclude $C_G(x) \cap N_G(C) \leq C$. Clearly, this inclusion is evident when $|W(G)| = 1$. Thus, if C is the only Borel subgroup containing $C_G(x)^\circ$, we obtain $C_G(x) \leq N_G(C_G(x)^\circ) \leq N_G(C)$ and $C_G(x) \leq C$, so we may assume that there is a Borel subgroup $B \neq C$ containing $C_G(x)^\circ$. We will show that this leads to the contradictory conclusion that $C_G(x)$ is connected.

In this vein, let B be a Borel containing $C_G(x)^\circ$ and assume by contradiction that $B \neq C$. First, we will show that $U_p(C) = 1$ for any prime p. This will then be used to conclude that $C_C(x)$ is connected. If $U_p(C) \neq 1$ for a prime p, then $U_p(Z(C)) \neq 1$ by Fact 2.6. Since $U_p(Z(C)) \leq C_C(x)^\circ$, it follows using Fact 2.34 (1) that $B = C$, a contradiction to $B \neq C$. Thus, C is divisible and $C_C(x)$ contains the torsion of C by Fact 2.6. It follows that $C_C(x)$ is connected.

Since $x \in C$, the conclusion that $C_C(x)$ is connected implies that $x \in C_C(x) \leq C_G(x)^\circ \leq B$. But by Corollary 4.5, $B \cap C$ is abelian, thus $C_C(x) = B \cap C$. Moreover, if B_0 is another Borel of G such that $B_0 \neq C$ and $B_0 \cap C \geq B \cap C$, then $B_0 \cap C$ is also abelian by Corollary 4.5, and hence, $B_0 \cap C = B \cap C$. It follows from Lemma 4.6 that $B \cap C$ is a Carter subgroup B and thus a Carter subgroup of $C_G(x)^\circ$.

By Fact 2.23 (3) and a Frattini argument, $C_G(x) = C_G(x)^\circ N_{C_G(x)}(C_C(x))$. But for each $g \in N_{C_G(x)}(C_C(x))$, we have $C_C(x) \leq C \cap C^g$. If $C \neq C^g$, then Lemma 4.6 and Corollary 4.5 imply as previously that $C \cap C^g$ is a proper Carter subgroup of C^g, a contradiction to the fact that C^g is nilpotent and connected. It follows that $g \in C_G(x) \cap N_G(C)$. But it has been already argued that this intersection is contained in C. Thus $g \in C_C(x) \leq C_G(x)^\circ$. This proves that $C_G(x)$ is connected, a contradiction. This final contradiction shows that $B = C$. This finishes the proof that C is the only Borel subgroup of G containing $C_G(x)^\circ$.

Finally, we will show that G is not of type (4) when $|W(G)|$ is odd. In this case, $|W(G)| > 1$, thus C has a non-trivial maximal decent torus, denoted T. Moreover, $C = C_G(T)$ by Corollary 3.14. If, toward a contradiction, G is of type (4), then C is abelian, $C_G(x)$ contains C, and we have $C_G(x)^\circ = C$ by Fact 3.12. Then there is a prime p dividing $|C_G(x)/C|$. In particular, p divides $|W(G)|$ by Fact 3.1. We consider a p-element a in $C_G(x) \setminus C$. Since $C = C_G(T)$, we obtain $a \in N_G(T)$ and $x \in C_{C_G(T)}(a) \setminus \{1\}$. Then Facts 3.4 and 3.5 (2) yield a contradiction. □

Corollary 5.4. – *Suppose G has a nilpotent Borel subgroup and a non-nilpotent one, with C a Carter subgroup of G. Let B be a Borel subgroup of G subject to one of the following conditions:*

(1) *B contains $N_G(U)^\circ$, where U is a definable connected subgroup of C;*
(2) *B contains $C_G(x)^\circ$ where $x \in C$.*

Then either $B = C$ or B is a major Borel subgroup. In the latter case, $B \cap C$ is a Carter subgroup of B contained in H, where H is either $N_G(U)^\circ$ as in (1) or $C_G(x)^\circ$ as in (2).

In the case where $H = C_G(x)^\circ$ with $x \in C$, we have $x \in B$.

Proof – In the case where $H = C_G(x)^\circ$ with $x \in C$, $x \in B$ by Lemma 5.3. Thus in both cases, since $B \cap C$ is abelian and divisible by Corollary 4.5, we have $B \cap C \leq H$. Let $A \neq C$ be a Borel subgroup containing $B \cap C$. Similarly $A \cap C \leq H$, and $rk(A \cap C) = rk(H \cap C) = rk(B \cap C)$. Proposition 4.7 yields the result. □

Lemma 5.5. – *Let B be a major Borel subgroup of G, and let C be a Carter subgroup of G such that $D = B \cap C$ is a Carter subgroup of B. Let H be a subgroup of B containing D. Then we have $H_u = H'$ and, for each element x of H, there exists a unique pair $(x_u, x_s) \in d(x)_u \times d(x)_s$ satisfying $x = x_u x_s = x_s x_u$ and such that $d(x) = d(x_u) \times d(x_s)$.*

Furthermore, if A is any subset of H formed by some semisimple elements and that generates a nilpotent subgroup, then A is conjugate in H with a subset of D. In particular, we have $H_s = \cup_{h \in H} D^h$.

Proof – By the Sylow structure description of H obtained in Corollary 4.10, we have $(H')_s = \{1\}$, so H' is contained in H_u.

We show that, for each element x of H, there exist $h \in H$ and $(x_u, x_s) \in (H' \cap d(x)) \times (D^h \cap d(x))$ satisfying $x = x_u x_s = x_s x_u$ and such that $d(x) = d(x_u) \times d(x_s)$. By Fact 2.27, the generalized centralizer $E_H(x)$ of x in H is definable and connected, x belongs to its Fitting subgroup $F(E_H(x))$, and, by Facts 2.25 (1) and 2.26, $E_H(x)$ contains a Carter subgroup Q of H. Moreover, there exists $h \in H$ such that $Q = D^h$ (Fact 2.23 (3)), and Corollary 4.10 yields $F(E_H(x)) = E_H(x)' \times Z(E_H(x))$ and $Z(E_H(x)) = F(E_H(x)) \cap D^h$. It follows from Fact 2.5 that $d(x) = d(x)^\circ \times U$ with U a finite cyclic subgroup, and $d(x)^\circ$ divisible. Also, by Fact 2.30 (6), if T denotes the maximal decent torus of $d(x)$, then $d(x)^\circ$ is the product of T by its Sylow $U_{0,r}$-subgroups for all the positive integers r. Let π be the set of primes p such that $E_H(x)'$ has a non-trivial p-element, and let π' be its complementary in the set of primes. Let S_1 be the set of π-elements of $d(x)$ and let S_2 be the set of π'-elements of $d(x)$. Then Corollary 4.10 (1) implies $S_1 \leq E_H(x)'$ and $S_2 \leq Z(E_H(x))$. Moreover, we have $T \leq d(S_1)d(S_2)$. Also, Corollary 4.10 (2) shows that, for each positive integer r, we have either $U_{0,r}(d(x)) \leq E_H(x)'$ or $U_{0,r}(d(x)) \leq Z(E_H(x))$. This implies $d(x) = (d(x) \cap E_H(x)') \times (d(x) \cap Z(E_H(x)))$. Since $E_H(x)'$ is contained in H', and since $Z(E_H(x))$ is contained in D^h, we obtain $(x_u, x_s) \in (H' \cap d(x)) \times (D^h \cap d(x))$ satisfying $x = x_u x_s = x_s x_u$ and such that $d(x) = d(x_u) \times d(x_s)$.

As for the uniqueness of (x_u, x_s), we note that the above argument of existence depends only on x and H. Indeed, the entire argument was carried out in H and used $d(x)$, $E_H(x)'$ and $Z(E_H(x))$.

Note that, since $\mathcal{U}$ contains H', we have $x_u \in d(x)_u$. On the other hand, since $\mathcal{S}$ contains $D^h \leq C^h$, we have $x_s \in d(x)_s$, and if x is semisimple, then we obtain $x_u = 1$, and $x = x_s$ belongs to $D^h \subseteq \cup_{k \in H} D^k$. This implies the equality $H_s = \cup_{k \in H} D^k$.

Now let $x \in H \setminus H'$. By the previous paragraph, there exists $h \in H$ and $(x_u, x_s) \in (H' \cap d(x)) \times (D^h \cap d(x))$ such that $x = x_u x_s$. In particular, x_s is a non-trivial semisimple element of $d(x)$, so x is not unipotent, and we obtain $H_u = H'$.

Let A be a subset of H formed by some semisimple elements and generating a nilpotent subgroup. Then, by Fact 2.27, the generalized centralizer $E_H(A)$ of A in H is definable and connected, $F(E_H(A))$ contains A and, by Facts 2.25 (1) and 2.26, there is a Carter subgroup P of H in $E_H(A)$. Moreover, since there exists $h \in H$ such that $P = D^h$ (Fact 2.23 (3)), Corollary 4.10 yields $F(E_H(A)) = E_H(A)' \times Z(E_H(A))$ and $Z(E_H(A)) = F(E_H(A)) \cap P$. But, by previous paragraphs, the semisimple elements of $E_H(A)$ are contained in $\cup_{k \in E_H(A)} P^k$. Thus, the ones in $F(E_H(A))$ are central in $E_H(A)$. Hence A is contained in a central subgroup of $E_H(A)$, and we obtain $A \subseteq D^h$, as desired. □

5.2. **Special case: groups of type (2).** In this subsection, we consider a minimal connected simple group G of finite Morley rank and of type (2). We recall that, by Fact 3.9, this group is torsion-free and it has neither a generous Carter subgroup, nor a generous Borel subgroup. It should be emphasized that the main raison d'être of this subsection is the class of groups of type (2) that do not have nilpotent Borel subgroups. Indeed, in the presence of a non-trivial Weyl group or nilpotent and non-nilpotent Borel subgroups, our methods using Major Borel subgroups are sufficient for the subsequent developments.

The following results describe the strong structural properties of nilpotent subgroups of such a group G.

Fact 5.6. – [Fré08, Corollaries 1.8 and 3.17, Fact 3.13] *Any nilpotent definable subgroup H of G has the following decomposition:*

$$H = U_{0,1}(H) \times \cdots \times U_{0,\overline{r}(H)}(H),$$

where $U_{0,i}(H)$ is a homogeneous $U_{0,i}$-subgroup for each $i \in \{1, \ldots, \overline{r}(H)\}$.

Fact 5.7. – [Fré08, Corollary 1.8] *There is an integer c such that any Carter subgroup C of G is a homogeneous $U_{0,c}$-subgroup.*

The following result follows from Facts 2.33 (5) and 5.7.

Corollary 5.8. – *Let c be the smallest integer such that $U_{0,c}(G)$ is nontrivial. Then*

- *the Carter subgroups of G are precisely its Sylow $U_{0,c}$-subgroups;*
- *the semisimple tori of G are precisely its nilpotent $U_{0,c}$-subgroups.*

By using Fact 2.33 (2) too, we obtain also the following result.

Corollary 5.9. – *In each proper definable connected subgroup H of G, the maximal semisimple tori of H are conjugate.*

Proof – Let c be the smallest integer such that $U_{0,c}(G)$ is nontrivial. By Corollary 5.8, the maximal semisimple tori of H are precisely its Sylow $U_{0,c}$-subgroups, and these ones are conjugate by Fact 2.33 (2). □

Proposition 5.10. – *Let H be a definable nilpotent subgroup of G. If c denotes the smallest integer such that $U_{0,c}(G)$ is nontrivial, then H_u and H_s are definable subgroups of G satisfying $H_s = U_{0,c}(H)$ and $H_u = U_{0,c+1}(H) \times \cdots \times U_{0,\overline{r}(H)}(H)$.*

In particular, we have $H = H_u \times H_s$.

Proof – By Corollary 5.8, an element s of G is semisimple if and only if $d(s)$ is a $U_{0,c}$-subgroup of G, and an element u of G is unipotent if and only if $U_{0,c}(d(u))$ is trivial, that is by Fact 5.6:

$$d(u) = U_{0,c+1}(d(u)) \times \cdots \times U_{0,\overline{r}(d(u))}(d(u)).$$

Now the result follows from Fact 5.6, which says that

$$H = U_{0,c}(H) \times \cdots \times U_{0,\overline{r}(H)}(H),$$

where $U_{0,i}(H)$ is a homogeneous $U_{0,i}$-subgroup for each $i \in \{c, \ldots, \overline{r}(H)\}$. □

Proposition 5.11. – *In each definable solvable subgroup H of G, the set H_u of unipotent elements is a definable subgroup such that $H = H_u \rtimes T$ for any maximal semisimple torus T of H.*

Proof – Let c be the smallest integer such that $U_{0,c}(G)$ is nontrivial. Let T be a maximal semisimple torus of H. By Corollary 5.8, it is a Sylow $U_{0,c}$-subgroup of H, and by Fact 2.33 (5), it is contained in a Carter subgroup C of H.

By Fact 5.6, the following decomposition holds

$$F(H) = U_{0,c}(F(H)) \times \cdots \times U_{0,\overline{r}(F(H))}(F(H)),$$

where $U_{0,i}(F(H))$ is a homogeneous $U_{0,i}$-subgroup for each $i \in \{c, \ldots, \overline{r}(F(H))\}$. Since T is a Sylow $U_{0,c}$-subgroup of H, it contains $U_{0,c}(F(H))$ (Fact 2.30 (2)).

By Fact 5.6, the following decomposition holds

$$C = U_{0,c}(C) \times \cdots \times U_{0,\overline{r}(C)}(C),$$

where $U_{0,i}(C)$ is a homogeneous $U_{0,i}$-subgroup for each $i \in \{c, \dots, \overline{r}(C)\}$. Since T is a Sylow $U_{0,c}$-subgroup of H contained in C, it is equal to $U_{0,c}(C)$. In particular, if we consider $D = U_{0,c+1}(C) \times \cdots \times U_{0,\overline{r}(C)}(C)$, then we have $C = T \times D$.

Let $F = U_{0,c+1}(F(H)) \times \cdots \times U_{0,\overline{r}(F(H))}(F(H))$. Since all the $U_{0,c}$-subgroups of DF/F and of F are trivial, Fact 2.31 shows that all the $U_{0,c}$-subgroups of DF are trivial. Thus, by Corollary 5.8, all the elements of DF are unipotent. In particular, we have $H = DF \rtimes T$.

Moreover, if u is a unipotent element of H, then $U_{0,c}(d(u))$ is trivial (Corollary 5.8), and since $H/DF \simeq T$, we find $d(u) \leq DF$ by Fact 2.31. Thus u belongs to $DF \subseteq \mathcal{U}$, and we may conclude $DF = H_u$, as desired. □

5.3. **Main theorem.** In this subsection, we will prove that for the groups that satisfy the negation of the (∗) hypothesis, the Jordan decomposition proposed in Definition 5.1 has the well-known properties of the usual Jordan decomposition in linear algebraic groups. In this vein, G will denote a group that satisfies the negation of the (∗) hypothesis.

Theorem 5.12. – **(Jordan decomposition)**

1. *For each $x \in G$, there exists a unique $(x_s, x_u) \in \mathcal{S} \times \mathcal{U}$ satisfying $x = x_s x_u = x_u x_s$.*
2. *For each $x \in G$, we have $d(x) = d(x_s) \times d(x_u)$.*
3. *For each $(x, y) \in G \times G$ such that $xy = yx$, we have $(xy)_u = x_u y_u$ and $(xy)_s = x_s y_s$.*

Proof – If G is a bad group, then there is nothing to do. Indeed, by Fact 2.12, all the elements of G are semisimple. Thus, we may assume that G is not bad.

We first prove (1) and (2). Let $x \in G \setminus \{1\}$. We show that there exists $(x_s, x_u) \in \mathcal{S} \times \mathcal{U}$ satisfying $x = x_s x_u = x_u x_s$, and such that $d(x) = d(x_s) \times d(x_u)$. If G is of type (2), then by Proposition 5.10, the sets $d(x)_u$ and $d(x)_s$ are definable subgroups of $d(x)$ satisfying $d(x) = d(x)_u \times d(x)_s$, so the existence of x_s and x_u is clear in this case. In the other cases either G has a nilpotent Borel subgroup or G is of type (4). The argument will eventually use this case devision. Note first that we may assume that x is neither semisimple, nor unipotent. In particular, there exists $y \in d(x) \setminus \{1\}$ such that y belongs to a Carter subgroup C_0 of G. Since $x \in C_G(y)$ is not semisimple, Lemma 5.3 shows that $C_G(y)$ is connected. Then, if G is of type (4), we have $C_G(y) \geq C_0$ as C_0 is abelian, and Lemma 5.5 proves the existence of (x_s, x_u). If G has a nilpotent Borel subgroup, then as $C_G(y)$ contains an element that is not semisimple, by Corollary 5.4 there exists a major Borel subgroup B_y containing $C_G(y)$ and such that $B_y \cap C_0$ is a Carter subgroup of B_y. Again, the existence of (x_s, x_u) follows from Lemma 5.5.

Now we show the uniqueness of (x_s, x_u) in the case where G is of type (2). Let $(x'_s, x'_u) \in \mathcal{S} \times \mathcal{U}$ satisfying $x = x'_s x'_u = x'_u x'_s$. We consider $H = C_G(x)$; recall that, G being torsion-free, H is a connected subgroup. We note that H contains x_s, x_u, x'_s and x'_u, and that x_s and x_u are central in H since they belong to $d(x)$. Let T_1 and T_2 be maximal semisimple tori of H containing x_s and x'_s respectively. By Corollary 5.9, there exists $h \in H$ such that $T_1^h = T_2$. Moreover, by Proposition 5.11, the set H_u of unipotent elements of H is a definable subgroup such that $H = H_u \rtimes T_1 = H_u \rtimes T_2$. In particular, we have $x \in x_s H_u = x'_s H_u$, and since x_s is central in H, we find

$$x'_s \in T_2 \cap x'_s H_u = T_1^h \cap x_s H_u = (T_1 \cap x_s H_u)^h = \{x_s\},$$

so $x'_s = x_s$. Now we have $x'_u = x_u x_s (x'_s)^{-1} = x_u$, finishing the proof of (1) and (2) in the special case where G is of type (2).

The uniqueness of (x_s, x_u) for the rest of the groups that do not satisfy the $(*)$ hypothesis is mainly a reduction to the solvable case, more precisely to Lemma 5.5. We assume $(x'_s, x'_u) \in \mathcal{S} \times \mathcal{U}$ be a pair that satisfies $x = x'_s x'_u = x'_u x'_s$. We first show that $x_s = 1$ if and only if $x'_s = 1$. Indeed, if $x_s = 1$ and $x'_s \neq 1$, then $x = x_u = x'_s x'_u$. Since $x, x_u, x'_u \in C_G(x'_s)$, by Lemma 5.3 $C_G(x'_s)$ is connected. Using Corollary 5.4 in the case G has a nilpotent Borel subgroup and the commutativity of the Carter subgroups of G when G is of type (4), we conclude that $C_G(x'_s)$ is contained in a major Borel subgroup of G and that it contains a Carter subgroup of it. It then suffices to apply Lemma 5.5 with $H = C_G(x_s)$ to reach a contradiction. Now, the uniqueness trivially follows when $x_s = 1$. We may thus assume $x_s \neq 1$; equivalently $x'_s \neq 1$.

If $x_u = x'_u = 1$ then the uniqueness is again trivial. If not, then $x_u \neq 1$ or $x'_u \neq 1$. Since $x_u, x'_u \in C_G(x_s)$ (equivalently, in $C_G(x'_s)$), $C_G(x_s)$ is connected. As in the preceding paragraph, $C_G(x_s)$ lies in a major Borel subgroup and contains a Carter subgroup of this major Borel. The uniqueness follows from Lemma 5.5 applied to $H = C_G(x_s)$.

In order to prove (3), it suffices to prove that the product of two commuting semisimple (resp. unipotent) elements x and y is semisimple (resp. unipotent). If G is of type (2), then $H = d(x)d(y)$ is an abelian subgroup of G, and Proposition 5.10 shows that $H = H_s$ (resp. $H = H_u$), so xy is a semisimple (resp. unipotent) element, as desired. Consequently we may assume that G is not of type (2). We suppose that x and y are two non-trivial semisimple elements that commute. We may assume $C_G(x) \not\subseteq \mathcal{S}$. In particular, Lemma 5.3 implies that $C_G(x)$ is connected. Then, using Corollary 5.4 when G has a nilpotent Borel subgroup, we apply Lemma 5.5 in $C_G(x)$, and find a Carter subgroup of G that contains both x and y.

Now suppose that x and y are two non-trivial unipotent elements that commute. We may assume $(xy)_s \neq 1$. Then by Lemma 5.3 $C_G((xy)_s)$ is connected and not contained in $\mathcal{S}$. Indeed, as $xy = (xy)_s (xy)_u$ such that $(xy)_s$ and $(xy)_u$ commute, either $(xy)_u \neq 1$ and $C_G((xy)_s) \not\subseteq \mathcal{S}$, or $xy = (xy)_s$. In the latter case, we still conclude $C_G((xy)_s) \not\subseteq \mathcal{S}$ because x and y commute with xy, therefore with $(xy)_s$ which is equal to xy. As a result, by using Corollary 5.4 when G has a nilpotent Borel subgroup, we may apply Lemma 5.5 in $C_G((xy)_s)$. It follows that x and y belong to $C_G((xy)_s)' \subseteq \mathcal{U}$, and the proof of (3) is finished. □

6. The structure of arbitrary Borel subgroups

In this section, we will prove the main results of this article, namely Theorem 6.16 and Corollary 6.17. Theorem 6.12 is an important step along the way. The development is relatively technical, but it follows a line reasoning that has already been encountered in the preceding sections. The underlying assumption throughout the entire section is that the ambient group G is connected minimal simple group G that satisfies the negation of the (*) hypothesis in the introduction. This covers the groups of types (2), (3) and (4) entirely, part of groups of type (1). As in the previous sections, we will analyze groups of type (2) separately. The rest of the arguments will follow the case division (N) and (W) of Theorem 4.9. In addition, G is assumed not to be a bad group. This last assumption does not limit the range of our results since bad groups vacuously satisfy the main conlusions.

The Jordan decomposition, established in the previous section, will provide an efficient setting and language for the entire development in this section. We will also try to emphasize where our notions of semisimple and unipotent deviate from the ones encountered in the realm of linear algebraic groups.

6.1. **Sylow subgroups.** It is well-known that in an algebraic group, the characteristic of the underlying field plays a decisive role on the nature of torsion elements, and this phenomenon is observed through the use of the Jordan decomposition in that torsion elements are either semisimple or unipotent. In Proposition 6.2, we will obtain a similar result for minimal connected simple groups satisfying the conditions (N) or (W) by proving that the Sylow p-subgroups of G are not of mixed type, in the sense that each Sylow p-subgroup is contained either in $\mathcal{U}$ or in $\mathcal{S}$. However, in a minimal connected simple group, it is not clear whether the elements of a p-unipotent group are unipotent, a well-known property of connected simple algebraic groups over algebraically closed fields (cf. Proposition 6.2 (2) (a)). This discussion will evidently not involve groups of type (2) since these are torsion-free.

Another well-known property in the algebraic category is that in minimal connected simple algebraic groups over algebraically closed fields, equivalently in $\mathrm{PSL}_2(K)$ with K algebraically closed the semisimple/unipotent dichotomy becomes global since every non-trivial element is either semisimple or unipotent. In Proposition 6.4, we will exhibit an analogous behaviour in the context of minimal connected simple groups, by proving a result similar to Proposition 6.2 for the Sylow $U_{0,r}$-subgroups of G.

The following conclusion from [BD09], in the spirit of Fact 2.17 (2), will be useful:

Fact 6.1. – [BD09, Corollary 4.7] *Let G be a minimal connected simple group and p a prime different from 2. Then the maximal p-subgroups of G are connected.*

Proposition 6.2. – *Let p be a prime number, and let S be a Sylow p-subgroup of G. Then one of the following three conditions is satisfied:*

(1) *$S \subseteq \mathcal{U}$ and S is p-unipotent;*
(2) *$S \subseteq \mathcal{S}$, S is contained in a Carter subgroup C of G, and it is connected; furthermore, we have two possibilities:*
   (a) *G satisfies the condition (N) and $S \cap B$ is a p-torus of Prüfer p-rank at most 1 for each Borel subgroup $B \neq C$;*
   (b) *G does not satisfy the condition (N) but (W), equivalently G is of type (4), and S is a p-torus of Prüfer p-rank at most 2;*
(3) *$S \subseteq \mathcal{S}$, $p = 2$, S° is a 2-torus of Prüfer 2-rank one, and $S = S^\circ \rtimes \langle i \rangle$ for an involution i inverting S°.*

Proof – We may assume that G is not of type (2) and that it is not isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K. If $p = 2$, by Fact 2.15, the classification of simple groups of even type and Fact 3.3, the group S° is a non-trivial 2-torus, and one of the following two conditions is satisfied:

(†) $|W(G)| = 2$, S° is a 2-torus of Prüfer 2-rank one, the involutions of G are conjugate, and G has an abelian Borel subgroup C_0 such that $N_G(C_0) = C_0 \rtimes \langle i \rangle$ for an involution i inverting C_0;
(††) $|W(G)| = 3$ and S° is a 2-torus of Prüfer 2-rank two.

The group S° is a maximal 2-torus of G, and even a maximal connected 2-subgroup of G by Fact 2.14. By Fact 3.13, $C_G(S^\circ)$ is a Carter subgroup of G. In particular, S°

is the only Sylow 2-subgroup of $C_G(S^\circ)$ by Fact 2.17 (1), so $N_G(S^\circ) = N_G(C_G(S^\circ))$. Thus, in case (†), Fact 2.23 (4) yields an involution j inverting $C_G(S^\circ)$ and such that $N_G(S^\circ) = C_G(S^\circ) \rtimes \langle j \rangle$. Then, by conjugacy of the Sylow 2-subgroups in $N_G(S^\circ)$ (Fact 2.14), we may decompose S in the form $S = S^\circ \rtimes \langle k \rangle$ for an involution k inverting S°. Moreover, since S° is a 2-torus, the elements of the coset kS° are some involutions, which are semisimple by conjugacy of the involutions in G. Hence S satisfies the assertion (3).

In case (††), by Fact 3.2 (2), $N_G(S^\circ)/C_G(S^\circ) \simeq W(G)$ has order 3, so $S \leq C_G(S^\circ)$. In particular, $S = S^\circ$ is connected and it is contained in the Carter subgroup $C_G(S^\circ)$ of G. On the other hand, the Carter subgroups of G are not Borel subgroups by Fact 3.3, consequently G is of type (4) by Fact 3.9, and S satisfies the assertion (2) (b) of our result. Hence we may assume $p \neq 2$.

We first show that if S is a p-unipotent subgroup then S satisfies (1) or (2) (a). We may assume that S contains a non-trivial semisimple element x. By Fact 2.34 (1), there is a unique Borel subgroup B of G containing $Z(S)^\circ$. In particular, B contains S and $C_G(x)^\circ$ and, by Fact 2.16, there is no non-trivial p-torus in B. Thus, x centralizes no non-trivial p-torus. If G is of type (4), then the Carter subgroups are abelian and divisible by Fact 3.11, and x belongs to a non-trivial p-torus. This contradicts that there is no non-trivial p-torus in $B \geq C_G(x)^\circ$. Hence G has a nilpotent Borel subgroup (condition (N)). Then, by Corollary 5.4, if C denotes a Carter subgroup containing x, we have either $B = C$ or B is a major Borel subgroup containing x, and $B \cap C$ is a Carter subgroup of B. In the latter case, $B \cap C$ is abelian and divisible by Corollary 4.5. Hence, $x \in B \cap C$ belongs to a p-torus. This is contradictory since there is no non-trivial p-torus in B. Hence we find $B = C$, and C contains no non-trivial p-torus. Since, for each Borel subgroup $B_0 \neq C$, the group $B_0 \cap C$ is abelian and divisible by Corollary 4.5, this implies that $B_0 \cap B = B_0 \cap C$ has no non-trivial p-element, so $S \cap B_0 = 1$. Thus S satisfies (2) (a), as desired.

From now on, we assume that S is not a p-unipotent subgroup. By Fact 2.13 (2), the maximal p-torus T of S is non-trivial, and $C_G(T)$ contains S. We set $C_T = C_G(T)$ and assume that G contains a nilpotent Borel subgroup (condition (N)). Then, by Fact 2.23 (2), $C_G(T)$ is a Carter subgroup of G and thus is a nilpotent Borel in G. Let $B \neq C_T$ be another Borel subgroup. We show that $S \cap B$ is a p-torus of Prüfer p-rank at most 1. By Lemma 4.6 and Proposition 4.7, we may assume that B is a major Borel subgroup, and that $B \cap C_T$ is a Carter subgroup of B. Let A be a B-minimal subgroup in B'. By Theorem 4.9, we have $A \cap C_T \leq B' \cap C_T = 1$, so $B \cap C_T$ does not centralize A. Consequently, Fact 2.9 provides a definable algebraically closed field K such that $(B \cap C_T)/C_{B \cap C_T}(A)$ is definably isomorphic to a subgroup of the multiplicative group K^*. By Corollary 4.5, $S \cap B$ is a p-torus. If $\mathrm{pr}_p(S \cap B) \geq 2$, then there is a non-trivial p-torus S_0 in $C_{B \cap C_T}(A)$. By Fact 2.6, S_0 centralizes C_T. It follows that $C_G(S_0)^\circ$ is a proper definable subgroup of G containing C_T and A. This contradicts that C_T is a Borel subgroup of G. Hence, $\mathrm{pr}_p(S \cap B) = 1$ and S satisfies (2) (a).

It remains to deal with the case when G does not satisfy the condition (N) but (W). Equivalently, G is of type (4). By Fact 3.4, p does not divide $|W(G)|$. By Fact 3.13, C_T is a Carter subgroup of G. Corollary 4.10 (1) shows that S is a p-torus. This p-torus has Prüfer p-rank at most 2 by Fact 3.11. Hence S satisfies (2) (b). □

Corollary 6.3. – *Let S be a Sylow p-subgroup of a solvable connected definable subgroup H of G. If H is non-nilpotent, then one of the following two conditions is satisfied:*

(1) *$S \subseteq \mathcal{U}$ and S is p-unipotent;*
(2) *$S \subseteq \mathcal{S}$ and S is a p-torus of Prüfer p-rank at most 2.*

Proof – Since S is connected by Fact 2.17 (2), the result follows from Fact 2.13 (2) and from Proposition 6.2. $\square$

We will now analyze Sylow $U_{0,r}$-subgroups.

Proposition 6.4. – *For each positive integer r and each Sylow $U_{0,r}$-subgroup S of G, one of the following two conditions is satisfied:*

(1) *$S \subseteq \mathcal{U}$ and S is a homogeneous $U_{0,r}$-subgroup;*
(2) *$S \subseteq \mathcal{S}$ and S is contained in a unique Carter subgroup of G.*

Proof – If G is of type (2), this result follows from Fact 5.6 and Corollary 5.8, hence we may assume that G is not of type (2). Otherwise, first we assume $S \subseteq \mathcal{U}$, and prove that S is a homogeneous $U_{0,r}$-subgroup. By Fact 2.23 (2), for each prime p, there is no non-trivial p-torus in S, and Fact 2.6 implies that S is torsion-free. We consider the subgroup S^* generated by the indecomposable subgroups A of S satisfying $rk(A/J(A)) \neq r$. In other words, S^* is generated by the subgroups of the form $U_{0,s}(S)$ for $s \neq r$. We will show that $S^* = \{1\}$. In this vein, we assume that S^* is non-trivial. By Fact 2.32 (1), the groups of the form $[N_G(S)^\circ, U_{0,s}(S)]$, where s is a positive integer, are some homogeneous $U_{0,s}$-subgroups. Since S is a $U_{0,r}$-subgroup, they are $U_{0,r}$-subgroup too. Hence $N_G(S)^\circ$ centralizes S^*.

On the other hand, $N_G(S)^\circ$ is a subgroup of $N_G(S^*)^\circ$ that contains a Carter subgroup D of $N_G(S^*)^\circ$ by Fact 2.33 (4). We show that $S^* = U_{0,r}(D)^*$, where $U_{0,r}(D)^*$ is the subgroup generated by the indecomposable subgroups A of $U_{0,r}(D)$ satisfying $rk(A/J(A)) \neq r$. Since S is the unique Sylow $U_{0,r}$-subgroup of $N_G(S)^\circ$ by Fact 2.33 (2), we have $U_{0,r}(D) \leq S$ and $U_{0,r}(D)^* \leq S^*$. In order to prove that $U_{0,r}(D)^*$ contains S^*, we have just to verify that $U_{0,r}(D)$ contains S^*. But D centralizes $S/[D,S]$, so $DS/[D,S]$ is a nilpotent group and Fact 2.23 (6) gives $DS = [D,S]D$. Hence we have $S = [D,S](S \cap D)$ and since $[D,S]$ is a homogeneous $U_{0,r}$-subgroup by Fact 2.32 (1), we obtain $S = [D,S]U_{0,r}(S \cap D)$ by Fact 2.30 (5). The homogeneity of $[D,S]$ implies $S \cap D = ([D,S] \cap D)U_{0,r}(S \cap D) = U_{0,r}(S \cap D)$, and since $[N_G(S)^\circ, S^*] = 1$, S^* is contained in D and thus in $S \cap D = U_{0,r}(S \cap D) \leq U_{0,r}(D)$. This is what was desired and proves that $S^* = U_{0,r}(D)^*$.

The previous paragraph implies that $N_G(D)^\circ$ normalizes S^*, so D is a Carter subgroup of G and $S^* \leq D$ is contained in $\mathcal{S}$. Consequently we have $S^* \subseteq S \cap \mathcal{S} \subseteq \mathcal{U} \cap \mathcal{S} = \{1\}$, and S is homogeneous.

From now on, we may assume that there is a Carter subgroup C of G with $S \cap C \neq 1$, and we have to prove that S is contained in a conjugate of C. We assume toward a contradiction that S is contained in no Carter subgroup of G. We may assume that C is chosen such that $rk(U_{0,r}(S \cap C))$ is maximal. We will now verify that $U_{0,r}(S \cap C) = 1$ and that as a result $[S, S \cap C] = 1$ (Fact 2.32 (1)). If $U_{0,r}(S \cap C)$ is non-trivial, we consider a Borel subgroup B containing $N_G(U_{0,r}(S \cap C))^\circ$. Then Fact 2.30 (4) gives $U_{0,r}(S \cap C) < U_{0,r}(S \cap B)$ and, by maximality of $rk(U_{0,r}(S \cap C))$, the subgroup $U_{0,r}(S \cap B)$ is contained in no conjugate of C. In particular, if G has a nilpotent Borel subgroup (condition (N)), then we have $B \neq C$ and Corollary 5.4

says that B is a major Borel subgroup such that $B \cap C$ is a Carter subgroup of B. Otherwise, G is of type (4), B contains C and B is a major Borel subgroup. Hence, in all the cases, Theorem 4.9 (2) gives $r > \overline{r}_0(B \cap C)$, contradicting that $U_{0,r}(S \cap C)$ is non-trivial. Thus $U_{0,r}(S \cap C)$ is trivial, and by Fact 2.32 (4) S centralizes $S \cap C$.

Let $x \in (S \cap C) \setminus \{1\}$, and let B be a Borel subgroup containing $C_G(x)^\circ$. In particular, B contains S, and we have $B \neq C$. Then, if G has a nilpotent Borel subgroup, Corollary 5.4 says that B is a major Borel subgroup and that $B \cap C$ is a Carter subgroup of B. Otherwise, G is of type (4). We then have $C \leq B$ and B is a major Borel subgroup too. Thus, in both cases, since S is contained in no Carter subgroup of G, Theorem 4.9 (2) gives $r > \overline{r}_0(B \cap C)$ and $B = B' \rtimes (B \cap C)$. This implies $S \leq B'$ and $S \cap C = 1$, contradicting $S \cap C \neq 1$. Hence S is contained in a conjugate of C, and we may assume $S \leq C$.

We will prove that no other Carter subgroup of G contains S. We first deal with the case when $W(G) \neq 1$. Since $S = U_{0,r}(C)$, $N_G(S) \geq N_G(C)$. As a result, $N_G(S) \geq C$. It follows from Facts 3.12 and 2.23 (5) that $N_G(S) = N_G(C)$, and in particular, $N_G(S)^\circ = C$. Since this equality holds for every Carter subgroup of G containing S, we conclude that C is unique. When G has nilpotent Borel subgroups (condition (N)), the conclusion follows from Corollary 5.4 and the uniqueness statement in Theorem 4.9. □

The previous result has the following consequence on the conjugacy of the Sylow $U_{0,r}$-subgroups.

Corollary 6.5. – *Let r be a positive integer, and let S be a Sylow $U_{0,r}$-subgroup of G. Then S is conjugate with any Sylow $U_{0,r}$-subgroup R of G satisfying $S \cap R \neq 1$.*

Proof – We assume toward a contradiction that R is a counterexample with $rk(S \cap R)$ maximal. In particular, by nilpotence of S and R, we have $S \cap R < N_S(S \cap R)$ and $S \cap R < N_R(S \cap R)$. Moreover, by Proposition 6.4 and by Fact 2.23 (4), the $U_{0,r}$-subgroups S and R are contained in $\mathcal{U}$ and they are homogeneous. Thus $S \cap R$ is a $U_{0,r}$-subgroup.

Let $H = N_G(S \cap R)^\circ$ and let S_1 (resp. R_1) be a Sylow $U_{0,r}$-subgroup of H containing $S \cap H$ (resp. $R \cap H$). By Fact 2.33 (2), there exists $h \in H$ such that $R_1^h = S_1$. Let S_2 be a Sylow $U_{0,r}$-subgroup of G containing S_1. Since $S \cap H > S \cap R$ is contained in $S \cap S_2$, there exists $g \in G$ such that $S_2^g = S$ by maximality of $rk(S \cap R)$. Then we obtain

$$(S \cap R)^{hg} < (R \cap H)^{hg} \leq R_1^{hg} = S_1^g \leq S_2^g = S.$$

But this forces

$$rk(S \cap R) < rk((R \cap H)^{hg}) \leq rk(R^{hg} \cap S).$$

Thus, R^{hg} and S are conjugate by maximality of $rk(S \cap R)$, a contradiction to our choice of R. □

6.2. **Structure of nilpotent subgroups.** The following result is similar to a classical result for algebraic groups [Hum81, Proposition 19.2].

Proposition 6.6. – *For each nilpotent definable subgroup H of G, the sets H_u and H_s are two definable subgroups satisfying $H = H_u \times H_s$.*

Moreover, either H_s is contained in a Carter subgroup of G, or $H = H_s$ is a finite 2-subgroup contained in no Borel subgroup.

Proof – For groups of type (2), this result follows from Corollary 5.8 and Proposition 5.10, hence we may suppose that G is not of type (2). First we assume that $Z(H)$ is not contained in $\mathcal{U}$, and we consider a non-trivial semisimple element x in $Z(H)$. Then $C_G(x)$ contains H. If $C_G(x)$ is not connected, then Lemma 5.3 gives $H = H_s$, and either H_s is contained in a Carter subgroup of G, or G is of odd type and of Prüfer 2-rank one, x is an involution, $C_G(x)^\circ$ is a Carter subgroup of G, and $C_G(x) = C_G(x)^\circ \rtimes \langle i \rangle$ for an involution i inverting $C_G(x)^\circ$. We may assume that we are in the second case, and that H is not contained in $C_G(x)^\circ$. Then we have $H = (H \cap C_G(x)^\circ) \rtimes \langle j \rangle$ for an involution j inverting $H \cap C_G(x)^\circ$. It follows from this that H is a finite 2-group. Indeed, if $z \in Z(H) \cap C_G(x)^\circ$, then $z = z^j = z^{-1}$, and $z^2 = 1$. Thus $Z(H)$ is an elementary abelian 2-group. But G is of odd type. Thus $Z(H)$ is finite. It follows from Fact 2.8 (2) that H is finite. Moreover, H has only 2-torsion elements since, H being nilpotent, any non-trivial Sylow p-subgroup intersects $Z(H)$ non-trivially. Since $x \in C_G(x)^\circ$ by Lemma 5.3, x and j are two distinct involutions of H, and they commute. Therefore, if H is contained in a Borel subgroup B of G, then the Sylow 2-subgroups of B are 2-tori of Prüfer 2-rank at least 2 since they are connected by Fact 2.17 (2), non-trivial. This contradicts that G has Prüfer 2-rank one. Hence H is contained in no Borel subgroup of G, as desired. Thus we may suppose that $C_G(x)$ is connected.

Let C be a Carter subgroup of G containing x, and let B be a Borel subgroup containing $C_G(x)$. Then either G has a nilpotent Borel subgroup (condition (N)), and Corollary 5.4 says that B is a major Borel subgroup such that $B \cap C$ is a Carter subgroup of B, or G is of type (4), and B is a major Borel subgroup containing C. Consequently, Lemma 5.5 says that H_s is conjugate in $C_G(x)$ with a subset of C, and we may assume $H_s \subseteq C$. This implies that C contains $d(H_s)$, so H_s is a definable subgroup of H. On the other hand, $H_u \subseteq C_G(x)' \subseteq \mathcal{U}$ by Lemma 5.5, so $C_G(x)'$ contains $d(H_u)$ and H_u is a definable subgroup of H. Now the equality $H = H_u \times H_s$ follows from the Jordan decomposition of each element of H (Theorem 5.12 (1) and (2)).

It remains the case when $Z(H)$ is contained in $\mathcal{U}$. We will prove that $H \subseteq \mathcal{U}$. By contradiction, we suppose that H is not contained in $\mathcal{U}$. Then we find $x \in H_s \setminus \{1\}$, and we may assume that x is chosen such that $C_H(x)$ is maximal for such an element x. By the previous paragraphs, $C_H(x)_u$ and $C_H(x)_s$ are two definable subgroups satisfying $C_H(x) = C_H(x)_u \times C_H(x)_s$. In particular, since $Z(H)$ is contained in $\mathcal{U}$, we have $C_H(x) < H$, and we obtain $C_H(x) < N_H(C_H(x))$. Since $C_H(x)_s$ is definably characteristic in $C_H(x)$, $N_H(C_H(x))$ normalizes $C_H(x)_s$, and there exists a non-trivial element z in $Z(N_H(C_H(x))) \cap C_H(x)_s$. Hence z is a non-trivial semisimple element of H such that $C_H(x) < N_H(C_H(x)) \leq C_H(z)$, which contradicts the maximality of $C_H(x)$. The proof is finished. □

6.3. **Tori.** In this subsection, we will derive an important ingredient, namely Theorem 6.12. The notion of semisimple torus (Definition 5.1 (3)) will play a major role.

Proposition 6.7. – *The maximal semisimple tori of G are Carter subgroups. In particular, they are conjugate and, if G is of type (4), they are abelian.*

Proof – If G is of type (2), this follows from Corollaries 5.8 and 5.9, so we may assume that G satisfies one of the conditions (N) or (W). Then G has a non-nilpotent major Borel subgroup B_0. Hence B_0' is a non-trivial subgroup of G

contained in $\mathcal{U}$ by Lemma 5.5, and thus G is not a semisimple torus. Consequently, the semisimple tori of G are solvable.

We consider a Carter subgroup C of G. By the previous paragraph, if G satisfies condition (N), then C is a maximal semisimple torus. Otherwise, G is of type (4), and there is a maximal semisimple torus T containing C. The elements of T' are unipotent by Lemma 5.5, and so T is abelian. Consequently we obtain $T = C$, and each Carter subgroup of G is a maximal semisimple torus.

Now, since the Carter subgroups of G are conjugate by Fact 2.23 (4) and they are abelian when G is of type (4), it remains to prove that each semisimple torus of G is contained in a Carter subgroup of G. Let T be a semisimple torus of G. If T is nilpotent, then it is contained in a Carter subgroup of G by Proposition 6.6, so we may assume that T is not nilpotent. Then T' is a non-trivial nilpotent semisimple torus by Fact 2.10, and T' is contained in a Carter subgroup C of G by Proposition 6.6. Let $H = N_G(T')^\circ$. Then H is a solvable non-nilpotent connected subgroup of G containing T. If G has a nilpotent Borel subgroup (condition (N)), then Corollary 5.4 and Lemma 5.5 give $T' \leq H' \subseteq \mathcal{U}$, contradicting that T' is a non-trivial semisimple torus. Otherwise, G is of type (4), and H contains C since C is abelian. Therefore we obtain $T' \leq H' \subseteq \mathcal{U}$ again, contradicting that T' is a non-trivial semisimple torus. Consequently, the maximal semisimple tori of G are Carter subgroups. □

Lemma 6.8. – *Let H be a definable connected solvable subgroup of G. Then $F(H)_s$ is a hypercentral subgroup of H. Furthermore, if G is not of type (2) and if H is not a semisimple torus, then $F(H)_s$ is a central subgroup of H.*

Proof – If G is of type (2), then G is torsion-free (Fact 3.9). By Proposition 6.7, the subgroup $F(H)_s$ is contained in a Carter subgroup of G, and by Fact 2.33 (5) and Corollary 5.8, $F(H)_s$ is contained in a Carter subgroup D of H. Since $H = F(H)D$ (Fact 2.23 (6)) and since $F(H) = F(H)_u \times F(H)_s$ (Proposition 5.10), we conclude that $F(H)_s$ is an hypercentral subgroup of H.

Otherwise we may assume that H is not a semisimple torus. By Proposition 6.6, $F(H)_s$ is a definable subgroup of a Carter subgroup C of G. We notice that we have $H \not\leq C$ since H is not a semisimple torus. Let x be a non-trivial p-element of $F(H)_s$ for a prime p, and let S be a Sylow p-subgroup of H containing x. Then S is a p-torus by Corollary 4.5 (in case G has a nilpotent Borel), Proposition 6.2 and Fact 2.17 (2), and x is central in H by Fact 2.19. Thus, to finish, it will suffice to prove that $F(H)_s^\circ$ is central in H. We may assume $F(H)_s^\circ \neq 1$.

Let B be a Borel subgroup of G containing $N_G(F(H)_s)^\circ$. Since H normalizes $F(H)_s$, it will suffice to prove that $F(H)_s \leq Z(N_G(F(H)_s)^\circ)$. If G is of type (4), then C is abelian, so $C \leq N_G(F(H)_s)^\circ$ and B is a major Borel subgroup. It follows from Corollary 4.10 that $F(H)_s \leq F(N_G(F(H)_s)^\circ) \cap C = Z(N_G(F(H)_s)^\circ)$.

We finish the proof handling the case when G has a nilpotent Borel. Since H is not a semisimple torus and $H \leq N_G(F(H)_s)^\circ \leq B$, necessarily $B \neq C$. Hence, by Corollary 5.4 B is a major Borel subgroup of G, and $B \cap C$ is a Carter subgroup of B contained $N_G(F(H)_s)^\circ$. Corollary 4.10 allows to finish as above. □

Corollary 6.9. – *We assume that G is not of type (2). Let H be a definable connected solvable subgroup of G. If $F(H)_s$ is non-trivial, then either H is a semisimple torus, or H is contained in a major Borel subgroup.*

Proof – We may assume that H is not a semisimple torus. Let $x \in F(H)_s \setminus \{1\}$. Therefore $C_G(x)^\circ$ contains H by Lemma 6.8. Now let B be a Borel subgroup containing $C_G(x)^\circ$. If G has a nilpotent Borel subgroup, then by Corollary 5.4, B is a major Borel subgroup. Otherwise, G is of type (4), and any Carter subgroup of G containing x is in $C_G(x)^\circ \leq B$. The result follows. □

Lemma 6.10. – *Let H be a solvable connected definable subgroup of G. If R is a subgroup of H formed by semisimple elements, then there is a Carter subgroup D of H such that R is contained in D_s.*

Proof – We may assume that R is non-trivial, and that R is maximal among the subgroups of H formed by some semisimple elements of H. Moreover, we may assume that H is non-nilpotent by Proposition 6.6. So H is not a semisimple torus by Proposition 6.7. Then, since $F(H)_s$ is a subgroup of H by Proposition 6.6, and that it is hypercentral in H by Lemma 6.8, it is an hypercentral subgroup of $F(H)_sR$. Now, since R' is contained in $F(H)_s$ by Fact 2.10, the subgroup $F(H)_sR$ is nilpotent, and it is formed by semisimple elements by Proposition 6.6. Thus R is a nilpotent group containing $F(H)_s$, and by maximality of R, it is definable (Proposition 6.6).

We let $E = E_H(R)$. Since by Fact 2.27 E is a connected definable subgroup of H and that $F(E)$ contains R, we have $R = F(E)_s$ by Proposition 6.6 and by maximality of R. Let D be a Carter subgroup of E (Fact 2.23 (1)). Since $R = F(E)_s$ is hypercentral in E by Lemma 6.8, it is contained in D. Since by Facts 2.25 (2), 2.26 and 2.23 (3), D is a Carter subgroup of H, we obtain the result. □

The conjugacy of maximal semisimple tori in H now follows from Fact 2.23 (3):

Corollary 6.11. – *In each proper definable connected subgroup H of G, the maximal tori of H are conjugate.*

Theorem 6.12. – *In each connected solvable definable subgroup H of G, the set H_u is a connected definable subgroup such that $H = H_u \rtimes T$ for any maximal semisimple torus T of H.*

Proof – By Proposition 5.11, we may assume that G is not of type (2). Moreover, we may assume that H is not a semisimple torus.

We claim that H_u contains H'. Since $F(H)$ contains H' by Fact 2.10, we may assume that H is contained in a major Borel subgroup by Corollary 6.9, and we obtain $H' \subseteq H_u$ by Corollary 4.10 and Proposition 6.6.

On the other hand, if T is any maximal semisimple torus of H, then Proposition 6.6 and Lemma 6.10 provide a Carter subgroup D of H such that $T = D_s$ and $D = D_u \times T$. Moreover, Fact 2.23 (6) gives $H = H'D = (H'D_u)T$. Thus, since D_u is definable and connected by Proposition 6.6, it remains to prove that $H'D_u = H_u$.

We claim that the subgroup $H'D_u$ contains only unipotent elements. Suppose towards a contradiction that there exists $x \in (\mathcal{S} \cap H'D_u) \setminus \{1\}$. By Lemma 6.10 and by conjugacy of Carter subgroups (Fact 2.23 (3)), we may assume $x \in T$. Then we have $x = hd$ for $h \in H' \subseteq \mathcal{U}$ and $d \in D_u \subseteq \mathcal{U}$. This implies $h = xd^{-1}$. Since $xd^{-1} = d^{-1}x$ with $x \in \mathcal{S}$ and $d^{-1} \in \mathcal{U}$, we obtain a contradiction to the Jordan decomposition of $h \in \mathcal{U}$ (Theorem 5.12 (1)).

The preceding paragraphs show that $(H'D_u) \subseteq H_u$. We will show now that these two sets are in fact equal. Indeed, for each $x \in H_u$ then, by Facts 2.27, 2.26, and 2.25 (2) the set $E_H(x)$ is a definable connected subgroup containing a

Carter subgroup of H, and such that x belongs to $F(E_H(x))$. By Fact 2.23 (3), we may assume $D \leq E_H(x)$. Since $H = H'D$, we have $x = hd$ for $d \in D$ and $h \in H' \cap E_H(x) \subseteq F(E_H(x))_u$. In particular, this implies $d \in F(E_H(x))$. But, by Proposition 6.6, the set $F(E_H(x))_u$ is a subgroup of $F(E_H(x))$. Hence, since x belongs to $F(E_H(x))_u$ as well, we conclude $d \in F(E_H(x))_u$, and $d \in D \cap \mathcal{U} = D_u \leq H'D_u$. This yields $x = hd \in H'D_u$ and $H_u = H'D_u$. □

6.4. **Structure of solvable subgroups.** In this final subsection, we prove the main theorem, namely Theorem 6.16. When G has abelian Carter subgroups, the theorem yields Corollary 6.17 that is much closer to the Borel subgroup description in simple algebraic groups.

Lemma 6.13. – *Let B be a Borel subgroup of G. If $B \subseteq \mathcal{U}$, then B is torsion-free.*

PROOF – By Fact 2.23 (2), each decent torus of B is trivial. Consequently, using Facts 2.13 (2) and 2.17 (2), we may assume that $U_p(B)$ is non-trivial for a prime p. We let $U = U_p(B)$. If a B-minimal section $\overline{A}$ of U is not centralized by B, then $B/C_B(\overline{A})$ is definably isomorphic to a definable subgroup of K^* for a definable algebraically closed field K of characteristic p by Fact 2.9, and Fact 2.20 shows that $B/C_B(\overline{A})$ is a decent torus. Then there is a non-trivial decent torus in B by Fact 2.22 (1), contradicting that each decent torus of B is trivial. Consequently each B-minimal section of U is centralized by B. This implies that, if C denotes a Carter subgroup of B, then C contains U, so $U = U_p(C)$.

Since $B \subseteq \mathcal{U}$, C is not a Carter subgroup of G by the definition of a semisimple element. Hence B does not contain $N_G(C)^\circ$. On the other hand, we have proven that $B = N_G(U)^\circ \geq N_G(C)^\circ$. This contradiction finishes the proof. □

Lemma 6.14. – *Let r be a positive integer, and let S be a Sylow $U_{0,r}$-subgroup of G. If $S \subseteq \mathcal{U}$, then $B = N_G(S)^\circ$ is a Borel subgroup of G, and S is contained in B'.*

PROOF – First we note that S is a homogeneous $U_{0,r}$-group by Proposition 6.4. Also, if S is contained in B' for a Borel subgroup B of G, the nilpotence of B' (Fact 2.10) as well as the unipotent structure of nilpotent groups of finite Morley rank (Facts 2.30 (6), (7) and 2.32 (2)) imply that $S = U_{0,r}(B')$ is normal in B and that $B = N_G(S)^\circ$. Then we may assume that, for each Borel subgroup B of G, we have $S \nleq B'$. We will assume towards a contradiction that r is a minimal counterexample to the statement of the lemma. Thus for each positive integer $s < r$ and for each $U_{0,s}$-Sylow subgroup R of G, the condition $R \subseteq \mathcal{U}$ implies the existence of a Borel subgroup A of G satisfying $R \leq A'$.

As a first step, we show that, for each Borel subgroup B of G such that $S \cap B$ is non-trivial, no Sylow $U_{0,r}$-subgroup of B is contained in B'. Indeed, by Fact 2.33 (2) and Corollary 6.5, we may assume that $S \cap B$ is a Sylow $U_{0,r}$-subgroup of B, and that $S \cap B$ is contained in B'. Then, the nilpotence of B' (Fact 2.10) and the unipotent structure of nilpotent groups of finite Morley rank (Facts 2.30 (6), (7) and 2.32 (2)) imply that $S \cap B = U_{0,r}(B')$ is normal in B and that $B = N_G(S \cap B)^\circ$. By the nilpotence of S, we obtain $S \leq B'$, contradicting our choice of S. Hence, no Sylow $U_{0,r}$-subgroup of B is contained in B'.

The second main step of the proof will consist in showing that $B \cap \mathcal{S} = \{1\}$ for each Borel subgroup B of G such that $S \cap B$ is non-trivial. We assume toward a contradiction that B is a Borel subgroup of G such that $B \cap \mathcal{S}$ and $S \cap B$ are non-trivial. Since S is homogeneous, we may assume that $S \cap B$ is a Sylow $U_{0,r}$-subgroup of B by Corollary 6.5. By the previous paragraph, $S \cap B$ is not contained in B'. By Fact 2.33 (4) there exists a Carter subgroup D of B in $N_B(S \cap B)^\circ$, and D_s is non-trivial by Lemma 6.10 and Fact 2.23 (3). Since D centralizes $(S \cap B)/[D, S \cap B]$, Fact 2.23 (6) gives $D(S \cap B) = [D, S \cap B]D$ and $S \cap B = [D, S \cap B](S \cap D)$. But $S \cap B$ is not contained in B', hence $S \cap D$ is non-trivial. Now we have to separate three cases.

- If G is not of type (2), let $x \in D_s \setminus \{1\}$. Then, by Proposition 6.6, we have $S \cap D \leq D_u \leq C_G(x)^\circ$. Moreover, $C_G(x)^\circ$ is contained in a major Borel subgroup A. Indeed, if G has a nilpotent Borel (condition (N)), then we have $C_G(x)^\circ \not\subseteq \mathcal{S}$ since $C_G(x)^\circ \geq S \cap D \neq 1$, and Corollary 5.4 justifies the existence of A. Otherwise, G is of type (4), and since Carter subgroups are abelian, A exists. Since $S \cap A \geq S \cap D$ is non-trivial, by Corollary 6.5 there exists $g \in G$ such that $S^g \cap A$ is a Sylow $U_{0,r}$-subgroup of A. Then, since $S \subseteq \mathcal{U}$, Lemma 5.5 yields $S^g \cap A \leq A'$, and contradicts the first step.
- If G is of type (2) and there is a Borel subgroup B_1 containing $N_G(D_s)$ and a Carter subgroup C of G, then $S \cap D$ is a nontrivial subgroup of $D_u \leq N_G(D_s) \leq B_1$. Since C is a Carter subgroup of B_1, it covers B_1/B_1' (Fact 2.23 (6)). Since C is a Carter subgroup of G, it is a homogeneous $U_{0,c}$-subgroup for an integer c (Fact 5.7), and we have $r < c$ by Corollary 5.8. This implies that B_1/B_1' is a homogeneous $U_{0,c}(G)$-group and that any Sylow $U_{0,r}$-subgroup of B_1 is contained in B_1', contradicting our first step.
- Otherwise, G is of type (2), and no Borel subgroup contains $N_G(D_s)$ and a Carter subgroup of G. By Corollary 5.8, there is a Carter subgroup C of G containing D_s. If $N_C(D_s)$ is not abelian, then it is contained in a unique Borel subgroup by Fact 2.35 (2), which necessarily contains $N_G(D_s)$ and C, contradicting our hypothesis over $N_G(D_s)$. Then $N_C(D_s)$ is abelian. Our hypothesis over $N_G(D_s)$ implies that C is not contained in $N_G(D_s)$, so C is not abelian. We consider $N = N_C(N_C(D_s))$. It is a non-abelian subgroup of C, so it is contained in a unique Borel subgroup B_1 of G (Fact 2.35 (2)). In particular C is a Carter subgroup of B_1. Now C covers B_1/B_1' (Fact 2.23 (6)), and since C is a Carter subgroup of G, it is a homogeneous $U_{0,c}$-subgroup for an integer c (Fact 5.7), and we have $r < c$ by Corollary 5.8. This implies that B_1/B_1' is a homogeneous $U_{0,c}(G)$-group and that any Sylow $U_{0,r}$-subgroup of B_1 is contained in B_1'. Our first step shows that $S^g \cap B_1$ is trivial for each $g \in G$. If B_2 denotes a Borel subgroup containing $N_G(D_s)$, then $S \cap D \leq D_u \leq N_G(D_s)$ is contained in B_2, and our first step implies that B_2/B_2' has a non-trivial $U_{0,r}$-subgroup. By Corollary 5.8 and Fact 2.33 (5), there is a Carter subgroup C_2 of B_2 containing $N_C(D_s)$, and since it covers B_2/B_2' (Fact 2.23 (6)), the subgroup $U_{0,r}(C_2)$ is non-trivial (Fact 2.31). Since $U_{0,r}(C_2)$ centralizes the $U_{0,c}$-group $N_C(D_s)$ (Fact 2.30 (6)), it is contained in $N_G(N_C(D_s)) \leq B_1$. But, by Corollary 6.5, there exists $g \in G$ such that $S^g \cap B_2$ is a Sylow $U_{0,r}$-subgroup of B_2, so there is $b \in B_2$ such that $S^{gb} \cap B_2$ contains $U_{0,r}(C_2)$ (Fact 2.33 (2)). Hence $S^{gb} \cap B_1$ is non-trivial, a contradiction.

Thus our three cases provides a contradiction, so we conclude that $B \cap \mathcal{S} = \{1\}$ for each Borel subgroup B of G such that $S \cap B$ is non-trivial. In particular, B is torsion-free by Lemma 6.13.

In the final step, we consider the smallest positive integer s such that there exists a Borel subgroup B with $S \cap B \neq 1$ and $U_{0,s}(B) \neq 1$. Then we fix such a Borel subgroup B whose Sylow $U_{0,s}$-subgroups have maximal Morley rank. By Corollary 6.5, we may choose B such that $S \cap B$ is a Sylow $U_{0,r}$-subgroup of B. In particular, by the first step, $S \cap B$ is not contained in B'. Also, by Facts 2.33 (2) and (3) there is a Carter subgroup D of B such that $U_{0,r}(D) = S \cap D$ and $S \cap B = (S \cap B')(S \cap D)$, so $S \cap D$ is non-trivial. Since s is minimal and B is torsion-free by the second step, $U_{0,s}(B')D$ is nilpotent by Fact 2.30 (2) and $U_{0,s}(D)$ is a Sylow $U_{0,s}$-subgroup of B by Fact 2.33 (3). We consider a Borel subgroup A of G containing $N_G(U_{0,s}(D))^\circ$. Then A contains D, so $S \cap A$ is non-trivial, and it follows from the second step that A is torsion-free. Moreover, the choice of s implies that $U_{0,t}(A)$ is trivial for each positive integer $t < s$. Since $U_{0,s}(D)$ is a Sylow $U_{0,s}$-subgroup of B contained in A, the choice of B implies that $U_{0,s}(D)$ is a Sylow $U_{0,s}$-subgroup of A too. Consequently, there is a Carter subgroup C of A in $N_A(U_{0,s}(D))$ by Fact 2.33 (4) and C contains $U_{0,s}(D)$ by Fact 2.30 (2). Now we have $U_{0,s}(C) = U_{0,s}(D)$, and $N_G(C)^\circ$ is contained in $N_G(U_{0,s}(C))^\circ = N_G(U_{0,s}(D))^\circ \leq A$, so C is a Carter subgroup of G. This contradicts the second step which implies $A \cap \mathcal{S} = \{1\}$, and completes the proof. □

Fact 6.15. – [Fré08, Lemma 10.1 and Theorem 11.1] *We assume that G is of type (2). Let B be a Borel subgroup of G containing a Carter subgroup C of G. Then $B = U \rtimes C$ for a definable definably characteristic nilpotent subgroup U.*

Theorem 6.16. – *Any Borel subgroup B of G satisfies the following decomposition*

$$B = U \rtimes D \quad \text{and} \quad Z_\infty(B) = F(B) \cap D,$$

where D is any Carter subgroup of B and U is a normal nilpotent connected definable subgroup of B.

Furthermore, if $B \neq D$, then D is divisible and the following properties hold:

1. *for each prime p, either $U_p(U)$ is the unique Sylow p-subgroup of B, or each Sylow p-subgroup of B is a p-torus contained in a conjugate of D;*
2. *there is at most one positive integer $r \leq \overline{r}_0(D)$ such that there is a Sylow $U_{0,r}$-subgroup S of B not of the form $U_{0,r}(D^b)$ for $b \in B$. In this case, S is a maximal abelian $U_{0,r}$-subgroup and is not a Sylow $U_{0,r}$-subgroup of G.*

Proof – We may assume that B is non-nilpotent, that is $B \neq D$. By Theorem 4.9, we may assume that, either G is of type (2), or B is not a major Borel subgroup.

If D is a Carter subgroup of G, then B is a major Borel subgroup of G, so G is of type (2). In particular, G is torsion-free and satisfies the assertion (1). Moreover, Fact 6.15 provides the decomposition $B = U \rtimes D$. Now, since D is a homogeneous Sylow $U_{0,c}$-subgroup of G for an integer c (Fact 5.7 and Corollary 5.8), the assertion (2) is satisfied, and the subgroup $F(B) \cap D$ is the Sylow $U_{0,c}$-subgroup of $F(B)$. Then we have $F(B) = U \times (F(B) \cap D)$ by Fact 5.6. Since D is a Carter subgroup of B, it contains the hypercenter of B, and we have $Z_\infty(B) \leq F(B) \cap D$. Since $F(B) \cap D$ centralizes U and since D is nilpotent, we find $Z_\infty(B) = F(B) \cap D$ as desired, and we may assume that D is not a Carter subgroup of G.

First we show that D is divisible. We may assume that G is not of type (2), since it is torsion-free in this case. If D is not divisible, then by Fact 2.6 $U_p(D) \neq 1$ for a prime p. By Lemma 6.13, we have $B \not\subseteq \mathcal{U}$. Fact 2.23 (3) and Lemma 6.10 imply $D_s \neq 1$. By Proposition 6.6, D_s is a connected definable subgroup of a Carter subgroup C of G and D_s centralizes D_u. Since D_u contains $U_p(D)$ (Corollary 6.3), the subgroup D_s centralizes $U_p(D)$. Moreover, by Corollary 5.4 if G has a nilpotent Borel, and by the commutativity of Carter subgroups otherwise (if G is of type (4)), we have $N_G(D_s)^\circ \not\leq B$ since B is not a major Borel subgroup. But Fact 2.34 (1) says that B is the only Borel subgroup containing $N_G(D_s)^\circ \geq U_p(D) \neq 1$, hence we have a contradiction, and D is divisible.

Secondly, D is abelian. Indeed, $D < N_G(D)^\circ$, and the conclusion follows from Fact 2.35 (2).

Thirdly, we show that $B = B' \rtimes D$. By Fact 2.23 (6), we have $B = B'D$, and DB''/B'' is a Carter subgroup of B/B''. Then, since D is abelian, Fact 2.23 (7) yields $B/B'' = B'/B'' \rtimes DB''/B''$, therefore $D \cap B'$ is contained in B''. By Facts 2.32 (2) and (3), we have

$$B' = A \times U_{0,1}(B') \times \cdots \times U_{0,\overline{r}_0(B')}(B'),$$

where A is definable, connected, definably characteristic and of bounded exponent, and where $U_{0,s}(B')$ is a homogeneous $U_{0,s}$-subgroup for each $s \in \{1, 2, \ldots, \overline{r}_0(B')\}$. If $D \cap A$ is non-trivial, there is a prime p such that $U_p(B')$ is non-trivial and, since D is abelian and divisible, D contains a non-trivial p-torus T. Then $U_p(B')T$ is a locally finite p-subgroup of G contradicting Corollary 6.3. Hence $D \cap A$ is trivial, and we may assume that $D \cap U_{0,r}(B')$ is non-trivial for a positive integer r. We notice that, since B' is contained in $\mathcal{U}$ (Theorem 6.12), each Sylow $U_{0,r}$-subgroup of B is contained in $\mathcal{U}$ by Fact 2.33 (2) and Proposition 6.4. On the other hand, since $D \cap B'$ is contained in B'', the structure of B' implies that $D \cap U_{0,r}(B')'$ is non-trivial. So B is the unique Borel subgroup containing $U_{0,r}(B')$ by Fact 2.35 (2), and $U_{0,r}(B')$ is a Sylow $U_{0,r}$-subgroup of G by Lemma 6.14 and Proposition 6.4 (1). Since D is not a Carter subgroup of G, we have $N_G(D)^\circ \not\leq B$, and $N_G(U_{0,r}(D))^\circ$ is contained in a Borel subgroup $A \neq B$. In particular, D is contained in A and is not a Carter subgroup of A. Let $S = N_{U_{0,r}(B')}(U_{0,r}(D))^\circ$. Then $S \leq A \cap B$ is abelian by Fact 2.35 (2), and since S contains $C_{U_{0,r}(B')}(U_{0,r}(D))^\circ$, it is a maximal abelian subgroup of $U_{0,r}(B')$. On the other hand, $D \cap U_{0,r}(B')'$ is non-trivial, so $U_{0,r}(B')$ is not abelian and we have $S < N_{U_{0,r}(B')}(S)^\circ$. By maximality of S in $U_{0,r}(B')$, the group $N_{U_{0,r}(B')}(S)^\circ$ is not abelian. This implies that B is the only Borel subgroup containing $N_G(S)^\circ$ (Fact 2.35 (2)). Now, if S_A is a Sylow $U_{0,r}$-subgroup of A containing S, then S_A is a homogeneous $U_{0,r}$-subgroup by Proposition 6.4 (1), and $N_{S_A}(S)^\circ$ is a $U_{0,r}$-subgroup. But $N_{S_A}(S)^\circ \leq N_G(S)^\circ$ is contained in B, hence it is contained in $U_{0,r}(B)$. Since $U_{0,r}(B')$ is a Sylow $U_{0,r}$-subgroup of G and that it is normal in B, we have $U_{0,r}(B) = U_{0,r}(B')$ by Fact 2.33 (2) and $N_{S_A}(S)^\circ$ is contained in $U_{0,r}(B')$. Thus, since S is a maximal abelian subgroup of $U_{0,r}(B')$, and since $N_{S_A}(S)^\circ \leq A \cap B$ is abelian by Fact 2.35 (2), we obtain $N_{S_A}(S)^\circ = S$. Therefore the nilpotence of S_A yields $S_A = S$ and S is a Sylow $U_{0,r}$-subgroup of A. Consequently, $N_G(S)^\circ$ contains a Carter subgroup of A by Fact 2.33 (4) and all the Carter subgroups of $N_G(S)^\circ$ are Carter subgroups of A (Fact 2.23 (3)). In particular, D is a Carter subgroup of A, contradicting that D is not a Carter subgroup of A. This proves $B = B' \rtimes D$.

Now we show that $Z_\infty(B) = Z(B) = F(B) \cap D$. Since we have $Z(B) \leq Z_\infty(B) \leq F(B) \cap D$ by Fact 2.23 (5), we have just to prove that $F(B) \cap D$ is central in B. Firstly we show that $F(B) \cap D$ is torsion-free. Indeed, we may assume that G is not of type (2), so $F(B)$ is contained in $\mathcal{U}$ by Corollary 6.9. Since D is abelian and divisible, for each prime p, each p-element x of $F(B) \cap D$ lies in a p-torus, and is semisimple by Fact 2.23 (2). Since $F(B)$ is contained in $\mathcal{U}$, this implies that $F(B) \cap D$ is torsion-free, as desired. On the other hand, for each positive integer r, if U_1 is a non-trivial $U_{0,r}$-subgroup in $F(B) \cap D$, then U_1 is contained in the Sylow $U_{0,r}$-subgroup S of $F(B)$. Since $S \geq U_1$ is not contained in B', Proposition 6.4 implies that either $S \subseteq \mathcal{S}$ and there is a Carter subgroup C of G containing S, or $S \subseteq \mathcal{U}$. In the last case, there is a Borel subgroup $B_0 \neq B$ containing S by Lemma 6.14, and S is abelian by Fact 2.35 (2). If $S \subseteq \mathcal{S}$ and if C is a Carter subgroup of G containing S, then C is not contained in B and S is abelian by Fact 2.35 (2). Now S is central in $F(B)$ by Fact 2.30 (7). Consequently, since D is abelian, U_1 centralizes $F(B)$ and D. Hence U_1 is central in B. Therefore Fact 2.30 (7) provides $F(B) \cap D \leq Z(B)$, and the equality $Z_\infty(B) = Z(B) = F(B) \cap D$ holds.

We verify assertion (1). We may assume that G is not of type (2), so we have $B' \leq \mathcal{U}$ by Lemma 6.8. Let p be a prime integer. If there is a p-element in $B \setminus B'$, then there is a non-trivial p-element in $D \simeq B/B'$. Since D is abelian and divisible, the maximal p-torus T of D contains all the p-elements of D. But Fact 2.23 (2) and (3) imply that T is a maximal p-torus of B, and Corollary 6.3 says that T is a Sylow p-subgroup of B. Hence the conjugacy of Sylow p-subgroups in B (Fact 2.16) allows to conclude (1) in this case. Thus we may assume that all the p-elements of B are contained in $B' \subseteq \mathcal{U}$, and Corollary 6.3 finishes the proof of (1).

Finally, we prove assertion (2). We may assume $\overline{r}_0(D) > 0$. Let A be a Borel subgroup containing $N_G(U_0(D))^\circ \geq N_G(D)^\circ > D$. In particular, we have $A \neq B$. By Fact 2.35 (1), there is a positive integer r such that $((A \cap B)^\circ)'$ is a homogeneous $U_{0,r}$-subgroup. Let $s \leq \overline{r}_0(D)$, and let S be a Sylow $U_{0,s}$-subgroup of B. By Fact 2.33 (3), there is a Carter subgroup Q of B such that $S = U_{0,s}(B')U_{0,s}(Q)$. We assume $S \neq U_{0,s}(Q)$, that is $U_{0,s}(B') \neq 1$. By Fact 2.23 (3), $Q = D^b$ for $b \in B$. On the other hand, by Fact 2.30 (2), the subgroup $SU_0(D^b)$ is nilpotent. If $s < \overline{r}_0(D)$, then $U_0(D^b)$ centralizes S (Fact 2.30 (6)), and $S \leq B \cap A^b$ is abelian by Fact 2.35 (2). If $s = \overline{r}_0(D)$ and $U_0(D) \subseteq \mathcal{S}$, then S is contained in a Carter subgroup of G by Proposition 6.4 (2). Since B does not contain a Carter subgroup of G, Fact 2.35 (2) implies that S is abelian. If $s = \overline{r}_0(D)$ and $U_0(D) \nsubseteq \mathcal{S}$, then we have $S \subseteq \mathcal{U}$ by Proposition 6.4 (1). In this case, S is contained in B'_S for a Borel subgroup B_S (Lemma 6.14). Since $s = \overline{r}_0(D) > 0$ and $D \cap B' = 1$, we have $B_S \neq B$. Again Fact 2.35 (2) implies that S is abelian. Thus, in all the cases, S is abelian and centralizes $U_0(D^b)$. Then S is contained in $(A^b \cap B)^\circ$. Let now, $H = (A^b \cap B)^\circ$. Since D^b is a Carter subgroup of H, we have $S = U_{0,s}(H')U_{0,s}(D^b)$ by Fact 2.33 (3). In particular, since $S \neq U_{0,s}(D^b)$, we have $U_{0,s}(H') \neq 1$. Hence, since $H' = (((A \cap B)^\circ)')^b$ is a homogeneous $U_{0,r}$-subgroup, we obtain $s = r$. In particular, this proves the uniqueness statement in assertion (2).

In order to complete the proof, it remains to prove that S is a maximal abelian $U_{0,r}$-subgroup and is not a Sylow $U_{0,r}$-subgroup of G. Before going any further, we verify that H is a maximal intersection of Borel subgroups in G with respect to containment. We will use condition (ii) of Fact 2.36 (1) to verify this. Since S is an abelian Sylow $U_{0,r}$-subgroup of B, all the Sylow $U_{0,r}$-subgroups of B are abelian by Fact 2.33 (2), and the Sylow $U_{0,r}$-subgroup of $F(B)$ is central in $F(B)$ by Fact 2.30 (7). Thus, since $F(B)$ contains B' by Fact 2.10, the $U_{0,r}$-group H' centralizes B'. On the other hand, since $D^b \leq H$, D^b normalizes H', and so $B = B' \rtimes D^b$ normalizes H'. This implies that $B = N_G(H')^\circ$. In particular, $B \geq C_G(H')^\circ$. The maximality follows.

An immediate consequence of the last paragraph is that S is a maximal abelian $U_{0,r}$-subgroup of G. Indeed, if S_A is a maximal abelian $U_{0,r}$-subgroup of G containing S, then $S_A \leq C_G(S)^\circ \leq C_G(H')^\circ \leq B$. Thus, $S = S_A$ by maximality of S in B.

It remains to prove that S is not a Sylow $U_{0,r}$-subgroup of G. Before proceeding towards this conclusion, we verify that B' does not contain S. If B' contains S, then S is normal in B and $B = N_G(S)^\circ$. By Fact 2.33 (1), S is a Sylow $U_{0,r}$-subgroup of G. Then $N_{A^b}(S)^\circ \leq B$ contains a Carter subgroup C_{A^b} of A^b by Fact 2.33 (4), and C_{A^b} is a Carter subgroup of H. Thus C_{A^b} and D^b are conjugate in H (Fact 2.23 (3)), and D^b is a Carter subgroup of A^b, contradicting that D is not a Carter subgroup of A. Hence B' does not contain S.

Finally, assume towards a contradiction that S is a Sylow $U_{0,r}$-subgroup of G. Since $H' \leq B' \subseteq \mathcal{U}$, $S \subseteq \mathcal{U}$ by Proposition 6.4 (1). Let $B_S = N_G(S)^\circ$. By Lemma 6.14, B_S is a Borel subgroup of G satisfying $S \leq B_S'$. It then follows using the conclusion of the preceding paragraph that $B \neq B_S$. Since $H' \leq S \leq H$, $H \leq N_G(S)^\circ = B_S$. Hence, $B \cap B_S$ is also a maximal intersection. Since $B \geq N_G(H')^\circ$, Fact 2.36 (2) implies that $\overline{r}_0(B) > \overline{r}_0(B_S)$. Since S is a Sylow $U_{0,r}$-subgroup of G, S is abelian and $S \lhd B_S$, we conclude that $S = U_{0,r}(F(B_S))$. Fact 2.36 (3) yields a contradiction. □

The following corollary is a direct consequence from Theorme 6.16.

Corollary 6.17. – *If a Borel subgroup B of G has an abelian Carter subgroup D, then it satisfies the following decomposition*

$$B = B' \rtimes D \quad \text{and} \quad Z(B) = F(B) \cap D.$$

7. Toward a Jordan decomposition for K^*-groups

The simple K^*-groups of finite Morley rank form the backbone of the inductive approach to the Cherlin-Zilber conjecture, and the geometric nature of the structural information conveyed by a Jordan decomposition is likely to allow to make advances towards the resolution of this problem. Our goal in starting this work was to establish a Jordan decomposition for connected minimal simple groups of finite Morley rank. Connected minimal simple groups form the basis of any inductive approach to the Cherlin-Zilber conjecture. Their structure is thus poor in terms of inductive information. Nevertheless, the theory of solvable groups of finite Morley rank is invaluable.

The progress made throughout the present article raises the following natural question: can we extend the Jordan decomposition to the entire class of simple K^*-groups of finite Morley rank? The reader should recall that a minimal counterexample to the Cherlin-Zil'ber conjecture is a simple K^*-group of finite Morley rank, equivalently a simple K^*-group of finite Morley rank is a group of finite Morley rank all of whose proper definable simple sections are algebraic groups over algebraically closed fields. In particular, connected minimal simple groups are K^*-groups. This generalization will involve only simple groups of odd type since the structure of simple groups of even type are known to be algebraic [ABC08].

Despite their partial character, our existing results suffice to form the basis of an induction. Indeed, in a non-minimal simple K^*-group, thanks to the presence of definable simple sections, there will be always be involutions, hence, infinite Sylow 2-subgroups. On the one hand, this eliminates a considerable number of technical problems encountered above, on the other hand, using inductive arguments based on the presence these definable simple sections one can show that the Weyl groups are not trivial. As a result, the above analysis of minimal simple groups of types (3) and (4) are sufficient to form an induction basis. Indeed, it is easy to prove the following dichotomy:

Lemma 7.1. – *Let G be a simple K^*-group of finite Morley rank. Then one of the following conditions is true:*

(1) *either G is minimal*
(2) *or the Weyl group of G is of even order.*

We should emphasize that the expression "the Weyl group" is justified in the context of non-minimal simple K^*-groups as well. Indeed, the presence of non-trivial divisible torsion (p-tori) in a non-minimal simple K^*-group implies the presence of non-trivial maximal decent tori, and these are conjugate. Moreover, the following lemma shows that the initial step of the minimal analysis is also available in general:

Lemma 7.2. – *Let G be a simple K^*-group of finite Morley rank of odd type. Then the following isomorphisms hold:*

$$W(G) \simeq N_G(C)/C \simeq N_G(S)/C_G(S) ,$$

where C is any Carter subgroup of G while S is a maximal 2-torus.

The proof is just the first part of the proof of Fact 3.1. It depends on another crucial fact still available in this context, namely the conjugacy of Carter subgroups for simple K^*-groups of finite Morley rank [Fré08], and on an inductive reasoning that implies that $C_G(S)$ is still solvable.

These motivate to undertake an analogue of the first subsequent major step in the minimal case, namely the self-normalization of Borel subgroups (Fact 3.8). In the general non-minimal context, it is likely that one will have to replace Borel subgroups by other classes of subgroups generalizing some of their properties, e.g. the maximal, definable, connected subgroups.

It is highly probable that the self-normalization conclusion will not be achieved fully, and one will be content with proving that there exist no involutions in the quotient of the normalizer by the subgroup in question. This restriction is caused by lack of torsion information in a general simple K^*-group. Indeed, the main definite numerical result known in this direction concerns only 2-tori:

Fact 7.3. – *Let G be a simple K^*-group of finite Morley rank of odd type, which is not algebraic. Then G has Prufer 2-rank at most two.*

On the other hand, this fact supported by other major works on semisimple torsion (e.g. [BC08b]) yields convincing evidence that the elimination of 2-torsion can be achieved. We expect that this partial information, supported by richer inductive information of the non-minimal case, will be sufficient to continue the analysis leading to the sought for generalization of the Jordan decomposition.

This generalization will necessitate an extended analysis around the following main lines:

- the analysis of intersections of maximal, definable, connected subgroups, i.e. an extension of the Bender method developed in [Bur07];

- the study of K-group configurations that arise in the analysis of a simple K^*-group of Prüfer 2-rank at most 2 and of the related simple group automorphisms;
- an extension of the work by Deloro in [Del08].

The presence of the last item in the preceding list is justified by our experience that, with sufficiently strong conditions, maximal, definable, connected subgroups tend to be solvable. For instance, if G is a simple K^*-group of finite Morley rank of Prüfer 2-rank 1 and H a maximal, definable, connected subgroup such that $N_G(H)/H$ is of even order, then one can easily show that H is solvable, a conclusion which yields a setting reminiscent of connected, minimal, simple groups. The configurations that arise when one replaces the Prüfer 2-rank assumption by 2 justify the analyses proposed in the first two items of the above list.

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