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JOINT PRODUCTION WITHOUT FREE DISPOSAL:

Sustainable Activity and Unsupportable Processes

ABSTRACT

Investigations of long-run sustainability of joint ecological-economic systems highlight the pertinence of reconsidering "free disposal" assumptions that underpin Sraffian and von Neumann approaches to value and growth theory. We investigate joint-production time-paths for square systems $\mathbf{A}$ and $\mathbf{B}$, defined by non-negative activity vectors $y_T, y_{T+1}, \ldots$ for periods $T, T+1, \ldots$ satisfying iteratively $y_T\mathbf{B} = y_{T+1}\mathbf{A}$, etc. Necessary and sufficient for existence of indefinitely sustainable "full resource utilisation" time paths (Hicks), is the condition that $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$ have a non-negative left-eigenvector with associated non-negative eigenvalue. Sustainable time-paths may then be balanced growth at $g > -1$, or convergent oscillations, or undamped cycles. Processes are classified as "essential", "non-essential", or "unsupportable" for sustainable activity, based on considerations of eigenvector (non)-negativity and (in)decomposability of $\mathbf{U}$.

KEY WORDS: Convergence, Environment, Free gifts of nature, Free disposal, Growth theory, Hicks, Joint production, Sraffa, Stability, Steady-state, Sustainability, Unsupportable processes, von Neumann

RESUME

L'examen des enjeux de viabilité à long terme des systèmes « joints » écologico-économiques met en exergue l'urgence de repenser la place des hypothèses de "décharge gratuite" sous-jacents aux approches de Sraffa et de von Neumann à la théorie de croissance économique et de valeur. Nous considérons les conditions pour l'existence, ou non, des sentiers multi-période des systèmes de production jointe avec matrices $\mathbf{A}$ *et* $\mathbf{B}$*, définis par des vecteurs d'activité non-négatifs* $y_T, y_{T+1}, \ldots$ *pour des périodes* $T, T+1, \ldots$ *obtenus en satisfaction, itérativement, de* $y_T\mathbf{B} = y_{T+1}\mathbf{A}$*, etc. La condition nécessaire et suffisante pour l'existence des sentiers de « pleine utilisation de ressources » viable sans limitation de durée, est que la matrice* $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$ *possède un eigen-vecteur à gauche non-négatif associé à une valeur propre elle aussi non-négative. Des sentiers durables peuvent alors être, ou bien de croissance équilibrée à taux* $g > -1$*, ou bien des oscillations convergentes, ou bien des cycles persistants. Les processus au sein d'un tel système sont ensuite classifiés comme « essentiels », « non-essentiels », or « insupportables » pour une activité durable, en fonction des considérations de la (non)négativité des eigen-vecteurs et de la (non) décomposabilité de la matrice* $\mathbf{U}$.

MOTS CLES : ***Convergence, Décharge gratuite, Développement durable, Environnement, Etat stationnaire, Hicks, Processus insupportables, Production jointe, Sraffa, Soutenabilité, Stabilité, Théorie de croissance, von Neumann***

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I. INTRODUCTION

This paper examines the structure of interdependencies within a set of production processes, with a view to the sustainability or not of production activity in the long-run under the assumption of no free disposal of resources. A typology is presented of technologically stationary time-paths in square joint-production systems, under strict assumptions of no "free gifts" and no "free disposal".

Sections II and III respectively lay out the formalism of the joint-production model, and give some background to the question of "free disposal" and the approach taken to typology of time-paths and interdependency. We seek to establish some formal solution concepts that are important for contemporary investigations (i) of sustainability prospects in joint economy-environment systems and (ii) of the value of ecological goods and services in the short and long terms.

The core of the paper is then devoted to algebraic exposition of solution classes. This exposition draws on established mathematical results in the Sraffa and von Neumann traditions, but adds some new and distinctive emphases for interpretation in todays sustainability context.. We investigate the algebraic structure comprised of a square non-negative matrix pair $\mathbf{A}$, $\mathbf{B}$ interpreted as a set of production process technologies (each row-pair corresponding to a distinct process), and the time-paths described for periods $T, T+1, T+2, \ldots$ by sequences of non-negative activity vectors $y_T, y_{T+1}, \ldots$ satisfying iteratively $y_T\mathbf{B} = y_{T+1}\mathbf{A} = q_{T+1}$, etc. These correspond to what Hicks (1965) called "full utilisation time-paths" that are "continually and ultimately viable". It is first shown, in ***Section IV,*** that the necessary and sufficient condition for sustainability without free disposal is that $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$ have a non-negative left eigenvector associated with a non-negative eigenvalue $\lambda = 1/(1+g)$. We then discuss, in ***Section V,*** how analysis of the spectral properties of $\mathbf{U}$ provides the basis for classification of square systems in terms of ability or inability to support various sorts of sustainable activity. Sustainable time-paths, if any, may be balanced growth (including negative or zero growth), convergent oscillations, or undamped cyclic behaviour. ***Section VI*** turns to the relation between decomposability of the coefficient matrices and process requirements for sustainable activity within a given system. Here we systematise and build on suggestive work by Pasinetti (1973), Schefold (1980), and Steedman (1989) concerning dispensable and indispensable processes as the mathematical dual concept to Sraffa's (1960) basic/non-basic goods. It is shown that decomposability of the matrix $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$ — which is a condition equivalent, subject to non-singularity restrictions, to decomposability of Pasinetti's (1973) $\mathbf{G}$ -matrix, $\mathbf{G} = \mathbf{A}[\mathbf{B}-\mathbf{A}]^{-1} = [\mathbf{U}-\mathbf{I}]^{-1}$ — is the correct basis for the distinction made by Schefold (1980) between indispensable and dispensable processes as dual to Sraffa's basics/nonbasics distinction for resources.

In ***Section VII*** the considerations of sustainability (existence or not of a non-negative activity eigenvector) and of indispensability (or not) are brought together, as the basis for a three-way classification of processes as ***essential, non-essential, or unsupportable*** within a given ensemble. ***Section VIII*** concludes by recapitulating the rationales for the classifications made and their pertinence for contemporary ecological economics theory problems.

II. THE JOINT PRODUCTION MODEL

The backdrop for this paper is the problematisation, since the 1970s, of (i) exponential economic growth based on "free gifts of nature", (ii) human economies dependent on an irreplaceable natural capital; (iii) perturbation of atmosphere and disruption of life-support functions on local and global levels through disposal of "waste" materials and pollutants; and thus (iv) the sustainability or not of a given pattern of economy-environment interdependent activity. In most analyses of commodity production in industrial economies, free-gift and free-disposal assumptions (henceforth FGFD) have been axiomatic, though often implicit. The cogency of excising the free disposal assumption in theoretical analysis, and the relevance of the notion of full resource utilisation time-paths, become obvious as soon as one extends the joint production framework to include environmental processes in the mass-closed "Spaceship Earth".

Since the 1980s, a number of analysts working in the classsical Sraffian tradition have given attention to the idea that disposal of wastes could prove difficult or costly. There are several notions involved here. One is "costly disposal", that is, the requirement for inputs of positively valued commodities in order to get rid of unwanted by products (see notably England 1986; Kurz 1986; Kurz & Salvadori 1992; O'Connor 1993b).[1] Another is the question of physical or political power to impose disposal. For example, Salvadori & Steedman (1988, p. 180) observed:

> *"To assume free disposal for even one product is to deny the principle of conservation of mass-energy, one of the most fundamental principles of Thermodynamics. Every process with some input must have some product. If that product has zero value, this may be related to the complete absence of property rights (e.g., smoke in the air, radioactive waste at the bottom of the ocean), to complex questions of externalities in the presence of partially defined property rights (e.g., creation of waste disposal sites), and so on."*

[1] A variety of more recent contributions including Baumgartner (2004), Klauer (2000) and Lager (2001), have explored related questions of value and growth potential in ecological-economic systems without, however, a systematic investigation of the algebraic properties of the "underlying" mass-closed system.

It is perhaps not so much that mass-energy conservation has wholly been denied in mainstream economics of the 20th century but, rather, that free gift and free disposal (FGFD) assumptions that have seemed reasonable when working on questions of commodity production and exchange within a commodity economy, become invalid when concern is turned to large-scale environmental repercussions of industrial activity. In our contemporary context of sustainability concerns, excising the free disposal underpinnings of modelling becomes a question of both theoretical and practical significance. This means abandonment not in a piecemeal sense — for example, with regard to a specific pollutant, as is done in neoclassical economics of the environment — but in a way that is systematic and across the board.

The notion of a mass-closed Spaceship Earth economy-environment system was made famous by Boulding (1966). Joint-production modelling applied to economy-environment systems by Perrings (1986, 1987) and by O'Connor (1993a, 1994, 1996), sought to tease out the significance of systematically rejecting FGFD for our understanding of economy-environment codependency, of the nature of technological change, and of the requirements for long-run sustainability of regimes of economic activity. Whereas Perrings focused on mass conservation, the work by O'Connor presented a unified treatment of mass-closure (and mass conservation) and energy conservation for a Spaceship Earth system. Several of the algebraic results given in the just cited papers by O'Connor (1993a, 1994, 1996) concerning technological requirements for long-run sustainability and the time structure of sustainable activity were, however, given without formal proofs.[2] The present paper gives a number of proofs and related typology of systems.

The model employed for the present demonstration, consists of an ensemble of interdependent ***physical production processes,*** involving exchange and transformation of ***material resources***. Description is in terms of process technologies (row-vector-pairs of coefficients) and a vector of activity levels for each production period. Bold upper-case letters designate matrices; and bold lower case letters designate vectors, row or column as specified.

2.1 There are N interdependent production processes, enumerated by subscript $i = 1, 2, ..., N$.

2.2 Production activity occurs in discrete time periods which are designated by index T, and is represented through specifying inputs and outputs of material resources for each process. There are G distinct types of resources produced and consumed, designated by $j = 1, 2, ..., G$. The ensemble as a whole is closed with respect to material resource transfers. Each process uses at least one material resource input, and produces at least one resource output.

[2] Work by Patterson (1998, 2002, 2006) also introduced mass and energy conservation as a motivation for value system solution concepts in joint production economy-environment models. However, his progress was hampered by the lack of adequate algebraic foundations for typology of systems and of solution classes such as this paper seeks to contribute.

2.3 Though in general we could have $N \neq G$, here we consider systems for which $N = G$.

2.4 A non-negative row-vector pair defines the levels of inputs and outputs for the ***unit activity*** of each production process. The technology set is represented by a non-negative matrix pair $\mathbf{A}, \mathbf{B}$, where each row-pair designates a distinct process. Each column-pair thus refers to a distinct resource type.[3]

2.5 In this paper, these coefficients will always be assumed time-invariant, so no time-index is written for $\mathbf{A}$ and $\mathbf{B}$. We do not adopt any explicit convention for normalisation of rows.

2.6 $\mathbf{A}$ and $\mathbf{B}$ are N-square and non-negative; we assume that also they are both non-singular. So both inverses $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist.

2.7 An ***activity vector*** in time period T, is the $(1 \times N)$ row-vector y_T, with elements y_T^i designating the level of activity of the i^{th} process during period T, as a multiple of the unit production activity denoted by the corresponding row-pairs of $\mathbf{A}$, $\mathbf{B}$.

2.8 Material resources utilised as ***inputs*** are allocated to processes at the outset of each period. For period T, total inputs are represented by elements q_T^j of a $(1 \times N)$ row vector: $q_T \equiv y_T A$; the superscript j designates the j^{th} resource type. Process ***outputs*** are the material resources present within a process at the end of a production period. The total outputs at the end of period T are $y_T B$.

2.9 Outputs from period T are available as inputs in period $T+1$. With no free gifts, we have $q_{T+1} \leq y_T B$. If, further, there is no free disposal, we have: $q_{T+1} = y_T B$.

III. SOLUTION CONCEPTS & FREE DISPOSAL

Turn now to description of production time-paths in the model, meaning sequences of activity vectors $y_T, y_{T+1}, \ldots$, etc., and the associated resource vectors $q_T, q_{T+1}, \ldots$ etc. In the lineage of growth theory (Hicks 1965; Morishima 1964), the most obvious properties to consider, in addition to non-negativity, are stationarity in the senses of (i) balanced growth (including negative or zero-growth); (ii) undamped regular oscillations; and (iii) convergence towards a stationary structure. Let us now define the four N-square matrices:

3.1(a) $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$ and so $\mathbf{U}^{-1} \equiv \mathbf{A}\mathbf{B}^{-1}$;

[3] The word technology usually has the connotation of purpose: the production process is considered a means to an end (the outputs). Nonetheless, it is quite possible to speak of a technology without this teleological overtone, as simply the relationship de facto between inputs and outputs in a transformation process.

(b) $\mathbf{Q} \equiv \mathbf{A}^{-1}\mathbf{B}$ and so $\mathbf{Q}^{-1} \equiv \mathbf{B}^{-1}\mathbf{A}$

U and **Q** have common eigenvalues, and similarly for the two inverses. Barring a few special cases (which are uninteresting in this context) all these matrices are non-singular, ***but they are not necessarily non-negative.*** With this notation, the Sraffian problematics of expanded reproduction (balanced growth at rate g) and profit-equalisation across all processes (with profit rate n), can be posed as:

3.2(a) $y_T\mathbf{U}^{-1} = \lambda y_T \Leftrightarrow (1+g)y_T = y_T\mathbf{U}$, where $1/\lambda = (1+g)$ is an eigenvalue of **U**;

(b) $\mathbf{Q}^{-1}p = \kappa p \Leftrightarrow (1+\pi)p = \mathbf{Q}p$, where $1/\kappa = (1+\pi)$ is an eigenvalue of **Q**.

Balanced resource expansion at rate g is achievable if, and only if, a non-negative activity vector y_T can be found for which $y_T\mathbf{B} = (1+g)y_T\mathbf{A}$, meaning the initial resource vector $q_T \equiv y_T\mathbf{A}$ generates outputs $(1+g)q_T$ in the same proportions. Such a y_T must be a left-eigenvector of $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$, associated with the eigenvalue $1/\lambda = (1+g)$; and the corresponding q_T is a left-eigenvector of $\mathbf{Q} \equiv \mathbf{A}^{-1}\mathbf{B}$ associated with the same eigenvalue.

Because **Q** and **U** are generally not non-negative (and nor are their respective inverses), there is no guarantee that there will exist any non-negative eigenvalue(s) of **U** and **Q**; nor that, even if a $1+g>0$ (so $g>-1$) is obtained, the corresponding y is non-negative. These complications of joint production systems have been noted since Manara (1968); however the possible absence of any "balanced activity" has usually been put aside as a curiosity. In part, this sidelining reflects the widespread supposition of the possibility of "free disposal" of resources surplus to the needs of balanced expansion of a multi-sector commodity economy, as is made axiomatic in von Neumann's (1945-46) formulation of "balanced growth" and economic equilibrium as inequality problems, with the equilibrium structure defined by expansion along the von-Neumann-ray. Where, by von Neumann's definition, there are non-scarce resources produced in the system, so-called "balanced growth" or Turnpike convergence towards the ray, relies on the free disposal of the surplus resources. This free-disposal is signaled in the value-relations by zero prices being attributed to such resources; but it also carries the implication that is somehow possible to dispose physically of the surpluses "outside the system".

In our "Spaceship Earth" perspective, however, all material resources must go somewhere and must come from somewhere within the ensemble of processes comprising the biosphere. In the model:

3.3 Total resources available at the end of period T, are $y_T\mathbf{B}$. There are no free gifts and no free disposal. So all the resources available for redeployment in period $(T+1)$ must actually be deployed: $q_{T+1} = y_T\mathbf{B} = y_{T+1}\mathbf{A}$.

Since also $q_T \equiv y_T \mathbf{A}$, then assuming matrix invertibility we obtain three equivalent forms of iterative equations:

3.4 (a) $y_{T+1} = y_T \mathbf{B} \mathbf{A}^{-1}$, and $q_{T+1} = q_T \mathbf{A}^{-1} \mathbf{B}$;

(b) $y_{T+1} = y_T \mathbf{U}$, and $q_{T+1} = q_T \mathbf{Q}$;

(c) $y_T = y_{T+1} \mathbf{U}^{-1}$ and $q_T = q_{T+1} \mathbf{Q}^{-1}$.

This motivates the idea of a ***feasible technologically stationary activity,*** what Hicks (1965, p.323) refers to as a full-resource-utilisation time-path, where disposal of unwanted resources outside the system is precluded. Let us define:

3.5 A feasible technologically stationary activity exists where, for a given $\mathbf{A}$, $\mathbf{B}$ and some initial non-negative resource vector q_T, for $\alpha + 1$ periods ($\alpha \geq 0$) there can be found non-negative activity vectors $y_T = q_T \mathbf{A}^{-1}, y_{T+1}, y_{T+2}, ..., y_{T+\alpha}$, satisfying Equations *(3.4b)*.

3.6 *A* ***sustainable activity*** is then a continually feasible technologically stationary activity — one that, from some initial non-negative q_T, is characterised by an unending sequence of non-negative activity vectors $y_T = q_T \mathbf{A}^{-1}$ and $y_{T+\tau} = y_{T+\tau-1} \mathbf{U}$, $\tau = 1, 2, 3, ..., \infty$.

In what follows, we give a typology of systems based on their ability or not to support various types of sustainable activities in line with *(3.6)*. The questions we address and their context have many different facets, and some initial caveats are warranted concerning the specific theoretical scope of this paper.

First, our formal treatment of a materially-closed ensemble (no free gifts, no free disposal), has abstracted away from all qualitative differences between different sorts of human production and consumption processes, and ecological and geological processes. This is not an inherent limitation, as auxiliary assumptions can be introduced when building particular models designed to explore and illuminate specific problem classes (see, for example, O'Connor 1993b; O'Connor, Andrew & Patterson 2009).

Second, we focus on excising FGFD assumptions as a stylized representation of the "mass closed" character of the Spaceship Earth system and we do not, in this paper, discuss the complementary and irreducible considerations of energy conservation and entropic irreversibility. The latter features have, however, been the object of our attention elsewhere and, the arguments made in this paper are carefully compatible with the lessons thermodynamic laws.[4]

[4] In economy-environment applications, the ensemble interpreted as the mass-closed biosphere (Spaceship Earth) interacts with its extra-terrestrial environment through energetic (heat and work) exchanges; and terrestrial processes also interact with each other through heat and work exchanges (O'Connor 1993a, 1994, 1998). These energy exchanges are simply repressed in the materials representation employed here.

Third, our attention has been confined to square (sometimes called quadratic) systems. Again this is not an inherent limitation, as the process classification put forward in terms of requirements for a sustainable time-path, can readily be extended to non-square systems based on considerations of matrix rank and decomposability explored here in the square-matrix context.[5]

Fourth, our focus is exclusively on technologically stationary time-paths, and on the identification of sustainable time-paths in that context. In looking at the real world, this exclusion of technological change may seem self-defeating. Yet, in theoretical terms, it is valid and useful as a heuristic procedure, furnishing us with clearly defined concepts of interdependency and with "benchmark" results from which we may depart in dialectical fashion (e.g., O'Connor 1993a, 1994a).

Finally (and as was remarked by a reviewer of a much earlier version of this paper), it seems obvious that, with reference to the mass-closed biosphere, the investigation of whole-system balanced growth should focus on the zero-growth rate. This is true, but there are several facets to the introduction of materials closure and mass conservation in joint production modeling. The analyses, already mentioned, by Perrings (1986, 1987) and by O'Connor (1993a, 1994a, 1996), introduced mass conservation at a high level of abstraction by requiring mass-balance between inputs and outputs of each process. Since, in the model, it is assumed that no resources cross process boundaries during a production period, this requirement is that the sum of the masses of the inputs of a process at the outset of production period T, be equal to the sum of the masses of the outputs of that process at the end of that period T. When all resources are measured in commensurate mass units, this imposes: $\mathbf{A}e = \mathbf{B}e \Leftrightarrow [\mathbf{B} - \mathbf{A}]e = 0 \Leftrightarrow \mathbf{Q}e = e$, where $\mathbf{Q} \equiv \mathbf{A}^{-1}\mathbf{B}$ and e is the column vector of ones. This establishes unity as an eigenvalue of both $\mathbf{Q}$ and $\mathbf{U}$, along with the corollary that $\det[\mathbf{B} - \mathbf{A}] = 0$. It follows directly that, *if* a long-run "full-utilisation balanced-growth" for the economy-environment ensemble is possible at all, it must be with zero-growth.[6]

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[5] The work by Patterson, cited above, presents some formulations of value problems for non-square systems, and insists rightly on the cogency of this "generalisation" for several classes of problems. However, it is difficult to formulate results for non-square systems without reference to the "special case" square systems, and so we stay, for present purposes, with the "special case".

[6] O'Connor (1996) gave a full exposition and discussion of zero-growth steady-state time-paths in a mass-closed technologically stationary Spaceship Earth system. However, and as had been exposed in counterpoint in O'Connor (1994), existence of a steady-state solution for a mass-closed system $\mathbf{A}$, $\mathbf{B}$ is not guaranteed. From the constraints $\mathbf{A}e = \mathbf{B}e \Leftrightarrow \mathbf{T}e = e$, there must exist at least one real solution y to the equations $y\mathbf{A} = y\mathbf{B}$; but such a $\mathbf{y}$ is not necessarily non-negative. Investigation into possible sustainable time-paths means determining the existence, or not, and various categories, of non-negative solutions y signalling steady-states involving one, some, or all of the processes. The steady-state solutions, when they exist, may also be asymptotes around which harmonic solutions oscillate, and towards which forward-narrowing solutions converge. Detailed algebraic exposition and interpretation of sub-classes (including illustrative special cases) is in unpublished work by O'Connor (1992).

The distinction to be made at length in this paper between essential/non-essential/unsupportable processes indeed has particular cogency in this Spaceship Earth context. But the ramifications are sufficiently rich that their full exposition requires separate discussions.

IV. TIME-PATHS WITH STRICT RESOURCE CLOSURE

In the sections that follow we will give a typology of systems based on their ability or not to support various types of sustainable activities. We first state the following theorem:

4.1 A system $\mathbf{A}$, $\mathbf{B}$ can support sustainable activity as defined in *(3.6)* above, if and only if there exist one or more non-negative solution time-paths for balanced growth: that is, if and only if there exists a non-negative activity vector y solving $y\mathbf{U} = (1+g)y$ with $g > -1$.

Sufficiency is already established by the definitions of y and $\mathbf{U}$ and of sustainable activity. We will show necessity under the assumption that $\mathbf{U}$ has N distinct eigenvalues, but omit the special case of recurring eigenvalues.[7] The critical consideration is the convergence (or cyclical) properties[8] of the vector time series defined by the iterative equation *3.4(b):* $y_{T+1} = y_T\mathbf{U}$, where $\mathbf{U} \equiv \mathbf{B}\mathbf{A}^{-1}$.

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[7] Existence of recurring eigenvalues for $\mathbf{U}$ is mathematically a special case, whose properties can be investigated by looking at limits as the ratio of two eigenvectors approaches 1. The nature of the time-path problem is not fundamentally altered, but there is the complication that sustainable time-paths (when these exist) may sometimes be linear combinations of time-paths properly associated with (i.e., on, converging towards, or oscillating around) each of the left-eigenvectors of $\mathbf{U}$ corresponding to the multiple eigenvalue in question. It follows that if, for a non-negative recurring eigenvalue, at least one left-eigenvector is non-negative, sustainable activity will be possible with at least one process activity at zero. These special properties may be of interest in considering process requirements for sustainability in Spaceship Earth assuming both no-free-disposal and mass-closure, but are fiddly to deal with in formal proofs and so are set aside here.

[8] Numerous proofs exist (e.g., Morishima 1964, pp.204-211) drawing on the Perron-Frobenius theory for non-negative indecomposable matrices, to the effect that starting from an arbitrary non-negative y_T and with non-negative indecomposable $\mathbf{U}$, the sequence $\mathbf{y}_{T+n} = \mathbf{y}_T\mathbf{U}^n$ converges towards the balanced-growth ray designated by strictly positive y_1 associated with the dominant eigenvalue $\mu_1 = 1+g^*$, where g^* happens to be the von-Neumann growth rate. But generally $\mathbf{U}$ is not non-negative. Perrings (1987, pp.24-27) set out to investigate limit properties for a technologically stationary system. However his formulation of the problem was inexact. He sought to prove convergence of a full-resource utilisation time-path for the system onto a balanced-growth path on the basis of analysing the limit behaviour as $n \to \infty$ of $\mathbf{B}^n$ for primitive non-negative $\mathbf{B}$, whereas he ought to have been investigating the limit behaviour as $n \to \infty$ of $\mathbf{U}^n$, where $\mathbf{U} = \mathbf{B}\mathbf{A}^{-1}$. So his conclusions about convergence were erroneous, although his approach to the proof was suggestive for the formulation that the present paper adopts. (The source of Perrings' error was that his equations 2.7 on his p.24 actually do not

Let $\mathbf{J}$ be the Jordan normal form of $\mathbf{U}$, that is, the matrix similar to $\mathbf{U}$ whose diagonal elements are the eigenvalues of $\mathbf{U}$. Let us assume:

4.2 Matrix $\mathbf{U}$ is N-square and has N distinct eigenvalues.

4.3 The Jordan normal form of $\mathbf{U}$ is $\mathbf{J} = \mathbf{PUP}^{-1} \Leftrightarrow \mathbf{U} = \mathbf{P}^{-1}\mathbf{JP}$, where $\mathbf{P}$ and $\mathbf{P}^{-1}$ are non-singular, and

4.4 *(a)* the rows of $\mathbf{P}$ are the left-eigenvectors of $\mathbf{U}$ ordered conformably with the μ_j; and we will write $y_{[j]}$ for the j^{th} row of $\mathbf{P}$;

(b) the columns of $\mathbf{P}^{-1}$ are the right-eigenvectors of $\mathbf{U}$ ordered conformably with the μ_j, and we will write $m^{[j]}$ for the j^{th} column of $\mathbf{P}^{-1}$. Further, $\mathbf{J}$ has non-zero off-diagonal elements only if $\mathbf{U}$ has recurring eigenvalues.[9] So, under assumption *(4.2)*

4.5 $\mathbf{J} = \text{diag}[\mu_1, \mu_2, ..., \mu_N]$, where the μ_j are the N distinct eigenvalues of $\mathbf{U}$, ordered in descending absolute value. There are now two categories to distinguish:

(a) All the μ_j have different modulus, in which case $\mathbf{P}$ is unique (up to a scalar multiple); or

(b) Some of the μ_j, have the same modulus, being a set (or sets) of multiple roots of some real number(s), in which case $\mathbf{P}$ is defined uniquely (up to a scalar multiple), with the proviso of an arbitrary ordering of eigen-values of the same modulus.

Recalling equations *(3.5b),* we now investigate the limit as $\tau \to \infty$ of $y_\tau = y_0\mathbf{U}^\tau$, for an arbitrary initial $y_0 \geq 0$. First we express this y_0, without loss of generality, as a linear combination of the $y_{[j]}$ defined in *(4.4a):*

4.6 $y_0 = \beta_1 y_{[1]} + \beta_2 y_{[2]} + ... + \beta_N y_{[N]} = [\beta_1, \beta_2, ..., \beta_N]\mathbf{P}$

where the vector, say $\mathbf{b}$ of coefficients β_j, is uniquely defined by $\mathbf{b} = y_0\mathbf{P}^{-1}$.

By components, we have thus: $\beta_j = y(0)m^{[j]}$. We have also defined $\mathbf{P}^{-1} = [m^{[1]}, m^{[2]}, ..., m^{[N]}]$; and may thus write the condition for orthogonality of eigenvectors as:

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correspond to a technologically stationary system, except in the cases where either $\mathbf{A}$ is diagonal or we are moving along the von-Neuman ray itself, or both. The way that he defines his coefficients makes the relative magnitudes of rows of $\mathbf{B}$ and $\mathbf{A}$ vary, except in the case of $\mathbf{A}$ diagonal, with the relative magnitudes of the resource vector $\mathbf{q}_T$ from period to period. This limitation to validity carries over to his discussion of decomposable systems.)

9 On the diagonal (or, with recurring eigenvalues, block-diagonal) incidence pattern of the Jordan normal form, see for example Gantmacher (1959) and Graham (1987).

4.7 $y_{[k]}m^{[j]} = 1$ if $j = k$, and zero otherwise.

Now, using definition *(4.3)* we have:

4.8 $y_\tau = y_T \mathbf{U}^\tau = y_0[\mathbf{P}^{-1}\mathbf{J}\mathbf{P}]^\tau = y_0\mathbf{P}^{-1}\mathbf{J}^\tau\mathbf{P}$.

The case (4.5a), systems in which all the μ_j have different modulus.

4.9(a) The μ_j will be all real, and the time-path of $y_{T+\tau}$ converges onto the sequence $\beta_k(\mu_k)^\tau y_{[k]}$ where μ_k is the highest-modulus eigen-value for which β_k in *(4.6)* is nonzero, with which is associated the eigenvector pair $y_{[k]}$ and $m^{[k]}$.

For the proof, recall by definition of the Jordan normal form:

$$P^{-1} = \begin{bmatrix} m^{[1]} & m^{[2]} & \cdots & m^{[N]} \end{bmatrix},$$

$$J^\tau = \operatorname{diag}\left(\begin{bmatrix} \mu_1^\tau & \mu_2^\tau & \cdots & \mu_N^\tau \end{bmatrix}\right),$$

$$P = \begin{bmatrix} y_{[1]} & y_{[2]} & \cdots & y_{[2]} \end{bmatrix}^T.$$

We have defined $\beta_j = y(0)m^{[j]}$. Therefore: $y(\tau) = \sum_i \beta_i \mu_i^\tau y_{[i]}$.

By convention, we have ordered the μ_j (the N distinct eigenvalues of $\mathbf{U}$) in descending absolute value. Take first the situation where $\beta_1 \neq 0$. Dividing through by μ_1^τ:

$$\frac{y(\tau)}{\mu_1^\tau} = \beta_1 \frac{\mu_1^\tau}{\mu_1^\tau} y_{[1]} + \cdots + \beta_k \frac{\mu_k^\tau}{\mu_1^\tau} y_{[k]} + \cdots + \beta_N \frac{\mu_N^\tau}{\mu_1^\tau} y_{[N]}.$$

But:

$$\lim_{\tau\to\infty} \frac{\mu_{i\neq 1}^\tau}{\mu_1^\tau} = 0 \text{ since } |\mu_1| > |\mu_{i\neq 1}|,$$

therefore: $\lim_{\tau\to\infty} \frac{y(\tau)}{\mu_1^\tau} = \beta_1 y_{[1]}$, and

4.10 $\lim_{\tau\to\infty} y(\tau) = \beta_1 \mu_1^\tau y_{[1]}$. where $y_{[1]}$ is the 1st row of $\mathbf{P}$, the left-eigenvector of $\mathbf{U}$ associated with the highest modulus eigenvalue μ_1.

Now alternatively:

4.9(b) Suppose $\beta_1 = \mathbf{y}_T\mathbf{m}^{[1]} = 0$.

Then the first term in $y_0 = \beta_1 y_{[1]} + \beta_2 y_{[2]} + ... + \beta_N y_{[N]} = [\beta_1, \beta_2, ..., \beta_N]\mathbf{P}$ is zero, and the above procedure in *(4.9a)* of dividing through by μ_1^τ yields an expression that tends towards the zero vector. To get round this, we now identify the highest-modulus eigenvalue μ_k (k>1) for which β_k in *(4.6)* is non-zero, with which is associated the eigenvector pair $y_{[k]}$ and $m^{[k]}$. Proceeding analogously to the above and using orthogonality properties *(4.7),* we get:

$$\frac{y(\tau)}{\mu_k^\tau} = 0\frac{\mu_1^\tau}{\mu_k^\tau}y_{[1]} + \cdots + \beta_k y_{[k]} + \cdots + \beta_N \frac{\mu_N^\tau}{\mu_k^\tau}y_{[N]}.$$

But:

$$\lim_{\tau\to\infty}\frac{\mu_{i\geq k}^\tau}{\mu_k^\tau} = 0 \text{ since } |\mu_k| > |\mu_{i\geq k}|,$$

therefore:

$$\lim_{\tau\to\infty}\frac{y(\tau)}{\mu_k^\tau} = \beta_k y_{[k]},$$

and

4.11 $$\lim_{\tau\to\infty} y(\tau) = \beta_k \mu_k^\tau y_{[k]}.$$

where $\mathbf{y}_{[k]}$ is the k^{th} row of **P**, the left-eigenvector of **U** associated with eigenvalue μ_k. So the time-path of y_τ converges towards the sequence of vectors $\beta_k(\mu_k)^\tau\mathbf{y}_{[k]}$.

The case (4.5b), systems in which some (or all) of the μ_j have the same modulus.

The situation defined by *(4.5b)* whereby some of the μ_j have the same modulus is, mathematically, a special class of systems, less tidy to treat but important for our ecological economics problematics because it but opens up potential for cyclical or oscillating solutions. The Jordan normal form is still $\mathbf{J} = \text{diag}[\mu_1, \mu_2, \ldots, \mu_N]$, but now the limit behaviour of $\mathbf{J}^n$ must be investigated with reference to the set(s) of the μ_j having the same modulus, and their corresponding left- and right-eigenvectors.

4.12 Suppose, for $\mathbf{U} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$ we have $H > 1$ distinct eigenvalues all of modulus r, including r itself. We apply condition *(4.2)* and so these roots are the set, $re^{i2\pi h/H}$, $h = 0, 1, ..., H-1$; that is, r times the H^{th} roots (generally complex) of unity.

Once again we consider a time-path under *(3.4b)* starting with an initial $y_0 \geq 0$. Limit properties are determined by the highest-modulus eigenvalue(s) for which a β_k in *(4.6)* is non-zero. So the case we must examine is that where this highest-modulus is the value $r = \mu_k$ for which there exist the H eigenvalues (all of modulus r) which we will designate μ_{k+h} where the subscript $k+h$ runs over $k,\ k+1,\ k+2,...,\ k+H-1$. We have:

$$|\mu_k| = |\mu_{k+1}| = \cdots = |\mu_{k+H-1}|, \text{ and}$$

associated with these eigen-values are the respective eigenvector pairs $\mathbf{y}_{[k+h]}$ and $\mathbf{m}^{[k+h]}$.

For the initial vector as written in *(4.6),* we have *either* that $\beta_1 \neq 0$ and $k = 1$, *or* that $\beta_1 = \beta_2 = \ldots = \beta_{k-1} = 0$. Taking the latter case for the sake of argument, write:

4.13 $$\mathbf{y}_T = 0\mathbf{y}_{[1]} + \ldots + 0\mathbf{y}_{[k-1]} + \beta_k \mathbf{y}_{[k]} + \beta_{k+1}\mathbf{y}_{[k+1]} + \ldots + \beta_{k+H-1}\mathbf{y}_{[k+H-1]} + \ldots + \beta_N \mathbf{y}_N .$$

We have

$$\lim_{\tau\to\infty} \frac{\mu_{i\geq\{k+H-1\}}^{\tau}}{\mu_k^{\tau}} = 0 \text{ since } |\mu_k| > |\mu_{i\geq\{k+H-1\}}|$$

and

$$\left| \lim_{\tau\to\infty} \frac{\mu_{i\geq\{k+H-1\}}^{\tau}}{\mu_k^{\tau}} \right| = 1 .$$

Therefore, for the time sequence of activity vectors:

$$\lim_{\tau\to\infty} \frac{y(\tau)}{\mu_k^{\tau}} = \beta_k y_{[k]} + \beta_{k+1} \frac{\mu_{k+1}^{\tau}}{\mu_k^{\tau}} y_{[k+1]} + \cdots + \beta_{k+H-1} \frac{\mu_{k+H-1}^{\tau}}{\mu_k^{\tau}} y_{[k+H-1]},$$

and

4.14 $$\lim_{\tau\to\infty}\ y(\tau) = \beta_k \mu_k^{\tau} y_{[k]} + \beta_{k+1}\mu_{k+1}^{\tau} y_{[k+1]} + \cdots + \beta_{k+H-1}\mu_{k+H-1}^{\tau} y_{[k+H-1]} .$$

Thus, for the case *(4.5b)*, the time-path of y_τ generally does not converge onto a ray, but rather towards a cycle of vectors, with a periodicity of H.[10] To see this, for the right-hand side of *(4.13)* now write:

[10] For completeness, we should mention the situation where the r characterizing the H eigenvalues is smaller than the highest-modulus eigenvalue for which $\beta_k > 0$, that is, $r < \mu_k$. The convergence is onto a ray

4.15 $\mathbf{c} = \beta_k \mathbf{y}_{[k]} + \beta_{k+1}\mathbf{y}_{[k+1]} + \ldots + \beta_{k+H-1}\mathbf{y}_{[k+H-1]}$,

We may consider, from the iterative equation *3.4(b):* $y_{T+1} = y_T \mathbf{U}$, the sequence of vectors $\mathbf{c}, \mathbf{cU}, \mathbf{cU}^2 \ldots$. Recalling the property stated in (4.12, that the eigenvalues are related together as the set $\{ re^{i2\pi h/H}, h = 0,1,...,H-1 \}$, it is easy to verify that this sequence of vectors follows a cyclical pattern: $\mathbf{c}, \mathbf{cU}, \ldots, \mathbf{cU}^{H-1}, \mathbf{cU}^H = r^H\mathbf{c}$, and so on endlessly. Moreover, it is straightforward to show that

4.16 $\mathbf{s} = \mathbf{c} + (1/r)\mathbf{cU} + \ldots + (1/r)^{H-1}\mathbf{cU}^{H-1}$ is an activity satisfying $r\mathbf{s} = \mathbf{sU}$.

This **s**, which under assumptions *(4.2 and 4.12)* is the unique (up to a scalar multiple) left-eigenvector of $\mathbf{U}$ associated with the positive eigenvalue r, designates a ray around which the cyclical time-paths spiral, with a "rate of growth" of $g = (r - 1)$, where $g > -1$.

Interpretation of sustainability potential and prospects from an arbitrary y_τ.

There are the several sub-cases to consider.

For the situation described in *(4.9a),* the time-path of y_τ converges onto the sequence $\beta_1(\mu_1)^\tau \mathbf{y}_{[1]}$. If $\mu_1 > 0$ and $\beta_1 > 0$, this is towards "balanced" exponential growth (or decay) of activity at rate $g_1 = (\mu_1 - 1)$, along the $y_{[1]}$ ray.

If also this activity eigenvector is non-negative, that is, $y_{[1]} \geq 0$, this is indefinitely sustainable; and so a balanced growth path with $\mathbf{y}_0 = \mathbf{y}_{[1]} \geq 0$ is itself feasible.

But, if this $y_{[1]}$ is not non-negative, the time-path indicated by *(3.4b)* converges on a ray for balanced growth with some negative process activities, which physically is not feasible.

Finally, if $\mu_1 < 0$ and $\beta_1 > 0$, the time-path of y_τ alternates in sign, meaning balanced growth is not feasible and the time-path indicated by *(3.4b)* is not feasible.

For the situation described in *(4.9b),* the argument is analogous. If $\mu_k > 0$ and $\beta_k > 0$, and also $y_{[k]} \geq 0$, this time-path converges towards a "balanced" exponential growth (or decay) of activity at rate $g_k = (\mu_k - 1)$, along the $y_{[k]}$ ray; and clearly the "balanced growth" path with $\mathbf{y}_0 = \mathbf{y}_{[k]} \geq 0$ is itself feasible. But if this $y_{[k]}$ is not non-negative, the time-path converges on a ray having "balanced growth" involving

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defined by $\mathbf{y}_{[k]}$ and, any 'cyclical' feature due to the existence of a set of H eigenvalues all with the same modulus becomes *relatively* less and less important over time.

some negative activities, which is not feasible. And finally, if $\mu_k < 0$ and $\beta_k > 0$, the time-path alternates in sign, meaning balanced growth is not feasible and the time-path indicated by *(3.4b)* is not feasible either.

These two sub-cases together show necessity for *Theorem 4.1* under assumption *(4.5a)*. Summing up for these two sub-cases, the necessary and sufficient condition for the existence of *sustainability potential* is that, for some $k \geq 1$, we have a positive μ_k and associated non-negative left-eigenvector $y_{[k]} \geq 0$. Then the system can support a sustainable balanced growth. But, this sustainability *potential* is circumscribed in two important ways.

First, in the case that $k > 1$ with the eigenvalues ordered as in *(4.5),* meaning that the non-negative left-eigenvector $y_{[k]} \geq 0$ is not associated with the highest-modulus eigenvector, a further *necessary* condition for time-path convergence from an initial activity y_0 onto the $y_{[k]}$ -ray, is that $\beta_1 = \beta_2 = ... = \beta_{k-1} = 0$ in *(4.6)*.

Second, the existence of sustainability potential *does not constitute sufficiency* for a convergent time-path onto a ray defined by $y_{[k]} \geq 0$ to be a sustainable activity. To be sure of viability of the "full resource utilization" path, we must also establish that the time-path starting from the selected non-negative vector y_0 passes through exclusively non-negative activity vectors $y_\tau \geq 0$ for $\tau = 1, 2, 3, \ldots, \infty$. Formally, this period-by-period requirement for sustainability can be written as follows (here stated without proof, see *Appendix One*):

4.17 $$\beta_k \mathbf{y}_{[k]} \geq -\{\beta_{k+1}(\mu_{k+1}/\mu_k)^\tau \mathbf{y}_{[k+1]} + \ldots + \beta_N(\mu_N/\mu_k)^\tau \mathbf{y}_{[N]}\}.$$

For the situation described under *(4.12 □ 4.14)*, in which some (or all) of the μ_j have the same modulus, the systems are capable of supporting cyclic or oscillatory timepaths and so it is tedious to spell out the time-path properties and their contingencies in rigorous detail; and there are various special cases. As already stated in *(4.15)* and *(4.16)*, the time-path of y_τ generally does not converge onto a ray, but rather towards a cycle of vectors, with a periodicity of H.

In particular, starting with some non-negative activity vector that generates a 'cyclical' timepath, say $\mathbf{c} = \beta_k \mathbf{y}_{[k]} + \beta_{k+1}\mathbf{y}_{[k+1]} + \ldots + \beta_{k+H-1}\mathbf{y}_{[k+H-1]}$, it is straightforward to solve for an eigenvector $\mathbf{s} = \mathbf{c} + (1/r)\mathbf{cU} + \ldots + (1/r)^{H-1}\mathbf{cU}^{H-1}$ is an activity satisfying $r\mathbf{s} = s\mathbf{U}$.

This **s,** the unique (up to a scalar multiple) left-eigenvector of $\mathbf{U}$ associated with the positive eigenvalue r, designates a ray around which the cyclical time-paths spiral, with a "rate of growth" of $g = (r - 1)$, where $g > -1$. If this **s** is non-negative, viz., $s \geq 0$, it represents a "sustainable balanced growth" time-

path. However, if there is no such $s \geq 0$, then it is impossible to have a set of vectors $\mathbf{c}, \mathbf{cU}, \ldots, \mathbf{cU}^{H-1}$ satisfying *(4.15) and (4.16)* that are all non-negative.

This shows necessity for *Theorem 4.1* under assumption *(4.5b)*. An important corollary is that when the $s \geq 0$, it is furthermore possible to choose values for the β_{k+h} so as to obtain H linearly independent and non-negative vectors $\mathbf{c}, \mathbf{cU}, \ldots, \mathbf{cU}^{H-1}$, thus providing the possibility of sustainable cyclical time-paths. [11]

V. A HICKSIAN TYPOLOGY OF SUSTAINABLE TIME-PATHS & SYSTEMS

Theorem *(4.1)* is the basis for a demarcation of systems in terms of the existence or not of full resource utilisation balanced-growth (or decay) time-paths. Systems that cannot support sustainable time-paths in the absence of "free disposal," could fall into either of two categories:

5.1 No positive eigenvalue of $\mathbf{U}$ exists at all.

5.2 One or more positive eigenvalue(s) for $\mathbf{U}$ exist(s), but none of the corresponding activity eigenvectors are non-negative.

Except for systems where one (or both) of $\mathbf{A}$, $\mathbf{B}$ is diagonal, it is not possible to tell simply by looking at matrix incidence patterns, whether or not a system falls into one of these categories *(5.1)* or *(5.2)*. Rather, one has to examine the characteristic equations and eigenvectors on a case-by-case basis — in other words, construct the Jordan normal form of $\mathbf{U}$.

In systems where one or more balanced-growth paths exist(s), the results summarised in *(4.11 and 4.17)* for systems whose eigenvalues are all of different modulus, and then in *(4.14 to 4.16)* for systems with families of equal-modulus but distinct eigenvalues, show the way that such analysis provides the basis for deciding what possibilities of sustainable activity the system can support, amongst: (i) balanced growth (or decay); (ii) convergence (monotonic, oscillatory, or spiral in character) onto a balanced growth path, from an initial resource proportion off a balanced growth ray; and (iii) non-convergent cyclical motion around a balanced growth configuration.

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[11] This is a feature that can readily be demonstrated by numerical examples using commonly available software. Note that we do not here formulate the restrictions — analogous to those discussed under assumption *(4.5a)* leading to condition *(4.17)* — that must be placed on the β_j for $j \geq k + H$, in order that a time-path converging onto a cycle of non-negative vectors from an initial $\mathbf{c} \geq \mathbf{0}$, has exclusively non-negative $y_{T+1}, y_{T+2}, \ldots, y_{T+H-1}, y_{T+H}$.

Our analysis has been couched in terms of the time-paths of activity vectors, referring to equation *(3.4b)*. Hicks (1965) has made a related classification of the behaviour types for joint production systems, based on considering the time-paths for resource vectors. Let:

Φ be the the set *of* ***feasible input proportions:*** all non-negative q_T such that $y_T = q_T A^{-1}$ is non-negative;

χ be the set of ***feasible output proportions:*** all $q_{T+1} = y_T \mathbf{B}$ obtainable with some non-negative activity y_T; and

$\chi = \Phi \cap \Psi$ be the intersection of the feasible input and feasible output proportions.

A time-path is unsustainable (or in Hicks' language, not continually and ultimately viable), if after one or more production periods, it gives rise to a set of outputs in proportions that do not fall within the set of feasible input proportions. Hicks' demarcations in terms of resource proportions (from Hicks 1965, pp.221-233 and pp.324-328) are:

5.3 $\chi = \emptyset$, the two sets are ***disjoint.*** So there can be no non-negative resource vector that supports non-negative "balanced growth" at $g > -1$, and hence no eigensolution $\mathbf{yU} = (1+g)\mathbf{y} \geq 0$ either. "Disjoint" systems *must* be in one *or* other of the categories *(5.1, 5.2)*.

5.4 $\Phi \subset \Psi$ (that is: $\chi = \Phi, \chi \subset \Psi$). The system is, in Hicks' terminology, ***backward-narrowing,*** meaning that all feasible input proportions are feasible output proportions, but some output proportions are ***not*** feasible input proportions. Off a balanced-growth path (or, where these are possible, stable oscillation paths), the trajectory will be unstable and unsustainable. Included in this category are those systems having N distinct eigenvalues at least some of different modulus, for which the sole non-negative eigenvector is $\mathbf{y}_{[N]}\mathbf{U} = (1+g)\mathbf{y}_{[N]}$, where $(1+g) = \mu_N > 0$ is the ***smallest-modulus*** eigenvalue of $\mathbf{U}$; oscillating paths will be possible when a set of $H \geq 2$ eigenvalues exist of modulus μ_N.

5.5 $\Psi \subset \Phi$ (that is: $\chi = \Psi, \chi \subset \Phi$). The system is ***forward-narrowing:*** all feasible output proportions are feasible input proportions, but some input proportions are ***not*** obtainable as output proportions. Such systems will support time-paths that are non-balanced but sustainable, being convergent towards full-utilisation balanced-growth or towards stable oscillation around a balanced-growth path. Included in this category are those systems having N distinct eigenvalues, at least some of different modulus, having a non-negative eigenvector $\mathbf{y}_{[1]}\mathbf{U} = (1+g)\mathbf{y}_{[1]}$, where $(1+g) = \mu_1$ is the ***largest-modulus*** eigenvalue of $\mathbf{U}$; oscillating paths can occur when $H \geq 2$ eigenvalues exist of modulus μ_1.

5.6 $\Psi = \Phi$ (that is: $\chi = \Psi = \Phi$). The system is ***strictly cyclical:*** all feasible output proportions are

feasible input proportions, and ***vice versa***. A balanced growth path will be possible; but from ***any*** other feasible initial resource proportions (not the balanced-growth proportions), there will be undamped sustainable oscillation about the balanced-growth configuration. This category comprises those systems having N distinct eigenvalues of $\mathbf{U}$, all of equal modulus, where the sole non-negative eigenvector is $\mathbf{y}_{[1]}\mathbf{U} = (1+g)\mathbf{y}_{[1]}$, where $(1+g) = \mu_1$ is this modulus.

5.7 $\Psi \neq \Phi$ and we have $\chi \subset \Psi$ and $\chi \subset \Phi$. Such systems are described by Hicks as ***overlapping:*** some, but not all, feasible output proportions are feasible input proportions, and some (but not all) feasible input proportions are obtainable as output proportions. These are systems where $\mathbf{U}$ has at least one positive eigenvalue μ_k (with associated non-negative $y_{[k]}$ whose magnitude is intermediate between the moduli of the largest and smallest eigenvalues. Such systems may be capable of both explosive (unsustainable) and sustainable convergent (onto a ray or a stable oscillation) behaviour. From some (but not all) feasible initial resource proportions, stable time-paths exist converging either towards a balanced-growth configuration or towards a stable non-damped oscillatory motion. But for other feasible initial resource proportions, the time-path is both unstable and unsustainable.

Hicks' classification is exhaustive, although evidently it lumps a great deal of variety under the catch-all category "*overlapping*".

VI. PROCESS INTERDEPENDENCY AND MATRIX (IN)DECOMPOSABILITY

We now turn to the question of process requirements for sustained activity in a joint production system. Schefold (1980, p.141) designated as ***indispensable processes,*** "those processes which have to be activated for whatever net output there is to be produced".

If there is no joint production, all resources are separately producible, and as Schefold intimated, the set of indispensable processes then coincides with the set of processes producing Sraffa "basic" resources. Equally, if in a square system there is no joint production, ***dispensable*** processes are those producing non-basic resources. The necessary and sufficient condition for presence of non-basic resources and of dispensable processes is that the input matrix $\mathbf{A}$ be decomposable.

But with joint outputs, there are no longer these exact coincidences, and the basis for identifying non-basic ***resources*** and dispensable ***processes*** becomes more complex.

Abraham-Frois & Berrebi (1979) and Schefold (1980, p.144), among others, have proposed an extension of the decomposability concept to joint production systems by designating as an ***indecomposable system*** one in which **A** and **B** are not both conformably decomposable (block-triangularisable).[12] An appealing feature of this definition is that, in such a system, it is always possible, in concept, to identify a chain of resource production (generation of an output) and consumption (use as an input) that involves each type of resource and links every process of the ensemble. With an appropriate ordering of processes and resources, a resource appearing as an output in the first process, is used as an input to the second process that, in turn, generates a third resource which is used as an input to the third process, whose output includes a fourth type of resource [and so on....], the N^{th} resource being used as an input in an N^{th} process which produces some of the original resource.[13] This feature suggests a high degree of inter-process dependency. Indeed, making some auxiliary assumptions to allow use of Perron-Frobenius theory, Schefold proceeded to show that "if all processes are indispensable, the system is indecomposable" (ibid.). But the converse does not necessarily hold. That is, system indecomposability as defined by Schefold and by Abraham-Frois & Berrebi is ***not a sufficient condition*** to establish that the system is ***all-indispensable*** in the intuitively meaningful sense that the continued activity of each process ***actually and necessarily*** depends, directly or indirectly, on the continuing activity of the other processes. It means only that the outputs of any process ***may*** contribute, directly or indirectly, to furnishing inputs of every other process. And it is possible — though evidently these are somewhat special cases — to have both **A** and **B** indecomposable, yet the system capable of supporting two (or more) orthogonal non-negative full-utilisation balanced-growth solutions of forms, respectively, $[0, y_2]$ and $[y_1, 0]$.[14]

We want to define a concept of decomposability for joint production systems that gives the exact basis for establishing whether or not a particular process is needed, directly or indirectly, in supporting sustainable production activity of each other process. Since the existence of a balanced-growth path with an activity $y_{[k]} \geq 0$ and $u_k = 1+g > 0$ is the necessary and sufficient condition permitting sustainable activity, this means finding the criterion for whether balanced-growth requires a particular process to be at non-zero activity; or at zero activity; or optionally at zero or non-zero activity. In other words, whether or not a particular process can or necessarily does appear with positive activity in the relevant eigensolution(s) for balanced growth in the system. The answer revolves around the decomposability

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[12] Abraham-Frois & Berrebi (1979) referred to this as a system that is both "technologically" [inputs matrix **A**] and "economically" [output coefficients **B**] decomposable.

[13] This is easily verified using graph theory, or with numerical examples, for example in the case of **A** indecomposable and **B** diagonal.

[14] The restriction on submatrices of **A** , **B** may be deduced by assuming simultaneous solution of two non-negative orthogonal activity vectors. It suffices to give an illustration: let **A** and **B** be both strictly positive and indecomposable with $\mathbf{A} = \alpha\mathbf{B}$ where $\alpha > 0$ is any scalar; each process can continue activity independently of the others.

properties of $\mathbf{U} = \mathbf{B}\mathbf{A}^{-1}$, or equivalently (subject to singularity considerations), decomposability of the Pasinetti $\mathbf{G}$-matrix (Pasinetti 1973).

Steedman (1989, pp.11-13) in a section titled "Reducible G Matrices", demonstrated that the question of process requirements for production of an arbitrary net output, can be resolved in terms of $\mathbf{G}$. Our approach here is to demonstrate that decomposability of $\mathbf{U}$ implies — as does decomposability of $\mathbf{G}$ — the presence of zero(es) in at least one eigenvector for balanced growth.

Retaking Schefold and Steedman's terminology, we shall distinguish between ***indispensable and dispensable processes*** as the exact mathematical dual of the distinction ***basic/non-basic resources***. Write $\mathbf{H} \equiv [\mathbf{B} - \mathbf{A}]^{-1}\mathbf{A}$, following Pasinetri's (1973) notation. Reconsider Sraffa's price-profit eigenproblem, Equation *3.2(b):* $\mathbf{A}\mathbf{p}(1+\pi) = \mathbf{B}\mathbf{p}$. This can be rewritten as: $[\mathbf{B} - \mathbf{A}]\mathbf{p} = \mathbf{A}\mathbf{p}\,\pi$; and if $[\mathbf{B} - \mathbf{A}]$ is non-singular and $\pi \neq 0$, we obtain:

6.1 $\mathbf{p} = [\mathbf{B} - \mathbf{A}]^{-1}\mathbf{A}\mathbf{p}\pi \Leftrightarrow (1/\pi)\mathbf{p} = \mathbf{H}\mathbf{p}$, where $\mathbf{H} \equiv [\mathbf{B} - \mathbf{A}]^{-1}\mathbf{A}$.

If A is non-singular, then: $\mathbf{Q}\mathbf{p} = (1+\pi)\mathbf{p}$, where $\mathbf{Q} = \mathbf{A}^{-1}\mathbf{B}$. The link between $\mathbf{H}$ and $\mathbf{Q}$ is immediate. By matrix inversion, when A is non-singular we obtain $[\mathbf{Q} - \mathbf{I}] \equiv \mathbf{H}^{-1}$. We may thus seek solutions for p, π as eigenvectors and eigenvalues of $\mathbf{H}$; or for p, $1+\pi$ with regard to $\mathbf{Q}$.

Pasinetti (1973, pp.31-36), did consider briefly the "dual counterpart" of his matrix $\mathbf{H}$. However, although he claimed to give "an application that brings out all the dual notions at once", he did not explore the niceties of the joint production situation. Recall Sraffa's growth equations *(3.2a):* $\mathbf{y}\mathbf{A}(1+\mathrm{g}) = \mathbf{y}\mathbf{B}$. These can be rewritten as: $\mathbf{y}[\mathbf{B} - \mathbf{A}] = \mathbf{y}\mathbf{A}\mathrm{g}$. Assuming that $\mathbf{A}^{-1}$ exists, we obtain $\mathbf{y}[\mathbf{B} - \mathbf{A}]\mathbf{A}^{-1} = \mathbf{y}\mathrm{g}$; and using $\mathbf{B}\mathbf{A}^{-1} \equiv \mathbf{U}$,

6.2 $\mathbf{y}[\mathbf{U} - \mathbf{I}] = \mathbf{y}\mathrm{g}$.

Then, assuming that $[\mathbf{B} - \mathbf{A}]$ is non-singular and that $\mathrm{g} \neq 0$, following Pasinetti (1973, pp.31-36) we obtain:

6.3 $\mathbf{y} = \mathrm{g}\mathbf{y}\mathbf{A}[\mathbf{B} - \mathbf{A}]^{-1} = \mathrm{g}\mathbf{y}\mathbf{G}$, where $\mathbf{G} \equiv \mathbf{A}[\mathbf{B} - \mathbf{A}]^{-1}$.

As long as $\mathbf{A}$ is non-singular,[15] we have:

[15] Strictly speaking, Pasinetti's $\mathbf{H}$ and $\mathbf{G}$ are not defined when $[\mathbf{B} - \mathbf{A}]$ is singular. Our $[\mathbf{T} - \mathbf{I}]$ and $[\mathbf{U} - \mathbf{I}]$ are defined when $[\mathbf{B} - \mathbf{A}]$ is singular, if $\mathbf{A}$ is non-singular. Significance of the restrictions depends on analysis context; but this is secondary to the questions being posed here.

6.4 $\mathbf{G}^{-1} \equiv \left[\mathbf{B}-\mathbf{A}\right]\mathbf{A}^{-1} = [\mathbf{U}-\mathbf{I}]$.

The inverse of a block-triangular matrix will be conformably block-triangular. So, subject to non-singularity, both $\mathbf{G}$ and its inverse $\mathbf{G}^{-1} = [\mathbf{U}-\mathbf{I}]$ are decomposable if and only if $\mathbf{U}$ and $\mathbf{U}^{-1}$ are decomposable. Let us define:

6.5 A system $\mathbf{A}$, $\mathbf{B}$ is ***decomposable with respect to processes*** if, where defined, the matrices $\mathbf{U}^{-1} = \mathbf{A}\mathbf{B}^{-1}$ and $\mathbf{U} = \mathbf{B}\mathbf{A}^{-1}$ are decomposable; and so too $\mathbf{G}$ and $\mathbf{G}^{-1}$ are decomposable.

6.6 A system $\mathbf{A}$, $\mathbf{B}$ is ***decomposable with respect to resources*** if, where defined, the matrices $\mathbf{Q}^{-1} = \mathbf{B}^{-1}\mathbf{A}$ and $\mathbf{Q} = \mathbf{A}^{-1}\mathbf{B}$ are decomposable; and so too $\mathbf{H}$ and $\mathbf{H}^{-1}$ are decomposable.

Suppose now, where both $\mathbf{A}$ and $\mathbf{B}$ are indecomposable, the matrices partitioned conformably as follows, with, $\mathbf{A}_{11}, \mathbf{A}_{22}$ and $\mathbf{B}_{11}, \mathbf{B}_{22}$ all being square:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

Manara (1968) and others since (e.g., Pasinetti 1973, 1980; Abraham-Frois & Berrebi 1979; Steedman 1980; Salvadori & Steedman 1988), have shown that the Sraffian basics/non-basics distinction can be made by constructing, where possible, a block-triangular $\mathbf{H}$.

6.7 Suppose there exists a matrix, say $\mathbf{\Omega}$, such that $\mathbf{A}_{21} = \mathbf{\Omega}\mathbf{A}_{11}$ and $\mathbf{B}_{21} = \mathbf{\Omega}\mathbf{B}_{11}$. Then $\mathbf{H}$ is decomposable with $\mathbf{H}_{21} = 0$

If we assume a solution with $\pi \neq 0$, then Equations *6.1* with $\mathbf{H}$ and p partitioned conformably are:

6.8(a) $\mathbf{H}_{11}\mathbf{p}_1 + \mathbf{H}_{12}\mathbf{p}_2 = (1/\pi)\mathbf{p}_1$;

(b) $\mathbf{H}_{22}\mathbf{p}_2 = (1/\pi)\mathbf{p}_2$.

The $\mathbf{p} = [\mathbf{p}_1, \mathbf{p}_2]$ thus obtained (solvable first for sub-vector $\mathbf{p}_2$ and π without reference to $\mathbf{p}_1$), which may or may not be non-negative, is the Sraffa price-profit solution for a system separable into ***basic and non-basic resources***. The resources corresponding to $\mathbf{p}_2$ are basics; the others are non-basics. Less remarked is a corollary that another solution to *(6.8)* is obtainable by first setting $\mathbf{p}_2 = 0$ and then solving for π_1 and $\mathbf{p}_1$ in Equation *6.8(a)* which becomes:

6.9 $\mathbf{H}_{11}\mathbf{p}_1 = (1/\pi_1)\mathbf{p}_1$

This subvector $\mathbf{p}_1$ is a solution to the upper-left sub-system eigen-equation $[\mathbf{B}_{11}]^{-1}\mathbf{A}_{11}\mathbf{p}_1 = (1+\pi_1)\mathbf{p}_1$. Thus, if for a given technology, the resources of a system can be divided into basic and non-basic resources à la Manara et alia, then there exists a solution to the price-profit equations in which prices of basics are all zero. (Since B^{-1}_{11}, is not necessarily non-negative, there may or may not be a $\mathbf{p} = [\mathbf{p}_1, 0] \geq 0$; but this is another matter.)[16]

Turning to the dual results for $\mathbf{G}$ and $\mathbf{U}$, we remark immediately that: ***Decomposability of*** $\mathbf{U}$ ***and*** $\mathbf{U}^{-1}$ ***or equivalently of*** $\mathbf{G}$ ***and*** $\mathbf{G}^{-1}$ ***implies the existence of solutions for balanced growth in the system with a non-negative activity vector in which some elements are zero.*** Formally:

6.10 Suppose there exists a matrix, say $\mathbf{\Theta}$, such that $\mathbf{A}_{21} = \mathbf{A}_{22}\mathbf{\Theta}$ and $\mathbf{B}_{21} = \mathbf{B}_{22}\mathbf{\Theta}$. Then the corresponding $\mathbf{U}$ and $\mathbf{U}^{-1}$, and $\mathbf{G}$ and $\mathbf{G}^{-1}$, are all decomposable, with a block $\mathbf{U}_{21} = 0$, etc.

Proof with respect to $\mathbf{G}$ is given by Steedman (1989); this is the analogous to the $\mathbf{H}$-problem of basics/non-basics. Proof for $\mathbf{U}$, etc., follows similar lines. Now let $\mathbf{U}$ have a block of zeroes in the lower left hand corner, say $\mathbf{U}_{21} = 0$. Write $\mathrm{y} \equiv [\mathrm{y}_1, \mathrm{y}_2]$, and Equations *(6.2)* are:

6-11
$$\mathbf{y}_1\mathbf{U}_{11} = (1+\mathrm{g})\mathbf{y}_1;$$
$$\mathbf{y}_1\mathbf{U}_{12} + \mathbf{y}_2\mathbf{U}_{22} = (1+\mathrm{g})\mathbf{y}_2.$$

The top equations can be solved for a y_1 and a growth rate, say g_1, prior to solution in the lower equations of the corresponding y_2. Thus a solution for the whole system, $\mathrm{y}\mathbf{A}(1+\mathrm{g}) = \mathrm{y}\mathbf{B}$ can be obtained where $\mathrm{g} = \mathrm{g}_1$ and the elements $\mathbf{y}_1$ are determinable prior to the remaining $\mathbf{y}_2$. This is the result properly dual to the Sraffa basics-prices determination.

If this $\mathbf{y} \geq 0$, it signals the feasibility of sustainable technologically stationary time-paths; but we are assured neither of a non-negative $\mathbf{y}_1$, associated with the g_1, nor of a corresponding non-negative $\mathbf{y}_2$.

Also from Equations *(6.11),* there exist one or more activity solutions obtained by first choosing $\mathbf{y}_1 = 0$, then solving for a growth-rate, say g_2, co-determined with the elements of $\mathbf{y}_2$ in:

6.12 $$\mathbf{y}_2\mathbf{U}_{22} = (1+\mathrm{g})\mathbf{y}_2.$$

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[16] Sraffa had seen this result, in the context of his "beans" problem (1960, p.91), but dismissed it as uninteresting. Yet it can find a very significant application in ecological economics, notably to express the idea of exploitation by a "luxury" economy (the non-basics-producing sector) of the basics-producing sector as an autonomous environment supplying "free gifts of nature" (see O'Connor 1993b and also O'Connor, Andrew & Patterson 2009).

Any such $\mathbf{y}_2$ is a solution to the lower-right sub-system eigenequation $\mathbf{y}_2\mathbf{A}_{22}[\mathbf{B}_{22}]^{-1} = \mathbf{y}_2(1+g_2)$. So the vector $\mathbf{y} = [0, \mathbf{y}_2]$ is an eigensolution associated with eigenvalue $\lambda_2 = (1+g_2)$ of $\mathbf{U}$.[17] Hence:

6.13 Decomposability of $\mathbf{U}$, or equivalently of $\mathbf{G}$, implies that there exist one or more solutions to the activity-growth equations in which activities of a sub-set of processes are all zero.

We designate the processes with identically zero activity under this decomposition as ***dispensable,*** the others as ***indispensable.*** Three final points can be made.

First, we can have *(6.7)* which establishes ***decomposability with respect to resources/prices,*** without having *(6.10)* which establishes ***decomposability with respect to processes/activities;*** or vice versa; or both simultaneously; or neither. The price-decomposability assumption imposes a particular linear dependency between the coefficients of inputs and outputs of ***non-basic resources*** in ***all processes.*** Process-decomposability depends on a particular linear dependency between the coefficients of inputs and outputs of ***all resources*** in ***the indispensable processes.*** Where both of $\mathbf{A}$, $\mathbf{B}$ are indecomposable, the restriction on technologies to permit price-decomposability appears, from a physical point of view, rather flukish. By contrast, the restriction on production technology to permit process-decomposability, has a very direct physical meaning even when $\mathbf{A}$, $\mathbf{B}$ are themselves indecomposable. If, for some such $g_2 > -1$, the $\mathbf{y}_2 \geq 0$, we have a solution for autonomous balanced growth ***involving the "indispensable" processes alone*** (though, importantly, perhaps involving all resources). Sustainable activity is then feasible within the indispensable subset of processes alone, irrespective of whether or not it is also feasible for a solution involving dispensable processes as well.

Second, the dispensable/indispensable distinction does not in any way refer to non-negativity of the activity vector solutions, and thus does not, in itself, provide an adequate basis for characterising process requirements for sustainable activity.

Third, and analogous to the situation with basics/non-basics, there may be more than one way of partitioning $\mathbf{A}$, $\mathbf{B}$ to permit choice of a $\mathbf{\Theta}$ satisfying Assumption *(6.10)*. In principle, one should consider all possible ways of block-triangularisation of $\mathbf{U}$ (or equivalently, of $\mathbf{G}$, etc.). In such cases, we might call ***truly indispensable*** those processes that have non-zero activity in ***all*** eigensolutions obtainable through block-triangularisation. A particular sub-vector $\mathbf{y}_2$ solving *(6.12)* will include all the truly ***indispensable processes*** of the ensemble, but may include other processes that are not indispensable to ***all*** growth solutions, and which should thus be classed as dispensable processes. In

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[17] To see that $[0, \mathbf{y}_2]$ is an eigenvector for the whole system that links back to Assumption *6.10,* rearrange and multiply this equation through on the right by $\mathbf{\Theta}$, to obtain $\mathbf{y}_2\mathbf{A}_{22}\mathbf{\Theta}(1+g_2) = \mathbf{y}_2\mathbf{B}_{22}\mathbf{\Theta}$. Since we have: $\mathbf{A}_{21} = \mathbf{A}_{22}\mathbf{\Theta}$ and $\mathbf{B}_{21} = \mathbf{B}_{22}\mathbf{\Theta}$, we obtain $\mathbf{y}_2\mathbf{A}_{21}(1+g_2) = \mathbf{y}_2\mathbf{B}_{21}$, exactly the desired form for resource-balance relations for balanced growth at g_2 of the left-hand set of resources within the lower set of processes.

systems having multiple sub-sets of processes capable of supporting a balanced-growth, if two or more such subsets can be chosen that are disjoint, there will be no truly indispensable processes.

VII. CHARACTERISING PROCESS REQUIREMENTS FOR SUSTAINABLE ACTIVITY

Let us now introduce some further terminology:

7.*1 (a)* a process is ***unsupportable*** if it cannot be present in any full-utilisation balanced growth (or balanced decay) activity of the ensemble designated by a non-negative **y** that is an eigenvector of **U** associated with an eigenvalue $\lambda = (1+g) > 0$.[18]

(b) a process is ***non-essential*** if, in a system that supports a full-utilisation balanced growth path, it may be, but is not necessarily present for balanced-growth/decay activity.

(c) a process is *essential* if it is necessarily present for balanced-growth/decay activity.

This three-way demarcation is system-specific; a process can be classified only with reference to a particular set of process technologies. The triple demarcation arises from distinguishing solutions involving negative activity levels from those with exclusively non-negative activities. (Note that in the dual problem of resource prices, for certain problems negative prices may be admissible).

Consider first process-indecomposable systems, those for which **U**, **G** (etc.), are indecomposable; such systems are "all-indispensable". For such systems, there is at most one non-negative eigenvector y; and if non-negative it is strictly positive. Two possibilities exist:

7.2(a) If the $\mathbf{y} > 0$ with $g > -1$, the system is ***all-essential;*** sustainable activity is possible at/around growth rate g, necessarily involving all processes.

(b) If there is no $\mathbf{y} > 0$, and/or no positive $\lambda = (1+g) > 0$, the system is ***all-unsupportable.***

All-essential systems may display a variety of capabilities. For illustration, first let **U** be non-negative and ***imprimitive*** (for example when **A** is diagonal and **B** is imprimitive). The sole solution for

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[18] Care with interpretation is required in some decomposable systems having strong forward-narrowing properties. If some processes can act as "sinks" for otherwise unusable resources, certain processes classed here as unsupportable may nonetheless still be sustainable indefinitely, but necessarily growing more slowly — or decaying faster — than the "sink" processes classified as essential. (A possible case in point would be an exponentially decaying, or shrinking, economy in a Spaceship Earth that is becoming a rubbish dump on a planetary scale.)

balanced growth is associated with the largest-modulus eigenvalue of $\mathbf{U}$, which is the von-Neumann root $1/\lambda^* = (1+g^*) > 0$; and there are other eigenvalues of the same modulus. Such systems are forward-narrowing, with potential for stable cyclic oscillations around the von-Neumann ray, for convergence onto either the ray, or convergence onto a stable oscillation around this ray.[19]

By contrast now consider those all-essential systems with ***no joint production,*** where $\mathbf{B}$ is diagonal. Assume $\mathbf{A}$ is indecomposable and primitive, so $\mathbf{U}^{-1}$ is non-negative and primitive. From the PF Theorem, $\mathbf{U}^{-1}$ has a positive dominant eigenvalue, say $\lambda^* = 1/\mu_N = 1/(1+g^*) > 0$, associated with a strictly positive eigenvector $\mathbf{y}_{[N]}\mathbf{U}^{-1} = (1/\mu_N)\mathbf{y}_{[N]}$. This $\boldsymbol{\mu}_N$ is the smallest-modulus eigenvalue of $\mathbf{U}$; and $g^* = (\mu_N - 1)$ is the von-Neumann growth rate for the system. Such systems are backward-narrowing. From *(4.18ii)* results, the only feasible sustainable time-path is strictly on the ray $\mathbf{y}_{[N]}$. So,

7.3 In the absence of free disposal, timepaths in process-indecomposable (and primitive) systems of no-joint-production are unsustainable unless they commence, and remain continually, on the von-Neumann ray with $\mathbf{q} = \mathbf{y}_{[N]}\mathbf{A}$, associated with the smallest-modulus eigenvalue of $\mathbf{U}$.

This point was in fact made by Hicks (1965, p.225, p.226), who commented that although the Turnpike theorem, a cornerstone of economic growth theory, holds for backward-narrowing as well as for forward-narrowing systems, "the character of the convergence to the Turnpike is not the same in the one case as in the other;" and thus "it will be wise.... to distinguish between the various kinds of narrowing."

The fact that the mechanics of convergence are "not the same" is, obviously, of paramount conceptual significance where free-disposal is axiomatically excluded. Only where we have forward-narrowing, can convergence (full or partial) take place ***without*** free disposal.

Turn now to process-decomposable systems, for which $\mathbf{U}$ and $\mathbf{G}$ are decomposable. One or more processes are dispensable. There may be one or more feasible full-utilisation balanced growth path(s), each one having its own characteristic growth rate (with special cases of common growth rates where eigenvalues are multiple); but conversely there may be no feasible balanced growth path at all. First consider systems for which $\mathbf{U}$ decomposes in only one way, so dispensable and indispensable processes are demarcated uniquely by the partition $[y_1, y_2]$ of *(6.10 to 6.13)*.

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[19] Where $\mathbf{A}$ is diagonal and $\mathbf{B}$ is primitive, each process transforms a unique input into a mixture of resources; hence no matter what the output proportions, the output mix will be usable by the system. Forward-narrowing means, intuitively, that processes working collectively, tend to transform extreme proportions of input resources into more balanced proportions ("extreme" and "balanced" being relative to the ray proportions). The opposite holds for backward-narrowing systems.

7.4(a) If both the all-process solution $[y_1, y_2]$ with $g_1 > -1$, and the indispensables-only solution $[0, y_2]$ with $g_2 > -1$, are non-negative, the lower sub-set of (indispensable) processes are ***essential,*** whereas the upper (dispensable) sub-set are ***nonessential.***

(b) If the g_1-solution for y is ***not*** non-negative and/or its g_1 is not > -1, while the indispensables-only solution has $y_2 > 0$ with its $g_2 > -1$, then the lower (indispensable) processes are ***essential*** but the upper (dispensable) processes are ***unsupportable.***[20]

(c) If neither activity solution is non-negative, and/or if neither growth rate is > -1, the whole ensemble of processes is ***unsupportable.***

Alternatively, if $\mathbf{U}$ were totally decomposable, we could put it in block diagonal form, with say $\mathbf{U}_{21} = 0$ and $\mathbf{U}_{12} = 0$, yielding two separate solutions: g_1 with $[y_1, 0]$, and g_2 with $[0, y_2]$. There are no truly indispensable processes. In terms of sustainable activity potential, the options are:

7.5(a) If $g_1 > -1$ with $[y_1, 0] \geq 0$, and $g_2 > -1$ with $[0, y_2] \geq 0$, then each sub-set of processes separately is capable of autonomous sustainable activity, and the ensemble may be considered as ***totally nonessential.***

(b) If, say, the g_1-solution for $[y_1, 0]$ is ***not*** non-negative and/or g_1 is not > -1, but the other solution has $y_2 > 0$ with its $g_2 > -1$, then the processes in the latter (lower) processes are ***essential*** but the processes in the other (upper) subset are ***unsupportable.***

(c) If neither activity solution is non-negative, and/or if neither growth rate is > -1, the whole ensemble of processes is ***unsupportable.***

Additional complications arise when $\mathbf{U}$ is decomposable in more than one way, with overlap (total or partial) between the several sets of what we might call "pseudo-indispensable" processes. But the various cases *(7.4 and 7.5)* already indicate the main variations of behaviour that might arise.

As intimated in ***Sections IV and V*** above, wherever balanced growth (with a $g > -1$) is feasible, there may also be possibilities for sustainable convergent or oscillatory activity, the spectral properties of $\mathbf{U}$ determining whether the (sub)system(s) in question have cyclical and/or forward-narrowing properties. With process-decomposable systems a number of interesting quirks can arise. For example, under category *7.3(a),* balanced-growth paths with g_2, $[0, y_2]$ involve zero activity of the nonessential

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[20] Although it is a marginal point, maybe one could construct a contrasting situation where there exists one strictly positive solution $y > 0$ with some $g_1 > -1$, while the indispensables-only solution $[0, y_2]$ with g_2 is not non-negative and/or the g_2 is not > -1. All processes would then be ***essential*** to sustainable activity even though the system is, mathematically speaking, process-decomposable and divisible into dispensable/indispensable processes. If this can occur, it would be only by virtue of a peculiar (and improbable) sort of relationship between technologies of dispensable and indispensable sectors.

processes. But if $g_2 > g_1$ sustainable time-paths can be found that converge towards the g_2-balanced-growth ***relative*** resource proportions and ***relative*** activity levels, but still involving persisting non-zero activity (converging on growth/decay at rate $g_1 < g_2$) for the nonessential activities (those whose levels are identically zero on the g_2-growth-path itself). This illustrates a wider point, that where overlapping or disjoint eigensolutions exist, as in categories *7.3(a) and 7.4(a)* respectively, there is the possibility of sustainable time-paths involving all processes at strictly positive activity levels, as linear combinations of separately feasible solutions. Where the separate solutions have differing growth rates, the relative importance of each solution to the activity will change over time. The system will converge in the limit towards activity (balanced growth or cyclical activity, as the case may be) associated with the largest associated growth-rate; but substantively this may amount to superposition of two or more distinct time-paths, each with its characteristic growth/decay rate.

VIII. CONCLUDING REMARKS

This paper has sought to bring out the ideas that, within a specified ensemble of process technologies, (i) there will often be processes which, relative to a defined sustainable activity path, are ***non-essential****;* and (ii) there will often be processes whose technological characteristics makes them ***unsupportable*** within any sustainable activity.

The idea of a process being non-essential has been, of course, widely acknowledged in the context of the von-Neumann analysis of non-square systems. The ideas of dispensability and non-essentiality also have important applications in the neo-Ricardian analysis of technology choice and technological change. More novel is our concept of an ***unsupportable*** process (or technology), which takes on meaning in the context of ***full-resource-utilisation*** time-paths, which in turn finds particular importance against the backdrop of the mass-closed "Spaceship Earth" in contemporary ecological economics.

This paper's expositions should therefore be understood dialectically. Emphasis has been on characterizing technologically stationary sustainable time-paths. This is one key step in exploring the changes of emphasis that follow from abandonment of free-gift/free-disposal assumptions in thinking about economy-environment codependency and long-run sustainability. Within the joint production model framework, we draw the general conclusion that sustainable activity will be feasible only if the "initial" relative abundances of resources present in the ensemble fall within certain limits. Conversely, forced technological change will be a feature of system activity whenever the actually available resources at the outset of any period are not reconcilable to pre-existing technologies.

For any period T, activity with a specified $\mathbf{A}$, $\mathbf{B}$ will be possible only for certain mixes of resources, those for which a non-negative y_T can be found such that $q_T = y_T\mathbf{A}$. We have seen that some systems $\mathbf{A}$, $\mathbf{B}$ do not permit any balanced-growth without free disposal; nor, therefore, any sustainable convergent or stable harmonic solutions. (Though a technologically stationary time-path may temporarily be possible for one or more periods, this is not sustainable in the long-run.) A similar unsustainability conclusion holds for systems having one or more balanced-growth solutions, but where the initial resource configuration means balanced-growth, convergent, or cyclical paths are inaccessible. Feasibility limits are breached when the notional activity vector $y_{T+1} = y_T\mathbf{A}^{-1}$ required to use the outputs $q_{T+1} = y_T\mathbf{B}$ from the previous period, ceases to be non-negative. The number of periods for which such activity can be sustained before technological change is forced, will depend on the $\mathbf{A}$, $\mathbf{B}$ and the initial resource vector; but technological change is an inevitability.[21]

Of course, there are many contexts where traditional FGFD assumptions remain valid methodological choices. But such assumptions are not "generally" pertinent for economic analysis investigations concerning our contemporary global challenges. And while this paper's results, once spelled out, may seem obvious enough, this should not be taken to mean that the implications of excising free disposal have already received the attention from economists warranted to them.

For example, our results about the absence, inaccessibility or unlikelihood of a long-run sustainable time-path within a mass-closed Spaceship Earth system, have important consequences for attempts at extending, or reforming, value theory for applications to environmental processes and the "services" obtained by economic processes from their environment. It already seems clear, based on early results obtained by O'Connor (1993b) and by Douguet, Noël & O'Connor (2000) for some didactic examples, that price systems (that is, vectors of relative prices) for joint production in the absence of FGFD assumptions must generally be obtained and interpreted with reference to unequal rates-of-return (across processes or across resources). This generic result gives a new lease on life to the classical political economy theme of class conflict over the definition and distribution (or appropriation) of the "surplus", now extending and revising these concepts in the formulation of an "ecological political economy" that extends across environmental assets and services up to the scale of Spaceship Earth.

[21] Unsustainability and forced technological change are the focus of the expositions in O'Connor (1993a, 1994), which incorporated energy conservation and entropic irreversibility considerations to extend results originally suggested by Perrings (1986, 1987). This forced technological change is, in general, imperfectly (or not at all) controlled by human agency. In real terms what we here call technological change may amount to such events as collapse of sub-system activity due to depletion of needed inputs, or perturbation and disruption due to invasion of unwanted resources (pollution, etc.). Changes taking place in one process or subsystem will often have repercussions in other subsystems….

For example, it has recently been established (O'Connor, Andrew & Patterson 2009) that some the least-squares solution concepts for relative prices and process rates-of-return proposed by Patterson et al. (2006) are closely related to algebraic concepts of unequal rates-of-return developed earlier by O'Connor (1993b) from a somewhat different point of view. These are convergences that, with the benefit of fuller mathematical foundations for investigating the mass-closed Spaceship Earth excised of FGFD assumptions, can now start to bear their ecological economic fruit.

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APPENDIX ONE:
TIME-PATH NON-NEGATIVITY CONDITIONS

Section IV of the paper proves the necessity for *Theorem 4.1* under assumption *(4.5a)*. Suppose that for some $k \geq 1$, we have a positive μ_k and associated non-negative left-eigenvector $y_{[k]} \geq 0$. Then the system can support a sustainable balanced growth.

It was further shown that a *necessary* condition for time-path convergence from an initial activity y_0 onto the $y_{[k]}$-ray, is that, with the eigenvalues ordered as in *(4.5),* we have $\beta_1 = \beta_2 = ... = \beta_{k-1} = 0$ in *(4.6)*. *H*owever, *this is not sufficiency* for a convergent time-path onto a ray defined by $y_{[k]} \geq 0$ to be a sustainable activity. To be sure of viability of the "full resource utilization" path, we must also establish that the time-path starting from the selected non-negative vector y_0 passes through exclusively non-negative activity vectors $y_\tau \geq 0$ for $\tau = 1, 2, 3, \ldots, \infty$. We will show here the result stated in the main text without proof, viz., that the period-by-period requirement for sustainability is:

4.17 $$\beta_k \mathbf{y}_{[k]} \geq -\{\beta_{k+1}(\mu_{k+1}/\mu_k)^\tau \mathbf{y}_{[k+1]} + \ldots + \beta_N(\mu_N/\mu_k)^\tau \mathbf{y}_{[N]}\}.$$

To investigate what this means, suppose we have an initial $\mathbf{y}_T$ such that $\beta_1 = \beta_2 = ... = \beta_{k-1} = 0$, and $\beta_k \neq 0$. Then $\lim_{\tau\to\infty}(1/\mu_k)^\tau \mathbf{y}_\tau = \beta_k \mathbf{y}_{[k]}$, and recalling *(4.4a)* we have for the τ^{th} term of the time-path sequence:

A.1 $$\mathbf{y}_\tau = \mathbf{y}_0\mathbf{U}^\tau = \mathbf{y}_0\mathbf{P}^{-1}\mathbf{J}^\tau\mathbf{P} = [0,\ldots,0,\beta_k,\beta_{k+1},\ldots,\beta_N]\,\mathrm{diag}[(\mu_1)^\tau,(\mu_2)^\tau,\ldots,(\mu_N)^\tau]\mathbf{P}$$

$$= 0y_{[1]} + 0y_{[2]} + \ldots + \beta_k(\mu_k)^\tau \mathbf{y}_{[k]} + \beta_{k+1}(\mu_{k+1})^\tau \mathbf{y}_{[k+1]} + \ldots + \beta_N(\mu_N)^\tau \mathbf{y}_{[N]}.$$

Dividing both sides through by $(\mu_k)^\tau$, we obtain:

A.2 $$\mathbf{y}_\tau/(\mu_k)^\tau = \beta_k \mathbf{y}_{[k]} + \left\{\beta_{k+1}(\mu_{k+1}/\mu_k)^\tau \mathbf{y}_{[k+1]} + \ldots + \beta_N(\mu_N/\mu_k)^\tau \mathbf{y}_{[N]}\right\}.$$

The first term on the right-hand side of *(A.2)* is the vector to which the sequence converges. The other term in the format $\{.(\tau).\}$ can be considered as the "residual" or (normalised) deviation from the ray for each successive value of τ. Convergence of the residual towards the zero vector is assured from the ordering *(4.5a)* meaning $|(\mu_m/\mu_k)| < 1$ for all $m > k$. We already have the necessity that, for sustainability potential,

A.3(a) $\beta_k \mathbf{y}_{[k]} \geq 0$.

Thus, viability along the way towards sustainability then further requires that no vector element in the sequence of residual vectors $\{.(\tau).\}$ defined by *(4.18)* be negative and larger in magnitude than the corresponding element in the limit vector. Write $\beta_k \mathbf{y}_{[k]}{}^{[j]}$ for the j^{th} element of $\beta_k \mathbf{y}_{[k]}$, and write $\{.(\tau).\}^{[j]}$ for the j^{th} element of the $T+\tau$ period residual. With $\tau = 0$, from *(4.19a)* we know that

A.3(b) $\beta_k \mathbf{y}_{[k]} > -\{.(0).\}$. Hence, for each j, we deduce

(b1) either: $\{.(0).\}^{[j]} \geq 0$

(b2) or: $|\{.(0).\}^{[j]}| \leq \beta_k \mathbf{y}_{[k]}{}^{[j]}$.

And so we deduce the stated result *(4.17)* that, for subsequent $\tau \geq 1$, sustainability requires:

A.4 (a) $\beta_k \mathbf{y}_{[k]} \geq -\{\beta_{k+1}(\mu_{k+1}/\mu_k)^\tau \mathbf{y}_{[k+1]} + \ldots + \beta_N(\mu_N/\mu_k)^\tau \mathbf{y}_{[N]}\}$, which implies

(b1) either: $\{.(\tau).\}^{[j]} \geq 0$

(b2) or: $|\{.(\tau).\}^{[j]}| \leq \beta_k \mathbf{y}_{[k]}{}^{[j]}$.

In the "residual" vector defined in *(A.2)*, the $\beta_m \mathbf{y}_{[m]}$ for m = k+1, k+2,..., N may each have both positive and negative elements, and whenever an eigenvalue μ_m is negative the coefficient $(\mu_m/\mu_k)^\tau$ alternates in sign for successive values of τ. So the elucidation of conditions *A.4(a,b1,b2)* is a rather idiosyncratic problem. We have not found any "intuitive" formulation of the mathematical condition (although, by exploring numerical simulations and variations with modern software, it is easy to get a feel for oscillations and convergence behaviours).

In what follows, we give a statement of some sufficient conditions obtainable by considering the signs of the β_m and μ_m, and the elements of the $\mathbf{y}_{[m]}$ for $m = k+1,\ldots, N$. (However, these are considerably more restrictive than the necessary conditions.)

Consider the behaviour of the sequence of $\{.(\tau).\}^{[j]}$ for each $j = 1,2,\ldots, N$, first for even τ, and then for odd τ. Here we give an upper bound, based on the fact that the sign of the sequence of the $(\mu_m/\mu_k)^\tau \beta_m \mathbf{y}_{[m]}{}^{[j]}$, $\tau = 0,1,2,\ldots,\infty$ that contribute to $\{.(\tau).\}$, is either invariant or alternating.

A.5 Let $R_{[-]}{}^{[j]}(\tau)$ be the sum of the negative-sign $(\mu_m/\mu_k)^\tau \beta_m \mathbf{y}_{[m]}{}^{[j]}$; and

let $R_{[+]}{}^{[j]}(\tau)$ be the sum of the positive-sign $(\mu_m/\mu_k)^\tau \beta_m \mathbf{y}_{[m]}{}^{[j]}$.

Then $\{.(0).\}^{[j]} = R_{[-]}^{[j]}(0) + R_{[+]}^{[j]}(0)$, and the condition *(A4a)* can be rewritten as:

A.6 $|R_{[-]}^{[j]}(0)| \leq R_{[+]}^{[j]}(0)$.

Using *(A.4b2),* a sufficient (but not necessary) condition that ensures *(A.6)* holds, is:

A.7 $|R_{[-]}^{[j]}(0)| \leq \beta_k \mathbf{y}_{[k]}^{[j]}$.

Similarly, a sufficient (but not necessary) condition that ensures that the *(A.4)* hold, is:

A.8 $|R_{[-]}^{[j]}(\tau)| \leq \beta_k \mathbf{y}_{[k]}^{[j]}$, for all $\tau = 1, 2, ..., \infty$.

Because $(\mu_m/\mu_k)^{\tau+2}\beta_m \mathbf{y}_{[m]}^{[j]}$ always has the same sign as $(\mu_m/\mu_k)^{\tau}\beta_m \mathbf{y}_{[m]}^{[j]}$ and also $|(\mu_m/\mu_k)| < 1$ for all $m > k$, we know that for all τ,

A.9 $|R_{[-]}^{[j]}(\tau+2)| < |R_{[-]}^{[j]}(\tau)|$. If *(A.9)* holds, then *(A.8)* implies that *(A.7)* holds for all τ even.

Now also suppose

A.10 $|R_{[-]}^{[j]}(1)| \leq \beta_k \mathbf{y}_{[k]}^{[j]}$. If *(A.10)* is true, then *(A.8)* implies that *(A.7)* holds for all odd τ.

So taken together, restrictions *(A.9 and A.10)* are sufficient bounds on the residual vector to ensure that the activity time-path is always non-negative. Although these bounds are considerably more restrictive than the necessary conditions *(A.4 = 4.17),* they have the advantage of being obtainable without having to calculate the elements of the residuals vectors for successive periods.

APPENDIX TWO:
THE VON NEUMANN PROBLEM AND FREE DISPOSAL

For a system consisting of N technologically distinct processes using $G \leq N$ resources as inputs and/or outputs, von Neumann (1945-46) sought to establish g^* and π^* as follows:

B.1 (a) the ***highest*** growth rate g^*, such that $\mathbf{y}_T\mathbf{B} \geq (1+g)\mathbf{y}_T\mathbf{A}$;

(b) the ***lowest*** profit rate π^*, such that $\mathbf{Bp} \leq (1+\pi)\mathbf{Ap}$.

(The inequalities permit full equality.) The problem posed in terms of inequalities is much less restrictive than the parallel Sraffian eigenproblem. More importantly, it has a quite different physical meaning. Inequalities *B.1 (a)* signal that the von Neumann solution concept is underpinned by a free disposal assumption. To summarise briefly, what von Neumann showed was that the following Assumptions *B.2(a,b,c,d)* are sufficient to ensure the existence of non-negative p and y associated respectively with a real "lowest" rate of profit, $\pi^* \geq -1$, and a real "highest" rate of growth, $g^* \geq -1$.

B.2 (a) $\mathbf{A}$ and $\mathbf{B}$ are each $(N \times G)$ and are non-negative.

(b) The number of processes, N, is not less than the number of resources, G. So: $N \geq G$.

(c) Each resource ***is produced*** by at least one process. So every column of $\mathbf{B}$ has at least one non-zero element.

(d) Each process ***uses*** at least one resource as an input. So every row of $\mathbf{A}$ has at least one non-zero element.

Von Neumann also showed that $g^* = \pi^*$ if, in addition to *(3.4),* we have: $\mathbf{A} + \mathbf{B} > 0$. Others since (e.g. Abraham-Frois & Berrebi 1979, pp.115-136) have shown that the result $g^* = \pi^* \geq -1$ holds if, with *(B.2),* we also make the slightly weaker assumption:

B.3 The system $\mathbf{A}$, $\mathbf{B}$ is technologically or economically ***indecomposable*** (or both), where:

(a) ***technologically decomposable*** means the input matrix $\mathbf{A}$ is decomposable (can be put in block-triangular form), and each column of $\mathbf{B}_{22}$ has at least one non-zero element;

(b) ***economically decomposable*** means the output matrix $\mathbf{B}$ is decomposable, and each

row of matrix $\mathbf{A}_{11}$ has at least one non-zero element.

In either case, the following statements can also be made:

B.4(a) ***The von-Neumann ray***. Where N is the number of processes, and G the number of resources, $N > G$, no more than G processes are required for optimal proportional growth at rate g^*; that is, an activity vector y can be chosen that has no more than G non-zero elements. However there may be multiple solutions for such a y.

(b) ***Unused processes***. If $\mathbf{a}_i\mathbf{p}(1+\pi^*) > \mathbf{b}_i\mathbf{p}$, where $\mathbf{a}_i$ is the i^{th} row of $\mathbf{A}$, and $\mathbf{b}_i$ is the i^{th} row of $\mathbf{B}$, then that process realises a "negative super-profit", and will not be used in a von-Neumann optimal proportionate growth regime.

(c) ***Non-scarce resources***. If $(1+g^*)\mathbf{y}\mathbf{a}^j < y\mathbf{b}^j$, where $\mathbf{a}^j$ is the j^{th} column of $\mathbf{A}$, $\mathbf{b}^j$ the j^{th} column of $\mathbf{B}$, then resource j is said to be ***non-scarce,*** as its availability is not a limiting factor in expansion along a von-Neuman "ray". It will be attributed a zero-price in the solution to the von-Neumann value equations.

Von Neumann's concern was with a particular concept of economic equilibrium, and these ($g^* = \pi^*$) solutions are by no means the only price or activity vectors that may be of interest in the analysis of joint-production systems. In this paper, they are put aside almost altogether, except in such cases that the von-Neumann-ray happens to coincide with a full-resource-utilisation balanced growth path. Where, by von Neumann's definition *(A.4c)* there are ***non-scarce*** resources produced in the system, the interpretation of the von-Neumann ray as "balanced growth" relies on *the* ***free disposal*** of the surplus resources. This free-disposal, while being formally signalled by the zero prices for such resources, also carries the implication that it is somehow possible to dispose physically of the surpluses "outside the system" — an assumption that we preclude in the present paper.

APPENDIX THREE:
ABRAHAM-FROIS & BERREBI'S "ANTI-BASICS"

With reference to Equations *(6.11)* in the main paper, Abraham-Frois & Berrebi (1979, pp. 160-161) would define the (upper) processes distinguished by y_1 as ***"non-antibasic processes",*** and the remainder of processes as ***"anti-basic processes",*** arguing a sort of analogy with the quasi-autonomous determination of prices for basics and a profit rate. However their analogy is askew and leads to trouble. Their definitions are couched in terms of resource requirements rather than the true dual of process requirements. They define an ***antibasic process*** as one that "uses all goods directly or indirectly"; and otherwise a process is ***non-antibasic.*** They would have been better to define as "antibasic" a process that ***uses all processes directly or indirectly.*** (This would coincide with their definition in the case of no-joint production). Their definitions are clear-cut and intuitively easy to understand with no joint production. But they are difficult to relate back to decomposability possibilities, as becomes clear in AF&B's own text. And as Sraffa himself noted, the question of defining "indirect" resource use is fraught for joint-production systems. The implicit reference point of Sraffa and of AF&B, it must be emphasised, is a viable system. In the case of a square system with no joint production, each resource is the unique output of a different process. So a process that does not utilise, directly or indirectly, all resources, must be one whose activity can be sustained indefinitely even while activity *of* ***some*** (one or more) other processes is zero. But, in a joint production system, processes generating resources that are basic by the Manara criteria, may nonetheless use inputs of all resources. This means that they would be ***antibasic*** even though they still might be ***indispensable*** in our sense of being needed in a balanced-growth whether or not a luxury sector (processes with only non-basic resources as outputs), is operating. Conversely, there may be cases where ***antibasic*** processes are ***dispensable,*** as when $\mathbf{A}$ is indecomposable and multiple non-negative but orthogonal solutions for y exist. Thus a process that we define as ***indispensable*** in a system of joint production where at least some processes are ***dispensable,*** may or may not be a ***non-antibasic process;*** and ***vice versa;*** and an ***antibasic*** process (that uses all resources) may or may not be ***indispensable;*** there are several contingencies. In short, AF&B seem to have created a monster hybrid, which should be dispensed with.

A system can be technologically indecomposable *or* economically decomposable, or both, or neither. In the main paper (section VI), we introduce a different (and, for our purposes, more useful) concept of system decomposability, based on whether either or both of $\mathbf{T} \equiv \mathbf{B}^{-1}\mathbf{A}$ and $\mathbf{U} \equiv \mathbf{A}\mathbf{B}^{-1}$ are decomposable. A sufficient, but not necessary, condition for both $\mathbf{T}$ and $\mathbf{U}$ being decomposable, is that the system is both economically and technologically decomposable in the AF&B sense.