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# What is the optimal shape of a fin for stationary heat conduction?

Gilles Marck\*      Grégoire Nadin<sup>†</sup>      Yannick Privat<sup>‡</sup>

## Abstract

This article is concerned with the shape of small devices used to control the heat flowing between a solid and a fluid phase, usually called *fin*. The temperature along a fin in stationary regime is modeled by a one-dimensional Sturm-Liouville equation whose coefficients strongly depend on its geometrical features. We are interested in the following issue: is there any optimal shape maximizing the heat flux at the inlet of the fin? Two relevant constraints are examined, by imposing either its volume or its perimeter, and nonexistence results are proved for both problems. Furthermore, using a kind of rearrangement argument, we explicitly compute the optimal values and construct maximizing sequences. We show in particular that the optimal heat flux at the inlet is infinite in the first case and finite in the second one. Finally, we provide several extensions of these results for more general models of heat conduction, as well as several numerical illustrations.

**Keywords:** calculus of variation, shape optimization, Sturm-Liouville equation, volume constraint, perimeter constraint.

**AMS classification:** 49J15, 49K15, 34E05.

## 1 Introduction

The increasing need for compact and efficient thermal systems leads to new challenges in the design of heat transfer devices. Across several engineering fields dealing with thermal management issues, a recurrent problematic concerns the optimal shape of several small elements aiming at locally increasing heat transfer. These systems, usually called fins, may adopt various shapes and designs, slightly extending the structure subjected to thermal loads.

For instance, fins are widely used to control the temperature of the heat exchangers taking place over the computational processing units (CPU), since thermal overloads can have devastating effects on their physical integrity. Fins are also extensively used in industrial forming processes, mainly to evacuate the heat handled by the molded material, and must be designed to guarantee that the bulk temperature evolves inside a range specified by manufacturing constraints.

Therefore, efficient fin shapes are required to either improve cooling or warming processes, shrewdly controlling the local temperature or increasing the heat transfer. Many engineering works focused on modeling the direct problem in order to assess the efficiency of different fin shapes [3]. Nevertheless, very few theoretical results and mathematical proofs do exist (see for

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instance [2, 7, 13]). The study proposed in the present article tackles this shape optimization problem, taking into account several relevant constraints on the admissible class of designs.

Engineering motivations behind the fin optimization are rooted in a category of physical problems related to efficient transport of a conservative flux. Optimizing the shape of a fin belongs to a larger class of problems aiming at reducing the thermal resistance occurring when the heat flows inside different media. Indeed, the fin thermally links a solid material subject to a heat flux with a fluid flow taking place around it. Its role consists in cooling down/warming up the solid phase by transferring heat to/from the fluid phase.

From the optimal design point of view, the main difficulties arise from the multi-physic aspects of this problem, depicted on Fig. 9 in Appendix A. Indeed, the heat flowing from a solid domain through a fin to a fluid part is only transported by conduction inside the structure, as far as it reaches its boundary. Then, according to heat flux conservativeness property, it is fully transferred into the fluid phase. Hence, it is at the same time transported by the fluid motion and conducted between fluid elements, which is referred to as *conducto-convection phenomenon* (see [21]). As a consequence, the heat flowing from the fin to the fluid depends on the flow pattern around boundaries. This multi-physics optimal design problem with a Navier-Stokes/heat coupling system has been numerically investigated for some academic configurations in [18]. For the needs of simple and robust modeling, physical/engineering works often only takes into consideration the conduction phenomenon occurring inside the bulk material, considering an average convective coefficient standing for the heat transfer at the solid/fluid interface [3, 21]. This is the choice we make in the sequel.

We also will assume that the fin is subject to a steady-state thermal regime and attached to a device at constant temperature. A fin is considered as thermally efficient if it conveys the largest amount of heat, while requiring the smallest volume of solid material. Two reasons motivate this claim: first, a fin is generally made of expensive components, because it requires high conductivity materials such as copper. Secondly, a fin is generally oriented orthogonally to the cross-flow direction, and we could expect that it is designed to generate the smallest possible perturbation in the fluid motion. Indeed, notice that a smaller fin produces a smaller perturbation and requires a smaller additional power to set the fluid into motion. This feature lead us to maximize the heat crossing the fin, by prescribing either its volume or its surface.

From a mathematical point of view, the problems settled within this article write as infinite dimensional optimization problems subject to constraints of several natures: (i) global ones such as volume or perimeter, (ii) pointwise ones, and (iii) ordinary differential equation ones, since the cost functional depends on the solution of a Sturm-Liouville equation, whose coefficients write as highly nonlinear functions of the unknown.

The idea of minimizing functionals depending on a Sturm-Liouville operator, such as eigenvalues or eigenfunctions, is a long story and goes back at least to M. Krein in [15], see also [11, 12, 16, 17] and [8] for a review on such problems. We also mention [9, 19] where two problems close to the ones addressed in this article have been solved in the context of Mathematical Biology (see Remark 3 for a comparison between these problems).

In [4], the authors considered a simplified model of fin, getting rid of the convective heat transfer from the lateral side of the structure. Using rearrangement techniques, they addressed and solved the problem of maximizing the cooling rate of the fin. The strong difference with the analysis developed in this article comes from the awareness of the convective heat transfer, making the optimal design problem harder to tackle.

This article is structured as follows: in Section 2.1, a Sturm-Liouville model of temperature along the fin is derived using physical arguments. Physical justifications are postponed in Appendix A for the sake of completeness. Section 2.2 is devoted to introducing the cost functional  $F(a)$ , standing for the heat flux at the inlet of the fin, and providing continuity properties about it. The

two main optimal design problems investigated are introduced in Section 3: in a nutshell, the first one consists in maximizing  $F(a)$  with a volume constraint. A nonexistence result for this problem is stated and proved in Section 3.2. We show in particular that the optimal value is infinite. The second one aims to maximize  $F(a)$  with a perimeter like constraint. Similarly to the first problem, we prove that it has no solution in Section 3.3 and we provide the optimal value for this problem, as well as a way to construct maximizing sequences. Finally, all these results are extended to a more general setting in Section 4. Several numerical simulations are also presented.

## 2 Modeling of a fin

### 2.1 Sturm-Liouville model of the temperature conduction

Let us consider an axisymmetric fin  $\Omega_a$  of length  $\ell > 0$  and radius  $a(x)$  at abscissa  $x$ , as displayed in Figure 9, defined in a Cartesian coordinate system by

$$\Omega_a = \{(r \cos \theta, r \sin \theta, x) \mid r \in [0, a(x)), \theta \in \mathbb{S}^1, x \in (0, \ell)\}.$$

We will assume in the sequel that

- (H<sub>1</sub>)  $a \in W^{1,\infty}(0, \ell)$  so that the surface element is defined almost everywhere;
- (H<sub>2</sub>) there exists  $a_0 > 0$  such that  $a(x) \geq a_0$  for every  $x \in [0, \ell]$  so that the fin cannot collapse.

Figure 1 sums-up the situation and the notations we will use throughout this article.

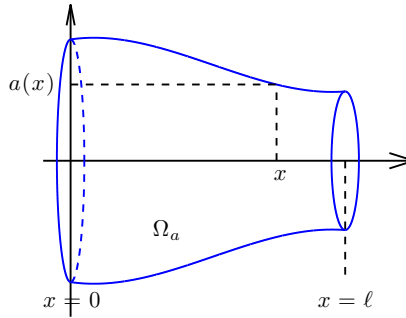


Figure 1: An axisymmetric fin

Some physical explanations about the derivation of the temperature model are provided in Appendix A. As a result, the temperature  $T$  along the fin is solution of the following ordinary differential equation

$$\begin{aligned} (a^2(x)T'(x))' &= \beta a(x)\sqrt{1+a'(x)^2}(T(x) - T_\infty) \quad x \in (0, \ell) \\ T(0) &= T_d \\ T'(\ell) &= -\beta_r(T(\ell) - T_\infty), \end{aligned} \tag{1}$$

where  $\beta = 2h/k$  is a positive real constant and  $\beta_r = h_r/k$  is a nonnegative real constant.

Since the function  $a$  is assumed to satisfy the assumptions  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$ , System (1) has a unique solution  $T \in H^1(0, \ell)$  by virtue of the Lax-Milgram lemma. The following lemma provides some precisions on this solution.

**Lemma 1.** *Let  $a$  be a function satisfying the assumptions  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$ . Then, the solution  $T(\cdot)$  of (1) is decreasing, and verifies*

$$T_\infty \leq T(x) \leq T_d$$

for every  $x \in [0, \ell]$ .

*Proof.* Let us use the standard change of variable for Sturm-Liouville problems (see for example [5])  $y = \int_0^x \frac{dt}{a(t)^2}$ . Setting  $\ell_1 = \int_0^\ell \frac{dt}{a(t)^2}$  and  $S(y) = T(x) - T_\infty$  for every  $x \in [0, \ell]$ , it follows from (1) that  $S$  is solution of the boundary value problem

$$\begin{aligned} S''(y) &= \beta b(y)S(y) & y \in (0, \ell_1) \\ S(0) &= T_d - T_\infty \\ S'(\ell_1) &= -\beta_r a(\ell)^2 S(\ell_1), \end{aligned} \tag{2}$$

where  $b(y) = a(x)^3 \sqrt{1 + a'(x)^2}$  for every  $x \in [0, \ell]$ . Let us now prove that the function  $S(\cdot)$  remains positive on  $[0, \ell_1]$ . We denote by contradiction  $y_0$  the largest zero of  $S(\cdot)$  on  $[0, \ell_1]$  whenever it exists. One has necessarily  $S'(y_0) \neq 0$ . Otherwise, the Cauchy-Lipschitz theorem would immediately yield the contradiction. In particular,  $y_0 \neq \ell_1$ . If  $S'(y_0) > 0$ , then  $S$  is positive on  $(y_0, \ell_1)$  since  $y_0$  is the largest zero of  $S$ , and thus  $S(\cdot)$  is strictly convex and increasing on  $(y_0, \ell_1)$ , which is incompatible with the boundary condition at  $\ell_1$ . In the same way, if  $S'(y_0) < 0$ ,  $S(\cdot)$  is concave negative and decreasing on  $[y_0, \ell_1]$  and the same conclusion follows. As a result, the function  $S(\cdot)$  is positive on  $[0, \ell_1]$ , and hence strictly convex according to (2). The function  $S(\cdot)$  is thus decreasing according to the boundary condition at  $\ell_1$ . The conclusion follows.  $\square$

## 2.2 The cost function $F(a)$

The cost function  $F(a)$  stands for the heat flux at the inlet of the fin. It is defined by

$$F(a) = -k\pi a(0)^2 T'(0), \tag{3}$$

where  $T(\cdot)$  denotes the solution of (1). Integrating the main equation of (1) yields the new (integral) expression

$$F(a) = k\pi\beta \int_0^\ell a(x) \sqrt{1 + a'(x)^2} (T(x) - T_\infty) dx + k\pi\beta_r a(\ell)^2 (T(\ell) - T_\infty). \tag{4}$$

In the forthcoming analysis of the optimal design problems we will investigate, we will need to use continuity properties of the criterion  $F$ . Let us write precisely these properties. Define the class of admissible designs

$$\mathcal{A}_{a_0, \ell} = \{a \in L^\infty(0, \ell), a \geq a_0 \text{ a.e. in } (0, \ell)\},$$

and the product space  $\widehat{\mathcal{A}}_{a_0, \ell}$  defined by

$$\widehat{\mathcal{A}}_{a_0, \ell} = \left\{ (a, b), a \in \mathcal{A}_{a_0, \ell} \text{ and } b = a\sqrt{1 + a'^2} \right\}.$$

Introduce the criterion  $\widehat{F}$  defined on  $\widehat{\mathcal{A}}_{a_0, \ell}$  by

$$\widehat{F}(a, a\sqrt{1 + a'^2}) = F(a), \tag{5}$$

for every  $a \in \mathcal{A}_{a_0, \ell}$ . Here and in the sequel, the notation  $\mathcal{M}(0, \ell)$  stands for the space of Radon measures on  $(0, \ell)$ .

**Definition 1.** Let  $(a_n, b_n)_{n \in \mathbb{N}}$  be a sequence of elements of  $\widehat{\mathcal{A}}_{a_0, \ell}$ . We will say that  $(a_n, b_n)_{n \in \mathbb{N}}$   $\tau$ -converges to  $(a, b) \in C^0([0, \ell]) \times \mathcal{M}(0, \ell)$  if

- $(a_n)_{n \in \mathbb{N}}$  converges to  $a$ , locally uniformly in  $(0, \ell]$ ;
- $(b_n)_{n \in \mathbb{N}}$  converges to  $b$  in the sense of measures.

We endow  $\widehat{\mathcal{A}}_{a_0, \ell}$  with the topology inherited from the  $\tau$ -convergence. One has the following continuity result of the criterion  $F$ .

**Proposition 1.** Let  $(a_n, b_n)_{n \in \mathbb{N}}$  be a sequence of elements of  $\widehat{\mathcal{A}}_{a_0, \ell}$  which  $\tau$ -converges to  $(a, b) \in C^0([0, \ell]) \times \mathcal{M}(0, \ell)$ . Then, the sequence  $(F(a_n))_{n \in \mathbb{N}}$  converges to  $\widehat{F}(a, b)$  defined by

$$\widehat{F}(a, b) = k\pi\beta\langle b, T - T_\infty \rangle_{\mathcal{M}(0, \ell), C^0([0, \ell])} + k\pi\beta_r a(\ell)^2 (T(\ell) - T_\infty), \quad (6)$$

where  $T = \tilde{T} + T_d$ , and  $\tilde{T}$  is the unique solution of the equation written under variational form: find  $\tilde{T} \in H^1(0, \ell)$  satisfying  $\tilde{T}(0) = 0$  such that for every test function  $\varphi \in H^1(0, \ell)$  satisfying  $\varphi(0) = 0$ , one has

$$\int_0^\ell a(x)^2 \tilde{T}'(x) \varphi'(x) dx + \beta \langle b, (\tilde{T} + T_d - T_\infty) \varphi \rangle_{\mathcal{M}(0, \ell), C^0([0, \ell])} + \beta_r a(\ell)^2 (\tilde{T}(\ell) + T_d - T_\infty) \varphi(\ell) = 0. \quad (7)$$

**Remark 1.** It follows from Proposition 1 that the functional  $\widehat{F}$  defined (with a slight abuse of notation) by (6) on the closure of  $\widehat{\mathcal{A}}_{a_0, \ell}$  for the topology associated to the  $\tau$ -convergence is a continuous extension of the function  $\widehat{F}$  defined by (5).

*Proof of Proposition 1.* Consider a sequence  $(a_n, b_n)_{n \in \mathbb{N}}$  of elements of  $\widehat{\mathcal{A}}_{a_0, \ell}$  as in the statement of Proposition 1. Denote by  $T_n$  the associated solution of (1). Let us multiply the main equation of (1) by  $T_n - T_\infty$  and then integrate by parts. One gets

$$\begin{aligned} a_0 \min\{\beta, a_0\} \|T_n - T_\infty\|_{H^1(0, \ell)}^2 &\leq \int_0^\ell (a_n(x)^2 T_n'(x)^2 + \beta b_n(x) (T_n(x) - T_\infty)^2) dx \\ &= -\beta_r a_n(\ell)^2 (T_n(\ell) - T_\infty)^2 - a_n(0)^2 T_n'(0) (T_d - T_\infty). \end{aligned}$$

Integrating Equation (1) on  $(0, \ell)$  yields

$$-\beta_r a_n(\ell)^2 (T_n(\ell) - T_\infty) - a_n(0)^2 T_n'(0) = \beta \int_0^\ell b_n(x) (T_n(x) - T_\infty) dx$$

and according to Lemma 1, it follows that

$$a_0 \min\{\beta, a_0\} \|T_n - T_\infty\|_{H^1(0, \ell)}^2 \leq \beta_r \|a_n\|_\infty^2 (T_d - T_\infty)^2 + \beta (T_d - T_\infty)^2 \langle b_n, 1 \rangle_{\mathcal{M}(0, \ell), C^0([0, \ell])}.$$

Since  $(a_n, b_n)_{n \in \mathbb{N}}$   $\tau$ -converges to  $(a, b)$ , we deduce from the previous estimate that the sequence  $(T_n)_{n \in \mathbb{N}}$  is uniformly bounded in  $H^1(0, \ell)$ . Hence, using a Rellich theorem, there exists  $T^* \in H^1(0, \ell)$  such that, up to a subsequence,  $(T_n)_{n \in \mathbb{N}}$  converges to  $T^*$ , weakly in  $H^1(0, \ell)$  and strongly in  $L^2(0, \ell)$ . Introduce for every  $n \in \mathbb{N}$ , the function  $\tilde{T}_n := T_n - T_d$ . The function  $\tilde{T}_n$  is the unique solution of the system whose variational form writes: find  $\tilde{T}_n \in H^1(0, \ell)$  such that  $\tilde{T}_n(0) = 0$  and for every test function  $\varphi \in H^1(0, \ell)$  satisfying  $\varphi(0) = 0$ , one has

$$\int_0^\ell (a_n(x)^2 \tilde{T}_n'(x) \varphi'(x) + \beta b_n(x) (\tilde{T}_n(x) + T_d - T_\infty) \varphi(x)) dx = -\beta_r a_n(\ell)^2 (\tilde{T}_n(\ell) + T_d - T_\infty) \varphi(\ell).$$

The conclusion follows hence easily, passing to the limit into this variational formulation, and noting that

$$\frac{\widehat{F}(a_n, b_n)}{k\pi} = -a_n(0)^2 T_n'(0) = \beta \int_0^\ell b_n(x)(T_n(x) - T_\infty) dx + \beta_r a_n(\ell)^2 (T_n(\ell) - T_\infty),$$

since the injection  $H^1(0, \ell) \hookrightarrow C^0([0, \ell])$  is compact.  $\square$

### 3 Optimal design problems for a simplified model of fin

#### 3.1 The optimal design problems

In this section we will introduce and solve the problems modeling the optimal shape of a fin for the Sturm-Liouville model (1). We will consider two kinds of constraints: a volume constraint or a lateral surface constraint. Notice that

$$\begin{aligned} \text{volume of } \Omega_a &= \pi \int_0^\ell a(x)^2 dx, \\ \text{lateral surface of } \Omega_a &= 2\pi \int_0^\ell a(x) \sqrt{1 + a'(x)^2} dx. \end{aligned}$$

**Optimal design problem with volume or lateral surface constraint.** *Let  $a_0$  and  $\ell$  denote two positive real numbers. Fix  $V_0 > \ell a_0^2$  and  $S_0 > \ell a_0$ . We investigate the problem of maximizing the functional  $a \mapsto F(a)$  defined by (3) either over the set*

$$\mathcal{V}_{a_0, \ell, V_0} = \{a \in W^{1, \infty}(0, \ell), a \geq a_0 \text{ a.e. in } [0, \ell] \text{ and } \pi \text{vol}(a) \leq \pi V_0\} \quad (8)$$

or the set

$$\mathcal{S}_{a_0, \ell, S_0} = \{a \in W^{1, \infty}(0, \ell), a \geq a_0 \text{ a.e. in } [0, \ell] \text{ and } 2\pi \text{surf}(a) \leq 2\pi S_0\}, \quad (9)$$

where the volume and lateral surface functionals are defined by

$$\text{vol}(a) = \int_0^\ell a(x)^2 dx \quad \text{and} \quad \text{surf}(a) = \int_0^\ell a(x) \sqrt{1 + a'(x)^2} dx.$$

**Remark 2.** Physically, the lateral surface constraint can be justified by considering the fluid flowing around the fin. In realistic engineering configurations, the fluid is put into motion by an external equipment, such as a pump or a fan. The pressure and kinetic energies provided by this system are dissipated all through the fluid flow, mainly because of the wall shear stress. In other words, one part of the dissipation is due to the fluid friction against the walls. As a consequence, limiting the wet surface of the fin is a suitable way to reduce the operating cost of the whole thermal system. In addition, limiting the fin surface may also help to reach easier and cheaper shapes to manufacture.

**Remark 3.** As underlined in the introduction, the optimal design problems settled here look similar to the ones addressed in [9, 19], devoted to the issue of understanding the nerve fibers shapes by solving an optimization problem. In these works, a Sturm-Liouville operator whose coefficients depended on the shape of the nerve fiber were also introduced and two criteria were considered: a kind of transfer function, and the first eigenvalue of a self-adjoint operator. However, the technics implemented in the present work are rather different and appear a bit more sophisticated. Indeed,

the ideas of the proofs in [9, 19] were all based on the standard change of variable for Sturm-Liouville problems used in Lemma 1, which permitted to consider auxiliary problems and thus get a lower bound of the optimal values. This trick was then used to construct minimizing sequences. Unfortunately, in the present work, we did not manage to adapt such technics, and the arguments of the proofs rest upon the use of particular perturbations that could be assimilated to a kind a rearrangement (see Remark 4 and the proof of Theorem 2).

Before solving these shape optimization problems, let us give some precisions on the topological nature of the classes  $\mathcal{V}_{a_0, \ell, V_0}$  and  $\mathcal{S}_{a_0, \ell, S_0}$  in  $L^\infty(0, \ell)$ . As it will be highlighted in the proofs of theorems 1 and 2, the classes  $\mathcal{V}_{a_0, \ell, V_0}$  and  $\mathcal{S}_{a_0, \ell, S_0}$  are not closed for the standard strong topology of  $W^{1, \infty}(0, \ell)$ . The following lemma investigates the  $L^\infty$ -boundedness of the elements of these classes.

**Lemma 2.** *Let  $a_0$  and  $\ell$  denote two positive real numbers, and let  $V_0 > \ell a_0^2$  and  $S_0 > \ell a_0$ . Then,*

- *the class  $\mathcal{V}_{a_0, \ell, V_0}$  is not a bounded set of  $L^\infty(0, \ell)$ ,*
- *the class  $\mathcal{S}_{a_0, \ell, S_0}$  is a bounded set of  $L^\infty(0, \ell)$  and for every  $a \in \mathcal{S}_{a_0, \ell, S_0}$ ,*

$$a_0 \leq a(x) \leq \sqrt{S_0^2/\ell^2 + 4S_0}.$$

*Proof.* First consider the sequence of functions  $(a_n)_{n \geq 1}$  defined by

$$a_n(x) = \begin{cases} \sqrt{n - \frac{n(n-a_0^2)}{2(V_0 - a_0^2 \ell)} x} & \text{if } x \in [0, 2(V_0 - a_0^2 \ell)/n] \\ a_0 & \text{if } x \in (2(V_0 - a_0^2 \ell)/n, \ell]. \end{cases}$$

By construction,

$$\int_0^\ell a_n(x)^2 dx = \ell a_0^2 + \frac{(n - a_0^2)(V_0 - a_0^2 \ell)}{n} < V_0,$$

so that  $a_n \in \mathcal{V}_{a_0, \ell, V_0}$  for every  $n \in \mathbb{N}^*$ . Since  $a_n \in C^0([0, \ell])$  and  $a_n(0) = \sqrt{n}$ , it follows obviously that  $\mathcal{V}_{a_0, \ell, V_0}$  is not bounded in  $L^\infty(0, \ell)$ .

Second, consider  $a \in \mathcal{S}_{a_0, \ell, S_0}$ . One has

$$S_0 \geq \int_x^y a(t) \sqrt{1 + a'(t)^2} dt \geq \int_x^y a(t) |a'(t)| dt \geq \left| \int_x^y a(t) a'(t) dt \right| = \frac{1}{2} |a(y)^2 - a(x)^2|,$$

for every  $0 < x < y < \ell$ . It follows that for every  $x \in [0, \ell]$ ,

$$a(0)^2 - 2S_0 \leq a(x)^2 \leq a(0)^2 + 2S_0. \tag{10}$$

Using this inequality, one gets

$$S_0 \geq \int_0^\ell a(t) dt \geq \ell \sqrt{a(0)^2 - 2S_0}$$

and thus,  $a(0)^2 \leq S_0^2/\ell^2 + 2S_0$ . The conclusion follows by combining this inequality with (10).  $\square$



### 3.2 Maximizing $F$ over the set $\mathcal{V}_{a_0, \ell, V_0}$ (volume constraint)

The following theorem highlights the ill-posed character of Problem (8).

**Theorem 1.** *Let  $a_0$  and  $\ell$  denote two positive real numbers, and let  $V_0 > \ell a_0^2$ . Problem (8) has no solution and*

$$\sup_{a \in \mathcal{V}_{a_0, \ell, V_0}} F(a) = +\infty.$$

*Proof.* To prove this theorem, we will exhibit an explicit maximizing sequence  $(a_n)_{n \in \mathbb{N}^*}$ . The construction of the sequence  $(a_n)_{n \in \mathbb{N}^*}$  is based on the use of an intermediate sequence denoted  $(a_{S,m})_{m \in \mathbb{N}^*}$  where  $S$  denotes a positive real number. Fix  $S > 0$ . The sequence  $(a_{S,m})_{m \in \mathbb{N}^*}$  is chosen to verify at the same time

- $(a_{S,m})_{m \in \mathbb{N}^*}$  converges strongly to  $a_0$  in  $L^\infty(0, \ell)$ ,
- the sequence  $(b_{S,m})_{m \in \mathbb{N}^*}$  defined by  $b_{S,m} = a_{S,m} \sqrt{1 + a_{S,m}'^2}$  for every  $m \in \mathbb{N}^*$  verifies

$$\text{surf}(a_{S,m}) = \int_0^\ell b_{S,m}(x) dx = S$$

for every  $m \in \mathbb{N}^*$  and converges in the sense of measures to  $a_0 + (S - a_0 \ell) \delta_0$ , where  $\delta_0$  denotes the Dirac measure at  $x = 0$ .

In particular, the sequence  $(a_{S,m}, a_{S,m} \sqrt{1 + a_{S,m}'^2})_{m \in \mathbb{N}^*}$   $\tau$ -converges to  $(a_0, a_0 + (S - a_0 \ell) \delta_0)$  as  $m$  tends to  $+\infty$ .

Let us now provide an example of such sequence. A possible choice of function  $b_{S,m}$  for  $m$  large enough is given by

$$b_{S,m}(x) = \begin{cases} a_0 + (S - a_0 \ell)m & \text{if } x \in [0, 1/m] \\ a_0 & \text{if } x \in (1/m, \ell]. \end{cases}$$

Solving the equation  $b_{S,m} = a_{S,m} \sqrt{1 + a_{S,m}'^2}$  comes to solve  $\frac{a_{S,m} |a_{S,m}'|}{\sqrt{b_{S,m}^2 - a_{S,m}^2}} = 1$  on each interval  $(0, 1/m)$  and  $(1/m, \ell)$ . This equation has obviously an infinite number of solutions. In particular, a family of solutions is obtained by making the function  $a_{S,m}$  oscillate an integer number of times in  $(0, 1/m)$ , each oscillation corresponding to two successive choices of the sign of  $a_{S,m}'$  on two same length intervals. The continuity of each function  $a_{S,m}$  determines then it in a unique way. As a consequence, it is possible to control the  $L^\infty$ -norm of  $a_{S,m}$  by choosing shrewdly the number of oscillations.

Introduce  $M_m = a_0 + (S - a_0 \ell)m$  for every  $m \in \mathbb{N}^*$ . We now construct the sequence  $(a_{S,m})_{n \in \mathbb{N}^*}$  so that each function oscillates  $m$  times on  $[0, 1/m]$  (see Figure 2). More precisely, we define  $a_{S,m}$  by

$$a_{S,m}(x) = \begin{cases} \sqrt{M_m^2 - (\sqrt{M_m^2 - a_0^2} - x)^2} & \text{on } [0, \frac{1}{2m^2}) ; \\ a_{S,m}(\frac{1}{m^2} - x) & \text{on } [\frac{1}{2m^2}, \frac{1}{m^2}) ; \\ a_{S,m}(x - \frac{i}{m^2}) & \text{on } [\frac{i}{m^2}, \frac{i+1}{m^2}), i \in \{1, \dots, m-1\} ; \\ a_0 & \text{on } [\frac{1}{m}, \ell]. \end{cases} \quad (11)$$

Notice in particular that

$$\|a_{S,m} - a_0\|_\infty = a_{S,m}\left(\frac{1}{2m^2}\right) - a_0 = o\left(\frac{1}{m^2}\right) \quad \text{as } m \rightarrow +\infty.$$

The sequence  $(a_{S,m})_{n \in \mathbb{N}^*}$  verifies then the expected properties above. Moreover, according to

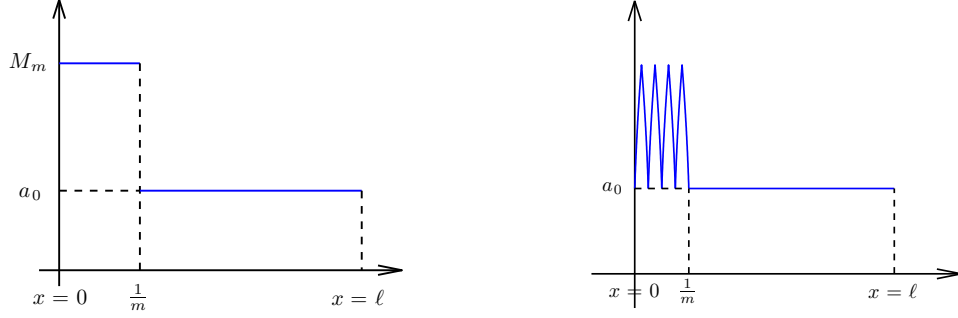


Figure 2: Left: graph of  $b_{S,m} = a_{S,m}\sqrt{1+a_{S,m}^{\prime 2}}$ . Right: Graph of  $a_{S,m}$ .

Proposition 1, one has

$$\lim_{m \rightarrow +\infty} F(a_{S,m}) = \lim_{m \rightarrow +\infty} \widehat{F}(a_{S,m}, a_{S,m}\sqrt{1+a_{S,m}^{\prime 2}}) = \widehat{F}(a_0, a_0 + (S - a_0\ell)\delta_0)$$

Notice that the solution  $T$  of (7) associated to the pair  $(a, b) = (a_0, a_0 + (S - a_0\ell)\delta_0)$  coincides with the solution of the same equation associated to the pair  $(a, b) = (a_0, a_0)$ . Explicit computations thus lead to

$$T(x) = T_\infty + (T_d - T_\infty) \left( \cosh \left( \sqrt{\frac{\beta}{a_0}} x \right) - \gamma \sinh \left( \sqrt{\frac{\beta}{a_0}} x \right) \right),$$

for every  $x \in [0, \ell]$ , where  $\gamma$  denotes the positive real number

$$\gamma = \frac{\sqrt{\frac{\beta}{a_0}} \sinh \left( \sqrt{\frac{\beta}{a_0}} \ell \right) + \beta_r \cosh \left( \sqrt{\frac{\beta}{a_0}} \ell \right)}{\sqrt{\frac{\beta}{a_0}} \cosh \left( \sqrt{\frac{\beta}{a_0}} \ell \right) + \beta_r \sinh \left( \sqrt{\frac{\beta}{a_0}} \ell \right)}. \quad (12)$$

We then compute

$$\begin{aligned} \lim_{m \rightarrow +\infty} \frac{F(a_{S,m})}{k\pi} &= \beta \langle a_0 + (S - a_0\ell)\delta_0, T - T_\infty \rangle_{\mathcal{M}(0,\ell), \mathcal{C}^0([0,\ell])} + \beta_r a_0^2 (T(\ell) - T_\infty) \\ &= \beta (T_d - T_\infty) \left( \frac{a_0^{3/2}}{\sqrt{\beta}} \gamma + (S - a_0\ell) \right). \end{aligned}$$

**Conclusion: construction of a maximizing sequence  $(a_n)_{n \in \mathbf{N}^*}$ .** Let  $n \in \mathbf{N}^*$  such that  $n \geq [a_0\ell] + 1$ . Consider the sequence  $(a_{n,m})_{m \in \mathbf{N}^*}$  introduced previously. Since  $(a_{n,m})_{m \in \mathbf{N}^*}$  converges strongly to  $a_0$  in  $L^\infty(0, \ell)$  and according to the previous convergence study, there exists  $p \in \mathbf{N}$  such that  $\text{vol}(a_{n,p}) \leq V_0 - \frac{1}{n}$  and

$$\frac{F(a_{n,p})}{k\pi\beta(T_d - T_\infty)} \geq \left( \frac{a_0^{3/2}\gamma}{\sqrt{\beta}} + (n - a_0\ell) \right) - \frac{1}{n}.$$

Let us denote by  $m_n$  the first integer for which this property is verified, and by  $a_n$  the function equal to  $a_0$  if  $n \leq [a_0\ell]$  and by  $a_{n,m_n}$  else. It follows not only that each element  $a_n$  belongs to  $\mathcal{V}_{a_0,\ell,V_0}$  and that

$$\sup_{a \in \mathcal{V}_{a_0,\ell,V_0}} \frac{F(a)}{k\pi\beta} \geq F(a_{n,m_n}) \geq (T_d - T_\infty) \left( \frac{a_0^{3/2}\gamma}{\sqrt{\beta}} + (n - a_0\ell) - \frac{1}{n} \right)$$

for every  $n \geq [a_0\ell] + 1$ . Letting  $n$  tend to  $+\infty$  yields the conclusion of the theorem.  $\square$

### 3.3 Maximizing $F$ over the set $\mathcal{S}_{a_0,\ell,S_0}$ (lateral surface constraint)

This section is devoted to the solving of Problem (9). We will prove that imposing a lateral surface constraint on  $\Omega_a$  is not enough to get the existence of a solution for this problem. Nevertheless, on the contrary of the previous case where a volume constraint were imposed (see Section 3.2, Theorem 1), the value of the supremum of  $F$  over the set  $\mathcal{S}_{a_0,\ell,S_0}$  is now finite.

The main result of this section is the following.

**Theorem 2.** *Let  $a_0$  and  $\ell$  denote two positive real numbers, and let  $S_0 > \ell a_0$ . Problem (9) has no solution and*

$$\sup_{a \in \mathcal{S}_{a_0,\ell,S_0}} F(a) = k\pi\beta(T_d - T_\infty) \left( \frac{a_0^{3/2}\gamma}{\sqrt{\beta}} + (S - a_0\ell) \right),$$

where  $\gamma$  is given by (12). Moreover, every sequence  $(a_n)_{n \in \mathbb{N}}$  of elements of  $\mathcal{S}_{a_0,\ell,S_0}$  such that  $(a_n, a_n \sqrt{1 + a_n'^2})_{n \in \mathbb{N}}$   $\tau$ -converges to  $(a_0, a_0 + (S - a_0\ell)\delta_0)$ , where  $\delta_0$  denotes the Dirac measure at  $x = 0$ , is a maximizing sequence for Problem (9).

**Remark 4.** It is standard, in shape optimization, to use rearrangements such as the so-called Steiner or Schwarz symmetrizations, to characterize the solutions of an optimal design problem (see for instance [14] for the definition and [8] for examples of use in shape optimization). In general, such an approach needs to (re)write the criterion as an energy infimum. In our case, note that one has for every  $a \in \mathcal{A}_{a_0,\ell}$ ,

$$F(a) = \frac{k\pi}{T_d - T_\infty} \min_{\check{T} \in H_{1,0}(0,\ell)} \mathcal{F}_a(\check{T}),$$

where  $H_{1,0}(0,\ell) = \{u \in H^1(0,\ell) \mid u(0) = 0\}$  and

$$\mathcal{F}_a(\check{T}) = \int_0^\ell (a(x)^2 \check{T}'(x)^2 + \beta a(x) \sqrt{1 + a'(x)^2} (\check{T}(x) + T_d - T_\infty)^2) dx + \beta_r a(\ell)^2 (\check{T}(\ell) + T_d - T_\infty)^2.$$

Unfortunately, a standard decreasing rearrangement of the functional  $\mathcal{F}_a$  does not permit to solve Problem (9), especially since it is *a priori* not possible using this technic to compare  $\mathcal{F}_a(\check{T})$  with  $\mathcal{F}_{a^*}(\check{T}^*)$ , the notation  $a^*$  (resp.  $\check{T}^*$ ) denoting the decreasing rearrangement of  $a$  (resp.  $\check{T}$ ). In a nutshell, the difficulties come from the facts that, if  $a \in \mathcal{A}_{a_0,\ell}$ , the function  $a^* \sqrt{1 + a^{*2}}$  does not coincide in general with the decreasing rearrangement of  $a \sqrt{1 + a'^2}$ , and moreover, we should apply a Polyà type inequality that would make the criterion decrease.

In the proof of Theorem 2, the key idea to show the nonexistence result and to construct maximizing sequences lies in applying a perturbation that can be assimilated to a kind of monotone rearrangement of the function  $a \sqrt{1 + a'^2}$ , and then to shrewdly deduce an element  $a \in \mathcal{A}_{a_0,\ell}$  belonging to the class of admissible designs, and that makes the job.

*Proof.* This proof is divided into several steps. Steps 1, 2 and 3 are devoted to showing that Problem (9) has no solution, whereas Step 4 focuses on building maximizing sequences using a kind of rearrangement inequality (see Remark 4). Let us argue by contradiction, assuming that Problem (9) has a solution  $a \in \mathcal{S}_{a_0, \ell, S_0}$ . Let us denote by  $b$  the function  $a\sqrt{1+a'^2}$ . The idea of the proof is to introduce an admissible perturbation  $a_\varepsilon$  of  $a$  in  $\mathcal{S}_{a_0, \ell, S_0}$ , suitably chosen to guarantee that  $F(a_\varepsilon) > F(a)$  for a given  $\varepsilon > 0$ . We will proceed in several steps.

**Step 1. Definition of the perturbation  $a_\varepsilon$ .** Since the constant function equal to  $a_0$  is obviously not a solution of Problem (9), there exists  $x_0 \in (0, \ell)$  and an interval  $(x_0 - \varepsilon/2, x_0 + \varepsilon/2)$  on which  $a > a_0$ . Introduce for such a choice of  $x_0$  and  $\varepsilon$ , the function

$$b_\varepsilon = b + c(\chi_{[0, \varepsilon]} - \chi_{[x_0 - \varepsilon/2, x_0 + \varepsilon/2]}),$$

where  $c$  is a positive constant, chosen small enough to guarantee that  $b_\varepsilon > a_0$  almost everywhere in  $(0, \ell)$ . We will use the following lemma to construct  $a_\varepsilon$ .

**Lemma 3.** *There exists a family  $(a_\varepsilon)_{\varepsilon > 0}$  such that  $a_\varepsilon \in \mathcal{S}_{a_0, \ell, S_0}$  for every  $\varepsilon > 0$ ,  $a_\varepsilon \sqrt{1+a_\varepsilon'^2} = b_\varepsilon$  for almost every  $\varepsilon > 0$ , and*

$$\|a_\varepsilon - a\|_{L^\infty(0, \ell)} = O(\varepsilon^2).$$

The proof of this lemma is a bit technical and is postponed to Section 3.4. We now consider a family  $(a_\varepsilon)_{\varepsilon > 0}$  chosen as in the statement of Lemma 3. In particular, and according to the construction of  $a_\varepsilon$  made in the proof of Lemma 3, we will assume without loss of generality that  $a_\varepsilon(\ell) = a(\ell)$ .

For  $\varepsilon > 0$ , we denote by  $T_\varepsilon$  the temperature associated to  $a_\varepsilon$ , *i.e.* the solution of (1) with  $a = a_\varepsilon$ .

**Step 2. Asymptotic development of  $T_\varepsilon$  at the first order.** Notice that  $\int_0^\ell b_\varepsilon(x) dx = \int_0^\ell b(x) dx$ ,  $(b_\varepsilon)_{\varepsilon > 0}$  converges in the sense of measures to  $b$ , and  $(a_\varepsilon)_{\varepsilon > 0}$  converges strongly to  $a$  in  $L^\infty(0, \ell)$  as  $\varepsilon \searrow 0$ , according to Lemma 3. Then, following the proof of Proposition 1, the family  $(T_\varepsilon)_{\varepsilon > 0}$  converges, up to a subsequence, weakly in  $H^1(0, \ell)$  and strongly in  $L^2(0, \ell)$  to  $T$ , the solution of (1) associated to the optimal radius  $a$ .

Let us write an asymptotic development of  $T_\varepsilon$  at the first order. For that purpose, introduce

$$\tilde{T}_\varepsilon = \frac{T_\varepsilon - T}{\varepsilon}.$$

The function  $\tilde{T}_\varepsilon$  is solution of the following ordinary differential equation

$$\begin{aligned} (a(x)^2 \tilde{T}_\varepsilon'(x))' &= \beta b(x) \tilde{T}_\varepsilon + \beta \frac{R_\varepsilon(x)}{\varepsilon} (T_\varepsilon(x) - T_\infty) - \left( \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T_\varepsilon'(x) \right)', \quad x \in (0, \ell) \\ \tilde{T}_\varepsilon(0) &= 0 \\ \tilde{T}_\varepsilon'(\ell) &= -\beta_r \tilde{T}_\varepsilon(\ell), \end{aligned} \tag{13}$$

where  $R_\varepsilon(x) = c(\chi_{[0, \varepsilon]} - \chi_{[x_0 - \varepsilon/2, x_0 + \varepsilon/2]})$ .

Let us first prove the convergence of  $\tilde{T}_\varepsilon$ . We multiply the main equation of System (13) by  $\tilde{T}_\varepsilon$  and then integrate. We get

$$\begin{aligned} \int_0^\ell \left( a(x)^2 \tilde{T}_\varepsilon'(x)^2 + \beta b(x) \tilde{T}_\varepsilon(x)^2 \right) dx + \beta_r a(\ell)^2 \tilde{T}_\varepsilon(\ell)^2 &= -\frac{\beta}{\varepsilon} \int_0^\varepsilon R_\varepsilon(x) \tilde{T}_\varepsilon(x) (T_\varepsilon(x) - T_\infty) dx \\ &\quad - \int_0^\ell \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T_\varepsilon'(x) \tilde{T}_\varepsilon(x) dx. \end{aligned}$$

First, notice that, using standard Sobolev imbedding results and the weak- $H^1$  convergence of the family  $(T_\varepsilon)_{\varepsilon>0}$ , there exists  $C_1 > 0$  such that

$$\begin{aligned} \frac{\beta}{\varepsilon} \left| \int_0^\varepsilon R_\varepsilon(x) \tilde{T}_\varepsilon(x) (T_\varepsilon(x) - T_\infty) dx \right| &= \frac{\beta c}{\varepsilon} \left| \int_0^\varepsilon \tilde{T}_\varepsilon (T_\varepsilon - T_\infty) dx - \int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} \tilde{T}_\varepsilon (T_\varepsilon - T_\infty) dx \right| \\ &\leq 2\beta c \|\tilde{T}_\varepsilon\|_{L^\infty(0,\ell)} (\|T_\varepsilon\|_{L^\infty(0,\ell)} + T_\infty) \\ &\leq C_1 \|\tilde{T}_\varepsilon\|_{H^1(0,\ell)}. \end{aligned}$$

Second, using at the same time Lemma 2, 3 and the Cauchy-Schwarz inequality, there exists  $C_2 > 0$  such that

$$\begin{aligned} \left| \int_0^\ell \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T'_\varepsilon(x) \tilde{T}'_\varepsilon(x) dx \right| &\leq 2\sqrt{\frac{S_0^2}{\ell^2} + 4S_0} \frac{\|a_\varepsilon - a\|_{L^\infty(0,\ell)}}{\varepsilon} \|\tilde{T}_\varepsilon\|_{H^1(0,\ell)} \|T_\varepsilon - T_\infty\|_{H^1(0,\ell)} \\ &\leq C_2 \varepsilon \|\tilde{T}_\varepsilon\|_{H^1(0,\ell)}. \end{aligned}$$

Combining the two previous estimates and using that

$$a_0 \min\{\beta, a_0\} \|\tilde{T}_\varepsilon\|_{H^1(0,\ell)}^2 \leq \int_0^\ell \left( a(x)^2 \tilde{T}'_\varepsilon(x)^2 + \beta b(x) \tilde{T}_\varepsilon(x)^2 \right) dx + \beta_r a(\ell)^2 \tilde{T}_\varepsilon(\ell)^2,$$

yields that  $(\tilde{T}_\varepsilon)_{\varepsilon>0}$  is bounded in  $H^1(0,\ell)$ . Then, using a Rellich theorem,  $(\tilde{T}_\varepsilon)_{\varepsilon>0}$  converges, up to a subsequence, to  $\tilde{T}$ , weakly in  $H^1(0,\ell)$  and strongly in  $L^2(0,\ell)$ .

Let us now write the system whose  $\tilde{T}$  is solution. The variational formulation of (13) writes: find  $\tilde{T}_\varepsilon \in H^1(0,\ell)$  satisfying  $\tilde{T}_\varepsilon(0) = 0$  such that for every test function  $\varphi \in H^1(0,\ell)$  satisfying  $\varphi(0) = 0$ , one has

$$\begin{aligned} &\int_0^\ell \left( a(x)^2 \tilde{T}'_\varepsilon(x) \varphi'(x) + \beta b(x) \tilde{T}_\varepsilon(x) \varphi(x) \right) dx + \beta_r a(\ell)^2 \tilde{T}_\varepsilon(\ell) \varphi(\ell) \\ &+ \beta \int_0^\ell \frac{R_\varepsilon(x)}{\varepsilon} (T_\varepsilon(x) - T_\infty) \varphi(x) dx - \int_0^\ell \left( \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T'_\varepsilon(x) \right)' \varphi(x) dx = 0. \end{aligned} \quad (14)$$

Note that

$$- \int_0^\ell \left( \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T'_\varepsilon(x) \right)' \varphi(x) dx = \int_0^\ell \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T'_\varepsilon(x) \varphi'(x) dx.$$

Then, since  $(T_\varepsilon)_{\varepsilon>0}$  is uniformly bounded in  $H^1(0,\ell)$  and using Lemma 3, one deduces that

$$\lim_{\varepsilon \searrow 0} \int_0^\ell \left( \left( \frac{a_\varepsilon(x)^2 - a(x)^2}{\varepsilon} \right) T'_\varepsilon(x) \right)' \varphi(x) dx = 0.$$

We let  $\varepsilon$  tend to zero in (14), and using the Lebesgue density theorem, we get

$$\int_0^\ell \left( a(x)^2 \tilde{T}'(x) \varphi'(x) + \beta b(x) \tilde{T}(x) \varphi(x) \right) dx - \beta c \varphi(x_0) (T(x_0) - T_\infty) + \beta_r a(\ell)^2 \tilde{T}(\ell) \varphi(\ell) = 0, \quad (15)$$

for every test function  $\varphi \in H^1(0,\ell)$  satisfying  $\varphi(0) = 0$ . We refer to Remark 5 for the characterization of  $\tilde{T}$ .

**Step 3. Asymptotic of the criterion when  $\varepsilon$  tends to 0 and conclusion.** We compute

$$\begin{aligned} \frac{F(a_\varepsilon) - F(a)}{k\pi\beta} &= \int_0^\ell (b_\varepsilon(x)(T_\varepsilon(x) - T_\infty) - b(x)(T(x) - T_\infty)) dx \\ &\quad + \frac{\beta_r}{\beta} a(\ell)^2 (T_\varepsilon(\ell) - T(\ell)) \\ &= \int_0^\ell b(x)(T_\varepsilon(x) - T(x)) dx + \int_0^\ell R_\varepsilon(x)(T_\varepsilon(x) - T_\infty) dx \\ &\quad + \frac{\beta_r}{\beta} a(\ell)^2 (T_\varepsilon(\ell) - T(\ell)). \end{aligned}$$

Dividing by  $\varepsilon$  and letting  $\varepsilon$  go to zero, one sees that

$$\lim_{\varepsilon \searrow 0} \frac{F(a_\varepsilon) - F(a)}{k\pi\beta\varepsilon} = \int_0^\ell b(x)\tilde{T}(x) dx + c(T_d - T(x_0)) + \frac{\beta_r}{\beta} a(\ell)^2 \tilde{T}(\ell). \quad (16)$$

The contradiction will follow by proving that the right hand-side of the last equality is positive. For that purpose, take  $\varphi = T - T_d$  in (15). We obtain

$$\begin{aligned} \int_0^\ell \left( a(x)^2 \tilde{T}'(x) T'(x) + \beta b(x) \tilde{T}(x) T(x) \right) dx &= \beta T_d \int_0^\ell b(x) \tilde{T}(x) dx + \beta c(T(x_0) - T_d)(T(x_0) - T_\infty) \\ &\quad - \beta_r a(\ell)^2 \tilde{T}(\ell)(T(\ell) - T_d). \end{aligned} \quad (17)$$

Multiply now Equation (1) by  $\tilde{T}$  and integrate then by parts. We obtain

$$\int_0^\ell \left( a(x)^2 \tilde{T}'(x) T'(x) + \beta b(x) \tilde{T}(x) T(x) \right) dx = \beta T_\infty \int_0^\ell b(x) \tilde{T}(x) dx - \beta_r a(\ell)^2 \tilde{T}(\ell)(T(\ell) - T_\infty). \quad (18)$$

Combining (17) and (18) yields

$$\int_0^\ell b(x) \tilde{T}(x) dx = c \frac{(T_d - T(x_0))(T(x_0) - T_\infty)}{T_d - T_\infty} - \frac{\beta_r}{\beta} a(\ell)^2 \tilde{T}(\ell).$$

According to (16), we compute

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{F(a_\varepsilon) - F(a)}{k\pi\beta\varepsilon} &= c \frac{(T_d - T(x_0))(T(x_0) - T_\infty)}{T_d - T_\infty} + c(T_d - T(x_0)) \\ &= c \frac{(T_d - T_\infty)^2 - (T(x_0) - T_\infty)^2}{T_d - T_\infty}. \end{aligned}$$

According to Lemma 1, the right hand sign is positive, and it follows that, for  $\varepsilon$  small enough,  $F(a_\varepsilon) > F(a)$ . This is a contradiction, and it proves that Problem (9) has no solution.

**Step 4. Convergence along maximizing sequences.** It remains now to prove the second claim of Theorem 2. To this aim, we will use an auxiliary optimal design problem obtained from Problem (9) by imposing a uniform upper bound on the surface density  $a\sqrt{1+a'^2}$ . Indeed, define for  $M > a_0$  the set

$$\mathcal{S}_{a_0, \ell, S_0}^M = \left\{ a \in \mathcal{S}_{a_0, \ell, S_0}, a_0 \leq a\sqrt{1+a'^2} \leq M \text{ a.e. in } (0, \ell) \right\}.$$

Notice that for a given  $M > a_0$ , the class  $\mathcal{S}_{a_0, \ell, S_0}^M$  is compact in  $W^{1, \infty}(0, \ell)$ . For  $M > a_0$ , let us introduce the auxiliary problem

$$\sup \{F(a), a \in \mathcal{S}_{a_0, \ell, S_0}^M\}. \quad (19)$$

In the following proposition, whose proof is postponed to Section 3.4, we perform a precise analysis of this problem.

**Proposition 2.** *Let  $a_0, \ell$  and  $M > 0$  denote three positive real numbers, with  $S_0 > la_0$  and  $M > a_0$ . Then, Problem (19) has a solution  $\underline{a}_M$ , satisfying necessarily (up to a zero Lebesgue measure subset)*

$$\underline{a}_M(x) \sqrt{1 + \underline{a}'_M(x)^2} = \begin{cases} M, & x \in (0, x_M), \\ a_0, & x \in (x_M, \ell), \end{cases} \quad (20)$$

with  $x_M = \frac{S - a_0 \ell}{M - a_0}$ .

Next lemma highlights how solutions of Problem (19) can be used to exhibit a maximizing sequence for Problem (9).

**Lemma 4.** *The family  $(\underline{a}_M)_{M > a_0}$  maximizes  $F$ .*

The proof of this elementary lemma is postponed to Section 3.4.

According to this result, consider such a family  $(\underline{a}_M)_{M > a_0}$ . Notice that the families  $(\underline{a}_M)_{M > a_0}$  and  $(\underline{b}_M)_{M > a_0}$  converge respectively uniformly in  $(0, \ell]$  to  $a_0$  and for the measures topology to a Dirac mass at  $x = 0$ , as  $M \rightarrow +\infty$ . Indeed, the convergence of  $(\underline{b}_M)_{M > a_0}$  is clear and one has  $\underline{b}_M(x) = a_0 = \underline{a}_M(x) \sqrt{1 + \underline{a}'_M(x)^2} \geq a_0$  for all  $x \in [x_M, \ell]$ , implying the  $L^\infty$ -convergence of  $\underline{a}_M(\cdot)$  in  $(0, \ell]$  to  $a_0$ , since  $x_M \rightarrow 0$  as  $M \rightarrow +\infty$ . Proposition 1 thus yields that  $(F(\underline{a}_M))_{M > a_0}$  converges to  $\widehat{F}(a_0, (S - a_0 \ell) \delta_0 + a_0)$  as  $M \rightarrow +\infty$ . In other words, using the same computations as those in the proof of Theorem 1, one gets

$$\lim_{M \rightarrow +\infty} F(\underline{a}_M) = 2h\pi(T_d - T_\infty) \left( \frac{a_0^{3/2} \gamma}{\sqrt{\beta}} + S_0 - a_0 \ell \right).$$

Moreover, each sequence  $(a_n)_{n \in \mathbb{N}}$  fulfilling the conditions of Theorem 2 admits the same limit by Proposition 1.  $\square$

**Remark 5.** System (15) must be understood in the following sense:  $\widetilde{T}$  solves the system

$$\begin{cases} (a(x)^2 \widetilde{T}'(x))' = \beta b(x) \widetilde{T}(x) & x \in (0, x_0) \\ (a(x)^2 \widetilde{T}'(x))' = \beta b(x) \widetilde{T}(x) & x \in (x_0, \ell) \\ \widetilde{T}(0) = 0 \\ \sigma(a(x_0)^2 \widetilde{T}'(x_0)) = \beta c(T(x_0) - T_\infty) \\ \widetilde{T}'(\ell) = -\beta_r \widetilde{T}(\ell), \end{cases}$$

where  $\sigma(a(x_0)^2 \widetilde{T}'(x_0)) = a(x_0^+)^2 \widetilde{T}'(x_0^+) - a(x_0^-)^2 \widetilde{T}'(x_0^-)$  denotes the discontinuity jump of the function  $a^2 \widetilde{T}'$  at  $x = x_0$ .

### 3.4 Proofs of Lemma 3, Proposition 2 and Lemma 4

*Proof of Lemma 3.* The proof consists in exhibiting an element  $a_\varepsilon \in \mathcal{S}_{a_0, \ell, S_0}$  so that the statements of Lemma 3 are satisfied. First, we impose  $a_\varepsilon = a$  on  $(\varepsilon, x_0 - \varepsilon/2) \cup (x_0 + \varepsilon/2, \ell)$ . Without loss of generality, let us now explain how to define  $a_\varepsilon$  on  $[0, \varepsilon]$ , the construction of  $a_\varepsilon$  on  $[x_0 - \varepsilon/2, x_0 + \varepsilon/2]$  being similar. The construction method is close to the one of the maximizing sequence presented in the proof of Theorem 1. The idea is to impose oscillations on  $a_\varepsilon$  to control the  $L^\infty$  distance between  $a$  and  $a_\varepsilon$ .

We first explain how to create one oscillation on an interval  $[\bar{x}, \bar{x} + \eta]$ , where  $0 < \bar{x} < \bar{x} + \eta < \varepsilon$  (see Figure 3). The function  $a_\varepsilon$  is chosen so that

$$a_\varepsilon = a_{\eta,1} \quad \text{on } (\bar{x}, \xi) \quad \text{and} \quad a_\varepsilon = a_{\eta,2} \quad \text{on } (\xi, \bar{x} + \eta),$$

where the function  $a_{\eta,1}$  is solution of the following Cauchy problem

$$\begin{aligned} a'_{\eta,1}(x) &= \frac{\sqrt{b_\varepsilon(x)^2 - a_{\eta,1}(x)^2}}{a_{\eta,1}(x)} & x \in (\bar{x}, \bar{x} + \eta) \\ a_{\eta,1}(\bar{x}) &= a(\bar{x}), \end{aligned}$$

the function  $a_{\eta,2}$  is solution of the following Cauchy problem

$$\begin{aligned} a'_{\eta,2}(x) &= -\frac{\sqrt{b_\varepsilon(x)^2 - a_{\eta,2}(x)^2}}{a_{\eta,2}(x)} & x \in (\bar{x}, \bar{x} + \eta) \\ a_{\eta,2}(\bar{x} + \eta) &= a(\bar{x} + \eta), \end{aligned}$$

and  $\xi \in (\bar{x}, \bar{x} + \eta)$  is chosen so that  $a_{\eta,1}(\xi) = a_{\eta,2}(\xi)$ . Notice that the functions  $a_{\eta,1}$  and  $a_{\eta,2}$  satisfy in particular  $a_{\eta,i} \sqrt{1 + a'^2_{\eta,i}} = b_\varepsilon$  for  $i \in \{1, 2\}$ ,  $a_{\eta,1}$  is increasing and  $a_{\eta,2}$  is decreasing. Such a construction is possible provided that the graphs of  $a_{\eta,1}$  and  $a_{\eta,2}$  intersect at a point whose abscissa belongs to  $(\bar{x}, \bar{x} + \eta)$ . The intermediate value theorem yields that it is enough to show that

$$a_{\eta,1}(\bar{x} + \eta) > a(\bar{x} + \eta) \quad \text{and} \quad a_{\eta,2}(\bar{x}) > a(\bar{x}).$$

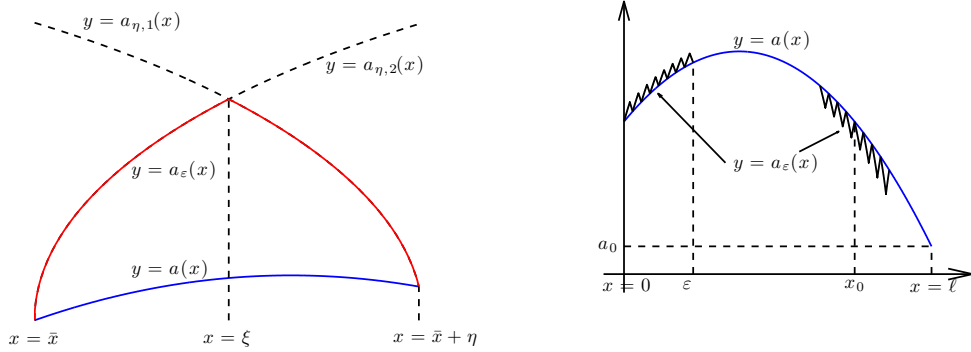


Figure 3: Left: Zoom on one oscillation. Right: the perturbation  $a_\varepsilon$

Without loss of generality, let us prove that the first assertion is true, the proof of the second one being similar. In fact, we will prove that  $a_{\eta,1} > a$  everywhere in  $(\bar{x}, \bar{x} + \eta]$ . Let us argue



by contradiction. Note that  $a'_{\eta,1}(\bar{x}) > a'(\bar{x})$  since  $\bar{x} < \varepsilon$  and thus  $b_\varepsilon(\bar{x}) > b(\bar{x})$ . Denote by  $\alpha$  the first point of  $(\bar{x}, \bar{x} + \eta)$  such that  $a_{\eta,1}(\alpha) = a(\alpha)$ , provided that it exists. Since  $b_\varepsilon > b$  on  $[\bar{x}, \bar{x} + \eta] \subset [0, \varepsilon]$ , there holds  $a_{\eta,1}\sqrt{1 + a'^2_{\eta,1}} > a\sqrt{1 + a'^2}$  almost everywhere in any neighborhood of  $\alpha$ , thus there exists necessarily a neighborhood  $\mathcal{O}_\alpha$  of  $\alpha$  such that  $|a'_{\eta,1}(x)| > |a'(x)|$  at every Lebesgue point of  $a'_{\eta,1}$  and  $a'$  in  $\mathcal{O}_\alpha$ . It implies the existence of  $\nu > 0$  such that  $a_{\eta,1}(x) < a(x)$  on  $(\alpha - \nu, \alpha)$ , which is absurd.

It remains now to find the number of oscillation on  $[0, \varepsilon]$  and  $[x_0 - \varepsilon/2, x_0 + \varepsilon/2]$  guaranteeing that  $a_\varepsilon$  is as close to  $a$  as desired (in the sense of the  $L^\infty$  distance). Notice that, according to the previous definition of  $a_\varepsilon$  on  $[\bar{x}, \bar{x} + \eta]$ , one has

$$\begin{aligned} a(x)^2 \leq a_\varepsilon(x)^2 &\leq 2 \int_{\bar{x}}^{\bar{x}+\eta} b_\varepsilon(x) + a(\bar{x})^2 \\ &\leq 2\eta(\|b + c\|_{L^\infty(0,\ell)} + 2\|a'\|_{L^\infty(0,\ell)}\|a\|_{L^\infty(0,\ell)}) + a(x)^2, \end{aligned}$$

for every  $x \in [\bar{x}, \bar{x} + \eta]$ . Moreover, since  $a \in W^{1,\infty}(0, \ell)$ ,

$$0 \leq a_\varepsilon(x) - a(x) \leq \frac{(\|b + c\|_{L^\infty(0,\ell)} + \|a'\|_{L^\infty(0,\ell)}\|a\|_{L^\infty(0,\ell)})\eta}{a_0}$$

for every  $x \in [\bar{x}, \bar{x} + \eta]$ . It suffices hence to fix for example  $\eta = \varepsilon/k_\varepsilon$  with  $k_\varepsilon = [1/\varepsilon] + 1$ , to ensure that  $\|a_\varepsilon - a\|_{L^\infty([\bar{x}, \bar{x} + \eta])} = \mathcal{O}(\varepsilon^2)$ .

**Conclusion.** We consider a regular subdivision of  $[0, \varepsilon]$  (resp.  $[x_0 - \varepsilon/2, x_0 + \varepsilon/2]$ ) into  $k_\varepsilon$  intervals, and we use the process described previously to construct the graph of  $a_\varepsilon$  by creating one oscillation on each of these intervals. This way, one gets:  $\|a_\varepsilon - a\|_{L^\infty([0,\ell])} = \mathcal{O}(\varepsilon^2)$ . The lemma is then proved.  $\square$

*Proof of Proposition 2.* Consider a maximizing sequence  $(a_n)_{n \in \mathbb{N}}$  in  $\mathcal{S}_{a_0, \ell, S_0}$ , with  $b_n = a_n\sqrt{1 + a'^2_n}$  satisfying  $a_0 \leq b_n \leq M$  for almost every  $n \in \mathbb{N}$ . Clearly  $a_0|a'_n(x)| \leq M$  for almost every  $n \in \mathbb{N}$  and  $x \in (0, \ell)$ . Hence  $(a_n)_{n \in \mathbb{N}}$  is uniformly Lipschitz-continuous and bounded. According to the Arzelà-Ascoli theorem, the sequence  $(a_n)_{n \in \mathbb{N}}$  converges, up to a subsequence, to some Lipschitz-continuous limit  $\underline{a}_M$ , satisfying  $a_0 \leq \underline{a}_M \leq M$  where  $\underline{a}_M = \underline{a}_M\sqrt{1 + \underline{a}_M'^2}$ .

On the other hand, Proposition 1 yields

$$\lim_{n \rightarrow +\infty} F(a_n) = \widehat{F}(\underline{a}_M, \underline{b}_M) = F(\underline{a}_M),$$

since  $\underline{b}_M = \underline{a}_M\sqrt{1 + \underline{a}_M'^2}$ . Hence  $\underline{a}_M$  solves Problem (19).

Assume by contradiction that  $\underline{a}_M$  does not satisfy (20). Then there exist  $0 < y_0 < x_0 < \ell$  and  $\varepsilon > 0$  such that the function  $b_\varepsilon$  defined by

$$b_\varepsilon = \underline{b}_M + c(\chi_{[y_0, y_0 + \varepsilon]} - \chi_{[x_0 - \varepsilon, x_0]}),$$

satisfies to  $a_0 \leq b_\varepsilon \leq M$  almost everywhere in  $[0, \ell]$ . The same arguments as in the proof of Theorem 2 yield that one can construct a family  $(a_\varepsilon)_{\varepsilon > 0}$  in  $\mathcal{S}_{a_0, \ell, S_0}$  such that  $b_\varepsilon = a_\varepsilon\sqrt{1 + a'^2_\varepsilon}$  for all  $\varepsilon > 0$  and  $\|\underline{a}_M - a_\varepsilon\|_{L^\infty(0,\ell)} = \mathcal{O}(\varepsilon^2)$ . Moreover, one computes

$$\lim_{\varepsilon \searrow 0} \frac{F(a_\varepsilon) - F(\underline{a}_M)}{k\pi\beta\varepsilon} = c \frac{(T_M(y_0) - T_\infty)^2 - (T_M(x_0) - T_\infty)^2}{T_d - T_\infty}$$

where  $T_M$  is the solution of (1) associated with  $\underline{a}_M$ , and is decreasing according to Lemma 1. Thus the right hand-side is positive and  $F(a_\varepsilon) > F(\underline{a}_M)$  provided that  $\varepsilon$  be small enough. It yields a contradiction since  $a_\varepsilon$  belongs to the class of admissible functions for the current maximization problem. It follows that necessarily, the function  $\underline{a}_M$  satisfies (20), whence the result.  $\square$

*Proof of Lemma 4.* Notice that constructed as well, the sequence of sets  $(\mathcal{S}_{a_0, \ell, S_0}^M)_{M > a_0}$  is increasing for the inclusion, and there holds

$$\mathcal{S}_{a_0, \ell, S_0} = \bigcup_{M > a_0} \mathcal{S}_{a_0, \ell, S_0}^M.$$

It thus follows that

$$\sup_{a \in \mathcal{S}_{a_0, \ell, S_0}} F(a) = \sup_{M > a_0} \sup_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} F(a) = \lim_{M \rightarrow +\infty} \max_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} F(a) = \lim_{M \rightarrow +\infty} F(\underline{a}_M),$$

whence the convergence of the maxima. As a result,  $(\underline{a}_M)_{M > a_0}$  is a maximizing family.  $\square$

## 4 Comments and conclusion

### 4.1 Extension of Theorems 1 and 2 to a more general setting

In this section we provide some generalizations of the results stated in Theorems 1 and 2 to a more general model. Indeed, the convection coefficient  $h$  strongly depends on the operating conditions and geometries driving the fluid flow around the fin. Consequently,  $h$  is now assumed to be a function of  $x$ , especially because the lower part of the fin lies in the boundary layer of the fluid characterized by lower velocities. Therefore, the temperature  $T$  along the fin is now assumed to solve the following ordinary differential equation,

$$\begin{aligned} (a^2(x)T'(x))' &= \beta(x)a(x)\sqrt{1+a'(x)^2}(T(x) - T_\infty) \quad x \in (0, \ell) \\ T(0) &= T_d \\ T'(\ell) &= -\beta_r(T(\ell) - T_\infty), \end{aligned} \tag{21}$$

where  $T_d, T_\infty$  and  $\beta_r$  are chosen as in Section 2.1, and  $\beta$  denotes a nonnegative continuous function satisfying

$$\beta(x) \geq \beta_0 > 0 \tag{22}$$

for every  $x \in [0, \ell]$  so that the fin surface cannot be insulated. Notice that, in this case, the statement of Lemma 1 still holds for the solution  $T(\cdot)$  of (21).

We investigate here the optimal design problems, generalizing those introduced in Section 3.1.

**Generalized optimal design problem with volume or lateral surface constraint.** Let  $a_0$  and  $\ell$  denote two positive real numbers. Fix  $V_0 > \ell a_0^2$  and  $S_0 > \ell a_0$ . We investigate the problem of maximizing the functional  $a \mapsto \tilde{F}(a)$  with  $\tilde{F}(a) = -ka(0)T'(0)$ , where  $T$  denotes the unique solution of (21), either over the set  $\mathcal{V}_{a_0, \ell, V_0}$  or  $\mathcal{S}_{a_0, \ell, S_0}$ , respectively defined by (8) and (9).

We obtain the following result.

**Theorem 3.** Let  $a_0, \beta_0$  and  $\ell$  denote three positive real numbers, and let  $V_0 > \ell a_0^2$  and  $S_0 > \ell a_0$ .

1. One has

$$\sup_{a \in \mathcal{V}_{a_0, \ell, V_0}} \tilde{F}(a) = +\infty.$$

2. Let us assume that  $\beta \in C^0([0, \ell])$  is nonconstant, satisfies (22) and

$$\max_{x \in [0, \ell]} \beta(x) = \beta(0). \tag{23}$$

Thus, the problem of maximizing  $\tilde{F}$  over  $\mathcal{S}_{a_0, \ell, S_0}$  has no solution and

$$\sup_{a \in \mathcal{S}_{a_0, \ell, S_0}} \tilde{F}(a) = k\pi a_0 \int_0^\ell \beta(x)(\tilde{T} - T_\infty) dx + k\pi(S - a_0\ell)\beta(0)(T_d - T_\infty) + k\beta_r a_0^2(\tilde{T}(\ell) - T_\infty),$$

where  $\tilde{T}$  is the unique solution of System (21) with  $a(\cdot) = a_0$ . Moreover, every sequence  $(a_n)_{n \in \mathbb{N}}$  of elements of  $\mathcal{S}_{a_0, \ell, S_0}$  defined as in the statement of Theorem 2 is a maximizing sequence for this problem.

**Remark 6.** The technic used in the proofs of Theorems 2 and 3 fails and cannot be easily adapted when the function  $\beta$  does not satisfy (23) anymore. The issue of solving the same problem in this case is discussed and commented in Section 4.2.

*Proof.* We do not give all details since the proof is very similar to the ones of Theorems 1 and 2. We only underline the slight differences in every step.

1. Consider the sequence  $(a_{S,m})_{m \in \mathbb{N}^*}$  introduced in the proof of Theorem 1. The same kind of computations show that

$$\begin{aligned} \lim_{m \rightarrow +\infty} \frac{\tilde{F}(a_{S,m})}{k\pi} &= \beta(0)(T_d - T_\infty)(S - a_0\ell) + a_0 \int_0^\ell (T(x) - T_\infty) dx + \beta_r a_0^2(T(\ell) - T_\infty) \\ &\geq \beta(0)(T_d - T_\infty)(S - a_0\ell), \end{aligned}$$

where  $T$  is the unique solution of System (21) with  $a(\cdot) = a_0$ . Then, we conclude similarly to the proof of Theorem 1.

2. Accordingly to the three first steps of the proof of Theorem 2, we first show that the aforementioned problem has no solution. We argue by contradiction, by assuming the existence of a solution  $a$ , introducing  $b = a\sqrt{1 + a'^2}$  and the perturbation

$$b_\varepsilon = b + c(\chi_{[0, \varepsilon]} - \chi_{[x_0 - \varepsilon/2, x_0 + \varepsilon/2]}),$$

where  $c$  is a positive constant, chosen small enough to guarantee that  $b_\varepsilon > a_0$  almost everywhere in  $(0, \ell)$ , and  $x_0 \in (0, \ell)$  is such that  $a > a_0$  on  $(x_0 - \varepsilon/2, x_0 + \varepsilon/2)$ . The same computations as those led in Steps 2 and 3 yield

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{\tilde{F}(a_\varepsilon) - \tilde{F}(a)}{k\pi\varepsilon} &= \int_0^\ell \beta(x)b(x)\tilde{T}(x) dx + c(\beta(0)(T_d - T_\infty) - \beta(x_0)(T(x_0) - T_\infty)) \\ &\quad + \beta_r a(\ell)^2 \tilde{T}(\ell), \end{aligned}$$

where  $\tilde{T} \in H^1(0, \ell)$  is the unique function such that  $\tilde{T}(0) = 0$  and

$$\begin{aligned} \int_0^\ell \left( a(x)^2 \tilde{T}'(x) \varphi'(x) + \beta(x)b(x)\tilde{T}(x)\varphi(x) \right) dx - \beta(x_0)c\varphi(x_0)(T(x_0) - T_\infty) \\ + \beta_r a(\ell)^2 \tilde{T}(\ell)\varphi(\ell) = 0, \end{aligned} \quad (24)$$

for every test function  $\varphi \in H^1(0, \ell)$  satisfying  $\varphi(0) = 0$ . An adequate choice of test function allows to reduce the last expression into

$$\lim_{\varepsilon \searrow 0} \frac{\tilde{F}(a_\varepsilon) - \tilde{F}(a)}{k\pi\varepsilon} = c \frac{\beta(0)(T_d - T_\infty)^2 - \beta(x_0)(T(x_0) - T_\infty)^2}{T_d - T_\infty}.$$

Since  $T - T_\infty$  is positive decreasing, the right-hand side is positive, which yields a contradiction.

It remains now to exhibit a maximizing sequence. For that purpose, we follow the lines of Step 4, by investigating the problem

$$\sup \left\{ \tilde{F}(a), a \in \mathcal{S}_{a_0, \ell, S_0}, a_0 \leq a\sqrt{1+a'^2} \leq M \text{ a.e. in } (0, \ell) \right\} \quad (25)$$

and showing that a solution  $\underline{a}_M$  verifies necessarily (20). Define the function  $\underline{b}_M$  by  $\underline{b}_M = \underline{a}_M \sqrt{1 + \underline{a}_M'^2}$ . As in Proposition 2, we argue by contradiction. Thus, it suffices to consider a particular perturbation  $b_\varepsilon$  of  $\underline{b}_M$  of the form

$$b_\varepsilon = \underline{b}_M + c(\chi_{[y_0, y_0 + \varepsilon]} - \chi_{[x_0 - \varepsilon, x_0]}),$$

with  $0 < y_0 < x_0 < \ell$ . The proof is identical, but we need to adapt slightly the proof of Proposition 2, and in particular, to make the choices of  $\varepsilon$ ,  $x_0$  and  $y_0$  precise, in order to guarantee that  $\lim_{\varepsilon \searrow 0} \frac{\tilde{F}(a_\varepsilon) - \tilde{F}(\underline{a}_M)}{k\pi\varepsilon}$  be positive, provided that  $\beta$  satisfies (23). According to this assumption, we consider  $\varepsilon > 0$  and  $x_0 \in (0, \ell)$  such that

$$\beta|_{[0, \varepsilon]} > \beta|_{[x_0 - \varepsilon/2, x_0 + \varepsilon/2]}.$$

Recall that  $x_M = \frac{S - a_0 \ell}{M - a_0}$  and notice that, for  $M$  large enough,  $x_M \in (0, \varepsilon)$ . We fix  $M_0 > a_0$  and  $y_0 \in (0, x_M)$  so that  $b|_{[y_0, y_0 + \varepsilon]} < M$  for  $M > M_0$ . Such conditions ensure that

$$\lim_{\varepsilon \searrow 0} \frac{\tilde{F}(a_\varepsilon) - \tilde{F}(\underline{a}_M)}{k\pi\varepsilon} = c \frac{\beta(y_0)(T(y_0) - T_\infty)^2 - \beta(x_0)(T(x_0) - T_\infty)^2}{T_d - T_\infty} > 0,$$

where  $T$  denotes the solution of System (21) with  $a(\cdot) = \underline{a}_M$ . Hence, the conclusion of Proposition 2 remains true in this case. It suffices then to mimic the rest of the proof, by considering the sequence  $(\underline{b}_M)_{M > M_0}$  and letting  $M$  go to  $+\infty$ . The end of the proof is similar to the one of Theorem 2 and can thus be easily adapted.  $\square$

## 4.2 Numerical investigations

In Proposition 2, we established an existence result for the auxiliary problem (19), close to the initial one, where a pointwise upper bound constraint on the term  $a\sqrt{1+a'^2}$  were added. Recall that this problem writes for a given  $M > a_0$ ,

$$\sup \left\{ F(a), a \in \mathcal{S}_{a_0, \ell, S_0}, a_0 \leq a\sqrt{1+a'^2} \leq M \text{ a.e. in } (0, \ell) \right\}.$$

According to Lemma 4, solving this problem and letting  $M$  go to  $+\infty$  yields an approximation of the optimal value for Problem (9). Furthermore, this problem is also used to construct maximizing sequences in the proof of Theorem 2. This is why we decided to focus the numerical investigations on the optimal design problem (19).

**Remark 7** (Brief precisions on the numerical simulations). The simulations were obtained with a direct method applied to the optimal design problem described previously, consisting in discretizing the underlying differential equations, the optimal design  $a(\cdot)$ , and to reduce the shape optimization problem to some finite-dimensional maximization problem with constraints. Equation (1) is discretized with the finite volume method, well adapted to that case because of its

heat flux conservativeness property. We used two staggered grids in order to avoid the so-called *checkerboard phenomenon* [1]. The resulting finite-dimensional optimization problem is solved by using an interior-point method. We used the code IPOPT (see [20]) combined with AMPL (see [6]) on a standard desktop machine. The resulting code works out the solution very quickly (for instance, around 3 seconds for the simulations below).

We provide hereafter several numerical simulations of Problem (19). Each structure has been discretized with 250 design elements and we chose as optimization constraints parameters:  $a_0 = 1$  mm and  $S = 3S_0$  with  $S_0 = 2\pi a_0 \ell$ . Figures 4 and 5 may be considered as numerical illustrations of the results stated in the theorems 2 and 3.

To the contrary, the numerical results provided on the figures 6, 7 and 8 provide some hints on physical cases that are not covered by the aforementioned theorems: on the two first ones,  $\beta$  is chosen to be a kind of regularized *step function* whereas on the third one, it is assumed to be an increasing function. These figures suggest that several situations may arise: according to the figures 6 and 7, one could expect a nonexistence result, since the term  $\underline{a}_M \sqrt{1 + \underline{a}_M'^2}$  seems to converge in the sense of measures, as  $M$  tends to  $+\infty$ , to the sum of a regular function and a Dirac measure at the point  $x = 60$  mm, where the step of the function  $\beta$  occurs. Notice however that the fins are assumed to be made of different raw materials having different thermal conductivities, respectively  $k = 10 \text{ W.m}^{-2}.\text{K}^{-1}$  on Figure 6 and  $k = 0.1 \text{ W.m}^{-2}.\text{K}^{-1}$  on Figure 7. This is the reason why the optimal profiles  $a$  strongly differ on each case.

At the opposite, one could maybe expect that the optimal design problem corresponding to the profile plotted on Figure 8, where  $\beta$  is an increasing affine function, has a solution.

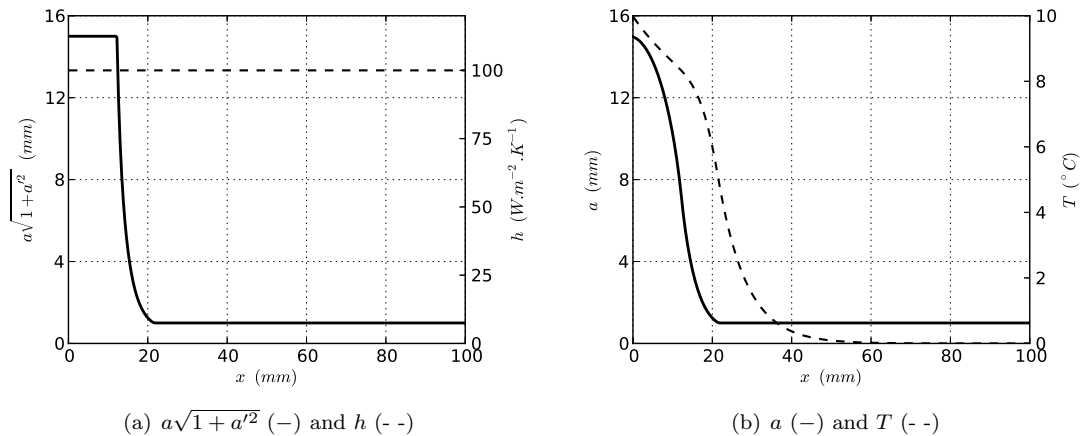


Figure 4: Optimal solution for a constant profile  $\beta = 2h/k$  with  $M = 15$  mm,  $\ell = 100$  mm,  $T_d = 10^\circ\text{C}$ ,  $T_\infty = 0^\circ\text{C}$ ,  $h_r = h(\ell)$  and  $k = 10 \text{ W.m}^{-2}.\text{K}^{-1}$ .

### 4.3 Conclusion and perspectives

In this article, we addressed the issue of finding the optimal shape of a fin, by assuming that its shape is axisymmetric and considering as physical model of the temperature along its axis a

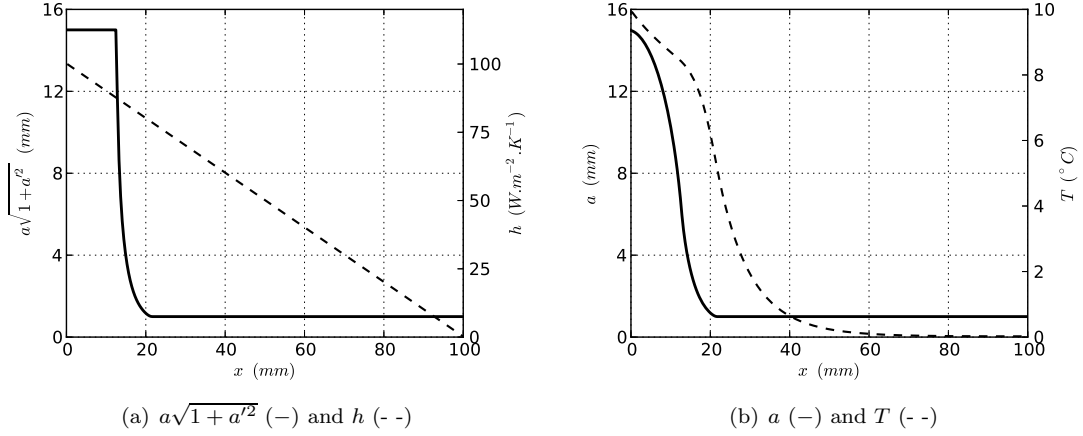


Figure 5: Optimal solution for a decreasing function  $\beta = 2h/k$  with  $M = 15$  mm,  $\ell = 100$  mm,  $T_d = 10^{\circ}C$ ,  $T_{\infty} = 0^{\circ}C$ ,  $h_r = h(\ell)$  and  $k = 10 W.m^{-2}.K^{-1}$ .

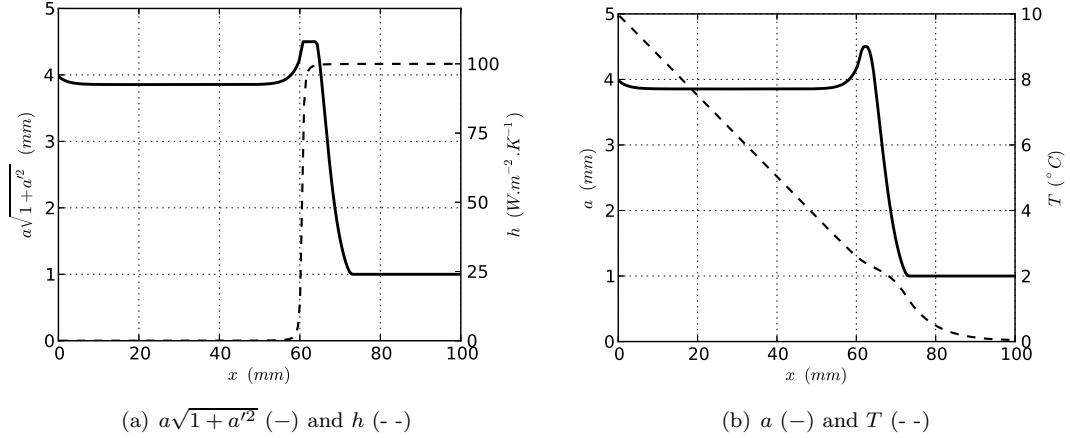


Figure 6: Optimal solution for a near step profile  $\beta = 2h/k$  at  $x = 60$  mm with  $M = 4.5$  mm,  $\ell = 100$  mm,  $T_d = 10^{\circ}C$ ,  $T_{\infty} = 0^{\circ}C$ ,  $h_r = h(\ell)$  and  $k = 10 W.m^{-2}.K^{-1}$ .

simplified one-dimensional Sturm-Liouville system, much used in the engineering literature [3, 21]. Two natural constraints for the shape optimization problem have been investigated, by imposing a maximal bound either on the volume or the lateral surface of the fin. In both cases, we proved in the theorems 1, 2 and 3 that the optimal design problems (8) and (9) have no solution, and we have exhibited maximizing sequences. More precisely, we showed that there is no optimal shape in the set of regular radii  $a$ , but that a nearly optimal shape is given by the function  $a_{S,M}$  defined by (11) and displayed on Figure 2: it is first a kind of accordion concentrated at  $x = 0$ , and then a flat fin. For such radii, the temperature inside the fin might not be independent of the polar variable  $r$  anymore: this hypothesis, from which we derived the model, becomes quite questionable.

It would thus be natural to investigate a more elaborated model taking into account the dependence of  $T$  with respect to  $r$ . Consider, as in Section 2.1, an axisymmetric fin  $\Omega_a$  of length  $\ell > 0$

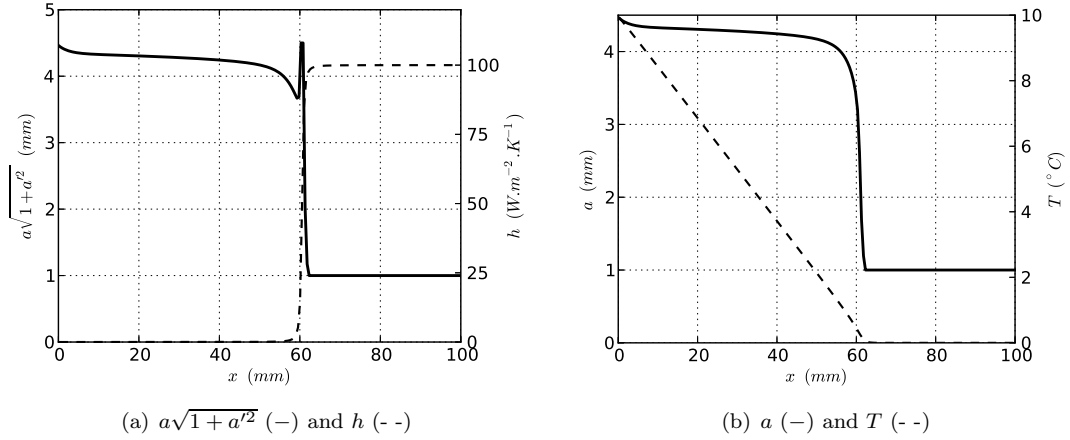


Figure 7: Optimal solution for a near step profile  $\beta = 2h/k$  at  $x = 60$  mm with  $M = 4.5$  mm,  $\ell = 100$  mm,  $T_d = 10^\circ\text{C}$ ,  $T_\infty = 0^\circ\text{C}$ ,  $h_r = h(\ell)$  and  $k = 0.1 \text{ W.m}^{-2}.\text{K}^{-1}$ .

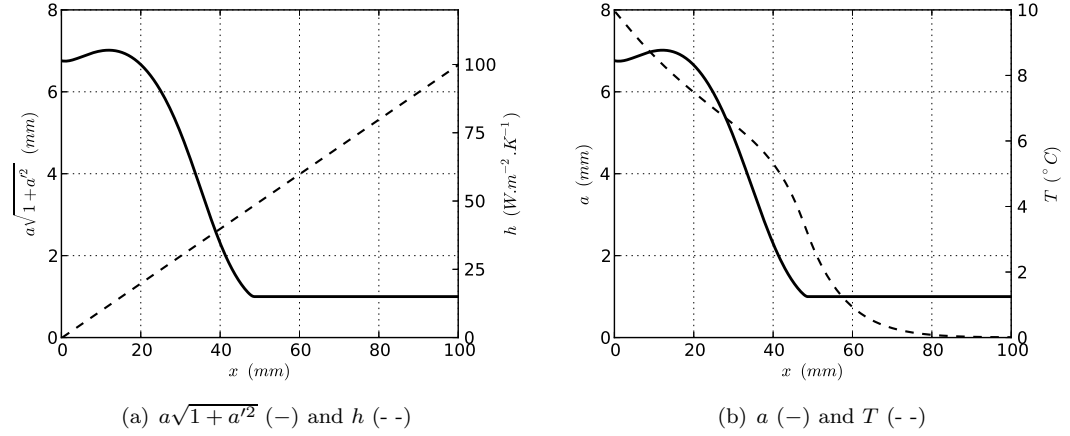


Figure 8: Optimal solution for an increasing function  $\beta = 2h/k$  with  $M = 10$  mm,  $\ell = 100$  mm,  $T_d = 10^\circ\text{C}$ ,  $T_\infty = 0^\circ\text{C}$ ,  $h_r = h(\ell)$  and  $k = 10 \text{ W.m}^{-2}.\text{K}^{-1}$ .

and radius  $a(x)$  at abscissa  $x$ , where  $a$  satisfies the hypothesis **(H<sub>1</sub>)** and **(H<sub>2</sub>)**. Introduce

$$\begin{aligned} \Sigma_a &= \{(r, \theta, x) \in \mathbb{R}_+ \times (0, 2\pi) \times (0, \ell) \mid \forall (x, y, z) \in \Omega_a, (y, z) = (r \cos \theta, r \sin \theta)\}, \\ \Gamma_{a,i} &= \{(a(\theta, x), \theta, x), (\theta, x) \in (0, 2\pi) \times (0, \ell)\}, \\ \Gamma_{a,lat} &= \{(r, \theta, 0), (r, \theta) \in (0, a(\theta, 0)) \times (0, 2\pi)\}, \\ \Gamma_{a,o} &= \{(r, \theta, \ell), (r, \theta) \in (0, a(\theta, \ell)) \times (0, 2\pi)\}, \end{aligned}$$

so that  $\partial\Omega_a = \Gamma_{a,i} \cup \Gamma_{a,lat} \cup \Gamma_{a,o}$ . A possible such model writes

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial x^2} &= 0 & (r, \theta, x) \in \Sigma_a \\ -k \left( \frac{\partial T}{\partial r} - \frac{1}{r^2} \frac{\partial a}{\partial \theta} \frac{\partial T}{\partial \theta} - \frac{\partial a}{\partial x} \frac{\partial T}{\partial x} \right) &= h \sqrt{1 + \frac{1}{r^2} \left( \frac{\partial a}{\partial \theta} \right)^2 + \left( \frac{\partial a}{\partial x} \right)^2} (T - T_\infty) & (r, \theta, x) \in \Gamma_{a,lat} \\ T(r, \theta, 0) &= T_d & (r, \theta, x) \in \Gamma_{a,i} \\ -k \frac{\partial T}{\partial x} (r, \theta, \ell) &= h_r (T(r, \theta, \ell) - T_\infty) & (r, \theta, x) \in \Gamma_{a,o} \end{aligned} \quad (26)$$

in cylindrical coordinates. As previously, we are led to maximize the heat flux at the inlet, given by

$$F(a) = -2\pi k \int_0^{a(\theta,0)} \frac{\partial T}{\partial x}(r, \theta, 0) r dr, \quad (27)$$

either over the class  $\mathcal{V}_{a_0, \ell, V_0}$  or the class  $\mathcal{S}_{a_0, \ell, S_0}$ . In particular, one of the challenging aspects of this problem lies in the fact that the technic used to construct maximizing sequences is based on the precise knowledge of the behaviour of the temperature  $T$ , combined with a kind of rearrangement argument on the function  $a\sqrt{1+a'^2}$ . For this reason, the extension to a partial differential equation model on the temperature is not immediate and will be investigated in an ongoing work.

We also mention the interesting mathematical issue of investigating the cases where Assumption (23) is not satisfied anymore. In that case, the perturbation  $b_\varepsilon$  introduced in the proof of Theorem 3 (and even its general version used in Proposition 2) does not permit to conclude to the nonexistence of solutions, and would probably need a specific study. And yet, it is not clear whether the related optimal design problem has a solution, or not.

## A Derivation of the one dimensional temperature model

Let us provide some explanations on the derivation of the one dimensional model for the temperature. The physical modeling of the heat transfer along the fin is based on the two main following assumptions:

- (i) the convective coefficient  $h$ , modeling the heat transfer between the fin surface and the fluid flow, does not depend<sup>1</sup> on the variable  $x$  and  $\theta$ . This hypothesis allows to reduce the three-dimensional problem to an axisymmetric one, which justifies that the temperature  $T$  along the fin can be considered as a function of  $r$  and  $x$  only.
- (ii) for most of its operating conditions, the fin can be viewed as thermally thin along the  $r$ -axis. As a consequence, its radial thermal resistance is low enough in comparison with the convective heat transfer  $h$  and it is relevant to claim that  $\partial T / \partial r \simeq 0$  almost everywhere in  $\Omega_a$ . This is why we will impose from now on that the temperature  $T$  is a function of the variable  $x$  only.

Using these assumptions, let us write the one dimensional model for the temperature. The energy balance equation under steady-state regime for a differential element of thickness  $dx$ , pictured in light gray on Figure 9, writes

$$\varphi(x) - \varphi(x + dx) - \varphi_s(x) = 0 \quad (28)$$

where  $\varphi(x)$  stands for the heat drained by conduction inside the fin at the position  $x \in (0, \ell)$  while  $\varphi_s$  stands for the thermal heat exchanged by convection along the lateral surface of the differential element, denoted  $dS$ . Each involved energy being computed as the product of the surface area by the heat flux crossing it, one gets according to Fourier's law for the conductive configuration, and to Newton's law for the convective heat transfer,

$$\varphi(x) = -k\pi a^2(x)T'(x) \quad \text{and} \quad \varphi_s(x) = h \left( \frac{T(x) + T(x + dx)}{2} - T_\infty \right) dS, \quad (29)$$

where  $k$  is a positive real number denoting the thermal conductivity of the fin. Since

$$dS = \pi (a(x) + a(x + dx)) \sqrt{dx^2 + (a(x + dx) - a(x))^2}, \quad (30)$$

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<sup>1</sup>Notice that a more general model where this assumption is weakened is investigated in Section 4.1.



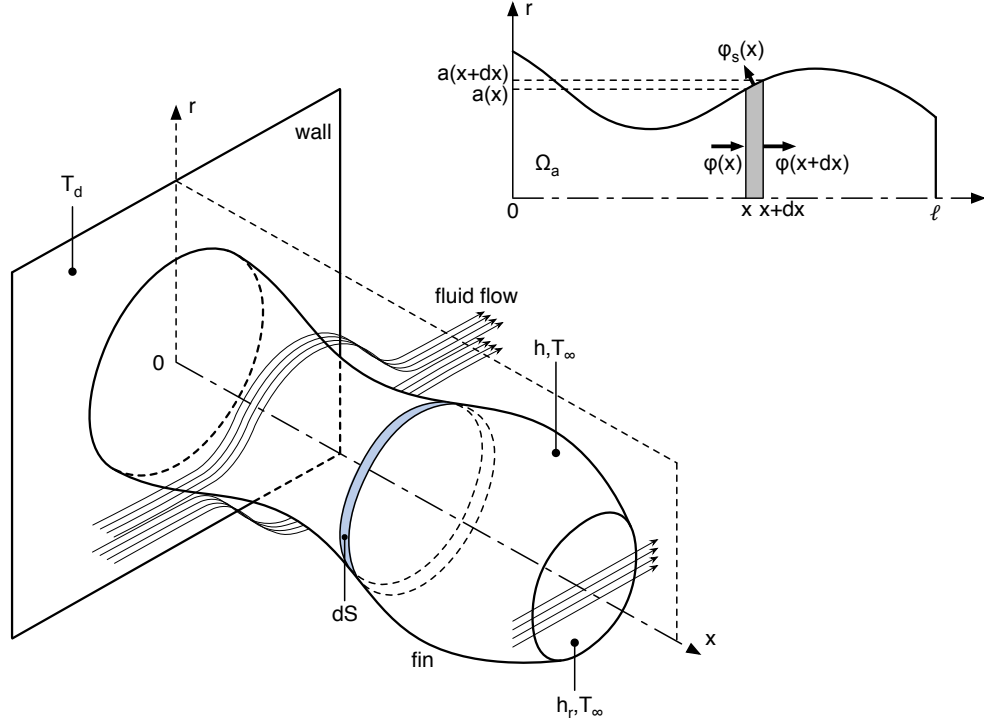


Figure 9: Scheme of the axisymmetric fin

one gets combining Equations (28), (29), and (30),

$$k \frac{1}{dx} (a^2(x+dx)T'(x+dx) - a^2(x)T'(x)) = h(a(x) + a(x+dx)) \sqrt{1 + \left(\frac{a(x+dx) - a(x)}{dx}\right)^2} \left(\frac{T(x) + T(x+dx)}{2} - T_\infty\right) \quad (31)$$

Letting  $dx$  tend to 0 leads to

$$k (a^2(x)T'(x))' = 2h a(x) \sqrt{1 + a'(x)^2} (T(x) - T_\infty). \quad (32)$$

The inlet of the fin is assumed to be at a constant temperature  $T_d$ . For the sake of simplicity, we will only consider processes where the fin aims at cooling a thermal system, *i.e.* where the heat flows from its basis towards the fluid. This assumption<sup>2</sup> comes to assume that  $0 < T_\infty < T_d$ .

The fin thermal model is completed with a Robin boundary condition at its tip ( $x = \ell$ ), modeling a convective heat transfer. It writes

$$-kT'(\ell) = h_r(T(\ell) - T_\infty)$$

<sup>2</sup>If a warming process had been considered, this inequality would have been reversed. It is worth noting that reversing the heat flowing through the fin will not affect the conclusions reached in this article.

where  $h_r$  is a nonnegative real number<sup>3</sup> representing the convective coefficient characterizing the heat transfer over the tip.

To sum-up, the temperature  $T$  along the fin is solution of the following ordinary differential equation

$$\begin{aligned}(a^2(x)T'(x))' &= \beta a(x)\sqrt{1+a'(x)^2}(T(x) - T_\infty) \quad x \in (0, \ell) \\ T(0) &= T_d \\ T'(\ell) &= -\beta_r(T(\ell) - T_\infty),\end{aligned}$$

where  $\beta = 2h/k$  is a positive real constant and  $\beta_r = h_r/k$  is a nonnegative real constant.

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<sup>3</sup>In the case where  $h_r > 0$ , there is convection occurring all over a fin. The surface all along the length as well as the surface of the tip of the fin are undergoing convection. In the case where  $h_r = 0$ , the tip is said *adiabatic* and is completely insulated. It means that there will be no heat loss from the tip.

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