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# Operator-valued spectral measures and large deviations

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## Abstract

Let  $\mathfrak{H}$  be a Hilbert space,  $U$  an unitary operator on  $\mathfrak{H}$  and  $\mathfrak{K}$  a cyclic subspace for  $U$ . The spectral measure of the pair  $(U, \mathfrak{K})$  is an operator-valued measure  $\mu_{\mathfrak{K}}$  on the unit circle  $\mathbb{T}$  such that

$$\int_{\mathbb{T}} z^k d\mu_{\mathfrak{K}}(z) = \left( P_{\mathfrak{K}} U^k \right)_{\upharpoonright_{\mathfrak{K}}}, \quad \forall k \geq 0$$

where  $P_{\mathfrak{K}}$  and  $\upharpoonright_{\mathfrak{K}}$  are the projection and restriction on  $\mathfrak{K}$ , respectively. When  $\mathfrak{K}$  is one dimensional,  $\mu$  is a scalar probability measure. In this case, if  $U$  is picked at random from the unitary group  $\mathbb{U}(N)$  under the Haar measure, then any fixed  $\mathfrak{K}$  is almost surely cyclic for  $U$ . Let  $\mu^{(N)}$  be the random spectral (scalar) measure of  $(U, \mathfrak{K})$ . The sequence  $(\mu^{(N)})$  was studied extensively, in the regime of large  $N$ . It converges to the Haar measure  $\lambda$  on  $\mathbb{T}$  and satisfies the Large Deviation Principle at scale  $N$  with a good rate function which is the reverse Kullback information with respect to  $\lambda$  ([20]). The purpose of the present paper is to give an extension of this result for general  $\mathfrak{K}$  (of fixed finite dimension  $p$ ) and eventually to offer a projective statement (all  $p$  simultaneously), at the level of operator-valued spectral measures in infinite dimensional spaces.

# 1 Introduction

## 1.1 The influence of the mathematical work of Studden on our research

A significant part of the mathematical contribution of W.J. Studden relies on moment problems or more generally on generalized moment problems for  $T$ -systems. The first author of the present paper first met the  $T$ -systems during his Ph.D preparation by the fascinating reading of two books on moments problem. The first one is the book of Krein and Nudel'man [26] dealing mainly with the Markov moment problem. The second one is the book of Karlin and Studden [24] that offers a beautiful journey inside the continent of  $T$ -systems properties. The reading of these two books has whetted our interest for the literature on moment problems and by the middle of the nineties we came across a very interesting paper of Chang, Kempermann and Studden [3] on the asymptotic behaviour of randomized moment sequences. This seminal paper gives a very nice Borel Poincaré like theorem for moment sequences of probability measures on the unit interval and has been quite motivating for at least the ten last years of our researches. The probabilized moment space frame developed therein led to many papers written by many authors (see for example [22], [17], [28],[10], [11], [19] ). One of the main ingredient tool for the study of probabilized moment spaces is the parametrization of these spaces by the canonical moments. Roughly speaking, under natural probability measures these parameters have a joint product law with beta marginal distributions. They are also very intriguing nice mathematical objects with a lot of properties that we learned in the excellent book of Dette of Studden [12]. Moreover, studying the exhaustive books of Simon ([34]), we realized that canonical moments, also called Verblunsky coefficients, are quite important objects in complex analysis and spectral theory. At this time the second author of the present paper was working on the asymptotic properties of the determinant of classical random matrix ensembles ([33]). Surprisingly, by the Bartlett formula, the distribution of these random determinants involves product of independent beta random variables having similar parameters as those found in the randomized moment problem. Observing this analogy, we discovered a connection between random moments and spectral measures of classical random matrix ensembles ([20], [21]). The present paper is a matricial extension of the asymptotic studies conducted in the latter papers dealing with scalar random spectral measures. This extension has been possible thanks to two significant contributions of Dette and Studden in the field of matricial moment problems (see [13], and [14]).

We never had the opportunity to meet W.J. Studden but we wish to pay here a tribute to this creative mathematician that had often enlighten the paths of our researches.

## 1.2 Introduction to this paper

To capture the asymptotic behavior of large dimensional unitary random matrices, the usual statistic is the empirical spectral distribution, providing equal weights to all eigenvalues

$$\mu_{\mathbf{u}} = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k}.$$

More recently, some authors used another random probability measure based on eigenvalues and eigenvectors ([25], [2],[20]). If  $U$  is a unitary matrix and  $e$  is a fixed vector (assumed to be cyclic), the so-called spectral measure  $\mu_{\mathbf{w},1}$  of the pair  $(U, e)$  may be defined through its algebraic moments. Indeed, for all  $n \in \mathbb{Z}$

$$\langle e, U^n e \rangle = \int_{\mathbb{T}} z^n d\mu_{\mathbf{w},1}(z).$$

Here,  $\mathbb{T}$  is the torus  $\{z \in \mathbb{C} : |z| = 1\}$ . The measure  $\mu_{\mathbf{w},1}$  is finitely supported on the eigenvalues of  $U$ , we may write

$$\mu_{\mathbf{w},1} = \sum_{k=1}^N \mathbf{w}_k \delta_{\lambda_k}$$

where  $\mathbf{w}_k$  is the square of the scalar product of  $e$  with a unitary eigenvector associated with  $\lambda_k$ . The latter measure carries more information than the former. The weights in  $\mu_{\mathbf{w},1}$  are blurred footprints of the eigenvectors of  $U$ . To make these footprints unblurred, it is then tempting to try to increase the *dimension* by *projecting*  $U$  on a fixed subspace of dimension  $p$  instead on the span generated by the single vector  $e$ . We obtain a matrix measure. This is what we will do in this paper. Actually we may even go back to the representation given by the spectral theorem (see [16])

$$U = \int_{\mathbb{T}} z E_U(dz)$$

where  $E_U$  is the spectral measure of  $U$  (or resolution of the identity for  $U$ ). In our work, we sample  $U$  according to the Haar distribution on  $\mathbb{U}(N)$  and we study the random object  $E_U$ .

The paper is organized as follows. In the next section, we first frame our paper by giving the main notations and definitions needed further. Then, we recall some facts on unitary matrices and matricial orthogonal polynomials. We also show technical results on these objects that will be useful later. In Section 3 we first study the effects of the randomization of the unitary matrices on the object defined in Section 2. In particular, our approach merely simplifies the proof of the asymptotic normality for a fixed corner extracted in the unitary ensemble given in [27]. Further, we show large deviation theorems both for matricial random spectral measure and their infinite dimensional lifting. All proofs are postponed to Section 4.

## 2 Preliminaries

### 2.1 Some notations and definitions

To begin with, let us give some definitions and notations. Let  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathcal{H} = \ell_{\mathbb{C}}^2(\mathbb{N})$ . For  $i \geq 1$ , let  $e_i = (0, \dots, 0, 1, 0, \dots)$  be the  $i$ -th element of the canonical basis of  $\mathcal{H}$  and for  $p \geq 1$  let  $\mathcal{H}_p$  be  $\text{span} \{e_1, \dots, e_p\}$ . We define several sets of matrices with complex entries:

- $\mathbb{M}_{p,n}$ , the set of  $p \times n$  matrices with complex entries,
- $\mathbb{M}_n^+$ , the set of  $n \times n$  Hermitian non-negative matrices,
- $\mathbb{U}(n)$ , the set of  $n \times n$  unitary matrices,

At last,  $I_p$  denotes the  $p \times p$  identity matrix on  $\mathcal{H}_p$ .

### 2.2 Operator-valued and spectral measures

Let  $\mathcal{B}(\mathbb{T})$  denote the Borel  $\sigma$ -algebra on  $\mathbb{T}$ . Let  $\mathfrak{H}$  be a separable Hilbert space,  $I_{\mathfrak{H}}$  be the identity in  $\mathfrak{H}$  and  $H(\mathfrak{H})$  be the algebra of bounded Hermitian endomorphism on  $\mathfrak{H}$ .

#### Definition 2.1

1. A mapping  $\Sigma : \mathcal{B}(\mathbb{T}) \rightarrow \mathcal{H}(\mathfrak{H})$  is called an operator-valued measure in  $\mathfrak{H}$  if
  - (a)  $\Sigma(\emptyset) = 0$ ;
  - (b)  $\Sigma$  is non-negative i.e. if  $\Sigma(\Delta) \geq 0$  for  $\Delta \in \mathcal{B}(\mathbb{T})$ ;
  - (c)  $\Sigma$  is strongly countably additive, i.e., if  $\Delta = \cup_{j=1}^{\infty} \Delta_j$  is a disjoint decomposition of  $\Delta \in \mathcal{B}(\mathbb{T})$ , then  $\Sigma(\Delta) = \sum_{j=1}^{\infty} \Sigma(\Delta_j)$  (in the strong sense);
2. An operator-valued measure  $\Sigma$  is an operator-valued probability if  $\Sigma(\mathbb{T}) = I_{\mathfrak{H}}$ ;
3. An operator-valued probability  $E$  is said to be spectral or orthogonal if it is projection-valued i.e. if for any  $\Delta \in \mathcal{B}(\mathbb{T})$ ,  $E(\Delta) = E(\Delta)^2$ .

Let  $P_{\mathfrak{L}}$  be the orthogonal projection onto the closed linear subspace  $\mathfrak{L}$ . The notation  $T|_{\mathfrak{L}}$  means the restriction of a linear operator  $T$  on the set  $\mathfrak{L}$ . In the sequel,  $\langle \cdot, \cdot \rangle$  will denote the standard

Hermitian product without mention of the ambient Hilbert space. For  $\alpha, \beta$  elements of an Hilbert space, the right bracket is a rank one endomorphism defined by

$$(|\alpha\rangle\langle\beta|)(\cdot) = \langle\cdot, \alpha\rangle\beta.$$

If  $\nu$  is an operator-valued measure on  $\mathbb{T}$  and  $\mathfrak{K}$  is a subspace of  $\mathfrak{H}$ , then  $\nu_{\uparrow\mathfrak{K}}$  denotes the map  $\mathcal{B}(\mathbb{T}) \rightarrow B(\mathfrak{K})$  such that  $\nu_{\uparrow\mathfrak{K}}(\Delta) = \nu(\Delta)_{\uparrow\mathfrak{K}}$ . Of course, if  $\mathfrak{L} \subset \mathfrak{K} \subset \mathfrak{H}$  we have

$$(2.1) \quad \nu_{\uparrow\mathfrak{L}} = (\nu_{\uparrow\mathfrak{K}})_{\uparrow\mathfrak{L}}.$$

$\mathcal{M}(\mathfrak{H})$  (resp.  $\mathcal{M}^1(\mathfrak{H})$ ) denotes the set of operator-valued measures in  $\mathfrak{H}$  (resp. operator-valued probability measures in  $\mathfrak{H}$ ). We equip  $\mathcal{M}(\mathfrak{H})$  with the following topology:  $\nu_n \rightarrow \nu$  if, and only if, for all  $f \in \mathfrak{H}$ , the sequence of positive (scalar) measures  $\langle f, \nu_n(\cdot)f \rangle$  converges weakly to  $\langle f, \nu(\cdot)f \rangle$ . In the finite dimensional space case, an operator-valued measure is a matrix measure. For  $p \in \mathbb{N}$ , we denote by  $\mathcal{M}_p$  (resp.  $\mathcal{M}_p^1$ ) the set of all Hermitian non-negative  $p \times p$  matrix measures (resp. matrix probability measures).

The operator-valued measure  $I_p\lambda$  is in some sense the reference measure. We need the notion of absolute continuity and Lebesgue decomposition for operator-valued measures. We refer to Robertson [32] and to Mandrekar [29].

If  $\nu$  is a non-negative  $\sigma$ -finite measure on  $\mathbb{T}$ , we say that a  $m \times n$  matrix measure  $M$  on  $\mathbb{T}$  is absolutely continuous (a.c. in short) with respect to  $\nu$  ( $M \ll \nu$ ) if each entry of  $M$  is a.c. with respect to  $\nu$ . In this case  $M'_\nu$  will denote the matricial function whose entries are the Radon-Nikodym derivatives. If  $M$  is  $m \times m$ , non-negative Hermitian, then there exist unique matrix measures  $M_a$  and  $M_s$  such that  $M = M_a + M_s$ ,  $M_a$  and  $M_s$  non-negative,  $M_a \ll \nu$ , and  $M_s$  singular with respect to  $\nu$ , i.e. nonzero only on a set of  $\nu$  measure zero.

Now, we say that  $N$  is strongly a.c. with respect to  $M$  ( $N \ll\ll M$ ) if and only if there exists a nonnegative measure  $\nu$  dominating  $M$  and  $N$  such that the range of  $N'_\nu(\omega)$  is a subset of the range of  $N'_M(\omega)$  for  $\nu$ -almost every  $\omega$ . Actually, if  $M$  is  $\ell \times n$  and  $N$  is  $m \times n$  then  $N \ll\ll M$  iff there exists a  $\ell \times m$  matricial function  $\Phi$  integrable with respect to  $M$  such that  $dN = \Phi dM$ . This  $\Phi$  is unique modulo  $M$  and  $\phi = N'_\nu (M'_\nu)^\sharp$  where  $\sharp$  denotes the pseudo-inverse.

### 2.3 Unitary operators and unitary matrices

For any unitary operator  $U$  in  $\mathfrak{H}$  the spectral theorem (see [16]) provides a spectral operator-valued probability  $E_U$  such that  $U = \int_{\mathbb{T}} z E_U(dz)$ . In other words, for any  $f, g \in \mathfrak{H}$  and  $k \in \mathbb{Z}$

$$(2.2) \quad \langle f, U^k g \rangle = \int_{\mathbb{T}} z^k \langle f, E_U(dz)g \rangle.$$

If  $\mathfrak{K}$  is a subspace of  $\mathfrak{H}$ , the spectral measure of the pair  $(U, \mathfrak{K})$  is by definition  $(E_U)_{|\mathfrak{K}}$ . If  $\mathfrak{K}$  is a one-dimensional subspace, the spectral measure is scalar (see the previous section).

Let  $N$  be a fixed integer. In the generic situation, an operator  $U \in \mathbb{U}(N)$  has distinct eigenvalues  $e^{i\theta_k}, k = 1, \dots, N$  and its spectral decomposition may be written

$$(2.3) \quad U = \sum_{k=1}^N e^{i\theta_k} |v_k\rangle\langle v_k| ,$$

where for  $k = 1, \dots, N$ ,  $v_k$  is a unitary eigenvector, associated with the eigenvalue  $e^{i\theta_k}$ . Obviously, for  $p \leq N$ , the spectral measure associated with the pair  $(U, \mathcal{H}_p)$  is

$$(2.4) \quad \mu_{\mathfrak{w},p} := \sum_{k=1}^N \mathfrak{w}_k \delta_{e^{i\theta_k}} ,$$

where, for  $k = 1, \dots, N$ ,  $\mathfrak{w}_k := |P_{\mathcal{H}_p} v_k\rangle\langle P_{\mathcal{H}_p} v_k|$ , is an Hermitian endomorphism on  $\mathcal{H}_p$ . This spectral measure is a matrix probability measure whose order  $n$  moment has generic entry  $\langle e_i, U^n e_j \rangle$  ( $i, j = 1, \dots, p$ ).

## 2.4 Orthogonal polynomials

### 2.4.1 Construction of matrix orthogonal polynomials

In spectral theory, orthogonal polynomials play a prominent role. Here, we will need to work with matrix-valued orthogonal polynomials with respect to matrix measures. We recall some useful facts and refer to [34] for more on the subject. To begin with, as in the scalar case, orthogonal polynomial satisfy a recursion and the matrices appearing in this recursion are the so-called matrix Verblunsky coefficients (see [6]). Let us give some more notations. Let  $p \in \mathbb{N}$  and  $\mu \in \mathcal{M}_p$ . To begin with, the moment  $m_\ell(\mu)$  of order  $\ell \in \mathbb{Z}$  of  $\mu$  is defined as

$$m_\ell(\mu) = \int_{\mathbb{T}} z^\ell d\mu .$$

Further, let  $F$  and  $G$  be measurable matrix valued functions :  $\mathbb{T} \rightarrow \mathbb{M}_{p,p}$ . We define two  $p \times p$  matrices by setting

$$\begin{aligned} \langle\langle F, G \rangle\rangle_R &= \int_{\mathbb{T}} F(z)^\dagger d\mu(z) G(z) \in \mathbb{M}_{p,p} \\ \langle\langle F, G \rangle\rangle_L &= \int_{\mathbb{T}} G(z) d\mu(z) F(z)^\dagger \in \mathbb{M}_{p,p} . \end{aligned}$$

A sequence of functions  $(\varphi_j)$  in  $\mathcal{H}_p$  is called right-orthonormal if and only if

$$\langle\langle \varphi_i, \varphi_j \rangle\rangle_R = \delta_{ij} I_p .$$

The orthogonal polynomial recursion is built as follows.

First, assume that the support of  $\mu$  is infinite. We define the right monic matrix orthogonal polynomials  $\Phi_n^R$  by applying Gram-Schmidt procedure to  $\{I_p, zI_p, z^2I_p, \dots\}$ . In other words,  $\Phi_n^R$  is the unique matrix polynomial  $\Phi_n^R(z) = z^n I_p +$  lower order terms such that  $\langle\langle z^k I_p, \Phi_n^R \rangle\rangle_R = 0$  for  $k = 0, \dots, n-1$ . The normalized orthogonal polynomials are defined by

$$\varphi_0 = I_p \quad , \quad \varphi_n^R = \Phi_n^R \kappa_n^R$$

where the sequence of  $p \times p$  matrix  $(\kappa_n^R)$  satisfy, for all  $n$ , the condition  $(\kappa_n^R)^{-1} \kappa_{n+1}^R > 0$  and is such that the set  $\{\varphi_n^R\}$  is orthonormal. We define the sequence of left-orthonormal polynomials  $\{\varphi_n^L\}$  in the same way except that the above condition is replaced by  $\kappa_{n+1}^L (\kappa_n^L)^{-1} > 0$ . The Szegő recursion is then

$$(2.5) \quad z\varphi_n^L - \rho_n^L \varphi_{n+1}^L = \alpha_n^\dagger (\varphi_n^R)^*$$

$$(2.6) \quad z\varphi_n^R - \varphi_{n+1}^R \rho_n^R = (\varphi_n^L)^* \alpha_n^\dagger ,$$

where for all  $n \in \mathbb{N}$ ,

- $\alpha_n$  belongs to  $\mathbb{B}_p$  the closed unit ball of  $\mathbb{M}_{p \times p}$  defined by

$$\mathbb{B}_p := \{V \in \mathbb{M}_{p,p} : VV^\dagger \leq I_p\} ,$$

- $\rho_n$  is the so-called defect matrix defined by

$$(2.7) \quad \rho_n^R := (I_p - \alpha_n \alpha_n^\dagger)^{1/2} \quad , \quad \rho_n^L := (I_p - \alpha_n^\dagger \alpha_n)^{1/2} ,$$

- for a matrix polynomial  $P$ , having degree  $n$ , the reversed polynomial  $P^*$  is defined by

$$P^*(z) := z^n P(1/\bar{z})^\dagger .$$

Notice that the construction of the recursion coefficients uses only the matricial moments. The Favard theorem establishes a one-to-one correspondance between matrix measures on  $\mathbb{T}$  with infinite support and sequences of elements of the interior of  $\mathbb{B}_p$ . For more details of the last construction, see Simon [34].

Now, for matrix measure having a finite support, the construction of the Verblunsky coefficients is not obvious. In [15], a sufficient condition on the moments for such a construction is provided. It is also close to the remark after 2.13.13 in [34] p.208.

**Lemma 2.2** *Let  $N = Qp + r$ , with  $Q \geq 1$  and  $0 \leq r < p$ . Let  $J \leq Q - 1$ . Then for almost every  $U \in \mathbb{U}(N)$  equipped with the Haar distribution, there exists a measure  $\nu$  with infinite support and satisfying*

$$m_k(\nu) = m_k(\mu_{\mathbf{w},p}), \quad k \leq J.$$

### 2.4.2 A unitary isomorphism

As in the scalar case, we can build a unitary equivalence between the linear space  $\mathbb{M}_{N,p}$  (of systems of  $p$  vectors of  $\mathcal{H}_N$ ) and the linear space  $\mathbb{P}_{N,p}$  of polynomials of degree  $\leq N - 1$  with coefficients in  $\mathbb{M}_{p,p}$ . Let  $e = [e_1 \cdots e_p]$  be a  $N \times p$  matrix consisting in  $p$  column vectors of dimension  $N$ . In the same way, we consider the  $n \times p$  matrix  $Ue = [Ue_1 \cdots Ue_p]$ . The pseudo-scalar (inner) product of  $e$  with  $f = [f_1 \cdots f_p]$  is a matrix  $p \times p$  denoted by  $\ll f, e \gg$  and defined by

$$\ll f, e \gg_{i,j} = \langle f_i, e_j \rangle \quad i, j = 1, \dots, p.$$

It is clear that if the system  $e_1, \dots, e_p$  is orthonormal, then  $\ll e, e \gg = I_p$ . It is clear also that if  $W \in \mathbb{M}_{N,N}$  and  $e, f$  are as above, then

$$(2.8) \quad \ll f, We \gg = \ll W^\dagger f, e \gg .$$

Elementary computations lead also to the useful following properties

$$(2.9) \quad \ll f, gW \gg = \ll f, g \gg W, \quad \ll fW, g \gg = W^\dagger \ll f, g \gg,$$

where  $W \in \mathbb{M}_{p,p}$ . Notice that  $s = [s_1 \cdots s_p]$  consists in elements of  $\text{span} \{e\}$  if and only if there exists a matrix  $\gamma \in \mathbb{M}_{p,N}$  such that  $s = e\gamma$ .

Let  $\varepsilon = [\varepsilon_1 \cdots \varepsilon_p]$  an orthonormal system,  $U$  unitary on  $\mathcal{H}_N$  and  $\mu$  the spectral measure of the pair  $(U, \text{span } \varepsilon)$ .

**Definition 2.3** *We say that  $\varepsilon$  is cyclic for  $U$ , if*

$$(2.10) \quad \text{span} \{U^k \varepsilon_j, 0 \leq k \leq N - 1, 1 \leq j \leq p\} = \mathbb{C}^N .$$

In this case, each element  $M$  of  $\mathbb{M}_{N,p}$  may be written as  $M = \sum_{k=0}^{N-1} U^k \varepsilon \gamma_k$  where  $\gamma_k \in \mathbb{M}_{p,p}$  and then we can associate the polynomial  $q(z) = \sum_{k=0}^{N-1} \gamma_k z^k \in \mathbb{P}_{N,p}$ .

**Remark 2.4** *The system  $\varepsilon = [\varepsilon_1, \dots, \varepsilon_p]$  is cyclic for  $U$  as soon as one of the  $\varepsilon_j$  is cyclic for  $U$ , but of course, it is not a necessary condition.*

If  $\varepsilon$  is cyclic for  $U$  we have then a one-to-one correspondance between  $\mathbb{M}_{N,p}$  and  $\mathbb{P}_{N,p}$  which preserves the pseudo-scalar products  $\ll, \gg$  and  $\langle\langle, \rangle\rangle_R$ :

- To  $q(z) = \sum_k \gamma_k z^k \in \mathbb{P}_{N,p}$  we associate  $M_q = \sum_k U^k \varepsilon \gamma_k \in \mathbb{M}_{N,p}$
- If  $p$  and  $q$  are two polynomials, we have

$$\ll M_p, M_q \gg = \langle\langle p, q \rangle\rangle_R,$$

(this property is straightforward for monomials and is extended easily).

**Lemma 2.5** *If  $\varepsilon$  is a cyclic system for  $U$ , then the first Verblunsky coefficient of the pair  $(U, \varepsilon)$  denoted by  $\alpha_0(U, \varepsilon)$  satisfies*

$$(2.11) \quad \alpha_0(U, \varepsilon)^\dagger = \ll \varepsilon, U\varepsilon \gg .$$

We construct now the GGT matrix  $\mathcal{G}^R$ , which is the matrix of the unitary operator

$$f \in L^2(d\mu) \mapsto (z \mapsto zf(z)) \in L^2(d\mu)$$

in the orthonormal basis  $\varphi_k^R$ , that is

$$\mathcal{G}_{k,\ell}^R = \langle\langle \varphi_k^R, z\varphi_\ell^R \rangle\rangle_R.$$

**Lemma 2.6** *We have*

$$(2.12) \quad \mathcal{G}_{k,\ell}^R = \begin{cases} -\alpha_{k-1} \rho_k^L \rho_{k+1}^L \cdots \rho_{\ell-1}^L \alpha_\ell^\dagger & 0 \leq k \leq \ell \\ \rho_\ell^R & k = \ell + 1 \\ 0 & k \geq \ell + 1 \end{cases}$$

(with  $\alpha_{-1} = -1_p$ ,  $\rho_{-1}^L = 0$ ).

It is convenient to call  $\mathcal{G}^R(\{\alpha_0, \alpha_1, \dots\})$  the above GGT matrix built from the Verblunsky coefficients  $(\alpha_0, \dots)$ . We have the easy following result, which is a replica of the Simon's Theorem 10.1.

**Proposition 2.7** *If  $\alpha$  is in the matrix ball, set*

$$(2.13) \quad \Theta(\alpha) := \begin{pmatrix} \alpha^\dagger & \rho^L \\ \rho^R & -\alpha \end{pmatrix},$$

and  $\tilde{\Theta}(\alpha)$  this matrix viewed as an operator on  $\mathbb{C}^{p \times N}$ , extending it by the identity operator. Then we have

$$(2.14) \quad \mathcal{G}^R(\{\alpha_0, \alpha_1, \dots\}) = \tilde{\Theta}(\alpha_0)[I_p \oplus \mathcal{G}^R(\{\alpha_1, \alpha_2, \dots\})].$$

For the sake of completeness, let us consider now the operator point of view.

**Proposition 2.8** *Assume that  $\varepsilon$  is cyclic for  $U$ . Let  $\xi(U, \varepsilon) \in \mathbb{M}_{N,p}$  be the image of  $\varphi_1$  in the isomorphism and set  $H_0 := \text{span} \{\varepsilon, U\varepsilon\} = \text{span} \{\varepsilon, \xi\}$ . Let  $V(U, \varepsilon)$  be the unitary transform which leaves invariant the subspace orthogonal to  $H_0$  and whose restriction to  $H_0$  has the matrix  $\Theta_0$  in the basis  $(\varepsilon, \xi)$ . Then  $\xi(U, \varepsilon)$  is cyclic for the restriction of  $W(U, \varepsilon) := V^{-1}(U, \varepsilon)U$  to  $\varepsilon^\perp$  and we have*

$$(2.15) \quad \alpha_1(\varepsilon, U) = \alpha_0(\xi(U, \varepsilon), W(U, \varepsilon)).$$

### 3 Randomization

#### 3.1 Distributions on $\mathbb{M}_{n,n}$

Along this paper, we use three probability distributions,

- The Haar measure on  $\mathbb{U}(n)$  for  $n \geq 1$ .
- $\text{Gin}(n)$ , the (Ginibre) distribution on  $\mathbb{M}_{n,n}$  which makes all elements independent and standard complex Gaussian. It has the density

$$\pi^{-n^2} \exp -\text{tr} GG^\dagger$$

with respect to the Lebesgue measure.

- $\text{Cor}(n, p)$  (for  $n > 2p$ ) the distribution on  $\mathbb{M}_{p,p}$  of the top-left corner of size  $p \times p$  of a Haar distributed random matrix. It has the density

$$(3.1) \quad V \mapsto K_{p,n} \det (I_p - VV^\dagger)^{n-2p}$$

on the unit ball  $\mathbb{B}_p$  where  $K_{p,n}$  is the normalization constant (see for instance [5] Theorem 5.1). Actually, [30] Lemma 1.4 gives

$$K_{p,n} = \pi^{-p^2} \frac{(n-2p)!(n-2p+1)! \dots (n-p-1)!}{(n-p)!(n-p+1)! \dots (n-1)!}.$$

#### 3.2 Preliminary results

Here, we describe the distribution of the matrix spectral measure. The first statement uses the encoding by weights and locations (2.4).

**Proposition 3.1** *Let  $U$  be Haar distributed in  $\mathbb{U}(N)$ .*

1. *The random variable  $(e^{i\theta_1}, \dots, e^{i\theta_N}) \in \mathbb{T}^N$  is independent of the random variable  $(\mathbf{w}_1, \dots, \mathbf{w}_N)$ .*
2. *The random variable  $(\mathbf{w}_1, \dots, \mathbf{w}_N)$  has the same distribution as*

$$(\mathbf{h}^{-1/2} \mathbf{v}_1 \mathbf{h}^{-1/2}, \dots, \mathbf{h}^{-1/2} \mathbf{v}_N \mathbf{h}^{-1/2})$$

where  $\mathbf{v}_k = |a_k\rangle a_k$ ,  $k = 1, \dots, N$  and

$$(3.2) \quad \mathbf{h} := \sum_{k=1}^N \mathbf{v}_k,$$

and the  $a_k$  for  $k = 1, \dots, N$  are independent  $p$ -dimensional random vectors with complex standard normal distribution.

The second statement describes the distribution of the Verblunsky coefficients.

**Theorem 3.2** *Let  $N = pQ + r$  with  $0 \leq r < p$  and  $Q > 2$ . Let  $U$  be chosen at random in  $\mathbb{U}(N)$  according to Haar measure. Let  $\mu_p^{(N)}$  be the matricial spectral measure for  $(U, e_1, \dots, e_p)$ . Then, the Verblunsky matricial coefficients  $\alpha_j^{(N)} := \alpha_j(\mu_p^{(N)})$  for  $j = 0, \dots, Q-2$  are independent. Moreover, for  $j \leq Q-2$ ,  $\alpha_j^{(N)}$  has in the matricial unit ball of  $\mathbb{M}_{p,p}$  the density  $\text{Cor}(N - pj, p)$ .*

**Remark 3.3** *Notice that for  $p = 1$  the last Verblunsky coefficient  $\alpha_{N-1}^{(1)}$  is uniformly distributed on  $\mathbb{T}$ . In the general case, the distribution of what could be the last coefficient  $\alpha_{j+1}^{(N)}$  is not obvious. Assume that  $N = p + r$  with  $r \leq p$ . Let us compute  $\alpha_1$ . Let us denote*

$$U = \begin{pmatrix} \alpha_0 & C \\ B & A \end{pmatrix}.$$

According to Arlinskii [1] Theorem 4.2, we have

$$(3.3) \quad \alpha_1 = (\rho_0^R)^{-1} C B (\rho_0^L)^{-1}$$

where  $(\rho_0^R)^{-1}$  and  $(\rho_0^L)^{-1}$  are the Moore-Penrose pseudo-inverses of  $\rho_0^R$  and  $\rho_0^L$  respectively. To see that we cannot use the true inverses let us look at  $(\rho_0^R)^2 = I_p - \alpha_0 \alpha_0^\dagger = C C^\dagger$ . To see if  $\rho_0^R$  is invertible, let us study its null-space, i.e. the set of  $x \in \mathbb{C}^p$  such that

$$(\rho_0^R)^2 x = 0 \Leftrightarrow C C^\dagger x = 0 \Rightarrow x^\dagger C C^\dagger x = 0 \Leftrightarrow \|C^\dagger x\| = 0 \Leftrightarrow C^\dagger x = 0$$

and since  $N - p < p$ , there are more unknown coordinates of  $x$  than equations, so the null-space is not trivial and  $\rho_0^R$  is not invertible. We did not succeed in compute the distribution of  $\alpha_1$  given by (3.3) in such a case. Notice that in formula (3.3) we can replace  $CB$  by  $\Gamma_2 - \Gamma_1^2$  where  $\Gamma_1 = \alpha_0$  and  $\Gamma_2$  are the first two moments of  $\mu$ . We recover a formula which fits with (2.19) in [15].

### 3.2.1 Asymptotics

#### Proposition 3.4 (CLT)

1. If  $V_p^{(N)}$  is a random matrix with distribution  $\mathbf{Cor}(N, p)$ , then for fixed  $p$

$$\sqrt{N}V_p^{(N)} \xrightarrow{\mathcal{D}} \mathbf{Gin}(p)$$

2. If  $k$  and  $p$  are fixed, the  $k$  first Verblunsky (matricial) coefficients satisfy

$$\sqrt{N}(\alpha_0^{(N)}, \dots, \alpha_k^{(N)}) \xrightarrow{\mathcal{D}} (G_1, \dots, G_k)$$

where  $G_1, \dots, G_k$  are independent and  $\mathbf{Gin}(p)$  distributed.

There are two proofs of 1) in [19]. 2) is a consequence of 1) and of Theorem 3.2 above.

Coming back to the moments of the measure  $\mu_w^{(N)}$  we recover a result of Krishnapur, so offering a proof shorter than the involved combinatorial original one.

**Corollary 3.5 ([27], Lemma 10 p.357)** *Let  $U \in \mathbb{U}(N)$  sampled from the Haar measure. Fix  $p \geq 1$  and  $n_0 \geq 1$ . Then the sequence of random variables  $\sqrt{N}[U^n]_{i,j}, 1 \leq n \leq n_0, i, j \leq p$ , converges as  $N \rightarrow \infty$  in distribution to independent standard complex Gaussian matrices.*

#### Theorem 3.6 (LDP)

1. If  $V_p^{(N)}$  is a random matrix with density (3.1) in the unit ball, then for fixed  $p$ , the sequence  $(V_p^{(N)})_N$  satisfies the LDP at scale  $N$  in  $\mathbb{M}_{p,p}$  with good rate function

$$(3.4) \quad v \in \mathbb{M}_{p,p} \mapsto \begin{cases} -\log \det(I_p - vv^\dagger) & \text{if } vv^\dagger < I_p, \\ \infty & \text{otherwise.} \end{cases}$$

$$I(v) = -\log \det(I - vv^\dagger)$$

2. For fixed  $p \geq 1$  and  $k \geq 0$ , the sequence  $(\alpha_0^{(N)}, \dots, \alpha_k^{(N)})_N$  satisfies the LDP at scale  $N$  in  $\mathbb{M}_{p,p} \times \dots \times \mathbb{M}_{p,p}$  with good rate function

$$I^k(\alpha_0, \dots, \alpha_k) = -\sum_{j=0}^k \log \det(I - \alpha_j^\dagger \alpha_j).$$

1. is a direct consequence of the explicit expression of the density and 2. comes from the independence of Verblunsky coefficients. These are arguments of [19]. It is worthwhile to quote the scalar case which was established in [28].

### 3.3 Large deviations for the spectral measure in fixed dimension

**Theorem 3.7** For fixed  $p \geq 1$ , the family of random matricial measures  $\left(\mu_{w,p}^{(N)}\right)_n$  satisfies the LDP in  $\mathcal{M}_p^1$  at scale  $N$  with good rate function

$$(3.5) \quad \mathcal{I}_p(\nu) = - \int_{\mathbb{T}} \log \det \nu'_a(z) dz$$

when  $d\nu(z) = \nu'_a(z)dz + d\nu_s(z)$  is the Lebesgue decomposition of  $\nu$ .

**Corollary 3.8** It is possible to rewrite the above quantity in the flavour of Kullback information with the notation of [29] or [32], i.e.

$$(3.6) \quad \mathcal{I}_p(\nu) = \begin{cases} \int_{\mathbb{T}} \log \det \frac{I_p dz}{d\nu(z)} dz & \text{if } I_p dz \lll d\nu(z), \\ \infty & \text{otherwise.} \end{cases}$$

where  $\lll$  denotes the strongly absolute continuity.

**Remark 3.9** The deviations are fom  $I_p dz$ , which is the spectral measure of the null operator. This corresponds to the fact that for fixed  $i, j$  we have  $U_{ij}$  tends to 0 as  $N \rightarrow \infty$ .

### 3.4 Large deviations - Operator-valued random measures

Every element  $U$  of  $\mathbb{U}(N)$  can be extended to an operator on  $\mathcal{H}$  by tensorisation with identity.

More precisely, if  $(e_i)$  denotes as above the canonical basis of  $\mathcal{H}$  we define

$$\widehat{U} \left( \sum_1^{\infty} h_i e_i \right) = \sum_1^N h_i U(e_i) + \sum_{N+1}^{\infty} h_j e_j.$$

When  $U$  is chosen according to the Haar measure in  $\mathbb{U}(N)$ , the (random) spectral measure associated with  $\widehat{U}$  is denoted  $\mu^{(N)}$ . It is of the form

$$\mu^{(N)} = E_{\widehat{U}} = \sum_{k=1}^N |\widehat{v}_k\rangle \langle \widehat{v}_k| \delta_{e^{i\theta_k}} + \left( \sum_{k=N+1}^{\infty} |e_k\rangle \langle e_k| \right) \delta_1,$$

where the  $\widehat{v}_k$  are the eigenvectors of  $U$  extended in  $\mathbb{C}^N$  by zeros. We establish now an LDP for this sequence.

**Theorem 3.10** *The family of random spectral measures  $(\mu^{(N)})_N$  satisfies the LDP in  $\mathcal{M}^1(\mathcal{H})$  at scale  $N$  with the good rate function*

$$(3.7) \quad \mathcal{I}_\infty(\mu) = \lim_k \uparrow \int_{\mathbb{T}} -\log \det \mu'_a{}^k(z) dz$$

when  $\mu = \mu'_a dz + \mu_s$  where  $\mu'_a$  is a measurable function from  $\mathbb{T}$  to  $H(\mathcal{H})$  and

$$\mu'_a{}^k = P_{\mathcal{H}_k}(\mu'_a(z))|_{\mathcal{H}_k}.$$

Moreover, if there is a constant  $C > 1$  such that for every  $k$  and  $z$

$$(3.8) \quad C^{-1} \leq \det \mu'_a{}^k(z) \leq C$$

and if for every  $z$  the operator  $I - \mu'_a(z)$  is trace class and  $z \mapsto \text{tr} \log \mu'_a(z) \in L^1(\mathbb{T})$ , then

$$(3.9) \quad \mathcal{I}_\infty(\mu) = \int_{\mathbb{T}} -\text{tr} \log \mu'_a(z) dz.$$

## 4 Proofs

### 4.1 Proof of Lemma 2.2

Let us first describe the result of Dette and Wagener [15] (up to a slight change of notation). Let for  $\ell = 1, 2, \dots$

$$F_\ell := \{(I_p, m_1(\mu), \dots, m_\ell(\mu)) \mid \mu \in \mathcal{M}_p^1\}$$

the  $\ell$ th moment space of matrix (probability) measures on the unit circle. Their Theorem 2.1 says that  $\lambda = (1, m_1, \dots, m_J) \in F_J$  if and only if

$$\sum_{i,j=0}^J \text{tr}(B_j^\dagger m_{i-j} B_i) \geq 0 \quad \text{for all } B_0, \dots, B_J \in \mathbb{C}^{p \times p}$$

where for  $k > 0$ ,  $m_{-k} = m_k^\dagger$ . Moreover  $\lambda = (1, m_1, \dots, m_J) \in \text{int } F_J$  if and only if there is a strict inequality above except if  $B_0 = \dots = B_\ell = 0$ .

Let us consider  $m_\ell = m_\ell(\mu_{\mathfrak{w},p})$  for  $\ell = 1, \dots, J$ . We have

$$(4.1) \quad \begin{aligned} \sum_{i,j=0}^J \text{tr}(B_j^\dagger m_{i-j} B_i) &= \sum_{k=1}^N \text{tr} \left( \sum_{i,j=0}^J B_j^\dagger P_{\mathcal{H}_p} v_k (P_{\mathcal{H}_p} v_k)^\dagger B_i e^{i\theta_k(i-j)} \right) \\ &= \sum_{k=1}^N \text{tr} A_k^\dagger A_k, \end{aligned}$$

where  $A_k = \sum_{j=0}^J e^{ij\theta_k} (P_{\mathcal{H}_p} v_k)^\dagger B_j$ . Let us prove that in our conditions  $(1, m_1, \dots, m_J) \in \text{int } F_J$  a.s. That means that a.s. we cannot find a system  $(B_0, \dots, B_J)$  non zero such that

$$(4.2) \quad \sum_{k=1}^N \text{tr } A_k^\dagger A_k = 0.$$

But since the matrices  $A_k^\dagger A_k$  are Hermitian nonnegative, (4.2) is equivalent to  $A_k = 0$  for all  $k = 1, \dots, N$  i.e.

$$\sum_{j=0}^J e^{ij\theta_k} (P_{\mathcal{H}_p} v_k)^\dagger B_j = 0 \text{ for all } k = 1, \dots, N.$$

This gives in the  $p^2(J+1)$  unknown variables  $(B_j)_{s,t}, j = 0, \dots, J, s, t = 1, \dots, p$  a linear system of  $pN$  equations. We have supposed that  $J \leq Q-1$ , so that in all cases  $p^2(J+1) \geq pN$ . If this system has a nontrivial solution, the following  $p^2(J+1) \times p^2(J+1)$  minor

$$\begin{pmatrix} I_p \otimes \Gamma_1 \\ \dots \\ I_p \otimes \Gamma_{(J+1)p} \end{pmatrix} \text{ where } \Gamma_k = \left( P_{\mathcal{H}_p} v_k^\dagger, P_{\mathcal{H}_p} v_k^\dagger e^{i\theta_k}, P_{\mathcal{H}_p} v_k^\dagger e^{2i\theta_k}, \dots, P_{\mathcal{H}_p} v_k^\dagger e^{Ji\theta_k} \right)$$

has a determinant zero. An easy permutation of rows shows that this determinant is, up to a change of sign,

$$\left[ \det \begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_{(J+1)p} \end{pmatrix} \right]^p$$

If our system has a non-trivial solution, we have

$$(4.3) \quad \det \begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_{(J+1)p} \end{pmatrix} = 0.$$

This equation is polynomial in the variables  $e^{i\theta_\ell}, (P_{\mathcal{H}_p} v_k)^T$  for  $\ell, k \leq (J+1)p$ . Besides, under the Haar probability on  $\mathbb{U}(N)$  the distribution of  $e^{i\theta_\ell}, \ell \leq (J+1)p$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{T}^{Jp}$ , the distribution of  $(P_{\mathcal{H}_p} v_k)^T, k \leq (J+1)p$  is also diffuse on the set  $\{\sum_{s=1}^{(J+1)p} f_s^T f_s \leq I_p\}$  and both arrays are independent. If the above polynomial is not identically zero, the set of its solutions will then be of measure zero. But if we choose  $P_{\mathcal{H}_p} v_1, \dots, P_{\mathcal{H}_p} v_p$  as the canonical basis  $(e_1, \dots, e_p)$  of  $\mathbb{C}^p$  and  $P_{\mathcal{H}_p} v_{kp+s} = P_{\mathcal{H}_p} v_s = e_s$  for

$k = 1, \dots, Jp$  and  $s = 1, \dots, p$ , equation (4.3) becomes:

$$(4.4) \quad \det \begin{pmatrix} e_1^T & e^{i\theta_1} e_1^T & e^{2i\theta_1} e_1^T & \dots & e^{Ji\theta_1} e_1^T \\ e_2^T & e^{i\theta_2} e_2^T & e^{2i\theta_2} e_2^T & \dots & e^{Ji\theta_2} e_2^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_p^T & e^{i\theta_p} e_p^T & e^{2i\theta_p} e_p^T & \dots & e^{Ji\theta_p} e_p^T \\ e_1^T & e^{i\theta_{p+1}} e_1^T & e^{2i\theta_{p+1}} e_1^T & \dots & e^{Ji\theta_{p+1}} e_1^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_p^T & e^{i\theta_{2p}} e_p^T & e^{2i\theta_{2p}} e_p^T & \dots & e^{Ji\theta_{2p}} e_p^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_1^T & e^{i\theta_{Jp+1}} e_1^T & e^{2i\theta_{Jp+1}} e_1^T & \dots & e^{Ji\theta_{Jp+1}} e_1^T \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_p^T & e^{i\theta_{(J+1)p}} e_p^T & e^{2i\theta_{(J+1)p}} e_p^T & \dots & e^{Ji\theta_{(J+1)p}} e_p^T \end{pmatrix} = 0.$$

We then repeat the classical calculation of Vandermonde determinant and get that the right hand side is (up to a sign)

$$\prod_{j=1}^p \prod_{k=1}^J (e^{i\theta_j} - e^{i\theta_{j+kp}})$$

So, it is then not identically zero and the proof that a.s.  $(I_p, m_1(\mu_{\mathbf{w},p}), \dots, m_J(\mu_{\mathbf{w},p})) \in \text{int } F_p$  is ended. We can then construct a measure  $\nu$  with infinite support whose  $J$  first moments fit with those of  $\mu_{\mathbf{w},p}$  using the Bernstein-Szegö construction (see [6] p.56 3.6). ■

## 4.2 Proof of Lemma 2.5

We start with  $\varphi_0^R = \varphi_0^L = I_p$ . Then by (2.5) and (2.6) with  $n = 0$

$$(4.5) \quad \begin{aligned} zI_p - \rho_0^L \varphi_1^L &= \alpha_0^\dagger \\ zI_p - \varphi_1^R \rho_0^R &= \alpha_0^\dagger. \end{aligned}$$

Since  $\varphi_1^R$  (resp.  $\varphi_1^L$ ) is orthogonal to  $\varphi_0^R$  (resp.  $\varphi_0^L$ ) we get  $\alpha_0^\dagger = \langle \langle I_p, zI_p \rangle \rangle_R$ , so that  $\alpha_0(U, \varepsilon)^\dagger = \langle \langle \varepsilon, U\varepsilon \rangle \rangle$ . ■

### 4.3 Proof of Lemma 2.6

Let us begin with the subdiagonal terms:

$$\mathcal{G}_{\ell+1,\ell} = \langle\langle \varphi_{\ell+1}^R, z\varphi_\ell^R \rangle\rangle.$$

From (2.6))

$$\langle\langle \varphi_{\ell+1}^R, z\varphi_\ell^R \rangle\rangle = \langle\langle \varphi_{\ell+1}^R, \varphi_{\ell+1}^R \rangle\rangle \rho_\ell^R - \langle\langle \varphi_{\ell+1}^R, (\varphi_\ell^L)^* \rangle\rangle \alpha_\ell^\dagger$$

and since  $(\varphi_\ell^L)^*$  is a polynomial of degree  $\ell$ , it is orthogonal to  $\varphi_{\ell+1}^R$ .

Suppose now  $0 \leq k \leq \ell$ . Again from (2.6) we have

$$\langle\langle \varphi_k^R, z\varphi_\ell^R \rangle\rangle_R = \langle\langle \varphi_k^R, \varphi_{\ell+1}^R \rangle\rangle \rho_\ell^R - \langle\langle \varphi_k^R, (\varphi_\ell^L)^* \rangle\rangle \alpha_\ell^\dagger$$

Now it is the first term which vanishes, so that it remains

$$(4.6) \quad \langle\langle \varphi_k^R, z\varphi_\ell^R \rangle\rangle_R = -\langle\langle \varphi_k^R, (\varphi_\ell^L)^* \rangle\rangle \alpha_\ell^\dagger$$

In the Christoffel-Darboux formula (see [6] Proposition 3.6 (b))

$$(1 - \bar{z}_1 z_2) \sum_{k=0}^n \varphi_k^R(z_2) \varphi_k^R(z_1)^\dagger = \varphi_{n+1}^{L,*}(z_2) \varphi_{n+1}^{L,*}(z_1)^\dagger - \varphi_{n+1}^R(z_2) \varphi_{n+1}^R(z_1)^\dagger$$

taking  $z_1 = 0, z_2 = z$  gives

$$(4.7) \quad \varphi_\ell^{L,*}(z) = \sum_{k=0}^{\ell} \varphi_k^R(z) g_k^\ell, \quad g_k^\ell = \varphi_k^R(0)^\dagger \varphi_\ell^{L,*}(0)^{-1}$$

and then

$$(4.8) \quad \langle\langle \varphi_k^R, (\varphi_\ell^L)^* \rangle\rangle = g_k^\ell$$

But we have

$$(4.9) \quad \varphi_k^R = \Phi_k^R \kappa_k^R; \quad \varphi_\ell^L = \kappa_\ell^L \Phi_\ell^L; \quad \Phi_\ell^{L,*}(0) = 1_p$$

and from the Szegő recursion

$$\varphi_k^R(0) = -\kappa_{k-1}^L \alpha_{k-1}^\dagger (\rho_{k-1}^R)^{-1}$$

so that

$$(4.10) \quad \begin{aligned} g_k^\ell &= -(\rho_{k-1}^R)^{-1} \alpha_{k-1}^\dagger (\kappa_{k-1}^L)^\dagger ((\kappa_\ell^L)^{-1})^\dagger = -\alpha_{k-1} (\rho_{k-1}^L)^{-1} (\kappa_{k-1}^L)^\dagger ((\kappa_\ell^L)^{-1})^\dagger \\ &= -\alpha_{k-1} (\kappa_k^L)^\dagger ((\kappa_\ell^L)^{-1})^\dagger \end{aligned}$$

(we have used  $\alpha_j \rho_j^L = \rho_j^R \alpha_j$  and  $\rho_{j-1}^L = \kappa_{j-1}^L (\kappa_j^L)^{-1}$ ). Gathering (4.10), (4.8) and (4.6) we get eventually  $\mathcal{G}_{k,\ell} = -\alpha_{k-1} \rho_k^L \cdots \rho_{\ell-1}^L \alpha_\ell^\dagger$ . ■

## 4.4 Proof of Proposition 2.8

First, let us rephrase the computation of  $\varphi_1^R$ . We look for  $\xi = [\xi_1 \cdots \xi_p] \in \mathbb{M}_{N,p}$ , *unitary* in the sense that  $\ll \xi, \xi \gg = I_p$ , "orthogonal" to  $\varepsilon$  in the sense that  $\ll \xi, \varepsilon \gg = 0_p$  and such that the vectors of  $\xi$  belong to  $\text{span} \{\varepsilon, U\varepsilon\}$ . In a first step, let us see that the matrix  $U\varepsilon - \varepsilon\gamma$  is orthogonal to  $\varepsilon$  if and only if

$$\ll U\varepsilon, \varepsilon \gg = \ll \varepsilon\gamma, \varepsilon \gg = \gamma^\dagger \ll \varepsilon, \varepsilon \gg$$

hence  $\gamma = \alpha^\dagger$ . Let us now normalize this vector. The square of its "norm" is

$$\ll U\varepsilon - \varepsilon\alpha^\dagger, U\varepsilon - \varepsilon\alpha^\dagger \gg = I_p - \alpha\alpha^\dagger,$$

so that, using the notation of Simon for the *defect* matrices we claim that the matrix

$$(4.11) \quad \xi = (U\varepsilon - \varepsilon\alpha^\dagger) (\rho^R)^{-1}$$

satisfies all the requirements above. Of course, we demand that  $\rho^R$  is invertible, but it is true in the generic case.

As in the scalar case, we define now an endomorphism  $V$  unitary letting invariant the subspace orthogonal to  $\text{span} \{\varepsilon, U\varepsilon\}$  and such that  $V\varepsilon = U\varepsilon$ . We know already from (4.11) that  $V\varepsilon = \xi\rho^R + \varepsilon\alpha^\dagger$ . In the "basis"  $(\varepsilon, \xi)$ , we can say that the matrix of the restriction of  $V$  will be  $\Theta(\alpha)$ , as defined in (2.13), in the sense that if  $w = \varepsilon a + \xi b$  then  $Vw = \varepsilon a' + \xi b'$  with

$$a'\alpha^\dagger a + \rho^L b, \quad b' = \rho^R a - \alpha b.$$

Now, the endomorphism

$$(4.12) \quad W = V^{-1}U$$

is unitary and it fixes  $\varepsilon$ . In the basis obtained by orthonormalization of  $\{\varepsilon, U\varepsilon, U^2\varepsilon, \dots\}$ , the endomorphism  $U$  has the block GGT matrix  $\mathcal{G}^R(\alpha_0, \dots)$ . In this basis  $V(U, \varepsilon)$  has the matrix  $\tilde{\Theta}(\alpha_0)$ . and by (2.14) the restriction of  $W(U, \varepsilon)$  to  $\varepsilon^\perp$  has the matrix  $\mathcal{G}^R(\alpha_1, \dots)$ .  $\blacksquare$

## 4.5 Proof of Proposition 3.1

The first assertion is a straightforward consequence of the invariance of the Haar measure.

To prove the second assertion, we will follow some notation of Collins doctoral thesis ([4]) Section 4.2. Let  $\pi$  be the canonical projection :  $\mathbb{M}_{N,N} \rightarrow \mathbb{M}_{p,N}$ . The set  $\pi(\mathbb{U}(N))$  is a real sub-manifold of  $\mathbb{M}_{p,N}$  of dimension  $p(2N - p)$ , characterized by

$$(4.13) \quad \pi(\mathbb{U}(N)) = \{V \in \mathbb{M}_{p,N} : VV^\dagger = I_p\}.$$

The pushforward  $\widehat{\pi}$  of the Haar measure on  $\mathbb{U}(N)$  by  $\pi$  is invariant under the natural action at left and right of  $\mathbb{U}(p)$  and  $\mathbb{U}(N)$ , respectively. Since the action of  $\mathbb{U}(p) \times \mathbb{U}(N)$  on  $\pi(\mathbb{U}(N))$  is transitive, this measure is the only normalised invariant one.

For  $M \in \mathbb{M}_{N,N}$  set

$$(4.14) \quad h(M) = \pi(M)\pi(M)^\dagger$$

and

$$(4.15) \quad v(M) = h(M)^{-1/2}\pi(M)$$

(when  $h(M) > 0$ ), where the square root is taken in the functional calculus sense.

From (4.13),  $v(M) \in \pi(\mathbb{U}(N))$ . Let us now provide  $\mathbb{M}_{N,N}$  with the distribution  $\mathbf{Gin}(N)$  and let us denote by  $\widetilde{\pi}$  the pushforward of this measure by  $v$ . Let us prove that

$$\widetilde{\pi} = \widehat{\pi}.$$

It is enough to show that  $\widetilde{\pi}$  is left and right invariant. Since  $\pi(MU) = \pi(M)U$  for  $U \in \mathbb{U}(N)$  and then  $h(MU) = h(M)$ , and since the Gaussian distribution is invariant by  $U$ , the right invariance by  $\mathbb{U}(N)$  is obvious. Let us consider the left invariance. If  $U \in \mathbb{U}(p)$  and if  $\widetilde{U}$  is defined by

$$\widetilde{U} = \begin{pmatrix} U & 0_{p,N-p} \\ 0_{N-p,p} & I_{N-p} \end{pmatrix},$$

then  $U\pi(M) = \pi(\widetilde{U}M)$ ,  $h(M) = U^\dagger h(\widetilde{U}M)U$ ,  $h(M)^{-1/2} = U^\dagger (h(\widetilde{U}M))^{-1/2}U$  and eventually  $Uv(M) = v(\widetilde{U}M)$ . The invariance of the Gaussian distribution by  $\widetilde{U}$  ends the job. (Let us notice that (4.14) is precisely the relation (3.2).) ■

## 4.6 Proof of Theorem 3.2

There are two approaches in the scalar case, that of [25] and that of [35] section 11. We follow the method of proof of Theorem 11.1 therein and extend it to the matricial case. The only difficulty comes from the noncommutativity. As said before, we will follow the tracks of the proof for the scalar case given in [35].

Let  $\varepsilon = [\varepsilon_1 \cdots \varepsilon_p]$  an orthonormal system and (see (2.11))

$$(4.16) \quad \alpha(U, \varepsilon) = \ll U\varepsilon, \varepsilon \gg.$$

If  $U$  is Haar distributed, we have to find the distribution of  $\alpha$  and to check that conditionally upon  $\alpha$ , the matrix  $W$  is Haar distributed on  $\mathbb{U}(N-p)$ . Actually,  $\alpha$  is nothing else than the

upper left corner of size  $p$  of  $U$  and its distribution is known from Collins to be the  $\text{Cor}(\mathbb{N}, \mathbf{p})$  one. To prove the remaining part, let us see how the different quantities depend on  $U$ . To be clear, let us write  $\alpha(U, \varepsilon)$  for  $\alpha$  and the same for  $\xi, V$  and  $W$  as defined in the proof of Proposition 2.8.. To characterize the Haar distribution, we use the criterion of invariance by left multiplication by an unitary. We have to prove that if  $\Gamma$  is a fixed unitary letting  $\varepsilon$  invariant and if  $F$  is a Borel function

$$(4.17) \quad \mathbb{E}F(W(U, \varepsilon)\Gamma, \alpha(U, \varepsilon)) = \mathbb{E}F(W(U, \varepsilon), \alpha(U, \varepsilon)).$$

We have

$$W(U, \varepsilon)\Gamma = V(U, \varepsilon)^{-1}U\Gamma$$

which leads to consider  $U' = U\Gamma$  which is a unitary fixing  $\varepsilon$  and to check successively

$$\alpha(U', \varepsilon) = \alpha(U, \varepsilon), \quad \xi(U', \varepsilon) = \xi(U, \varepsilon).$$

It should be clear that  $V(U', \varepsilon) = V(U, \varepsilon)$  since they coincide on  $\text{span}\{\varepsilon, \xi\}$  (see (2.13) and leave invariant its orthogonal subspace. We have then

$$W(U, \varepsilon)\Gamma = V(U', \varepsilon)^{-1}U' = W(U', \varepsilon)$$

and then

$$\mathbb{E}F(W(U, \varepsilon)\Gamma, \alpha(U, \varepsilon)) = \mathbb{E}F(W(U', \varepsilon), \alpha(U', \varepsilon)).$$

Since  $U'$  and  $U$  have the same distribution, we have checked (4.17) and the proof of the first step of the iteration is complete. Now we have to consider a matrix Haar distributed in  $\mathbb{U}(N - p)$ . It's the same reasoning. We stop the recursion at  $j = J$ . ■

## 4.7 Proof of Corollary 3.5

In [19] Lemma 3.2 gives a Taylor expansion of the moments in function of the canonical moments. Then it is enough to use the so-called delta-method. Notice that for  $n_0 = 1$  the result goes back to [31]. ■

## 4.8 Proof of Theorem 3.7

There is actually two possible proofs. The first one (that is presented here), is short and use directly the Verblunsky coefficients. The second one is quite longer and does not use the Verblunsky coefficients but directly the representation on the eigenvalues of the matricial

measure. This second proof is much more general as it may be applied to a general sequence of matrix-valued random measures and is useful to obtain general sum rules. This point of view will be developed in the forthcoming paper [18].

First of all, invoking Dawson-Grtner's theorem on projective limits ([8] Th.4.6.1), we get the LDP for the random matricial measure at scale  $N$  with good rate function

$$I(\mu) = - \sum_{j=0}^{\infty} \log \det(I - \alpha_j^\dagger \alpha_j).$$

To conclude, we use the matricial Verblunsky form of the Szegő theorem (see originally [7], [34] Theorem 2.13.5 and more recently [9]):

$$I(\mu) = - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det W(\theta) d\theta$$

if  $d\mu(\theta) = W(\theta) \frac{d\theta}{2\pi} + d\mu_s(\theta)$  (Lebesgue decomposition). ■

## 4.9 Proof of Corollary 3.8

Starting from

$$\nu = \nu_a + \nu_s, \quad \nu_a = \nu'_a dz$$

we see that  $\nu_a \ll dz$  yields  $\nu_a \lll I_p dz$ . If  $\nu'_a > 0$  a.e. then, by theorem 5.5 in [32] we have  $I_p dz \lll \nu_a$ , hence  $I_p dz \lll \nu$  and the Radon-Nikodym derivative of  $I_p dz$  with respect to  $\nu$  is  $(\nu'_a)^{-1}$  and (3.6) is valid.

Conversely, if  $I_p dz \lll \nu$ , then there exists a finite measure  $\gamma$  on  $\mathbb{T}$  such that  $I_p dz \ll \gamma$ ,  $\nu \ll \gamma$  and  $\text{range}(I_p dz/d\gamma) \subset \text{range}(\nu'_\gamma)$ . But  $I_p dz \ll \gamma$  implies  $dz \ll \gamma$ , so that  $dz = g(z)d\gamma$ . Since  $I_p dz/d\nu = g(z)(\nu'_\gamma(z))^\sharp$  the finiteness of the integral in (3.6) has two consequences:

- $g(z) > 0$  for a.e.  $z$  and then

$$d\gamma = (g(z))^{-1} 1_{g(z)>0} dz + d\gamma_s \quad \gamma_s \perp dz$$

- $(\nu'_\gamma)^\sharp(z) \neq 0$  for a.e.  $z$ . From the definition of the pseudo-inverse, this last requirement needs  $\nu'_\gamma \neq 0$  for a.e.  $z$ .

This yields

$$d\nu = \nu'_\gamma d\gamma = \nu'_\gamma(z)(g(z))^{-1} 1_{g(z)>0} dz + \nu'_\gamma(z) d\gamma_s$$

and then  $\nu'_a(z) = \nu'_\gamma(z)(g(z))^{-1}$  for a.e.  $z$  and (3.5) is valid. ■

## 4.10 Proof of Theorem 3.10

As we noticed in (2.1) the structure of spectral measures is projective. We may apply the Dawson-Grtner theorem ([8] Th. 4.6.1) and we get the rate function

$$(4.18) \quad \sup_k \int -\log \det \mu'_a{}^k(z) dz .$$

Then we use the following lemma whose proof is slightly postponed. ■

### Lemma 4.1

1. The sequence  $\int -\log \det \mu'_a{}^k(z) dz$  is increasing in  $k$ .
2. We have

$$(4.19) \quad \lim_k \det \mu'_a{}^k(z) = \mu'_a(z) .$$

The point 1. entails that the supremum in (4.18) is actually an increasing limit. The point 2. gives a limit for the integrand. But, in general, it is not possible to commute limit and integral in (3.7). Assumption (3.8) ensures a dominated convergence. The equality  $\text{tr} \log = \log \det$  is classical (see [9]).

### Proof of Lemma 4.1

1. For fixed  $k$ , the Hermitian non-negative matrix  $\mu'_a{}^k(z)$  admits a Cholesky decomposition

$$(4.20) \quad \mu'_a{}^k(z) = L_k(z)L_k(z)^\dagger ,$$

and it is straightforward to see that the  $(k-1)$ -section of  $L_k(z)$  is  $L_{k-1}(z)$ , so that we have an infinite Cholesky matrix  $L(z)$  whose generic entry will be denoted by  $\ell_{i,j}(z)$ . From (4.20), we have the relation

$$(4.21) \quad \det \mu'_a{}^k(z) = \det \mu'_a{}^{k-1}(z) |\ell_{kk}(z)|^2 .$$

Taking logarithm and integrating the last display in  $z$  we get

$$(4.22) \quad \int_{\mathbb{T}} -\log \det \mu'_a{}^k(z) dz + \int_{\mathbb{T}} \log \det \mu'_a{}^{k-1}(z) = \int_{\mathbb{T}} -\log |\ell_{kk}(z)|^2 dz$$

which, by Jensen inequality gives

$$(4.23) \quad \int_{\mathbb{T}} -\log \det \mu'_a{}^k(z) dz + \int_{\mathbb{T}} \log \det \mu'_a{}^{k-1}(z) \geq -\log \int |\ell_{kk}(z)|^2 dz$$

From(4.20) we have also

$$(4.24) \quad |\ell_{kk}(z)|^2 = (\mu'_a(z))_{kk} - \sum_{j=0}^{k-1} |\ell_{kj}(z)|^2,$$

so that using the fact that  $\mu'_a$  is a sub-density matrix, we have

$$(4.25) \quad \int |\ell_{kk}(z)|^2 dz \leq 1 - \sum_{j=0}^{k-1} \int |\ell_{kj}(z)|^2 dz \leq 1.$$

Plugging in (4.23) ends the proof of the first part of Lemma 4.1.

2. It is more or less classical. A reference is in Theorem 6.1 of [23].

■

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