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DISTRIBUTION FUNCTIONS OF THE SEQUENCE $\varphi(n)/n$, $n \in (k, k + N]$

V. BALÁŽ – P. LIARDET – O. STRAUCH

ABSTRACT. It is well known that the sequence $\varphi(n)/n$, $n = 1, 2, \dots$ has a singular asymptotic distribution function. P. Erdős in 1946 found a sufficient condition on the sequence of intervals $(k, k + N]$, such that $\varphi(n)/n$, $n \in (k, k + N]$, has the same singular function. In this note we prove a sufficient and necessary condition. For simplification of necessary condition we express the sum $\sum_{k < n \leq k+N} (\omega(n) - \log \log N)^2$, where $\omega(n)$ is the number of different primes divided n .

1. INTRODUCTION

Many papers have been devoted to the study of the distribution of the sequence

$$\frac{\varphi(n)}{n}, \quad n = 1, 2, \dots,$$

where φ denotes the Euler phi function. I. J. Schoenberg [S1, S2] established that this sequence has a continuous and strictly increasing asymptotic distribution function (basic properties of distribution functions can be found in [KN, p. 53], [DT, p. 138–157] and [SP, p. 1–7]) and P. Erdős [E1] showed that this function is singular (i.e. has vanishing derivative almost everywhere on $[0, 1]$, see [SP, p. 2–191]). Here the asymptotic distribution function $g_0(x)$ of the sequence $\varphi(n)/n$, $n = 1, 2, \dots$, is defined as

$$g_0(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_{[0,x)} \left(\frac{\varphi(n)}{n} \right), \quad \text{for every } x \in [0, 1],$$

where $c_{[0,x)}(t)$ is the characteristic function of a subinterval $[0, x)$ of $[0, 1]$. For an interval $(k, k + N]$ and $x \in [0, 1]$ define the step distribution function

$$F_{(k,k+N]}(x) = \frac{1}{N} \sum_{k < n \leq k+N} c_{[0,x)} \left(\frac{\varphi(n)}{n} \right).$$

P. Erdős [E2] proved that the limit

$$\lim_{N \rightarrow \infty} \frac{\log \log \log k}{N} = 0$$

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implies that

$$F_{(k,k+N]}(x) \rightarrow g_0(x), \quad (x \in [0, 1]) \quad (1)$$

as $N \rightarrow \infty$. In opposite case he found N and k such that $\lim_{N \rightarrow \infty} \frac{\log \log \log k}{N} = \frac{1}{2}$ and $F_{(k,k+N]}(x) \not\rightarrow g_0(x)$. In this note we give necessary and suffice condition for (1).

2. NECESSARY AND SUFFICIENT CONDITION

Theorem 1. *For any two sequences N and k of positive integers we have (1) if and only if for every $s = 1, 2, \dots$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n} \Phi(d) = 0, \quad (2)$$

where

$$\Phi(d) = \prod_{p|d} \left(\left(1 - \frac{1}{p}\right)^s - 1 \right) \quad (3)$$

for square-free d , $\Phi(d) = 0$ otherwise, and p are primes.

Proof. Applying Weyl's limit relation (see [SP, p. 1–12, Th. 1.8.1.1]) we see that (1) holds if and only if, for every $s = 1, 2, \dots$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n} \right)^s = \int_0^1 x^s dg_0(x), \quad (4)$$

where [P, p. 363]

$$\int_0^1 x^s dg_0(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(\frac{\varphi(n)}{n} \right)^s = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p}\right)^s \right).$$

We express $\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n} \right)^s$ by means of

$$\sum_{d|n} \Phi(d) = \left(\frac{\varphi(n)}{n} \right)^s.$$

We have

$$\begin{aligned} \sum_{k < n \leq k+N} \sum_{d|n} \Phi(d) &= \sum_{d=1}^{k+N} \Phi(d) \left(\left[\frac{k+N}{d} \right] - \left[\frac{k}{d} \right] \right) \\ &= \sum_{d=1}^{k+N} N \frac{\Phi(d)}{d} + \sum_{d=1}^{k+N} \Phi(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right), \end{aligned}$$

where $[x]$ is integer part and $\{x\}$ fractional part of x . Since

$$\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} = \begin{cases} -\left\{ \frac{N}{d} \right\} & \text{if } \left\{ \frac{k}{d} \right\} + \left\{ \frac{N}{d} \right\} < 1 \\ 1 - \left\{ \frac{N}{d} \right\} & \text{others} \end{cases} \quad (5)$$

the boundary $k + N$ of summations can be reduced to N and following equality holds

$$\begin{aligned} \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n} \right)^s &= \sum_{d=1}^N \frac{\Phi(d)}{d} + \frac{1}{N} \sum_{d=1}^N \Phi(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right) \\ &+ \frac{1}{N} \sum_{\substack{N < d \leq k+N \\ \left\{ \frac{k}{d} \right\} + \frac{N}{d} \geq 1}} \Phi(d). \end{aligned} \quad (6)$$

We begin by proving

$$\sum_{\substack{N < d \leq k+N \\ \left\{ \frac{k}{d} \right\} + \frac{N}{d} \geq 1}} \Phi(d) = \sum_{j=1}^N \sum_{\substack{d|k+j \\ d > N}} \Phi(d) \quad (7)$$

for every $k, N = 0, 1, 2, \dots$. Proof: Expressing $k = m_d d + r_d$, where $0 \leq r_d < d$, $m_d \geq 0$ are integers, we see that

$$\left\{ \frac{k}{d} \right\} + \frac{N}{d} \geq 1 \iff r_d + N \geq d \iff r_d = d - N + i_d$$

where $0 \leq i_d < N$. This gives $k = (m_d + 1)d - N + i_d$, thus $d(m_d + 1) = k + N - i_d$ and, for $d > N$, we have $\left\{ \frac{k}{d} \right\} + \frac{N}{d} \geq 1$ if and only if

$$d \left(\left[\frac{k}{d} \right] + 1 \right) = k + j \quad (8)$$

for some $j = 1, 2, \dots, N$. In the following we see that for every $d|k + j$ we have that (8) $\iff j \leq d$. Really, put $d = k + j - x$ and express $k + j = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, $k + j - x = p_1^{\beta_1} \dots p_n^{\beta_n}$, where p_i are primes. Now d satisfies (8) if and only if

$$\begin{aligned} (k + j - x) \left(\left[\frac{k}{k + j - x} \right] + 1 \right) &= k + j \iff \left[\frac{k}{k + j - x} \right] = \frac{x}{k + j - x} \\ \iff \left[p_1^{\alpha_1 - \beta_1} \dots p_n^{\alpha_n - \beta_n} - \frac{j}{p_1^{\beta_1} \dots p_n^{\beta_n}} \right] &= p_1^{\alpha_1 - \beta_1} \dots p_n^{\alpha_n - \beta_n} - 1 \\ \iff j &\leq p_1^{\beta_1} \dots p_n^{\beta_n} \end{aligned}$$

Since $j \leq N$ and $d > N$, we have $j < d$ and we conclude that (8) holds and applying it in (6) we get the following basic equality

$$\begin{aligned} \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n} \right)^s &= \sum_{d=1}^N \frac{\Phi(d)}{d} + \frac{1}{N} \sum_{d=1}^N \Phi(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right) \\ &+ \frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n} \Phi(d). \end{aligned} \quad (9)$$

Because (see A. G. Postnikov [P, p. 361–363])

$$\begin{aligned}
|\Phi(d)| &\leq \frac{s^{\omega(d)}}{d}, \text{ where } \omega(d) \text{ is the number of different primes which divide } d, \\
\sum_{d=1}^N |\Phi(d)| &= O((1 + \log N)^s), \\
\sum_{d=N+1}^{\infty} \frac{|\Phi(d)|}{d} &\leq \frac{3^s(1 + \log N)^s}{N}, \\
\sum_{d=1}^{\infty} \frac{\Phi(d)}{d} &= \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p}\right)^s\right), \tag{10}
\end{aligned}$$

(9) shows that (1) holds if and only if

$$\frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n} \Phi(d) \rightarrow 0$$

as $N \rightarrow \infty$ and the proof of Theorem 1 is complete. \square

Notes 1. Using the basic equation (9) and

$$\frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n} \Phi(d) + \frac{1}{N} \sum_{k < n \leq k+N} \sum_{d|n, d \leq N} \Phi(d) = \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n}\right)^s$$

we obtain

$$\frac{1}{N} \sum_{k < n \leq k+N} \sum_{d|n, d \leq N} \Phi(d) = \sum_{d=1}^N \frac{\Phi(d)}{d} + O\left(\frac{(1 + \log N)^s}{N}\right)$$

and thus, as $N \rightarrow \infty$, the left hand side converges to $\prod_p \left(1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p}\right)^s\right)$ uniformly with respect to k .

3. ERDŐS' APPROACH

For any positive integer n and real $t \geq 2$, denote

$$n(t) = \prod_{\substack{p|n \\ p \leq t}} p, \quad n'(t) = \prod_{\substack{p|n \\ p > t}} p, \quad \text{and } P(t) = \prod_{p \leq t} p, \tag{11}$$

where p are primes and the empty product is 1. P. Erdős in [E2] proved (without an explicit error term and for $s = 1$) the following lemma:

Lemma 1. *For every integer k, N and $t = N$ we have*

$$\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)} \right)^s = \frac{1}{N} \sum_{n=1}^N \left(\frac{\varphi(n)}{n} \right)^s + O\left(\frac{3^s(1 + \log N)^s}{N} \right) \quad (12)$$

for $s = 1, 2, \dots$

Proof. We have

$$\begin{aligned} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)} \right)^s &= \sum_{k < n \leq k+N} \sum_{d|n(t)} \Phi(d) = \sum_{d|P(t)} \Phi(d) \left(\left[\frac{k+N}{d} \right] - \left[\frac{k}{d} \right] \right) \\ &= N \sum_{d|P(t)} \frac{\Phi(d)}{d} + \sum_{d|P(t)} \Phi(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right). \end{aligned}$$

Bearing in mind

$$\begin{aligned} \sum_{d|P(t)} \frac{\Phi(d)}{d} &= \prod_{p \leq t} \left(1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p} \right)^s \right), \\ \left| \sum_{d|P(t)} \Phi(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right) \right| &\leq \sum_{d|P(t)} \frac{s^{\omega(d)}}{d} = \prod_{p \leq t} \left(1 + \frac{s}{p} \right) \\ &= A(s)(\log t)^s \left(1 + O\left(\frac{1}{\log t} \right) \right) \quad (\text{see [N, p. 110]}), \\ \frac{1}{N} \sum_{n=1}^N \left(\frac{\varphi(n)}{n} \right)^s &= \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p} \right)^s \right) + O\left(\frac{3^s(1 + \log N)^s}{N} \right) \\ & \quad (\text{see [P, p. 363]}), \\ \left| 1 - \prod_{p > N} \left(1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p} \right)^s \right) \right| &\leq \sum_{n > N} \frac{|\Phi(n)|}{n} \leq \frac{3^s(1 + \log N)^s}{N} \quad (13) \end{aligned}$$

we obtain (12). \square

Next in his method Erdős used implicitly the following theorem:

Theorem 2. *For every two sequences k and N and $t = N$ we have*

$$\left(\prod_{k < n \leq k+N} \frac{\varphi(n'(t))}{n'(t)} \right)^{\frac{1}{N}} \rightarrow 1 \implies F_{(k, k+N]}(x) \rightarrow g_0(x) \quad (14)$$

for all $x \in [0, 1]$.

Proof. Let x_n , $n = 1, 2, \dots$, be a sequence in the interval $(0, 1)$ and define the step distribution function $F_N(x) = \frac{\#\{n \leq N; x_n \in [0, x]\}}{N}$. By Riemann-Stieltjes integration for

every continuous function $f(x)$ on $[0, 1]$ we have $\frac{1}{N} \sum_{n=1}^N f(x_n) = \int_0^1 f(x) dF_N(x)$. By Helly theorem, if $F_N(x) \rightarrow g(x)$, then $\int_0^1 f(x) dF_N(x) \rightarrow \int_0^1 f(x) dg(x)$.

Now, assume that $\frac{1}{N} \sum_{n=1}^N x_n \rightarrow 1$. If $F_N(x) \rightarrow g(x)$, then $\int_0^1 x dg(x) = 1$, which is equivalent to $g(x) = c_0(x)$ (it has step 1 in $x = 1$). But $F_N(x) \rightarrow c_0(x)$ is equivalent to statistical convergence $x_n \rightarrow 1$, i.e. for every $\varepsilon > 0$ and $A_\varepsilon = \{n \leq N; x_n \geq 1 - \varepsilon\}$ we have $\frac{\#A_\varepsilon}{N} \rightarrow 0$. From statistical convergence $x_n \rightarrow 1$ follows statistical convergence $f(x_n) \rightarrow f(1)$ for every continuous $f(x)$. Furthermore,

$$\frac{1}{N} \sum_{n=1}^N f(x_n) = \int_0^1 f(x) dF_N(x) \rightarrow \int_0^1 f(x) dc_0(x) = f(1).$$

Thus the limit $\frac{1}{N} \sum_{n=1}^N x_n \rightarrow 1$ is equivalent any of the following limits

(i) $\frac{1}{N} \sum_{n=1}^N \log(x_n) \rightarrow 0$,

(ii) $\left(\prod_{n=1}^N x_n\right)^{1/N} \rightarrow 1$,

(iii) $\frac{1}{N} \sum_{n=1}^N x_n^s \rightarrow 1$.

The limits (i)–(iii) also hold assuming restriction $n \in (k, k + N]$.

Put $x_n = \frac{\varphi(n'(t))}{n'(t)}$. Then

$$\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n'(t))}{n'(t)}\right) \rightarrow 1 \iff \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s \rightarrow 1$$

and for $A_\varepsilon = \left\{n \in (k, k + N]; \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s < 1 - \varepsilon\right\}$ we have $\frac{\#A_\varepsilon}{N} \rightarrow 0$ for every $\varepsilon > 0$.

Replace in $\frac{\varphi(n)}{n} = \frac{\varphi(n(t))}{n(t)} \frac{\varphi(n'(t))}{n'(t)}$ the $\frac{\varphi(n'(t))}{n'(t)}$ by $1 - \varepsilon$ we see

$$\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n}\right)^s \geq \left(\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)}\right)^s\right) (1 - \varepsilon) - (1 - \varepsilon) \frac{\#A_\varepsilon}{N} \quad (15)$$

and the other hand $\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n}\right)^s \leq \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)}\right)^s$ and Lemma 1 gives

$$(1 - \varepsilon) \int_0^1 x^s dg_0(x) \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n}\right)^s \leq \int_0^1 x^s dg_0(x).$$

Thus we have

$$\frac{1}{N} \sum_{k < n \leq k+N} \frac{\varphi(n'(t))}{n'(t)} \rightarrow 1 \implies F_{(k, k+N]}(x) \rightarrow g_0(x) \quad (16)$$

and then we use (ii). \square

Note 2. In (14) we have only implication, since the right-hand side of (15) has the following precise form

$$(1 - \varepsilon) \left(\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)} \right)^s \right) - (1 - \varepsilon) \left(\frac{1}{N} \sum_{k < n \leq k+N, n \in A_\varepsilon} \left(\frac{\varphi(n(t))}{n(t)} \right)^s \right)$$

and it can be $\left(\frac{1}{N} \sum_{k < n \leq k+N, n \in A_\varepsilon} \left(\frac{\varphi(n(t))}{n(t)} \right)^s \right) \rightarrow 0$ and $\frac{\#A_\varepsilon}{N} \rightarrow \delta > 0$.

Finally Erdős prove

Theorem 3. *For every sequence of intervals $(k, k + N]$ we have*

$$\frac{\log \log \log k}{N} \rightarrow 0 \implies F_{(k, k+N]}(x) \rightarrow g_0(x). \quad (17)$$

Proof. For $t = N$, the $n'(t)$, $k < n \leq k + N$ are pairwise relatively prime, because the interval $(k, k + N]$ cannot contain two different positive integers divisible by the same prime $p > N$. Denote $M(t) = \prod_{k < n \leq k+N} n'(t)$ and $x = \omega(M(t))$. Using the expression

$$\prod_{p \leq x} \left(1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x} \right) \right) \quad (18)$$

see [MSC, p. 259, VII. 29]) we have

$$\frac{\varphi(M(t))}{M(t)} \geq \prod_{N < p \leq x} \left(1 - \frac{1}{p} \right) \geq c_1 \frac{\log N}{\log x}$$

and (a) $\left(\frac{\log N}{\log x} \right)^{1/N} \rightarrow 1$ implies (b) $\left(\frac{\varphi(M(t))}{M(t)} \right)^{1/N} \rightarrow 1$. Since

$$e^{c_2 x} \leq \prod_{N < p \leq x} p \leq (k+1)(k+2) \dots (k+N) < (k+N)^N$$

we have $c_2 x < N \log(k+N)$ and (c) $\left(\frac{\log N}{\log(N \log(k+N))} \right)^{1/N} \rightarrow 1$ implies (a) and (b) and finally $F_{(k, k+N]}(x) \rightarrow g_0(x)$. But

$$(c) \iff \frac{1}{N} \left(\log \frac{\log N}{\log(N \log(k+N))} \right) \rightarrow 0 \iff \frac{\log \log \log k}{N} \rightarrow 0.$$

□

Note 3. Assume that $P(t)|k$, where $P(t) = \prod_{p \leq t} p$ and $t = N$. Analogically as in (11), for a divisor $d|n$ denote $d(t) = \prod_{p|d, p \leq t} p$ and $d'(t) = \prod_{p|d, p > t} p$. Since $d(t)|n = k+j$, $d(t)|k$ then $d(t) \leq N$ and if we assume $d > N$, then must be $d'(t) > 1$. Hence

$$\sum_{N < d|n} \Phi(d) = \sum_{d(t)|n(t)} \Phi(d(t)) \sum_{\substack{d'(t)|n'(t) \\ d'(t) > 1}} \Phi(d'(t)) = \left(\frac{\varphi(n(t))}{n(t)} \right)^s \left(\left(\frac{\varphi(n'(t))}{n'(t)} \right)^s - 1 \right)$$

which gives

$$\left| \sum_{N < d|n} \Phi(d) \right| \leq 1 - \left(\frac{\varphi(n'(t))}{n'(t)} \right)^s.$$

Applying Theorem 1 we see that $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n'(t))}{n'(t)} \right)^s = 1$ implies (1). Since $\frac{\varphi(M(t))}{M(t)} \leq \frac{\varphi(n(t))}{n(t)}$ also $\frac{\varphi(M(t))}{M(t)} \rightarrow 1$ implies (1). These results directly follows from Erdős Theorem 2 and moreover for general k .

Using Erdős' Lemma 1 we prove the following quantitative form of Theorem 1.

Theorem 4. *For any two sequences N and k of positive integers and for every $s = 1, 2, \dots$, we have*

$$\begin{aligned} \frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n} \Phi(d) &= \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n)}{n} \right)^s - \frac{1}{N} \sum_{n=1}^N \left(\frac{\varphi(n)}{n} \right)^s \\ &\quad + O\left(\frac{3^s (1 + \log N)^s}{N} \right) \end{aligned} \quad (19)$$

Proof. Replace n by $n(t)$ in (9). Then

$$\begin{aligned} \frac{1}{N} \sum_{k < n \leq k+N} \sum_{d|n(t)} \Phi(d) &= \frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)} \right)^s = \\ &= \sum_{d=1}^N \frac{\Phi(d)}{d} + \frac{1}{N} \sum_{d=1}^N \Phi(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right) \\ &\quad + \frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n(t)} \Phi(d). \end{aligned}$$

Erdős' Lemma 1 implies that for every $k, N, N \rightarrow \infty$

$$\frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d|n(t)} \Phi(d) \rightarrow 0. \quad (20)$$

Since

$$\sum_{N < d|n} \Phi(d) = \sum_{N < d|n(t)} \Phi(d) + \sum_{\substack{d|n(t)n'(t) \\ (d, n'(t)) > 1}} \Phi(d)$$

and the second sum is equal to $\left(\frac{\varphi(n(t))}{n(t)} \right)^s \left(\left(\frac{\varphi(n'(t))}{n'(t)} \right)^s - 1 \right)$ we see (19). \square

4. EXAMPLES

For the optimality of $\log \log \log k$ in Theorem 3, Erdős gave the following example.

Example 1. Divide $P(t) = \prod_{p \leq t} p$ into N numbers A_1, A_2, \dots, A_N such that

- (i) A_i , $i = 1, 2, \dots$, are relatively prime,
- (ii) $\frac{\varphi(A_i)}{A_i} < \frac{1}{2}$ for $i = 1, 2, \dots$,
- (iii) if p is the maximal prime in A_i , then for $A'_i = A_i/p$ we have $\frac{\varphi(A'_i)}{A'_i} > \frac{1}{2}$.

The part (iii) implies $\frac{\varphi(A_i)}{A_i} > \frac{1}{4}$ and thus

$$\left(\frac{1}{4}\right)^N < \prod_{p \leq t} \left(1 - \frac{1}{p}\right) = \frac{\varphi(A_1)}{A_1} \dots \frac{\varphi(A_N)}{A_N} < \left(\frac{1}{2}\right)^N.$$

From it, applying (18), we find $N < c_1 \log \log t$. By Chinese theorem there exists $k_0 < A_1 \dots A_N$ such that $k_0 \equiv -i \pmod{A_i}$ for $i = 1, 2, \dots, N$. Put $k = k_0 + A_1 \dots A_N$, then we have

$$e^{c_2 t} < P(t) = A_1 \dots A_N < k$$

which implies $t < c_3 \log k$ and $\log \log t < c_4 \log \log \log k$. Thus

$$\frac{\log \log \log k}{N} > \frac{1}{c_1 c_4} \frac{\log \log t}{\log \log t}.$$

Furthermore for these k and N we have $F_{(k, k+N]}(x) \not\rightarrow g_0(x)$ since by (ii)

$$\frac{1}{N} \sum_{k < n \leq k+N} \frac{\varphi(n)}{n} < \frac{1}{2} < \frac{1}{N} \sum_{n=1}^N \frac{\varphi(n)}{n} = \frac{6}{\pi^2} + O\left(\frac{\log N}{N}\right).$$

In the following Example 2 we find integer sequences k, N , for which (1) holds and $\lim_{N \rightarrow \infty} \frac{\log \log \log k}{N} = \infty$.

Example 2. Let $x = x(N)$ be increases arbitrary quickly as $N \rightarrow \infty$, e.g. $x(N) = e^{e^{e^N}}$. The left boundary point $k = k(N)$ of the interval $(k, k + N]$ we put as $k = \prod_{p \leq x} p$, where p are primes. Let $M^* = \prod_{x < p \leq x+y(x)} p$ having the same number of prime divisors as $M(t)$ ($t = N$). Clearly, to prove $\frac{\varphi(M(t))}{M(t)} \rightarrow 1$ it suffices to show

$$\frac{\varphi(M^*)}{M^*} = \prod_{x < p \leq x+y(x)} \left(1 - \frac{1}{p}\right) \rightarrow 1 \quad (21)$$

as $x \rightarrow \infty$. Using the expression (18) we have that (21) holds if

$$\frac{\log \left(1 + \frac{y(x)}{x}\right)}{\log x} \rightarrow 0 \quad (22)$$

as $x \rightarrow \infty$. The inequality

$$M^* \leq M = \prod_{k < n \leq k+N} n'(t) \leq (k + N)^N \leq (2k)^N$$

leads to $\sum_{x < p \leq x+y(x)} \log p \leq 2N \sum_{p \leq x} \log p$ and thus

$$\sum_{p \leq x+y(x)} \log p \leq (2N+1) \sum_{p \leq x} \log p. \quad (23)$$

Applying the well known inequality (see [L, p. 83])

$$\log 2 \leq \liminf_{x \rightarrow \infty} \frac{\sum_{p \leq x} \log p}{x} \leq \limsup_{x \rightarrow \infty} \frac{\sum_{p \leq x} \log p}{x} \leq 2 \log 2$$

into (23), then we have, for $x \geq x_0(\varepsilon)$, that $y(x)$ satisfies

$$(\log 2 - \varepsilon)(x + y(x)) \leq (2N + 1)(2 \log 2 + \varepsilon)x$$

which implies $\frac{y(x)}{x} \leq cN$, where c is a constant. Thus (21) and consequently (1) holds, if $x = x(N) \geq e^N$. Since $k(N) = \prod_{p \leq x(N)} p \geq e^{c_1 x(N)}$, for $x(N) = e^{e^N}$ we have

$$\lim_{N \rightarrow \infty} \frac{\log \log \log k}{N} = \infty,$$

what we asked for.

4. NORMAL ORDER OF $\omega(n)$, $n \in (k, k + N]$

To simplify (2) we study the normal ordering of $\omega(n)$ -the number of distinct prime divisors of n - in the interval $(k, k + N]$ (compare with V.A. Plaksin, see [MSC, p. 156]).

Theorem 4. *For every positive integers k and N we have*

$$\sum_{k < n \leq k+N} (\omega(n) - \log \log N)^2 = O(N \log \log N) + \sum_{k < n \leq k+N} \sum_{\substack{N < p, q | k+n \\ p \neq q}} 1.$$

A proof follows from the following lemmas.

Lemma 2. *For every positive integers k and N we have*

$$\sum_{k < n \leq k+N} \omega(n) = N \log \log N + B.N + O\left(\frac{N}{\log N}\right) + \sum_{k < n \leq k+N} \sum_{N < p | n} 1,$$

where the constant $B = \lim_{N \rightarrow \infty} \left(\left(\sum_{p \leq N} \frac{1}{p} \right) - \log \log N \right)$.

Lemma 3. For every positive integers k and N we have

$$\begin{aligned} \sum_{k < n \leq k+N} (\omega(n))^2 &= N(\log \log N)^2 + O(N \log \log N) + \sum_{k < n \leq k+N} \sum_{\substack{N < p, q | n \\ p \neq q}}^N 1 \\ &\quad + \sum_{k < n \leq k+N} \sum_{N < p | n} 1. \end{aligned}$$

Proof of Lemma 1.

$$\sum_{k < n \leq k+N} \omega(n) = \sum_{p \leq k+N} \left(\left[\frac{k+N}{p} \right] - \left[\frac{k}{p} \right] \right) = \sum_{p \leq k+N} \frac{N}{p} + \sum_{p \leq k+N} \left\{ \frac{k}{p} \right\} - \left\{ \frac{k+N}{p} \right\}.$$

The final sum we divide into two parts; $\sum_{p \leq N}$ and $\sum_{N < p \leq k+N}$. The first sum is $O(\pi(N))$ and for the second we use (5) and the summation method in (7) which gives

$$\sum_{N < p \leq k+N} \left\{ \frac{k}{p} \right\} - \left\{ \frac{k+N}{p} \right\} = \sum_{N < p \leq k+N} -\frac{N}{p} + \sum_{k < n \leq k+N} \sum_{N < p | n} 1.$$

Thus

$$\sum_{k < n \leq k+N} \omega(n) = \sum_{p \leq N} \frac{N}{p} + O(\pi(N)) + \sum_{k < n \leq k+N} \sum_{N < p | n} 1. \quad (24)$$

Bearing in mind

$$\sum_{p \leq N} \frac{1}{p} = \log \log N + B + O\left(\frac{1}{\log N}\right)$$

the proof is finished. \square

Notes 2. Similarly to Notes 1 we have

$$\sum_{j=1}^N \sum_{\substack{p | k+j \\ p \leq N}} 1 = \sum_{p \leq N} \frac{N}{p} + O(\pi(N))$$

uniformly on k .

Proof of Lemma 2.

$$\begin{aligned} \sum_{k < n \leq k+N} \omega^2(n) &= \sum_{\substack{\text{all pairs of primes } (p, q) \\ p, q \leq k+N}} \sum_{\substack{k < n \leq k+N \\ p | n \text{ and } q | n}} 1 \\ &= \sum_{\substack{p, q \leq k+N \\ p \neq q}} \left[\frac{k+N}{p \cdot q} \right] - \left[\frac{k}{p \cdot q} \right] + \sum_{p \leq k+N} \left[\frac{k+N}{p} \right] - \left[\frac{k}{p} \right] \\ &= \sum_{\substack{p, q \leq k+N \\ p \neq q}} \frac{N}{p \cdot q} + \sum_{\substack{p, q \leq k+N \\ p \neq q}} \left\{ \frac{k}{p \cdot q} \right\} - \left\{ \frac{k+N}{p \cdot q} \right\} \\ &\quad + \sum_{p \leq k+N} \frac{N}{p} + \sum_{p \leq k+N} \left\{ \frac{k}{p} \right\} - \left\{ \frac{k+N}{p} \right\} \end{aligned}$$

Since (using (5) and (7))

$$\begin{aligned} \sum_{\substack{p \cdot q \leq N \\ p \neq q}} \left\{ \frac{k}{p \cdot q} \right\} - \left\{ \frac{k+N}{p \cdot q} \right\} &= O(N \log \log N), \\ \sum_{\substack{N < p \cdot q \leq k+N \\ p \neq q}} \left\{ \frac{k}{p \cdot q} \right\} - \left\{ \frac{k+N}{p \cdot q} \right\} &= \sum_{\substack{N < p \cdot q \leq k+N \\ p \neq q}} -\frac{N}{p \cdot q} + \sum_{j=1}^N \sum_{\substack{N < p \cdot q | k+j \\ p \neq q}} 1, \\ \sum_{p \leq N} \left\{ \frac{k}{p} \right\} - \left\{ \frac{k+N}{p} \right\} &= O\left(\frac{N}{\log N}\right), \\ \sum_{N < p \leq k+N} \left\{ \frac{k}{p} \right\} - \left\{ \frac{k+N}{p} \right\} &= \sum_{N < p \leq k+N} -\frac{N}{p} + \sum_{j=1}^N \sum_{N < p | k+j} 1, \end{aligned}$$

and moreover

$$\sum_{\substack{p \cdot q \leq N \\ p \neq q}} \frac{1}{p \cdot q} = (\log \log N)^2 + O(\log \log N),$$

thus the proof of Lemma 2 is finished. \square

Finally, the proof of Theorem 4 follows directly from the expression

$$\begin{aligned} \sum_{k < n \leq k+N} (\omega(n) - \log \log N)^2 &= \sum_{k < n \leq k+N} \omega^2(n) - 2 \log \log N \sum_{k < n \leq k+N} \omega(n) \\ &\quad + N(\log \log N)^2, \end{aligned}$$

from the Lemma 2 and 3 and from

$$\sum_{k < n \leq k+N} \sum_{N < p | n} 1 - 2 \log \log N \sum_{k < n \leq k+N} \sum_{N < p | n} 1 \leq 0.$$

For every integers k, N , the sum $\frac{1}{N} \sum_{k < n \leq k+N} \sum_{N < d | n} \Phi(d)$ can be approximate by using the following steps.

a) Let $\omega(n) \leq c \cdot \log \log N$ for fixed $c > 0$. Since (see [P, p. 361])

$$\left| \sum_{N < d | n} \Phi(d) \right| \leq 2^{\omega(n)} \frac{s^{\omega(n)}}{N}$$

and $2^{\omega(n)} \leq (e^{\log \log N})^{c \log 2} = (\log N)^{c_1}$, $s^{\omega(n)} \leq (\log N)^{c_2}$, thus

$$\left| \sum_{N < d | n} \Phi(d) \right| \leq \frac{(\log N)^{c_3}}{N}.$$

b) Let $\omega(k + j) > c \cdot \log \log N$. Then $|\omega(k + j) - \log \log N| > (c - 1) \log \log N$. Denote by (as in a classical proof of normal order of $\omega(n)$)

$$R_\varepsilon((k, k + N]) = \#\{n \in (k, k + N]; |\omega(n) - \log \log N| > \varepsilon \log \log N\}$$

then

$$R_\varepsilon((k, k + N]) \varepsilon^2 (\log \log N)^2 \leq \sum_{k < n \leq k + N} (\omega(n) - \log \log N)^2.$$

Applying Theorem 4 we obtain

$$R_\varepsilon((k, k + N]) \leq \frac{1}{\varepsilon^2} \left(O\left(\frac{N}{\log \log N}\right) + \frac{1}{(\log \log N)^2} \sum_{k < n \leq k + N} \sum_{\substack{N < p, q | n \\ p \neq q}} 1 \right).$$

Thus we need to estimate the following

$$\frac{R_{c-1}((k, k + N])}{N} \cdot \max_{k < n \leq k + N} \left| \sum_{\substack{N < d | n \\ \omega(n) \geq c \log \log N}} \Phi(d) \right|.$$

5. CONCLUDING REMARKS

1. Lemma 1 implies that every d.f. $g(x)$, $F_{(k, k + N]} \rightarrow g(x)$ a.e. on $[0, 1]$ must satisfies

$$\int_0^1 x^s dg(x) \leq \int_0^1 x^s dg_0(x),$$

for every $s = 1, 2, \dots$

2. Replaced $1/2$ by $1/N$ in (ii) in Example 1, then by Chinese theorem we can find k, N such that $F_{(k, k + N]}(x) \rightarrow c_1(x)$ where d.f. $c_1(x)$ has a step 1 in $x = 1$.

3. A. Schinzel and Y. Wang [SW] proved that for every fixed N the $N - 1$ -dimensional sequence

$$\left(\frac{\varphi(k + 2)}{\varphi(k + 1)}, \frac{\varphi(k + 3)}{\varphi(k + 2)}, \dots, \frac{\varphi(k + N)}{\varphi(k + N - 1)} \right), \quad k = 1, 2, \dots, N$$

is dense in $[0, \infty)^{N-1}$. Thus, for any given $(\alpha_1, \alpha_2, \dots, \alpha_{N-1}) \in [0, \infty)^{N-1}$ we can select a sequence of k such that

$$\left(\frac{\varphi(k + 2)}{\varphi(k + 1)}, \frac{\varphi(k + 3)}{\varphi(k + 2)}, \dots, \frac{\varphi(k + N)}{\varphi(k + N - 1)} \right) \rightarrow (\alpha_1, \alpha_2, \dots, \alpha_{N-1}).$$

Select a subsequence of k such that $\frac{\varphi(k+1)}{k+1} \rightarrow \alpha$. Then

$$\left(\frac{\varphi(k + 1)}{k + 1}, \frac{\varphi(k + 2)}{k + 2}, \dots, \frac{\varphi(k + N)}{k + N} \right) \rightarrow (\alpha, \alpha \alpha_1, \alpha \alpha_1 \alpha_2, \dots, \alpha \alpha_1 \alpha_2 \dots \alpha_{N-1}).$$

Summary, if α_n , $n = 1, 2, \dots$ is an infinite sequence in $[0, \infty)$ then there exists a sequence $k = k(N)$ and $\alpha \in [0, 1]$ such that $F_{(k, k+N]}(x) \rightarrow g(x)$ and $g(x)$ is the asymptotic distribution function of the sequence $\alpha_1 \dots \alpha_n$, $n = 1, 2, \dots$. The constant α is not arbitrary, since

$$\lim_{N \rightarrow \infty} \alpha \frac{1}{N} \sum_{n=0}^{N-1} \alpha_1 \dots \alpha_n \leq \frac{6}{\pi^2}.$$

4. For completeness we refer to some known properties of the d.f. $g_0(x)$. P. Erdős [E3] estimates modulus of continuity of $g_0(x)$. The explicit construction of $g_0(x)$ is given in H. Davenport [D].

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