At the origins and in the vanguard of peridynamics, non-local and higher gradient continuum mechanics. An underestimated and still topical contribution of Gabrio Piola

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Abstract
Gabrio Piola's scientific papers have been underestimated in mathematical physics literature. Indeed, a careful reading of them proves that they are original, deep and far-reaching. Actually, even if his contribution to the mechanical sciences is not completely ignored, one can undoubtedly say that the greatest part of his novel contributions to mechanics, although having provided a great impetus to and substantial influence on the work of many preeminent mechanicians, is in fact generally ignored. It has to be remarked that authors Capecchi and Ruta dedicated many efforts to the aim of unveiling the true value of Gabrio Piola as a scientist; however, some deep parts of his scientific results remain not yet sufficiently illustrated. Our aim is to prove that non-local and higher-gradient continuum mechanics were conceived already in Piola's works and to try to explain the reasons for the unfortunate circumstance which caused the erasure of the memory of this aspect of Piola's contribution. Some relevant differential relationships obtained in Piola (*Memoria intorno alle equazioni fondamentali del movimento di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione*, 1846) are carefully discussed, as they are still too often ignored in the continuum mechanics literature and can be considered the starting point of Levi-Civita's theory of connection for Riemannian manifolds.

Keywords
Piola, peridynamics, higher gradient, non-local, Riemannian manifolds.

1. Introduction

Piola's contribution to the mechanical sciences is not completely ignored: indeed his contribution to the formulation of balance equations of force in Lagrangian description is universally recognized (when and how this first rediscovery of Piola occurred will be the object of another investigation). In this context the spirit of Piola's works can be recognized in many modern contributions (see e.g. [1]). One can undoubtedly say that the greatest

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part of his novel contributions to mechanics, although having imparted a great momentum to and substantial influence on the work of many prominent mechanicians, is in fact generally ignored.

Although the last statement may seem exaggerated at first sight, the aim of the present paper is to prove it while presenting the evidence of a circumstance which may seem surprising: some parts the works of Gabrio Piola represent a topical contribution as late as the year 2013.

Those who have appreciated the works of Russo [2, 3] will not be at all shocked by such a statement, as there is evidence that many scientific contributions remained unsurpassed for centuries, if not millennia. Therefore one thesis that we want to put forward in this paper is that the contribution of Gabrio Piola should not be studied with the attitude of a historian of science but rather with the mathematical rigour needed to understand a contemporary textbook or a research paper.

On the other hand, the authors question the concept of a 'historical method' especially when applied to the history of science and history of mathematics. We claim that there is not any peculiar 'historical method' to be distinguished from the generic 'scientific method' which has to be applied to describe any other kind of phenomenon, although the subject of the investigation is as complex as those involved in the transmission, storage and advancement of scientific knowledge. A fortiori, however, imagine that one could determine precisely what constitutes such a 'historical method': then it *must* include the capability of the historian to understand, master and rigorously reconstruct the mathematical theories which he has decided to study from the historical point of view. In other words: a historian of a particular branch of mathematics has to completely master the theory whose historical development he wants to describe. It is impossible, for instance, that somebody who does not know the theory of integration could recognize that (see [3]) Archimedes actually used rigorous arguments leading to the proof of the existence of the integral of a quadratic function. Moreover, together with the linguistic barriers (one has to know Doric Greek to understand Archimedes and 19th-century Italian to understand Gabrio Piola), there are also notational difficulties: one should not naively believe that *one's own* notation is advanced and modern while the notation found in the sources is clumsy and primitive. Actually, notation is a matter of 'arbitrary choice' and from this point of view (remember that mathematics is based on axiomatic definition of abstract concepts to which the mathematician assigns a meaning by means of axioms and definitions) all notation is equally acceptable. Very often historians of mathematics[1] decide that a theory is much moremodern than it actually is, simply because they do not find 'the modern symbols' or the 'modern nomenclature' in old textbooks. For instance, if one does not find in a textbook the symbol $\int$, this does not mean that the integral was not known to the author of that textbook. It could simply mean that the technology of printing at the age of that textbook required the use of another symbol or of another symbolic method. Indeed, some formulae by Lagrange or Piola seem at first sight to the authors of the present paper to resemble, for their complicated length, lines of commands for LaTeX. Actually, the historian has to carefully *read* the textbooks which he wants to assess and interpret: when these books are books whose content is a mathematical theory, reading them implies reading all the fundamental definitions, lemmas and properties, which are needed to follow its logical development.

In the authors' opinion, in Truesdell and Toupin [4] the contribution of Piola to mechanical sciences is accounted for only partially while in Truesdell [5] it is simply overlooked. It has to be remarked that authors Capecchi and Ruta [6] dedicated many efforts to the aim of unveiling the true value of Gabrio Piola as a scientist; however, some deep parts of his scientific results remain not yet sufficiently illustrated.

Our aim is:

- To prove that non-local and higher-gradient continuum mechanics were conceived already in Piola's works starting from a clever use of the principle of virtual work;
- To explain the unfortunate circumstances which caused the erasure from memory of this aspect of Piola's contribution, although his pupils respected his scientific standing so greatly that they managed to dedicate an important square to him in Milan (close to the Politecnico) to be named after him, while a statue celebrating him was erected in the Brera Palace, also in Milan.

Finally, some differential relationships obtained in [7] are carefully discussed, as they are still too often used without proper attribution in the continuum mechanics literature and can be considered the starting point of the Levi-Civita theory of connection in Riemannian manifolds.

The main source for the present paper is the work *Memoria intorno alle equazioni fondamentali del movimento di corpi qualsivogliono considerati secondo la naturale loro forma e costituzione* [7] but the authors have also consulted other works by Piola [8–11].

In all the above-cited papers by Piola the kinematical descriptor used is simply the placement field defined on the reference configuration: in these works there is no trace of more generalized models of the type introduced by

Cosserat and Cosserat [12]. However, the spirit of Piola's variational formulation (see e.g. [13–22]) and his methods for introducing generalized stress tensors can be found in the papers by Green and Rivlin [23–26] and also many modern works authored for instance by Neff and his co-workers [27–29] and by Forest and his co-workers [30, 31].

2. Linguistic, ideological and cultural barriers impeding the transmission of knowledge

It is evident to many authors and it is very often recognized in the scientific literature that linguistic barriers may play a negative role in the transmission and advancement of science. We recall here, for instance, that Peano [32] in 1903, being aware of the serious consequences which a Babel effect can have on effective collaboration among scientists, tried to push the scientific community towards the use of Latin or of a specially constructed lingua franca in scientific literature, that is, the so-called latino sine flexione.

Actually, in Russo [2, 3] the author clearly analyses the consequences of the existence of those linguistic ideological and cultural barriers which did not permit the Latin-speaking scholars to understand the depth of Hellenistic science: the beginning of the economical and social processes leading to the Middle Ages.

2.1. Gabrio Piola as a protagonist of the Italian Risorgimento (Resurgence)

It is surprising that some important contributions of a well known scientist to mechanics remained unnoticed and have been neglected for so long a time. Actually, after a careful observation of distinct traces and by gathering hints and evidence, one can propose a well founded conjecture: Gabrio Piola was a leading cultural and scientific protagonist of the Italian Risorgimento (Resurgence).

The main evidence for this statement has been found, for example, in his eulogy in memoriam of his 'maestro' Vincenzo Brunacci. This eulogy was written in 1818 [33] (three years after the famous Rimini Proclamation by Giocchino Murat that, to give an idea of its content, started with 'Italians! The hour has come to engage in your highest destiny' and which is generally considered as the beginning of the Italian Resurgence). In this eulogy (completely translated in Appendix B from reference [33]) there is a continuous reference to the Italian nation which, in that time, could pursue some serious legal difficulties for the author of such a eulogy, leading eventually to the loss of his personal freedom.

The eulogy starts with the words 'It is extremely painful for us to announce in this document the death of a truly great man, who, as during his life, was a glory for Italy', and ends with the words 'May these last achievements of such an inventive and ingenious geometer be delivered up to a capable and educated scholar, who could enlighten them as they deserve, for the advancement of SCIENCES, for the glory of the AUTHOR and for the prestige of ITALY'.

In the body of the eulogy one can find the following statements:

- 'It seemed that the Spirit of Italy, who was in great sufferance because in that time the most brilliant star of all mathematical sciences, the illustrious Lagrangia, had left the Nation, that Spirit wanted to have the rise of another star which, being born on the banks of the river Arno, was bound to become the successor of the first one. This consideration is presenting itself even more spontaneous because we will remark that Brunacci was the first admirer in Italy of the luminous Lagrangian doctrines, the first scientist who diffused and supported them, the first scientist who in his studies was always a very creative innovator in their applications. His first Maestri were two famous Italians, Father Canovai and the great geometer Pietro Paoli.'

We remark that here Piola refers to Lagrange by his true and original Italian name, Lagrangia, that he refers to Italy as a unique cultural entity, that he deemed the 'Spirit of Italy' to exist, that he refers to two scientists who were professors in Pisa as Italians (outside the Kingdom of Lombardy–Venetia, where Piola lived and worked).

- 'It is not licit for me neglecting to indicate another subject in which – with honored efforts – our professor distinguished himself. The Journal of Physical Chemistry of Pavia was illustrated in many of his pages by his erudite pen; I will content myself to indicate here three Memoirs where he examines the doctrine of capillary attraction of Monsieur Laplace, comparing it with that of Pessuti and where, with his usual frankness which is originated by his being persuaded of how well-founded was his case, he proves with his firm reasonings, whatever it is said by the French geometers, some propositions which are of great praise for the mentioned Italian geometer.'

For the purposes of this paper, we note that Piola remarked that Brunacci gave in these memoirs of the Journal of Physical Chemistry the role of the champion of Italian science to the Italian Pesutti as opposed to the French geometer M Laplace.

Brunacci greatly influenced Piola's scientific formation and rigorously cultivated his ingenious spirit, as Piola himself recognized in many places in his works. Piola was initiated by Brunacci to mathematical analysis but was immediately attracted, since his first original creations, to mathematical physics, which he based on the principle of virtual velocities (as Lagrange called what has been later called the principle of virtual work).

Actually the aim of the whole scientific activity of Gabrio Piola was to demonstrate that such a principle can be considered the basis of the postulation of every mechanical theory; see for example the papers [34–43] for the inclusion of the dissipative effects. Indeed he developed (by using the Lagrangian postulation) modern continuum mechanics, being, to our knowledge, the first author who introduced the dual in power of the gradient of velocity in the referential description of a continuous body. The coefficients of what will be recognized to be a distribution in the modern sense (as defined by Schwartz) were to be identified later, after the revolutionary theories introduced by Ricci and Levi-Civita, as a double tensor, the *Piola stress tensor*.

Some of the results presented in Piola's works (e.g. those concerning continua the strain energy of which depends on higher gradients of the strain measures) can be regarded even nowadays as among the most advanced available in the literature.

2.2. Piola's works did not receive their due attention because they are written in Italian

It is clear that the strongest limiting factor of the full recognition of Piola's contribution to continuum mechanics must be found in his 'ideological' choice: the use of the Italian language. Moreover, he used a very elegant and erudite style which can be understood and appreciated only by a few specialists, and he did not care if his works were translated into other languages (as later was decided by Levi-Civita who, instead, cared to have some of his works translated into English, and who wrote some others in French directly; see the works by Ricci-Curbastro and Levi-Civita [44–47]).

It is clear that, in a historical period when all scientists of a given nation were using their own language in higher studies, when in every university the official spoken language was the national one and where all textbooks, essays and scientific memoirs were written in the mother language of the authors, Piola could not accept admitting the inferiority of his own mother language and decided to use it for publishing his works.

A well founded conjecture about this linguistic choice can be advanced: although Piola was surely fluent in French (he edited some works by Cauchy in Italian and cites long French excerpts by Poisson) he decided ('per la gloria dell'Italia') for the glory of Italy to use his mother language, in a historical climate in which the Italian nation was not yet united and independent and therefore was not able to self-determine its destiny. This was a patriotic choice which was repaid by a nearly complete neglect of his contribution to mechanical science, exacerbated by the fact that Italian authors seem to have underestimated his contributions (for a detailed analysis of this point see [6]).

From a general point of view, the linguistic barriers often play a very puzzling role in the diffusion of ideas and theories. As discussed in [2, 3] the diffusion of Hellenistic science was actually slowed by the great barrier represented by the ignorance of the language used, but not stopped. The information slowly flowed from East to West, and although it needed some centuries, in the end, maybe translated into a Latin difficult to understand and still keeping Greek nomenclature and terminology, this science managed to pollinate the Italian and European Renaissance; however, the linguistic transfer corresponded to a nearly complete loss of the knowledge about the identity of the scientists who had first formulated the ideas at the basis of the scientific revolution. Even the true period of the appearance of the scientific method was postponed for more than a millennium.

It must not be considered astonishing, then, that the contribution of Piola still is permeating the modern continuum mechanics literature, but is generally misunderstood, even by those who know his contributions better.

Indeed, linguistic barriers are very often insurmountable.

2.3. The mathematics used by Piola in his mechanics treatises

The mathematics used by Piola is modern in every aspect, except in a very important point. Indeed as Levi-Civita's absolute calculus was invented many years later, Piola's presentation proceeds firmly and rigorously but encumbered by a very heavy component-wise notation, which in the eyes of a modern mechanician conveys an undeserved appearance of primitiveness. The reader should not believe that Piola would refuse (as some mechanicians still do!) to use the powerful tools given to us by Levi-Civita. Indeed (again as proven in the eulogy he wrote for honouring his 'maestro' Vincenzo Brunacci) Gabrio Piola knew how important the choice of the right notation and the conceptual tools for the advancement of science are. Furthermore, he calls those who refused the nominalistic and conceptual improvements introduced by Lagrange in mathematical analysis 'obscurantists'.

Unfortunately Piola did not have available to him the tools he needed to progress more quickly in his research. It is astonishing to discover how many results he managed to obtain notwithstanding this limitation.

3. Non-local continuum theories in Piola's works

In the work by Piola [7] the homogenized theory which is deduced by means of the identification of powers in the discrete micro-model and in the continuous macro-model can be called (in the language used by Eringen [48, 49]) a non-local theory. Also, some Italian authors (see e.g. [50]) who contributed to the field with important papers seem not to give explicit recognition that they were reformulating (and extending) the results already found by Piola.

In Appendix A we translate those parts of Piola's work which are most relevant in the present context and in this section we translate into modern symbols the formulae which the reader may find in that appendix in their original form. Moreover, we will recall in a less suggestive, but more direct and modern, language the statements made by Piola.

It is our opinion that some of Piola's arguments can compete in depth and generality, even nowadays, with those which can be found in some of the most advanced modern presentations. Postponing the analysis of Piola's homogenization process to a subsequent investigation, we limit ourselves here to describing the continuum model which he deduces from the principle of virtual velocities for a discrete mechanical system constituted of a finite set of molecules, which he considers to be (or, because of his controversy with Poisson, he must accept as) the most fundamental principle in his postulation process.

In Piola [7, Capo I, p. 8] the reference configuration of the considered deformable body is introduced by labelling each material particle with the three Cartesian coordinates (a, b, c). It is suggestive that the same notation is used in Hellinger [51]: see for example p. 605. We will denote by the symbol X the position occupied by each of the considered material particles in the reference configuration. The placement of the body is then described by the set of three scalar functions (Capo I, p. 8 and then pp. 11–14)

$$x\,(a, b, c)\,, y\,(a, b, c)\,, z\,(a, b, c)\,,$$

which, by using a compact notation, we will denote by the symbol χ mapping any point in the reference configuration to its position in the actual one.

3.1. Piola's non-local internal interactions

In Capo VI, on p. 149 Piola introduces the following.

The quantity ρ (equations (3), (5) and (6)) has the value given by the equation

$$\begin{aligned}\rho^2 &= [x\,(a+f, b+g, c+k) - x\,(a, b, c)]^2 \\ &\quad + [y\,(a+f, b+g, c+k) - y\,(a, b, c)]^2 \\ &\quad + [z\,(a+f, b+g, c+k) - z\,(a, b, c)]^2\,.\end{aligned} \tag{8}$$

So by denoting by the symbol $\bar{X}$ the particle labelled by Piola with the coordinates $(a+f,\, b+g,\, c+k)$ we have, in modern notation, that

$$\rho^2(X, \bar{X}) = \left\|\chi(\bar{X}) - \chi(X)\right\|^2. \tag{8b}$$

In Capo VI on p. 150 we read the following expression for the internal work, relative to a virtual displacement $\delta\chi$, followed by a very clear remark:

$$\int da \int db \int dc \int df \int dg \int dk \cdot \frac{1}{2} K\,\delta\rho \tag{10}$$

‘In it the integration limits for the variables f, g, k will depend on the surfaces which bound the body in the antecedent configuration, and also on the position of the molecule m, which is kept constant, that is by the variables a, b, c which after the first three will also vary in the same domain.’

Here the scalar quantity K is introduced as the *intensity* of the force (see p. 147 of the translated part of [7] in Appendix A) exerted by the particle $\bar{X}$ on the particle X and the 1/2 is present as the action–reaction principle holds. The quantity K is assumed to depend on $\bar{X}, X$ and ρ and manifestly it is measured in $N\,(m)^{-6}$ (SI units). In sect. 72 starting on p. 150 (translated completely in Appendix A) Piola discusses the physical meaning of this scalar quantity and consequently establishes some restrictions on the constitutive equations which have to be assigned to it. Indeed he refrains from any effort to obtain for it an expression in terms of microscopic quantities and limits himself to require its objectivity by assuming its dependence on ρ, an assumption which will have some important consequences in the sequel. Moreover he argues that if one wants to deal with continua more general than fluids (for a discussion of this point one can have a look an the recent paper [52]) then it may depend (in a symmetric way) also on the Lagrangian coordinates of both $\bar{X}$ and X; therefore

$$K(\bar{X}, X, \rho) = K(X, \bar{X}, \rho).$$

On pp. 151–152 we then read some statements which cannot be rendered more clearly:

As a consequence of what was said up to now we can, by adding up the two integrals (1), (10), and by replacing the obtained sum in the first two parts of the general equation (1) sect. 16, formulate the equation which includes the whole molecular mechanics. Before doing so we will remark that it is convenient to introduce the following definition

$$\Lambda = \frac{1}{4}\frac{K}{\rho} \tag{11}$$

by means of which it will be possible to introduce the quantity $\Lambda\delta\rho^2$ instead of the quantity $(1/2)K\delta\rho$ in the sextuple integral (10); and that inside this sextuple integral it is suitable to isolate the part relative to the triple integral relative to the variables f, g, k, placing it under the same sign of triple integral with respect to the variables a, b, c which includes the first part of the equation: which is manifestly allowed. In this way the aforementioned general equation becomes

$$\int da \int db \int dc \cdot \left\{ \left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z \right.$$
$$\left. + \int df \int dg \int dk \cdot \Lambda\delta\rho^2 \right\} + \Omega = 0 \tag{12}$$

where it is intended that (as mentioned at the beginning of sect. 71) it is included in the Ω those parts which may be introduced because of the forces applied to surfaces, lines or well-determined points and also because of particular conditions which may oblige some points to belong to some given curve or surface.

Piola is aware of the technical difficulty which he could be obliged to confront in order to calculate the first variation of a square root: as he knows that these difficulties have no physical counterparts, instead of K he introduces another constitutive quantity Λ which is the dual in work of the variation $\delta\rho^2$.

Remark 1 ***Boundedness and attenuation assumptions on*** K ***and*** Λ. *Note that Piola explicitly assumes the summability of the function*

$$\Lambda\delta\rho^2 = \frac{1}{4}\frac{K}{\rho}\delta\rho^2 = \frac{1}{2}K\delta\rho$$

and the boundedness of the function K. *As a consequence when* ρ *is increasing then* Λ *decreases.*

Remark 2 ***Objectivity of virtual work.*** *Note that* $\delta\rho^2$ *and* $\Lambda(X, \bar{X}, \rho)$ *are invariant (see [53]) under any change of observer and as Piola had repeatedly remarked, see for example Capo IV, sect. 48, pp. 86–87, the expression for virtual work has to verify this condition. Remark also that, as the work is a scalar, in this point Piola’s reasoning is rendered difficult by his ignorance of Levi-Civita’s tensor calculus. In another formalism the previous formula can be written as follows:*

$$\int_{\mathcal{B}} \left[(b_m(X) - a(X))\,\delta\chi(X) + \left(\int_{\mathcal{B}} \Lambda(X, \bar{X}, \rho)\delta\rho^2 \mu(\bar{X})\,d\bar{X} \right) \right] \mu(X)\,dX + \delta W(\partial\mathcal{B}) = 0 \tag{12b}$$

where $\mathcal{B}$ is the considered body, $\partial\mathcal{B}$ its boundary, μ is the volume mass density, $b_m(X)$ is the (volumic) mass-specific externally applied density of force, $a(X)$ the acceleration of material point X and $\delta W(\partial\mathcal{B})$ the work expended on the virtual displacement by actions on the boundary $\partial\mathcal{B}$ and eventually the first variations of the equations expressing the applied constraints on that boundary times the corresponding Lagrange multipliers.

In Eringen [48, 49, 54], the non-local continuum mechanics is founded on a postulation based on principles of balance of mass, linear and angular momentum, energy and entropy. However, in [54] a Capo on variational principles is presented.

One can easily recognize by comparing, for example, the presentation in [54] with (12b) that in the works by Piola the functional

$$\left(\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho)\delta\rho^2\mu(\bar{X})\,d\bar{X}\right) \tag{N1}$$

is assumed to satisfy a slightly generalized version of what in [54, p. 34] is called the 'smooth neighbourhood hypothesis' which reads as follows (in Eringen's work the symbol V is used with the same meaning as our symbol $\mathcal{B}$, X' is used instead of $\bar{X}$, x instead of χ, t' denotes a time instant, the symbol $()_{,K_i}$ denotes the partial derivatives with respect to the K_ith coordinate of X, and we assume the convention of sums over repeated indices):

> Suppose that in a region $V_0 \subset V$, appropriate to each material body, the independent variables admit Taylor series expansions in $X' - X$ in V_0 [...] terminating with gradients of order P, Q, etc.,
>
> $$\begin{aligned} x(X',t') = x(t') &+ \left(X'_{K_1} - X_{K_1}\right) x_{,K_1}(t') \\ &+ \ldots + \frac{1}{P}\left(X'_{K_1} - X_{K_1}\right)\ldots\left(X'_{K_P} - X_{K_P}\right) x_{,K_1\ldots K_P}(t'), \end{aligned}$$
>
> and [...]. If the response functionals are sufficiently smooth so that they can be approximated by the functionals in the field of real functions
>
> $$x(t'), x_{,K_1}(t'), \ldots, x_{,K_1\ldots K_P}(t'), [\ldots] \tag{3.1.6}$$
>
> we say that the material at X [...] satisfies a *smooth neighbourhood hypothesis. Materials of this type, for $P > 1$, $Q > 1$* are called *nonsimple materials of gradient type.*

Actually Piola is not truncating the series and keeps calculating the integrals on the whole body $\mathcal{B}$. Although no explicit mention can be found in the text of Piola, because of the arguments presented in Remark 1, it is clear that he uses a weaker form of the *attenuating neighbourhood hypotheses* stated in [54, p. 34].

To be persuaded of this statement the reader will need to proceed to the next section.

To conclude this section we need to remark (see Appendix C) that in very recent times, like a karstic river, the ideas of Piola are back on the stage of continuum mechanics.

The idea of an internal interaction which does not fall in the framework of Cauchy continuum mechanics is again attracting the attention of many researchers. Following Piola's original ideas modern 'peridynamics'[2] assumes that the force applied on a material particle of a continuum actually depends on the deformation state of a whole neighbourhood of the particle.

3.2. An explicit calculation of the strong form of the variational principle (12b)

A more detailed discussion about the eventual novelties contained in the formulation of peridynamics when compared with for example Eringen's non-local continuum mechanics is postponed to further investigations. In this section we limit ourselves to explicitly computing the Euler–Lagrange equation corresponding to the variational principle (12b). To this end we need to treat the following expression algebraically:

$$\int_{\mathcal{B}}\left(\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho)\delta\rho^2\mu(\bar{X})\,d\bar{X}\right)\mu(X)\,dX \tag{N2}$$

by explicitly calculating

$$\delta\rho^2 = \delta\left(\sum_{i=1}^{3}\left(\chi_i(\bar{X}) - \chi_i(X)\right)\left(\chi_i(\bar{X}) - \chi_i(X)\right)\right).$$

With simple calculations we obtain that (Einstein convention is applied from now on)

$$\delta\rho^2 = \left(2\left(\chi^i(\bar{X}) - \chi^i(X)\right)\left(\delta\chi_i(\bar{X}) - \delta\chi_i(X)\right)\right),$$

which once placed in (N2) produces

$$\int_{\mathcal{B}}\int_{\mathcal{B}} \left(2\Lambda(X,\bar{X},\rho)\mu(\bar{X})\mu(X)\left(\chi^i(\bar{X}) - \chi^i(X)\right)\right)\left(\delta\chi_i(\bar{X}) - \delta\chi_i(X)\right)\, d\bar{X}\, dX =$$
$$= \frac{1}{2}\left(\int_{\mathcal{B}} f^i(\bar{X})\delta\chi_i(\bar{X})\, d\bar{X} + \int_{\mathcal{B}} f^i(X)\delta\chi_i(X)\, dX\right),$$

where we have introduced the internal interaction force (recall that Piola, and we agree with his considerations as presented in sect. 72 on pages 150–151, assumes that $\Lambda(X,\bar{X},\rho) = \Lambda(\bar{X},X,\rho)$) by means of the definition

$$f^i(\bar{X}) := \int_{\mathcal{B}} \left(4\Lambda(X,\bar{X},\rho)\mu(\bar{X})\mu(X)\left(\chi^i(\bar{X}) - \chi^i(X)\right)\right)\, dX.$$

By a standard localization argument one easily gets that (12b) implies that

$$a^i(X) = b^i_m(X) + f^i(X) \tag{N3}$$

which (also see Appendix C) is exactly the starting point of modern 'peridynamics'.

Many non-local continuum theories have been formulated since the first formulation by Piola: we cite here for instance [48, 49, 54, 55]. Remarkable also are the more modern papers [56–67].

The non-local interaction described by the integral operators introduced in the present subsections are not to be considered exclusively as interactions of a mechanical nature: indeed recently a model of biologically driven tissue growth has been introduced (see e.g. [68–70]) where such a non-local operator is conceived to model the biological stimulus to growth.

3.3. *Piola's higher-gradient continua*

The state of deformation of a continuum in the neighbourhood of one of its material points can be approximated by means of the Green deformation measure and of all its derivatives with respect to Lagrangian referential coordinates. Piola never considers the particular case of linearized deformation measures (which is physically rather unnatural): his spirit has been recovered in many modern works, among which we cite [71–72].

Indeed in Capo VI, on p. 152, Piola develops Taylor series $\delta\rho^2$ (also by using his regularity assumptions about the function $\Lambda(X,\bar{X},\rho)$ and definition (11)) and replaces the obtained development in (N1).

In a more modern notation (see Appendix A for the word-by-word translation) starting from

$$\chi_i(\bar{X}) - \chi_i(X) = \sum_{N=1}^{\infty} \frac{1}{N!}\left(\frac{\partial^N \chi_i(X)}{\partial X_{i_1}\ldots\partial X_{i_N}}(\bar{X}_{i_1} - X_{i_1})\ldots(\bar{X}_{i_N} - X_{i_N})\right).$$

Piola gets an expression for the Taylor expansion with respect to the variable $\bar{X}$ of centre X for the function

$$\rho^2(\bar{X},X) = \left(\chi^i(\bar{X}) - \chi^i(X)\right)\left(\chi_i(\bar{X}) - \chi_i(X)\right).$$

He estimates and explicitly writes first, second and third derivatives of ρ^2 with respect to the variable $\bar{X}$. This is what we will do in the sequel, repeating his algebraic procedure with the only difference consisting of the use of Levi-Civita tensor notation.

We start with the first derivative,

$$\frac{1}{2}\frac{\partial\rho^2(\bar{X},X)}{\partial\bar{X}_\alpha} = \left(\chi^i(\bar{X}) - \chi^i(X)\right)\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}_\alpha}. \tag{N4}$$

We remark that when $\bar{X} = X$ this derivative vanishes. Therefore the first item of the Taylor series for ρ^2 vanishes. We now proceed by calculating the second- and third-order derivatives,

$$\frac{1}{2}\frac{\partial^2\rho^2(\bar{X},X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta}=\frac{\partial\chi^i(\bar{X})}{\partial\bar{X}_\beta}\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}_\alpha}+\left(\chi^i(\bar{X})-\chi^i(X)\right)\frac{\partial^2\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta}=$$
$$=: C_{\alpha\beta}(\bar{X})+\left(\chi^i(\bar{X})-\chi^i(X)\right)\frac{\partial^2\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta};$$

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X},X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma}=\frac{\partial C_{\alpha\beta}(\bar{X})}{\partial\bar{X}_\gamma}+\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}_\gamma}\frac{\partial^2\chi^i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta}+\left(\chi^i(\bar{X})-\chi^i(X)\right)\frac{\partial^3\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma}. \tag{N5}$$

The quantities of this last equation are exactly those described in [7, p. 157] concerning the quantities appearing in formulae (14) on p. 153.

We now introduce the result (formula (N12)) found in Appendix D (in order to remain closer to Piola's presentation we refrain here from using the Levi-Civita alternating symbol),

$$F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta}=\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta}+\frac{\partial C_{\beta\gamma}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right),$$

so that by replacing in (N5) we get

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X},X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma}=\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta}+\frac{\partial C_{\beta\gamma}}{\partial X^\alpha}+\frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right)+\left(\chi^i(\bar{X})-\chi^i(X)\right)\frac{\partial^3\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma} \tag{N6}$$

so that when $\bar{X}=X$ we get that the third-order derivatives of ρ^2 can be expressed in terms of the first derivatives of $C_{\gamma\beta}$.

Now we go back to read in Capo VI, sect. 73, pp. 152–153:

> 73. What remains to be done in order to deduce useful consequences from equation (12) is simply a calculation process. Once having recalled equation (8), it is seen, transforming into series the functions in the brackets, that one has
>
> $$\begin{aligned}\rho^2=&\left(f\frac{dx}{da}+g\frac{dx}{db}+k\frac{dx}{dc}+\frac{f^2}{2}\frac{d^2x}{da^2}+\text{etc.}\right)^2\\&+\left(f\frac{dy}{da}+g\frac{dy}{db}+k\frac{dy}{dc}+\frac{f^2}{2}\frac{d^2y}{da^2}+\text{etc.}\right)^2\\&+\left(f\frac{dz}{da}+g\frac{dz}{db}+k\frac{dz}{dc}+\frac{f^2}{2}\frac{d^2z}{da^2}+\text{etc.}\right)^2;\end{aligned}$$
>
> and by calculating the squares and gathering the terms which have equal coefficients:
>
> $$\begin{aligned}\rho^2=&f^2t_1+g^2t_2+k^2t_3+2fgt_4+2fkt_5+2gkt_6\\&+f^3T_1+2f^2gT_2+2f^2kT_3+fg^2T_4+\text{etc.}\end{aligned} \tag{13}$$
>
> in this expression the quantities t_1,t_2,t_3,t_4,t_5,t_6 represent the six trinomials which are already familiar to us, as we have adopted such definitions from equations (6) in sect. 34; and the quantities T_1,T_2,T_3,T_4, etc. where the index goes to infinity, represent trinomials of the same nature in which derivatives of higher and higher order appear.

Then the presentation of Piola continues with the study of the algebraic structure of the trinomial constituting the quantities T_1, T_2, T_3, etc. as shown by the formula appearing in Capo VI, sect. 73 on pp. 153–160. The reader will painfully recognize that these huge component-wise formulae actually have the same structure which becomes easily evident in formula (N6) and in all formulae deduced, with Levi-Civita tensor calculus, in Appendices D and E.

What Piola manages to recognize (also with a courageous conjecture; see Appendices D and E) is that in the expression of virtual work all the quantities which undergo infinitesimal variation (which are naturally to be chosen as 'measures of deformation') are indeed either components of the deformation measure C or components of one of its gradients.

Indeed in sect. 74, p. 156 one reads:

74. A new proposition, to which the reader should pay much attention, is that all the trinomials T_1, T_2, T_3, etc. where the subscript goes to infinity, which [trinomials] appear in the precedent equation (17), can only be expressed by means of the first six $t_1, t_2, t_3, t_4, t_5, t_6,$ and of their derivatives with respect to the variables a, b, c of all orders. I started to suspect this analytical truth because of the necessary correspondence which must hold between the results which are obtained via the method considered in this Capo and those results obtained via the method considered in Capo III and IV.

This statement is true and its importance is perfectly clear to Piola: for a discussion of the mathematical rigour of his proof the reader is referred to Appendix E.

In order to transform the integral expression (N1)

$$\left(\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho)\delta\rho^2(X,\bar{X})\mu(\bar{X})\,d\bar{X}\right),$$

Piola remarks that (pp. 155–156):

When using the equation (13) to deduce the value of the variation $\delta\rho^2$, it is clear that the characteristic δ will need to be applied only to the trinomials we have discussed up to now, so that we will have:

$$\begin{aligned}\delta\rho^2 &= f^{\,2}\delta t_1 + g^{\,2}\delta t_2 + k^{\,2}\delta t_3 + 2fg\,\delta t_4 + 2fk\,\delta t_5 + 2gk\,\delta t_6\\ &\quad + f^{\,3}\delta T_1 + 2f^{\,2}g\delta T_2 + 2f^{\,2}k\delta T_3 + f\,g^2\delta T_4 + \text{etc.}\end{aligned} \tag{16}$$

Indeed the coefficients $f^{\,2}, g^{\,2}, k^{\,2}, 2fg$, etc. are always the same for every functions giving the variables x, y, z in terms of the variables a, b, c, and therefore cannot be affected by that operation whose aim is simply to change the form of these functions. Vice versa, by multiplying the precedent equation (16) times Λ and then integrating with respect to the variables f, g, k in order to deduce the value to be given to the fourth term under the triple integral of the equation (12), such an operation is affecting only the quantities $\Lambda f^{\,2}, \Lambda g^{\,2}$, etc. and the variations $\delta t_1, \delta t_2, \delta t_3 \ldots \delta T_1, \delta T_2$, etc. cannot be affected by it, as the trinomials $t_1, t_2, t_3 \ldots T_1, T_2$, etc. (one has to consider carefully which is their origin) do not contain the variables f, g, k: therefore such variations result in being constant factors, which are to be multiplied by the integrals to be calculated in the subsequent terms of the series.

Using modern notation we have that

$$\rho^2(\bar{X},X) = \sum_{N=1}^{\infty} \frac{1}{N!} \left.\frac{\partial^N \rho^2(\bar{X},X)}{\partial \bar{X}_{i_1} \ldots \partial \bar{X}_{i_N}}\right|_{X=\bar{X}} (\bar{X}_{i_1} - X_{i_1}) \ldots (\bar{X}_{i_N} - X_{i_N})$$

and therefore that

$$\delta\rho^2(\bar{X},X) = \sum_{N=1}^{\infty} \frac{1}{N!} \left(\delta \left.\frac{\partial^N \rho^2(\bar{X},X)}{\partial \bar{X}_{i_1} \ldots \partial \bar{X}_{i_N}}\right|_{X=\bar{X}}\right) (\bar{X}_{i_1} - X_{i_1}) \ldots (\bar{X}_{i_N} - X_{i_N}).$$

As a consequence

$$\begin{aligned}&\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho)\delta\rho^2(\bar{X},X)\mu(\bar{X})\,d\bar{X} =\\ &= \sum_{N=1}^{\infty} \frac{1}{N!} \left(\delta \left.\frac{\partial^N \rho^2(\bar{X},X)}{\partial \bar{X}^{i_1} \ldots \partial \bar{X}^{i_N}}\right|_{X=\bar{X}}\right) \left(\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho) \left((\bar{X}^{i_1} - X^{i_1}) \ldots (\bar{X}^{i_N} - X^{i_N})\right) \mu(\bar{X})\,d\bar{X}\right).\end{aligned}$$

If we introduce the tensors

$$T^{i_1 \ldots i_N}_{\cdot}(X) := \left(\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho) \left((\bar{X}^{i_1} - X^{i_1}) \ldots (\bar{X}^{i_N} - X^{i_N})\right) \mu(\bar{X})\,d\bar{X}\right)$$

we get (also by recalling formula (N18) from Appendix E)

$$\int_{\mathcal{B}} \Lambda(X,\bar{X},\rho)\delta\rho^2(\bar{X},X)\mu(\bar{X})\,d\bar{X} = \sum_{N=1}^{\infty} \frac{1}{N!} \left(\delta L_{i_1 \ldots i_N}\left(C(X), \ldots, \nabla^{n-2}C(X)\right)\right) T^{i_1 \ldots i_N}_{\cdot}(X).$$

Piola then states that:

After these considerations the truth of the following equation is manifest:

$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{17}$$

$$\begin{aligned}&(1)\ \delta t_1 + (2)\ \delta t_2 + (3)\ \delta t_3 + (4)\ \delta t_4 + (5)\ \delta t_5 + (6)\ \delta t_6\\&+ (7)\ \delta T_1 + (8)\ \delta T_2 + (9)\ \delta T_3 + (10)\ \delta T_4 + \text{etc.}\end{aligned}$$

where the coefficients (1), (2), etc. indicated by means of numbers in between brackets, must each be regarded to be a function of the variables a, b, c, [the same functions] which are obtained by calculating the aforementioned definite integrals.

In order to establish the correct identification between Piola's notation and the more modern notation which we have introduced the reader may simply consider the following table ($i = 1, 2, \ldots, n, \ldots$):

$$T_{\cdot}^{i_1 \ldots i_N} \leftrightarrows (1), (2), \text{etc.} \quad \delta L_{i_1 \ldots i_N}\left(C, \ldots, \nabla^{n-2}C\right) \leftrightarrows \delta T_i.$$

After having accepted Piola's assumptions the identity (12b) becomes

$$\int_{\mathcal{B}} \left((b_m(X) - a(X))\, \delta\chi(X) + \sum_{N=1}^{\infty} \frac{1}{N!} \left(\delta L_{\alpha i} \ldots i_N \left(C(X), \ldots, \nabla^{n-2}C(X)\right)\right) T_{\cdot}^{i_1 \ldots i_N}(X) \right) \mu(X)\, dX$$
$$+ \delta W(\partial\mathcal{B}) = 0.$$

By a simple rearrangement and by introducing suitable notation the last formula becomes

$$\int_{\mathcal{B}} \left((b_m(X) - a(X))\, \delta\chi(X) + \sum_{N=1}^{\infty} \left\langle \nabla^N \delta C(X) | S_{\cdot}(X) \right\rangle \right) \mu(X)\, dX + \delta W(\partial\mathcal{B}) = 0 \tag{12c}$$

where S is an Nth-order contravariant totally symmetric tensor[3] and the symbol $\langle\rangle$ denotes the total saturation (inner product) of a pair of totally symmetric contravariant and covariant tensors.
Indeed on pp. 159–160 of [7] we read:

75. Once the proposition of the precedent sect. has been given, it is manifest that equation (17) can assume the following other form

$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{18}$$

$$\begin{aligned}&(\alpha)\ \delta t_1 + (\beta)\ \delta t_2 + (\gamma)\ \delta t_3 + \ldots + (\epsilon)\ \frac{\delta dt_1}{da} + (\zeta)\ \frac{\delta dt_1}{db} + (\eta)\ \frac{\delta dt_1}{dc}\\&+ (\vartheta)\ \frac{\delta dt_2}{da} + \ldots + (\lambda)\ \frac{\delta d^2 t_1}{da^2} + (\mu)\ \frac{\delta d^2 t_1}{da db} + \ldots + (\xi)\ \frac{\delta d^2 t_2}{da^2} + (o)\ \frac{\delta d^2 t_2}{da db} + \text{etc.}\end{aligned}$$

in which the coefficients $(\alpha), (\beta) \ldots (\epsilon) \ldots (\lambda) \ldots$ etc. are suitable quantities given in terms of the coefficients $(1), (2) \ldots (7), (8) \ldots$ of equation (17), of the six trinomials $t_1, t_2 \ldots t_6$, and of the derivatives of any order of these trinomials with respect to the variables a, b, c. Besides, the variations δt_1, $\delta t_2 \ldots$ (with the subscript varying up to infinity) and the variations of all their derivatives of all the required orders $\delta dt_1/da, \delta dt_1/db$, etc. appear in (18) only linearly.

4. Weak and strong evolution equations for Piola continua

To our knowledge a formulation of the principle of virtual work for Nth-gradient Piola continua equivalent to (12c) is found in the literature only in [73], but the authors were unaware of the previous work of Piola.

The reader is referred to the aforementioned paper for the detailed presentation of the needed postulation process and the subsequent procedure of integration by parts needed for transforming the weak formulation of evolution equations given by (12c) into a strong formulation in which suitable bulk equations and the corresponding boundary conditions are considered.

We comment in short here about the relative role of weak and strong formulations, framing it in a historical perspective.

Since at least the pioneering works by Lagrange the postulation process for mechanical theories was based on the least-action principle or on the principle of virtual work.

One can call both these principles 'variational' as the stationarity condition for least-action requires that for all admissible variations of motion the first variation of action must vanish, a statement which, as already recognized by Lagrange himself, implies a form of the principle of virtual work.

However in order to 'compute' the motions relative to given initial data the initiators of physical theories needed to integrate by parts the stationarity condition which they had to handle.

In this way they derived some PDEs with some boundary conditions which sometimes were solved by using analytical or semi-analytical methods.

From the mathematical point of view this procedure is applicable when the searched solution has a stronger regularity than the one strictly needed to formulate the basic variational principle.

It is a rather ironic circumstance that nowadays very often those mathematicians who want to prove well-posedness theorems for PDEs (which originally were obtained by means of an integration-by-parts procedure) start their reasonings by applying the same integration-by-parts process in the reverse direction: indeed, very often the originating variational principle of all PDEs is forgotten. Some examples of mathematical results which exploit in an efficient way the power of variational methods are those presented in Neff [74–76].

Actually, even if one refuses to accept the idea of basing all physical theories on variational principles, one is indeed obliged, in order to find the correct mathematical frame for one's models, to 'prove' the validity of a weak form applicable to his painfully formulated balance laws. In reality (see [77]) his model will not be acceptable until he has been able to reformulate it in a 'weak' form.

It seems that the process which occurred in mathematical geography, described in Russo [2, 3], occurs very frequently in science. While the reader is referred to the cited works for all details, we recall here the crucial point of Russo's argument, as needed for our considerations. Ptolemy presented in his Almagest a useful tool for astronomical calculations: actually his *Handy tables* tabulate all the data needed to compute the positions of the sun, moon and planets, the rising and setting of the stars, and eclipses of the sun and moon. The main calculation tools in Ptolemy's treatise are the deferents and epicycles, which were introduced by Apollonius of Perga and Hipparchus of Rhodes in the framework of astronomic theories much more advanced than the one formulated by Ptolemy (if Russo's conjecture is true). Unfortunately Ptolomy misunderstood the most ancient (and much deeper) theories and badly re-organized the knowledge, observations and theories presented in the treatise by Hipparchus (a treatise which has been lost); indeed, Ptolemy being 'a practical scientist' gives too high an importance to 'the calculation tools' by blurring in a list of logical incongruence the rigorous and deep (and eliocentric!) theories formulated by Hipparchus four centuries before him. Actually, in Ptolemy's vision the calculation tools become the fundamental ingredients of the mathematical model which he presents.

This seems to also have occurred in continuum mechanics: the Euler–Lagrange equations, obtained by means of a process of integration by parts, were originally written, starting from a variational principle, to supply a 'calculation tool' to applied scientists. They soon became (for simplifying) the 'bulk' of the theories and often the originating variational principles were forgotten (or despised as too 'mathematical'). For a period balance equations were (with some difficulties which are discussed for example in [77]) postulated 'on physical grounds'.

When the need to prove rigorous existence and uniqueness theorems met the need to develop suitable numerical methods, and when the many failures of the finite difference schemes became evident, the variational principles were rediscovered *starting from the balance equations.*

The variational principles represent at first the starting point of mechanical theories and were used to get, by means of algebraic manipulation, some tools for performing 'practical calculations', that is, the associated Euler–Lagrange equations or (using another name) balance equations. However, with a strange exchange of roles, if their basic role is forgotten and balance equations are regarded as the basic principles from which one has to start the formulation of the theories, then variational principles need to be recovered as a computational tool.

One question needs to be answered: why, in the modern paper [73], was a strong formulation of the evolution equation for Nth-gradient continua searched for? The answer is: because of the need for finding for those theories the most suitable boundary conditions.

This point is also discussed in [7] as remarked already in [52].
In [7, pp. 160–161] it is claimed that:

> Now it is a fundamental principle of the calculus of variations (and we used it also in this memoir in sect. 36 and elsewhere) that a series, as the precedent one, where the variations of some quantities and the variations of their derivatives with respect to the fundamental variables a, b, c appear linearly, can always be transformed into an expression which contains that quantities without any sign of derivation, with the addition of other terms which are their exact derivatives with respect to one of the three simple independent variables. As a consequence of such a principle, the expression which follows can be given to the equation (18):
>
> $$\int df \int dg \int dk \cdot \Lambda \delta \rho^2 = \tag{19}$$
>
> $$A \delta t_1 + B\, \delta t_2 + C\, \delta t_3 + D \delta t_4 + E \delta t_5 + F\, \delta t_6 + \frac{d\Delta}{da} + \frac{d\Theta}{db} + \frac{d\Upsilon}{dc}.$$
>
> The values of the six coefficients A, B, C, D, E, F are series constructed with the coefficients $(\alpha), (\beta), (\gamma) \ldots (\epsilon), (\zeta) \ldots (\lambda)$, etc. of equation (18) which appear linearly, with alternating signs and affected by derivations of higher and higher order when we move ahead in the terms of said series: the quantities Δ, Θ, Υ are series of the same form as the terms which are transformed, in which the coefficients of the variations have a composition similar to the one which we have described for the six coefficients A, B, C, D, E, F.
>
> Once – instead of the quantity under the integral sign in the left-hand side of the equation (12) – one introduces the quantities which are on the right-hand side of the equation (19), it is clear to everybody that an integration is possible for each of the last three terms of the sum appearing in it and that, as a consequence, these terms only give quantities which supply boundary conditions. What remains under the triple integral is the only sextinomial which is absolutely similar to the sextinomial already used in equation (10) sect. 35 for rigid systems. Therefore after having remarked on the aforementioned similarity, the analytical procedure to be used here will result as perfectly equal to the one used in sect. 35, the procedure which led to the equations (26), (29) in sect. 38 and it will become possible to demonstrate the extension of said equations to every kind of bodies which does not respect the constraint of rigidity, as was mentioned at the end of sect. 38. It will also be visible the coincidence of the obtained results with those which are expressed in the equations (23) of sect. 50 which hold for every kind of systems and which were shown in the Capo IV by means of those intermediate coordinates p, q, r, whose consideration, when using the approach used in this Capo, will not be needed.

The novel content in [73] consists in the determination of:

- The exact structure of the tensorial quantity whose components are called A, B, C, D, E, F by [7];
- The exact structure of the boundary conditions resulting when applying Gauss's theorem to the divergence field called by [7],

  $$\frac{d\Delta}{da} + \frac{d\Theta}{db} + \frac{d\Upsilon}{dc},$$

  on a suitable class of contact surfaces.

The considerations sketched about the history of celestial mechanics should persuade the reader that it is not too unlikely that some ideas by Piola needed 167 years to be further developed (even if the authors did not manage to find any intermediate reference it does not mean that such a reference does not exist, maybe in a language even less understandable than Italian).

Earlier papers (nowadays considered fundamental) by Mindlin [78–82] had developed a more complete study of Piola continua, at least up to those whose deformation energy depends on the third gradient, completely characterizing the nature of contact actions in these cases, or for continua having a kinematics richer than that considered by Piola, including microdeformations and micro-rotations.

Many important applications can be conceived for higher-gradient materials, as for instance those involving the phenomena described for instance in [57, 69, 83–99].

5. One- and two-dimensional continua and micro–macro identification procedure as introduced by Piola [7]

On p. 19 Piola justifies the introduction of one-dimensional or two-dimensional bodies as follows:

> 11. Sometimes mathematicians are used considering the matter configured not in a volume with three dimensions but [configured] in a line or in a surface: in these cases we have the so-called linear or surface systems. Indeed [these systems] are nothing other than abstractions and it is just for this reason that the geometer should pay major attention to three-dimensional systems. Nevertheless, it is useful to consider [these systems] because several analyses for the three kinds of systems provide feedback that make [such analyses] clear, and moreover [such analyses] are useful for physical applications, even though always in an approximate way, because the bodies, rigorously speaking are being never deprived in Nature of one or two dimensions.
>
> Although for both linear and surface systems we need special considerations in order to represent the distribution of the molecules, and [in order] to form the idea of the density and of the measure of the mass, [the idea and the measure] are still similar to the above referred for three-dimensional systems: thus, I will expound on them shortly.

On p. 39, sect. 24 and on p. 46, sect. 29 of [7] the structure of the principle of virtual work is studied in the case for which one or two dimensions of the considered body can be neglected in the description of its motion.

Piola uses these parts to prepare the reader for the micro–macro identification process for three-dimensional bodies which he will study later in full detail.

This identification process:

- Starts from a discrete system of material particles which are placed in a reference configuration at the nodes of a suitably introduced mesh;
- Proceeds with the introduction of a suitable placement field χ having all the needed regularity properties;
- Assumes that the values of χ at the aforementioned nodes can be considered an approximation of the displacements of the discrete system of material particles;
- Is based on the identification of virtual work expressions in discrete and continuous models.

While the detailed description of aforementioned identification (see [83, 100, 101]) process is postponed to further studies, we want to remark here that non-local and higher-gradient theories for beams and shells are already implicitly formulated in [7], although the main subject there is the study of three-dimensional bodies.

The authors have found interesting connections in this context with many of the subsequent works and the most suggestive are those concerning the theory of shells and plates, namely [102–107], where interesting phenomena involving phase transition are considered, and the papers by Neff [27, 28, 74, 76, 108, 109].

Moreover the methods started by Piola are used also when describing two-dimensional surfaces carrying material properties as for instance in [110–117].

Also, interesting analogies for what concerns the connections between discrete and continuous models can be found in papers dealing with one-dimensional continua and their stability, as for instance [118] and [119], where the dynamics of beams or chains of beams are studied; [120], where the nonlinear equations for inextensible cables deduced by Piola are applied to very interesting special motions; [121], where the case of pre-stressed networks is considered; and [122, 123], where the spirit of Piola's contribution is adapted to the context of spatial rods and the nonlinear theory for spatial lattices. Concerning the micro–macro identification procedure in recent literature one can find many continuators of Piola's works. Notable are the works [124–126] in which Piola continua are obtained by means of homogenization procedures starting from lattice beam microstructures. It is possible to also cite some studies which consider visco-elastic continuum theories with damage (see [127–131]) for microscopically granular or discrete systems, as for instance [132–137], or other studies of phenomena involving multiscale coupling (see e.g. [138]).

6. A conclusion: Piola as precursor of the Italian school of differential geometry

The most important contribution of Gabrio Piola to mechanical sciences is the universally recognized Piola transformation, which allows for the transformation of some equations in a conservative form from Lagrangian to Eulerian description. The differential geometric content of this contribution does not need to be discussed, as it has been treated in many works and textbooks: we simply refer to [139] and to the references cited there for a detailed discussion of this point and more considerations about the relationship between continuum mechanics and differential geometry (see also [140]).

In the present paper we have shown that there are other major contributions to mechanics by Gabrio Piola which have been underestimated; we also have tried a first analysis of the reasons for which this circumstance occurred.

In this concluding section we want to remark that the results by Piola which we have described in the present paper also have a strong connection to differential geometry (in this context see also [141, 142]).

The reader is referred to the discussion about 'historical method' which was developed in the introduction: knowledge of the basic ideas of differential geometry is required to follow the considerations which we present here. The criticism usually given to the kind of reconstruction which we want to present is usually based on the following statement: the historian wanted to read something which could not be written in such an early stage of knowledge.

We dismiss this criticism a priori on the basis of the following statements:

- The inaugural lecture by Riemann dates to 1854, therefore Piola's results are surely antecedent but very close in time.
- Riemann is considered one of the founders of Riemannian geometry even if he did not write any formula using the indicial notation developed by Ricci and Levi-Civita.
- The Riemannian tensor is named after Riemann even though there is no formal definition of the concept of tensor in Riemann's works.

In his inaugural lecture Riemann discusses one of his main contributions to geometry, that is, the condition for which a Riemannian manifold is flat. This study (indirectly influenced by Gauss) started a flow of investigations in which the Italian school has played a dominant role. We recall here for example Ricci's lemma and identities, the concept of Levi-Civita parallel transport and the Levi-Civita theorem about parallel transports compatible with a Riemannian structure. Also, referring to Appendix F to substantiate our statement, we claim that it was indeed continuum mechanics which originated differential geometry and that the Italian school in differential geometry may have originated in the works of Piola. Indeed, in Appendix D we have proven that Piola has obtained (component-wise, exactly in the same form in which Riemann obtained all his results) the equation (N14),

$$F_{i\gamma}\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\beta} = \frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial C_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right).$$

This equation is equivalent (see [143, 144, vol. 2, page 184]) to the Riemannian condition of flatness.

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Conflict of interest

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Notes

1. See for instance [3, p. 53ff] concerning the difficulties encountered by historians who did not know trigonometry in recognizing that Hellenistic science had formulated it but with different fundamental variables and notation.
2. We remark that (luckily!) the habit of inventing new names (although sometimes the related concepts are not so novel) is not lost in modern science (see [3] for a discussion of the importance of this attitude in science) and that the tradition of using Greek roots for inventing new names is still alive.
3. The constitutive equations for such tensors must verify the condition of frame invariance. When these tensors are defined in terms of a deformation energy (that is, when the principle of virtual work is obtained as the first variation of a least-action principle) the objectivity becomes a restriction on such an energy. The generalization of the results in [53] to the Nth-gradient continua still needs to be found.

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Appendix A Verbatim translation of excerpts from Piola [7]

In the following appendix subsections, one page in each subsection, we translate the page indicated in the respective title from Italian to English.

A.1 Capo VI On the motion of a generic [deformable] body following the ideas of the modern scientists about the molecular actions, p. 146

At the beginning of the Capo IV it was said that there are two ways for taking into account – in the general equation of motion of a generic body – the effect of the constraints established among its molecules by internal forces. A [first] way which was introduced consisted in expressing such effects by means of equations of condition, and therefore by means of the third part of the most general equation (1) in sect. 16: this was the way which we used in the preceding two Capos. A second way consisted in considering – following the ideas of modern scientists – the molecular actions by making use of the second part of the aforementioned equation (1), where are to be included all the terms introduced by internal active forces: I will discuss now about this second way. This effort will be performed also because we will see that the two different ways actually lead to the same results at least for that part of the solution which is the most important and fundamental (and this agreement is really very reassuring): on the other hand, it has to be remarked that the two said ways are completing each other, and one sheds light on the other so that what was complicated and difficult in one way becomes easy in the other one.

A.2 P. 147

71. Recalling what was said in the sections 31, 32 to show how, in the case of systems having three dimensions, the first part of the general equation (1) sect. 16. due to external forces, is modified to be represented as follows:

$$\int da \int db \int dc \cdot \left\{ \left(X - \frac{d^2 x}{dt^2} \right) \delta x + \text{etc.} \right\}; \tag{1}$$

we see now how it has to be modified the second part $Sm_i m_j \, K\delta\rho$, which is that one we want to consider now, while, at the same, time in the third part we equate to zero all terms expressing actions applied to all the mass [of considered body] and only retain those terms related to forces concentrated on surfaces, lines and points.

This second part, once assuming that for each pair of molecules there is acting always an internal force K, when the number of points is equal to n, when expressed explicitly can be represented as follows:

$$\begin{aligned} & m_1 m_2 K_{1,2} \delta\rho_{1,2} + m_1 m_3 K_{1,3} \delta\rho_{1,3} + \ldots + m_1 m_n K_{1,n} \delta\rho_{1,n} \\ & \qquad + m_2 m_3 K_{2,3} \delta\rho_{2,3} + \ldots + m_2 m_n K_{2,n} \delta\rho_{2,n} \\ & \qquad\qquad \vdots \qquad\qquad \vdots \\ & \qquad\qquad\qquad + m_{n-1} m_n K_{n-1,n} \delta\rho_{n-1,n} \end{aligned} \tag{2}$$

being in general:

$$\rho_{i,j} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}. \tag{3}$$

It can be however seen that the subsequent horizontal lines appearing in it [equation (2)], which one after another have a lacking term with respect to the previous line, can be rewritten in such a way that all have exactly n terms, by writing, at the place of the equation (2) the quantity

$$\begin{aligned}
&\frac{1}{2}m_1m_1K_{1,1}\delta\rho_{1,1} + \frac{1}{2}m_1m_2K_{1,2}\delta\rho_{1,2} + \ldots. + \frac{1}{2}m_1m_nK_{1,n}\delta\rho_{1,n}\\
&\quad + \frac{1}{2}m_2m_1K_{2,1}\delta\rho_{2,1} + \frac{1}{2}m_2m_2K_{2,2}\delta\rho_{2,2} + \ldots. + \frac{1}{2}m_2m_nK_{2,n}\delta\rho_{2,n}\\
&\qquad\quad \vdots \qquad\qquad \vdots\\
&\quad + \frac{1}{2}m_im_1K_{i,1}\delta\rho_{i,1} + \frac{1}{2}m_im_2K_{i,2}\delta\rho_{i,2} + \ldots + m_im_nK_{i,n}\delta\rho_{i,n}\\
&\qquad\quad \vdots \qquad\qquad \vdots\\
&\quad + \frac{1}{2}m_nm_1K_{n,1}\delta\rho_{n,1} + \frac{1}{2}m_nm_2K_{n,2}\delta\rho_{n,2} + \ldots + \frac{1}{2}m_nm_nK_{n,n}\delta\rho_{n,n}.
\end{aligned} \tag{4}$$

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To recognize the equality of the two quantities (2), (4), it is enough to observe first that in the second one the terms containing the variations

$$\delta\rho_{1,1}, \delta\rho_{2,2} \ldots \delta\rho_{i,i} \ldots \delta\rho_{n,n}$$

are introduced only for maintaining the regularity in the progression of the indices, however it is as if they were not present, because the radicals $\rho_{1,1}, \rho_{2,2} \ldots \rho_{n,n}$ and their variations are vanishing, as it is manifest from the generic expression (3). Secondly it has to be observed that the remaining terms can be pair-wise added: therefore the two terms $(1/2)m_1m_2K_{1,2}\delta\rho_{1,2} + (1/2)m_2m_1K_{2,1}\delta\rho_{2,1}$ are equivalent to the following one $m_1m_2K_{1,2}\delta\rho_{1,2}$. Indeed it is clear that, because of (3), the quantity $\rho_{1,2}$ equals the quantity $\rho_{2,1}$: and that the force $K_{1,2}$ equals the force $K_{2,1}$, as it is implied by the principle of action and reaction and as it will become even clearer for what we will add about the way in which the generic composition of the internal action K has to be understood. In a similar way the two terms $(1/2)m_1m_3K_{1,3}\delta\rho_{1,3} + (1/2)m_3m_1K_{3,1}\delta\rho_{3,1}$ will gather into a single one $m_1m_3K_{1,3}\delta\rho_{1,3}$; and so on for the other terms. After the previous observations it is easy to persuade oneself that the quantity (4) is equivalent to the shorter form given by (2).

Any whatsoever of the horizontal series which compose the quantity (4) can be reduced by means of a triple summation. To be persuaded of the truth of this statement it is needed to recall the idea of the precedently introduced disposition of the molecules by means of the coordinates a, b, c which allows us to represent the coordinates of the generic molecule m_i as given by the following functions:

$$x_i = x(a, b, c); \quad y_i = y(a, b, c); \quad z_i = z(a, b, c). \tag{5}$$

Once we have also represented as follows:

$$x_j = x(a+f, b+g, c+k); \quad y_j = y(a+f, b+g, c+k); \quad z_j = z(a+f, b+g, c+k) \tag{6}$$

the coordinates x_j, y_j, z_j of the other whatsoever molecule m_j, which (if the molecule m_i is kept as fixed) will subsequently pass to mean all the other molecules of considered mass; and we mean that these analytical values (6) will vary following the variation of the molecules m_j when in them the variables f, g, k are changed, while the variables a, b, c are kept fixed. This is done as if we immagine to introduce in the preceding ideal configuration three new axes that have as origin the molecule m_i and [directed] parallel to those[axes] relative to the variables a, b, c,

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and as if we call, f, g, k the coordinates of a molecule whatsoever [, m_j,] with respect to said new axes. Now it is convenient to recall what was said in sect. 31, when the first part of the general equation was treated, about the way in which one can imagine how the n points of the system are distributed according to the positions relatively to the three axes, which lead to give to the ensemble of n terms [appearing in that general equation] a structure of triple series: and it will not be difficult to understand that the (i)th [series] of the horizontal series which compose the quantity (4) can be represented by means of the following finite triple integral

$$\Sigma\Delta f\Sigma\Delta g\Sigma\Delta k\cdot\frac{1}{2}m_i m_j K\delta\rho, \tag{7}$$

where the quantity ρ (equations (3), (5), (6)) has the value given by the equation

$$\begin{aligned}\rho^2 &= [x(a+f,b+g,c+k)-x(a,b,c)]^2\\ &+[y(a+f,b+g,c+k)-y(a,b,c)]^2\\ &+[z(a+f,b+g,c+k)-z(a,b,c)]^2.\end{aligned} \tag{8}$$

The limits of the precedent finite integrations will depend, as it was said in sect. 31, by the surfaces which are the boundaries of the body in the configuration preceding the real one. The expression (7) will then be adapted to represent the first, the second, the (n)th [series] of the horizontal series which are composing the quantity (4), by changing in it the coordinates a, b, c of the generic molecule m_i, i.e. by giving to these variables [, a, b, c,] those suitable values such that it [the molecule m_i] becomes one after the others the molecules $m_1, m_2 \ldots m_n$; and as the said horizontal series are exactly n (and also the terms of each of these series are n) the sum of all sums will be given by the finite sextuple integral

$$\Sigma\delta a\Sigma\delta b\Sigma\delta c\Sigma\Delta f\Sigma\Delta g\Sigma\Delta k\cdot\frac{1}{2}m_i m_j K\delta\rho. \tag{9}$$

Let us recall now what we said in the last lines of sect. 21 about the need of assigning the value σ^3 to the letter m which expresses the mass of the generic molecule: and as in the precedent integral there appears the product of two similar m it will appear manifest that this product must be replaced by the expression $\sigma^3\cdot\sigma^3$. Once we will have also recalled the analytical theorem written in equation (17) in o. sect. 26.

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theorem of which we will repeat here six times the application, we will be ready to admit that the preceding finite sextuple integral is transformed into the continuous sextuple integral

$$\int da\int db\int dc\int df\int dg\int dk\cdot\frac{1}{2}K\delta\rho \tag{10}$$

with the addition of other terms, which then must be neglected. In it the integration limits for the variables f, g, k will depend on the surfaces which bound the body in the antecedent configuration, and also on the position of the molecule m, which is kept constant, that is by the variables a, b, c which after the first three will also vary in the same domain.

72. Let us now spend some time developing some considerations about the internal force K which is exerted between one molecule and another molecule, being either attractive or repulsive forces, which would have acted both independently of the applied external forces and because of the presence of these external forces. To assume that, as it was at first suggested, it [, the internal force,] is a function $K(\rho)$ of the molecular distance only, it is admissible only in the case of fluid bodies, as in that case the part of the action due to the shape of the molecules is not present. In general (the reader is invited to read once more what said in sect. 54) when the action due to the shape of molecules cannot be neglected, there [in the expression for K] must appear also those cosines a_1, a_2, a_3, a_4, etc. which are fixing the directions of the edges or axes of the two [involved] molecules relatively to the three orthogonal axes, cosines whose values are changing from one molecule to the other and therefore must result to be functions of the corresponding coordinates. It could be very difficult to find how such functions have to be assigned (and it is sufficient to this aim only to imagine that said directions could be normal to curved surfaces having various and unknown nature for different bodies), and beyond the ignorance about the internal structure of these functions, it is not known how the[function] K depends on them. As a consequence the K, if one wants to keep its most general possible use, must be a function of the six coordinates, whose values are expressed by the equations (5), (6), and we cannot presume to express its form, as we can only argue that it has to be symmetric relatively to said six

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variables when taken three by three: that is, that when changing the x_i, y_i, z_i into the x_j, y_j, z_j, and these last into the first ones the [function K] will remain the same. This is true because it is known (as there is no reason for the contrary) that one half of K expresses the action of the molecule m_j on the molecule m_i, and the other half of K expresses the reciprocal action: it is possible to assume that the roles of the two molecules are exchanged, and notwithstanding this the analytical values must remain the same: this observation leads us to conclude the stated property of the analytical expression, as we mentioned also in the precedent section. The impossibility of assigning the function $K(x_i, y_i, z_i, x_j, y_j, z_j)$ can be deduced also by means of other arguments which I wish to omit: only I remark how, also from this point of view, the superiority of the methods which we have in our hands is emerging: with them one can continue safely to proceed in the argumentation notwithstanding an ignorance which cannot be defeated. We will add another observation about the smallness of this molecular force K, by recalling what we said about this subject at the end of sect. 22. Similarly to what expounded in sect. 18 and following ones, it has to be regarded as an elementary force which is so small that once considering the resultant of such forces on a single physical point as acted by all the other molecules of the mass, we have still a very small force of the same order of those forces $\sigma^3 X, \sigma^3 Y, \sigma^3 Z$ considered in sect.18. This concept is perfectly corresponded by the scaling given by the factor σ^6, which we will see to result from the sextuple integral (9) due to the product $m_i m_j$ of the two elementary masses.

As a consequence of what was said up to now we can, by adding up the two integrals (1), (10), and by replacing the obtained sum in the first two parts of the general equation (1) sect. 16 formulate the equation which includes the whole molecular mechanics. Before doing so we will remark that it is convenient to introduce the following definition

$$\Lambda = \frac{1}{4}\frac{K}{\rho} \tag{11}$$

by means of which it will be possible to introduce the quantity $\Lambda \delta \rho^2$ instead of the quantity $(1/2) K \delta \rho$ in the sextuple integral (10); and that

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inside this sextuple integral it is suitable to isolate the part relative to the triple integral relative to the variables f, g, k, placing it under the same sign of triple integral with respect to the variables a, b, c which includes the first part of the equation: which is manifestly allowed. In this way the aforementioned general equation becomes

$$\int da \int db \int dc \cdot \left\{ \left(X - \frac{d^2x}{dt^2} \right) \delta x + \left(Y - \frac{d^2y}{dt^2} \right) \delta y + \left(Z - \frac{d^2z}{dt^2} \right) \delta z \right.$$

$$\left. + \int df \int dg \int dk \cdot \Lambda \delta \rho^2 \right\} + \Omega = 0 \tag{12}$$

where it is intended that (as mentioned at the beginning of sect. 71) it is included in the Ω those parts which may be introduced because of the forces applied to surfaces, lines or well-determined points and also because of particular conditions which may oblige some points to belong to some given curve or surface. This equation (12) replaces the equation (1) of sect. 44 or the equation (12) of sect. 46 and it is seen how it is expressed differently (while the remaining parts are left the same) the part introduced by the reciprocal actions of the molecules which in those equations were taken into account by means of equations of conditions to hold in the whole body.

73. What remains to be done in order to deduce useful consequences from equation (12) is simply a calculation process. Once having recalled equation (8), it is seen, transforming into series the functions in the brackets,

that one has

$$\rho^2 = \left(f\frac{dx}{da} + g\frac{dx}{db} + k\frac{dx}{dc} + \frac{f^2}{2}\frac{d^2x}{da^2} + \text{etc.}\right)^2$$
$$+ \left(f\frac{dy}{da} + g\frac{dy}{db} + k\frac{dy}{dc} + \frac{f^2}{2}\frac{d^2y}{da^2} + \text{etc.}\right)^2$$
$$+ \left(f\frac{dz}{da} + g\frac{dz}{db} + k\frac{dz}{dc} + \frac{f^2}{2}\frac{d^2z}{da^2} + \text{etc.}\right)^2;$$

and by calculating the squares and gathering the terms which have equal coefficients:

$$\begin{aligned}\rho^2 = &f^2 t_1 + g^2 t_2 + k^2 t_3 + 2fgt_4 + 2fkt_5 + 2gkt_6 \\ &+ f^3 T_1 + 2f^2 gT_2 + 2f^2 kT_3 + f g^2 T_4 + \text{etc.}\end{aligned} \tag{13}$$

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in this expression the quantities $t_1, t_2, t_3, t_4, t_5, t_6$ represent the six trinomials which are already familiar to us, as we have adopted such definitions from equations (6) in sect. 34; and the quantities T_1, T_2, T_3, T_4, etc. where the index goes to infinity, represent trinomials of the same nature in which derivatives of higher and higher order appear. In all these trinomials the last two terms are always similar to the first but they differ in having the letters y, z at the place of the letter x. Those in which the second derivatives appear are of two kinds. There are those which are composed with first order and second order derivatives, and these are exactly 18 in number, which are listed in the following formula:

$$\begin{aligned}
&\frac{dx}{da}\frac{d^2x}{da^2} + \frac{dy}{da}\frac{d^2y}{da^2} + \frac{dz}{da}\frac{d^2z}{da^2}\\
&\frac{dx}{da}\frac{d^2x}{dadb} + \frac{dy}{da}\frac{d^2y}{dadb} + \frac{dz}{da}\frac{d^2z}{dadb}\\
&\frac{dx}{da}\frac{d^2x}{dadc} + \frac{dy}{da}\frac{d^2y}{dadc} + \frac{dz}{da}\frac{d^2z}{dadc}\\
&\frac{dx}{db}\frac{d^2x}{dadb} + \frac{dy}{db}\frac{d^2y}{dadb} + \frac{dz}{db}\frac{d^2z}{dadb}\\
&\frac{dx}{db}\frac{d^2x}{db^2} + \frac{dy}{db}\frac{d^2y}{db^2} + \frac{dz}{db}\frac{d^2z}{db^2}\\
&\frac{dx}{db}\frac{d^2x}{dbdc} + \frac{dy}{db}\frac{d^2y}{dbdc} + \frac{dz}{db}\frac{d^2z}{dbdc}\\
&\frac{dx}{dc}\frac{d^2x}{dadc} + \frac{dy}{dc}\frac{d^2y}{dadc} + \frac{dz}{dc}\frac{d^2z}{dadc}\\
&\frac{dx}{dc}\frac{d^2x}{dbdc} + \frac{dy}{dc}\frac{d^2y}{dbdc} + \frac{dz}{dc}\frac{d^2z}{dbdc}\\
&\frac{dx}{dc}\frac{d^2x}{dc^2} + \frac{dy}{dc}\frac{d^2y}{dc^2} + \frac{dz}{dc}\frac{d^2z}{dc^2}\\
&\frac{dx}{db}\frac{d^2x}{da^2} + \frac{dy}{db}\frac{d^2y}{da^2} + \frac{dz}{db}\frac{d^2z}{da^2}\\
&\frac{dx}{da}\frac{d^2x}{db^2} + \frac{dy}{da}\frac{d^2y}{db^2} + \frac{dz}{da}\frac{d^2z}{db^2}\\
&\frac{dx}{da}\frac{d^2x}{dbdc} + \frac{dy}{da}\frac{d^2y}{dbdc} + \frac{dz}{da}\frac{d^2z}{dbdc}
\end{aligned} \tag{14}$$

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$$\frac{dx}{dc}\frac{d^2x}{da^2}+\frac{dy}{dc}\frac{d^2y}{da^2}+\frac{dz}{dc}\frac{d^2z}{da^2}$$
$$\frac{dx}{db}\frac{d^2x}{dadc}+\frac{dy}{db}\frac{d^2y}{dadc}+\frac{dz}{db}\frac{d^2z}{dadc}$$
$$\frac{dx}{da}\frac{d^2x}{dc^2}+\frac{dy}{da}\frac{d^2y}{dc^2}+\frac{dz}{da}\frac{d^2z}{dc^2}$$
$$\frac{dx}{dc}\frac{d^2x}{dadb}+\frac{dy}{dc}\frac{d^2y}{dadb}+\frac{dz}{dc}\frac{d^2z}{dadb}$$
$$\frac{dx}{dc}\frac{d^2x}{db^2}+\frac{dy}{dc}\frac{d^2y}{db^2}+\frac{dz}{dc}\frac{d^2z}{db^2}$$
$$\frac{dx}{db}\frac{d^2x}{dc^2}+\frac{dy}{db}\frac{d^2y}{dc^2}+\frac{dz}{db}\frac{d^2z}{dc^2}$$

Then we have the trinomial constituted with second order derivatives only, and these last are 21 in number, and are listed in the following formula

$$\begin{gathered}
\left(\frac{d^2x}{da^2}\right)^2+\left(\frac{d^2y}{da^2}\right)^2+\left(\frac{d^2z}{da^2}\right)^2\\
\left(\frac{d^2x}{db^2}\right)^2+\left(\frac{d^2y}{db^2}\right)^2+\left(\frac{d^2z}{db^2}\right)^2\\
\left(\frac{d^2x}{dc^2}\right)^2+\left(\frac{d^2y}{dc^2}\right)^2+\left(\frac{d^2z}{dc^2}\right)^2\\
\left(\frac{d^2x}{dadb}\right)^2+\left(\frac{d^2y}{dadb}\right)^2+\left(\frac{d^2z}{dadb}\right)^2\\
\left(\frac{d^2x}{dadc}\right)^2+\left(\frac{d^2y}{dadc}\right)^2+\left(\frac{d^2z}{dadc}\right)^2\\
\left(\frac{d^2x}{dbdc}\right)^2+\left(\frac{d^2y}{dbdc}\right)^2+\left(\frac{d^2z}{dbdc}\right)^2\\
\frac{d^2x}{da^2}\frac{d^2x}{db^2}+\frac{d^2y}{da^2}\frac{d^2y}{db^2}+\frac{d^2z}{da^2}\frac{d^2z}{db^2}\\
\frac{d^2x}{da^2}\frac{d^2x}{dc^2}+\frac{d^2y}{da^2}\frac{d^2y}{dc^2}+\frac{d^2z}{da^2}\frac{d^2z}{dc^2}\\
\frac{d^2x}{db^2}\frac{d^2x}{dc^2}+\frac{d^2y}{db^2}\frac{d^2y}{dc^2}+\frac{d^2z}{db^2}\frac{d^2z}{dc^2}\\
\frac{d^2x}{da^2}\frac{d^2x}{dadb}+\frac{d^2y}{da^2}\frac{d^2y}{dadb}+\frac{d^2z}{da^2}\frac{d^2z}{dadb}\\
\frac{d^2x}{da^2}\frac{d^2x}{dadc}+\frac{d^2y}{da^2}\frac{d^2y}{dadc}+\frac{d^2z}{da^2}\frac{d^2z}{dadc}
\end{gathered}\tag{15}$$

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$$\frac{d^2x}{da^2}\frac{d^2x}{dbdc}+\frac{d^2y}{da^2}\frac{d^2y}{dbdc}+\frac{d^2z}{da^2}\frac{d^2z}{dbdc}$$

$$\frac{d^2x}{db^2}\frac{d^2x}{dadb}+\frac{d^2y}{db^2}\frac{d^2y}{dadb}+\frac{d^2z}{db^2}\frac{d^2z}{dadb}$$
$$\frac{d^2x}{db^2}\frac{d^2x}{dadc}+\frac{d^2y}{db^2}\frac{d^2y}{dadc}+\frac{d^2z}{db^2}\frac{d^2z}{dadc}$$
$$\frac{d^2x}{db^2}\frac{d^2x}{dbdc}+\frac{d^2y}{db^2}\frac{d^2y}{dbdc}+\frac{d^2z}{db^2}\frac{d^2z}{dbdc}$$
$$\frac{d^2x}{dc^2}\frac{d^2x}{dadb}+\frac{d^2y}{dc^2}\frac{d^2y}{dadb}+\frac{d^2z}{dc^2}\frac{d^2z}{dadb}$$
$$\frac{d^2x}{dc^2}\frac{d^2x}{dadc}+\frac{d^2y}{dc^2}\frac{d^2y}{dadc}+\frac{d^2z}{dc^2}\frac{d^2z}{dadc}$$
$$\frac{d^2x}{dc^2}\frac{d^2x}{dbdc}+\frac{d^2y}{dc^2}\frac{d^2y}{dbdc}+\frac{d^2z}{dc^2}\frac{d^2z}{dbdc}$$
$$\frac{d^2x}{dadb}\frac{d^2x}{dadc}+\frac{d^2y}{dadb}\frac{d^2y}{dadc}+\frac{d^2z}{dadb}\frac{d^2z}{dadc}$$
$$\frac{d^2x}{dadb}\frac{d^2x}{dbdc}+\frac{d^2y}{dadb}\frac{d^2y}{dbdc}+\frac{d^2z}{dadb}\frac{d^2z}{dbdc}$$
$$\frac{d^2x}{dadc}\frac{d^2x}{dbdc}+\frac{d^2y}{dadc}\frac{d^2y}{dbdc}+\frac{d^2z}{dadc}\frac{d^2z}{dbdc}.$$

The trinomials with third order derivatives are of three kinds: there are those constituted by derivatives of first and third order, and one can count 30 of them: there are those constituted by derivatives of second and third order, and they are 60 in number: and there are those which contain only third order derivatives and they are 55 in number. I am not writing them, as everybody who is given the needed patience can easily calculate them by himself, as it can be also done for those trinomials containing derivatives of higher order.

When using the equation (13) to deduce the value of the variation $\delta\rho^2$, it is clear that the characteristic δ will need to be applied only to the trinomials we have discussed up to now, so that we will have:

$$\begin{aligned}\delta\rho^2 =& f^2\delta t_1 + g^2\delta t_2 + k^2\delta t_3 + 2fg\,\delta t_4 + 2fk\,\delta t_5 + 2gk\,\delta t_6\\ &+f^3\delta T_1 + 2f^2g\delta T_2 + 2f^2k\delta T_3 + f g^2\delta T_4 + \text{etc.}\end{aligned} \tag{16}$$

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Indeed the coefficients f^2, g^2, k^2, $2fg$, etc. are always the same for every functions giving the variables x, y, z in terms of the variables a, b, c, and therefore cannot be affected by that operation whose aim is simply to change the form of these functions. Vice versa, by multiplying the precedent equation (16) times Λ and then integrating with respect to the variables f, g, k in order to deduce the fourth term under the triple integral of the equation (12), such an operation is affecting only the quantities $\Lambda f^2, \Lambda g^2$, etc. and the variations $\delta t_1, \delta t_2, \delta t_3 \ldots \delta T_1, \delta T_2$, etc. cannot be affected by it, as the trinomials $t_1, t_2, t_3 \ldots T_1, T_2$, etc. (one has to consider carefully which is their origin) do not contain the variables f, g, k: therefore such variations result in being constant factors, which are to be multiplied by the integrals to be calculated in the subsequent terms of the series. After these considerations the truth of the following equation is manifest:

$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{17}$$

$$\begin{aligned}&(1)\ \delta t_1 + (2)\ \delta t_2 + (3)\ \delta t_3 + (4)\ \delta t_4 + (5)\ \delta t_5 + (6)\ \delta t_6\\ &+ (7)\ \delta T_1 + (8)\ \delta T_2 + (9)\ \delta T_3 + (10)\ \delta T_4 + \text{etc.}\end{aligned}$$

where the coefficients (1), (2), etc. indicated by means of numbers in between brackets, must each be regarded to be a function of the variables a, b, c, [the same function] that is obtained calculating the aforementioned definite integrals. This is the equivalent quantity which should be introduced in the equation (12) at the place of the fourth term under the triplicate integral.

74. A new proposition, to which the reader should pay much attention, is that all the trinomials T_1, T_2, T_3, etc. where the subscript goes to infinity, which[trinomials] appear in the precedent equation (17), can only be expressed by means of the only first six $t_1, t_2, t_3, t_4, t_5, t_6$, and of their derivatives with respect to the variables a, b, c of all orders. I started to suspect this analytical truth because of the necessary correspondence which must hold between the results which are obtained via the method considered in this Capo and those results obtained via the method considered in the Capos III and IV. I have then verified the stated property for 39 terms of the precedent

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series (17), beyond the first six, that is for all trinomials written in the tables (14), and (15), and after these calculations I abandoned myself to the analogy: and this will sooner or later be unavoidable, because our series is infinite and it will be impossible to check all its terms. Now I will say how I performed the stated verification and the importance of the conclusions will justify the lengthiness of the calculations, which, except for the prolixity, do not present any difficulty. Checking the values of the variables $t_1, t_2, t_3, t_4, t_5, t_6$ (equations (6) sect. 34) it is immediate to recognize that the first nine trinomials of the table (14) respectively have the values:

$$\begin{array}{ccc}
\frac{1}{2}\frac{dt_1}{da}; & \frac{1}{2}\frac{dt_1}{db}; & \frac{1}{2}\frac{dt_1}{dc} \\
\frac{1}{2}\frac{dt_2}{da}; & \frac{1}{2}\frac{dt_2}{db}; & \frac{1}{2}\frac{dt_2}{dc} \\
\frac{1}{2}\frac{dt_3}{da}; & \frac{1}{2}\frac{dt_3}{db}; & \frac{1}{2}\frac{dt_3}{dc}.
\end{array}$$

It is then possible to find that (and this can be verified by simple substitution of known values) the trinomials labelled with the number ten, eleven, thirteen, fifteen, seventeen and eighteen are equivalent respectively to the following binomials:

$$\begin{array}{ccc}
\frac{dt_4}{da} - \frac{1}{2}\frac{dt_1}{db}; & \frac{dt_4}{db} - \frac{1}{2}\frac{dt_2}{da}; & \frac{dt_5}{da} - \frac{1}{2}\frac{dt_1}{dc} \\
\frac{dt_5}{dc} - \frac{1}{2}\frac{dt_3}{da}; & \frac{dt_6}{db} - \frac{1}{2}\frac{dt_2}{dc}; & \frac{dt_6}{dc} - \frac{1}{2}\frac{dt_3}{db},
\end{array}$$

and that the trinomials labelled with the numbers twelve, fourteen and sixteen have respectively these other values:

$$\frac{1}{2}\frac{dt_4}{dc} + \frac{1}{2}\frac{dt_5}{db} - \frac{1}{2}\frac{dt_6}{da}; \quad \frac{1}{2}\frac{dt_4}{dc} - \frac{1}{2}\frac{dt_5}{db} + \frac{1}{2}\frac{dt_6}{da}; \quad -\frac{1}{2}\frac{dt_4}{dc} + \frac{1}{2}\frac{dt_5}{db} + \frac{1}{2}\frac{dt_6}{da}.$$

In this way the stated proposition is proven relatively to the first 18 trinomials.

Let us now imagine to have formed 18 equations having each as left-hand sides the trinomials of the table (14) and as right-hand side respectively the values to which we have proven they are equal. Among these equations let us consider the

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first, the tenth and the thirteenth ones and let us multiply them orderly times l_1, m_1, n_1, and then let us sum the obtained results: then let us multiply the same equations again times l_2, m_2, n_2, and again sum the obtained results: finally let us multiply the same three equations times l_3, m_3, n_3, and again sum the obtained results: by considering the nine equations of sect. 14 labelled (28), we will manage to get an expression for the values of the three second order derivatives d^2x/da^2, d^2y/da^2, d^2z/da^2. Following the same procedure, suitably choosing among the aforementioned 18 equations, we will determine the values of the other second order derivatives and we will get

$$
\begin{aligned}
2H\,\frac{d^2x}{da^2} &= l_1\frac{dt_1}{da} + m_1\left(2\frac{dt_4}{da} - \frac{dt_1}{db}\right) + n_1\left(2\frac{dt_5}{da} - \frac{dt_1}{dc}\right)\\
2H\,\frac{d^2y}{da^2} &= l_2\frac{dt_1}{da} + m_2\left(2\frac{dt_4}{da} - \frac{dt_1}{db}\right) + n_2\left(2\frac{dt_5}{da} - \frac{dt_1}{dc}\right)\\
2H\,\frac{d^2z}{da^2} &= l_3\frac{dt_1}{da} + m_3\left(2\frac{dt_4}{da} - \frac{dt_1}{db}\right) + n_3\left(2\frac{dt_5}{da} - \frac{dt_1}{dc}\right)\\
2H\,\frac{d^2x}{db^2} &= l_1\left(2\frac{dt_4}{db} - \frac{dt_2}{da}\right) + m_1\frac{dt_2}{db} + n_1\left(2\frac{dt_6}{db} - \frac{dt_2}{dc}\right)\\
2H\,\frac{d^2y}{db^2} &= l_2\left(2\frac{dt_4}{db} - \frac{dt_2}{da}\right) + m_3\frac{dt_2}{db} + n_2\left(2\frac{dt_6}{db} - \frac{dt_2}{dc}\right)\\
2H\,\frac{d^2z}{db^2} &= l_3\left(2\frac{dt_4}{db} - \frac{dt_2}{da}\right) + m_3\frac{dt_2}{db} + n_3\left(2\frac{dt_6}{db} - \frac{dt_2}{dc}\right)\\
2H\,\frac{d^2x}{dc^2} &= l_1\left(2\frac{dt_5}{dc} - \frac{dt_3}{da}\right) + m_1\left(2\frac{dt_6}{dc} - \frac{dt_3}{db}\right) + n_1\frac{dt_3}{dc}\\
2H\,\frac{d^2y}{dc^2} &= l_2\left(2\frac{dt_5}{dc} - \frac{dt_3}{da}\right) + m_2\left(2\frac{dt_6}{dc} - \frac{dt_3}{db}\right) + n_2\frac{dt_3}{dc}\\
2H\,\frac{d^2z}{dc^2} &= l_3\left(2\frac{dt_5}{dc} - \frac{dt_3}{da}\right) + m_3\left(2\frac{dt_6}{dc} - \frac{dt_3}{db}\right) + n_3\frac{dt_3}{dc}
\end{aligned}
$$

$$
\begin{aligned}
2H\,\frac{d^2x}{dadb} &= l_1\frac{dt_1}{db} + m_1\frac{dt_2}{da} + n_1\left(\frac{dt_5}{db} + \frac{dt_6}{da} - \frac{dt_4}{dc}\right)\\
2H\,\frac{d^2y}{dadb} &= l_2\frac{dt_1}{db} + m_2\frac{dt_2}{da} + n_2\left(\frac{dt_5}{db} + \frac{dt_6}{da} - \frac{dt_4}{dc}\right)\\
2H\,\frac{d^2z}{dadb} &= l_3\frac{dt_1}{db} + m_3\frac{dt_2}{da} + n_3\left(\frac{dt_5}{db} + \frac{dt_6}{da} - \frac{dt_4}{dc}\right)\\
2H\,\frac{d^2x}{dadc} &= l_1\frac{dt_1}{dc} + m_1\left(\frac{dt_4}{dc} + \frac{dt_6}{da} - \frac{dt_5}{db}\right) + n_1\frac{dt_3}{da}
\end{aligned}
$$

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$$
\begin{aligned}
2H\,\frac{d^2y}{dadc} &= l_2\frac{dt_1}{dc} + m_2\left(\frac{dt_4}{dc} + \frac{dt_6}{da} - \frac{dt_6}{db}\right) + n_2\frac{dt_3}{da}\\
2H\,\frac{d^2z}{dadc} &= l_3\frac{dt_1}{dc} + m_3\left(\frac{dt_4}{dc} + \frac{dt_6}{da} - \frac{dt_5}{db}\right) + n_3\frac{dt_3}{da}
\end{aligned}
$$

$$
\begin{aligned}
2H\,\frac{d^2x}{dbdc} &= l_1\left(\frac{dt_4}{dc} + \frac{dt_5}{db} - \frac{dt_6}{da}\right) + m_1\frac{dt_2}{dc} + n_1\frac{dt_3}{db}\\
2H\,\frac{d^2y}{dbdc} &= l_2\left(\frac{dt_4}{dc} + \frac{dt_5}{db} - \frac{dt_6}{da}\right) + m_2\frac{dt_2}{dc} + n_2\frac{dt_3}{db}\\
2H\,\frac{d^2z}{dbdc} &= l_3\left(\frac{dt_4}{dc} + \frac{dt_5}{db} - \frac{dt_6}{da}\right) + m_3\frac{dt_2}{dc} + n_3\frac{dt_3}{db}.
\end{aligned}
$$

Now, by means of these values, let us look for the values of the trinomials of the table (15). Recalling the equations (31), (33), (34) in sect. 67 we will see that these values result to be constituted uniquely by the variables $t_1, t_2 \ldots t_6$ and by their first order derivatives, and this is exactly what we wanted to prove. For instance the value of the first trinomial

$$\left(\frac{d^2x}{da^2}\right)^2 + \left(\frac{d^2y}{da^2}\right)^2 + \left(\frac{d^2z}{da^2}\right)^2$$

can be proven to be equal to a fraction whose numerator is

$$\begin{aligned}&\left(t_2t_3 - t_6^2\right)\left(\frac{dt_1}{da}\right)^2 + \left(t_1t_3 - t_5^2\right)\left(2\frac{dt_4}{da} - \frac{dt_1}{db}\right)^2 + \left(t_1t_2 - t_4^2\right)\left(2\frac{dt_5}{da} - \frac{dt_1}{dc}\right)^2\\ &+ 2\left(t_5t_6 - t_3t_4\right)\frac{dt_1}{da}\left(2\frac{dt_4}{da} - \frac{dt_1}{db}\right) + 2\left(t_4t_6 - t_2t_5\right)\frac{dt_1}{da}\left(2\frac{dt_6}{da} - \frac{dt_1}{dc}\right)\\ &+ 2\left(t_4t_5 - t_1t_6\right)\left(2\frac{dt_4}{da} - \frac{dt_1}{db}\right)\left(2\frac{dt_5}{da} - \frac{dt_1}{dc}\right)\end{aligned}$$

and whose denominator is:

$$4\left(t_1t_2t_3 + 2t_4t_5t_6 - t_1t_6^2 - t_2t_5^2 - t_3t_4^2\right).$$

Similarly forms can be found for the values of the other twenty trinomials of the table (15): therefore it was not exaggerated to affirm that the stated analytical property has been actually verified for 39 trinomials.

75. Once the proposition of the precedent section. has been given, it is manifest that equation (17) can assume the following other form

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$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{18}$$

$$\begin{aligned}&(\alpha)\ \delta t_1 + (\beta)\ \delta t_2 + (\gamma)\ \delta t_3 + \ldots + (\epsilon)\ \frac{\delta dt_1}{da} + (\zeta)\ \frac{\delta dt_1}{db} + (\eta)\ \frac{\delta dt_1}{dc}\\ &+ (\vartheta)\ \frac{\delta dt_2}{da} + \ldots + (\lambda)\ \frac{\delta d^2t_1}{da^2} + (\mu)\ \frac{\delta d^2t_1}{dadb} + \ldots + (\xi)\ \frac{\delta d^2t_2}{da^2} + (o)\ \frac{\delta d^2t_2}{dadb} + \text{etc.}\end{aligned}$$

in which the coefficients $(\alpha), (\beta) \ldots (\epsilon) \ldots (\lambda) \ldots$ etc. are suitable quantities given in terms of coefficients $(1), (2) \ldots (7), (8) \ldots$ of equation (17), of the quantities $t_1, t_2 \ldots t_6$, and of the derivatives of any order of these trinomials with respect to the variables a, b, c . Besides, the variations δt_1, $\delta t_2 \ldots$.(with the index varying up to infinity) and the variations of all their derivatives of all orders $\delta dt_1/da, \delta dt_1/db$, etc. appear in the (18) only linearly. Now it is a fundamental principle of the calculus of variations (and we used it also in this memoir in sect. 36 and elsewhere) that a series, as the precedent one, where the variations of some quantities and the variations of their derivatives with respect to the fundamental variables a, b, c appear linearly, can always be transformed into an expression which contains that quantities without any sign of derivation, with the addition of other terms which are their exact derivatives with respect to one of the three simple independent variables. As a consequence of this principle applied to the equation (18), the expression which follows can be given

$$\int df \int dg \int dk \cdot \Lambda\delta\rho^2 = \tag{19}$$

$$\begin{aligned}&A\delta t_1 + B\,\delta t_2 + C\,\delta t_3 + D\delta t_4 + E\delta t_5 + F\,\delta t_6\\ &+ \frac{d\Delta}{da} + \frac{d\Theta}{db} + \frac{d\Upsilon}{dc}.\end{aligned}$$

The values of the six coefficients A, B, C, D, E, F are series constructed with the coefficients $(\alpha), (\beta), (\gamma) \ldots (\epsilon), (\zeta) \ldots (\lambda)$, etc. of equation (18) which appear linearly, with alternating signs and affected by derivations of higher and higher order when we move ahead in the terms of said series: the quantities Δ, Θ, Υ are series of the same form as the terms which are transformed, in which the coefficients of the variations have a composition

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similar to the one which we have described for the six coefficients A, B, C, D, E, F.

Once – instead of the quantity under the integral sign in the left-hand side of the equation (12) – one introduces the quantities which are on the right-hand side of the equation (19), it is clear to everybody that an integration is possible for each of the last three terms of the sum appearing in it and that, as a consequence, these terms only give quantities which supply boundary conditions. What remains under the triple integral is the only sextinomial which is absolutely similar to the sextinomial already used in equation (10) sect. 35 for rigid systems. Therefore after having remarked on the aforementioned similarity, the analytical procedure to be used here will result as perfectly equal to the one used in sect. 35, the procedure which led to the equations (26), (29) in sect. 38 and it will become possible to demonstrate the extension of said equations to every kind of bodies which do not respect the constraint of rigidity, as was mentioned at the end of sect. 38. It will also be visible the coincidence of the obtained results with those which are expressed in the equations (23) of sect. 50 which hold for every kind of systems and which were shown in the Capo IV by means of those intermediate coordinates p, q, r, whose consideration, when using the approach used in this Capo, will not be needed.

76. The preceding analysis allows for many useful considerations. First of all I will mention those which are needed to clarify the doubts to which we already tried to answer in sect. 63 when we promised to add more explanations later in this section. Starting from the antecedent ideal configuration of the molecules described by means of the coordinates a, b, c, and arriving at the real configuration, we intend this second configuration as described by means of two different reference triples of orthogonal axes, those of the variables p, q, r, and of the variables x, y, z. To express the real configuration by means of the variables p, q, r we need simply to copy the preceding analysis writing everywhere p, q, r where previously we had written x, y, z. Now if we want to pass from the coordinates p, q, r to the coordinates x, y, z, we will observe that in the case of the fluids, as it was said in sect. 72 the internal force K, or equivalently the quantity Λ,

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is uniquely a function of the molecular distance ρ; and if one considers carefully the [algebraic and differential] form of the coefficients (1), (2), (3) . . . . . . in the equation (17), and that of the coefficients $(\alpha), (\beta), (\gamma) \ldots\ldots$ of the equation (18), and finally that of the six coefficients A, B, C, D, E, F in the equation (19), we will be easily persuaded that in these last six quantities, when introducing the first reference frame relative to the variables p, q, r, will contain such variables p, q, r only because these last appear in the radical ρ (equation (8)), and in the six trinomials $t_1, t_2, \ldots t_6$, having written everywhere the letters p, q, r at the place of the letters x, y, z. Therefore the six quantities A, B, C, D, E, F will enjoy the analytical property which we have discussed many times, which consists in changing when undergoing the substitution of the values (31) sect. 40 in quantities which depends in an equal way on the variables x, y, z, as any trace of the nine angular quantities $\alpha_1, \beta_1, \gamma_1, \alpha_2$, etc., disappears because such property is verified by the radical ρ, and by each of the six trinomials $t_1, t_2 \ldots t_6$. Now in this case one has only to perform the indicated operations and he will be persuaded that the situation is exactly the one which we have described, once the equations among the angular quantities introduced in sect. 33 are recalled. Here is one of those results forecasting which, at the beginning of the present Capo, we stated preliminarily that the two conceived ways for calculating the internal constraints among the molecules were illustrating each other.

Vice versa the procedure used in the Capo IV makes possible the treatment of the quantities at the boundaries which would be very difficult with the present method. We have seen there (sect. 52) as in the boundary conditions the same six quantities appear which express the effect of internal forces in the three [bulk] equations valid in all points of the body mass: here, however, the boundary conditions would be complicated because of the role played by those other quantities Δ, Θ, Υ which appear in the last terms of the equation (19). It is convenient to say that the whole part introduced by such terms in the limit quantities is playing only an apparent role: and this analytical fact is often encountered in the calculus of variations. In the questions related to the calculus of variations which can be referred to formulae involving definite triple integrals, if one has an equation of condition $L = 0$, and considers the differential equations

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$$\frac{dL}{da} = 0, \frac{dL}{db} = 0, \frac{dL}{dc} = 0; \frac{d^2L}{da^2} = 0, \text{etc.}$$

of all orders, and if he treats them as if they were many new equations of conditions, multiplying their variations times undetermined coefficients and introducing the obtained products in the equations for maximum or minimum under the integral sign and operating the usual transformations he will add (to the quantity which would have been left without doing the further operations which we have said) a quantity of the same nature of the trinomial which is at the end of the equation (19). In such a case one understands that the novelty of the appearance of quantities at the boundaries can be only apparent, because the aforesaid differentiated equations do not have any new meaning which was not already expressed by the equation $L = 0$ alone.

However, the requirement that the said boundary quantity must vanish, as here we are not dealing with a result concerning equations of conditions, which cannot be used more than really necessary, but simply with a necessary consequence of the comparison of the two presented methods, [that requirement] could be a tool which can lead us to the discovery of new truths, which could lead us, for instance, to more detailed investigations about the nature of molecular interactions. I limit myself to a simple hint: but concerning the molecular action I cannot avoid to remark something which seems to me to be not irrelevant. The reader will have discovered that in the preceding analysis to come to the most important conclusions, that is to the equations which hold for all points of the mass, it was not needed to use the hypotheses accepted by modern geometers, and by myself in the §.V in the memoir which was published in the Tome XXI of these Proceedings. It was said that the molecular action must be appreciable at distances which cannot be sensed and not appreciable for distances which can be sensed: this statement may be true, but whatever it will be, one can ignore, in our discussion, such a hypothesis: the integrals with respect to the variables s, g, k in the equations (17), (18), (19) can be regarded to be extended up at the boundaries of the body, and not only for very small distances, what is done with some effort.

I will conclude the Capo by imagining an objection. Somebody could say that: 'you have here calculated the internal actions between

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one molecule and another molecule, and we can agree about this point: but why you have not done the same also in the Capo IV, where you left out all the second part of the general equation (1) sect. 16? Such omission can nullify all the deductions obtained with those analytical procedures'. I answer as follows: 'I have not taken into account then all the internal actions between pairs of molecules because at their place I considered the six equations of condition which were treated there: indeed considering also said actions would have meant to include their effects two times. Also when one considers the motion of rigid bodies the action between each pair of molecules is effective, but everybody who knows the spirit of analytical mechanics will be persuaded that all of these actions were accounted for by the six equations (8) in sect. 34 which are valid for all physical points: on the contrary if one doubt could be raised it could concern the fact that the six equations of conditions were redundant, and for this reason the considerations in the section 39 became necessary.

The same has to be said in the general case of the equations of conditions (14) in sect. 47 which also hold for all points of the mass: the only fact of considering and calculating them is equivalent to account for all internal forces, although it is not explicitly detailed how this occurs. To somebody it could seem – in some aspects – more persuasive the method used in this Capo, as it allows us to have an explanation about the way in which the molecular actions are acting: however I believe that those who think in the best way will prefer the method presented in the Capo IV, which is a method more direct and more powerful, because it is based on that geometrical principle which assumes the indifference of the choice of the reference frames with respect to the system, principle which we will need also in the next Capo, and which contains the reason of so many mechanical truths. Moreover I see that it is possible – by following the flow of reasonings expounded in the Capo IV – to explain the true meaning of that other Lagrangian principle, which at the beginning of the same Capo we said to be too abstract, and to firmly establish its extension and the effective way of using it. Aforementioned explication, however, would not be maybe of great utility, and I state this because I believe that all advantages to which Lagrange was aiming by postulating that principle could be obtained more directly and naturally by using the procedure described in the same Capo IV.

Appendix B Eulogy in memoriam of Vincenzo Brunacci by Gabrio Piola

It is extremely painful for us to announce in this document the death of a truly great man, who, as during his life was a glory for Italy so now moves, because of his loss, everybody inconsolably to tears. One of the most eminent mathematicians of our time, the illustrious professor Vincenzo Brunacci of Pavia, suffering for many years because of a painful disease, on the 16th day of the current month of June was attacked by those very strong convulsions which were consequence of it, ceased to live surrounded by friendship and religion. To recall the merits of the deceased person, during the time while still everybody cries on his tomb, it is really a way for increasing painful laments, even if it will be simply a meagre tribute of praise which will be written by our pen. However, our intention is not that of presenting a formal elogium; this kind of encomium soon will be heard in the most erudite academies and in all palaces of sciences.

Vincenzo Brunacci was born in the fatherland of Galileo the 3rd day of March 1768, his father first name being Ignazio Maria and his mother being named Elisabetta Danielli.

It seemed that the Spirit of Italy, who was in great sufferance because in that time the most brilliant star of all mathematical sciences, the illustrious Lagrangia, had left the Nation, that Spirit wanted to have the rise of another star which, being born on the banks of the river Arno, was bound to become the successor of the first one. This consideration is presenting itself even more spontaneous as soon as we will remark that Brunacci was the first admirer in Italy of the luminous Lagrangian doctrines, the first scientist who diffused and supported them, the first scientist who in his studies was always a very creative innovator in their applications.

His first maestri were two famous Italians, Father Canovai and the great geometer Pietro Paoli. Although in his first youth he was diverted by other studies, which were opposed to his natural inclinations and from which, because of due respect he could not subtract himself, he still was able to cultivate at the same time those studies for which he had been born.Very soon he was pupil only of the classic textbooks and of himself. Very soon, as he did not allow to any man to see his genius while being born or in his first childhood, in the 'opuscolo analitico', printed in Livorno in the year 1792, he showed his fully developed creative ingenuity in that part of 'sublime calculus' in which he was bound to find the subject for great discoveries.

Called as Professor of Nautical Sciences in the College of Naval cadets in Livorno in the year 1796 he published the Navigation Treatise of which were printed three more editions more and more improved and detailed. This work was and still is the only Italian textbook which is really suitable to educate the practical pilot.

In the year 1798 the work entitled 'Calculus of linear equations' was printed in Florence. In this oeuvre our author showed that he could successfully compete with the most eminent geometers of Europe. Postponing to a later discussion, as it will be suitable to do so while talking about another book, the exposition of the many merits of this book, we will limit ourselves here to say that while Laplace was calling falsely not-integrable certain linear equations in which second order partial differentials appear, while Paoli and Lacroix were investigating the same subject and started to doubt about the statements of the mentioned French geometer, Brunacci gave a method for integrating similar equations, being able to generalize it to all differential orders. Paoli himself, by exposing this method for a particular case in the third of the parts which form the supplement to his book *Elements of Algebra*, calls illustrious geometer that scientist [i.e. Brunacci himself] who had been his student. A voice finally was uttered from that place which had given birth to such eminent scientists as Cavalieri, Frisi, Agnesi, and Oriani and [this voice] called Brunacci to occupy the empty chair [of mathematics] in Pavia. He arrived there in the year 1800 and although he arrived in a place where the mathematical sciences were not ignored he met the greatest expectations and advanced the fame [of that chair] to a yet unrivalled dignity.

Indeed it is not sufficient to be erudite in science to become its professor, it is necessary to have the gift of the word, the capacity of finding the right way to explain it. These gifts were given to him in the highest, unrivalled level. Whoever heard him will admit that my expressions although admired however are not enough to reveal the truth. The mathematical teaching, when coming from his lips, was destituted of every difficulty and bitterness, and was presented with a peculiar charm and incantation resulting to be at the same time education for the mind and pleasure for the ears. It was then that the mathematical schools on the banks of Ticino river reached the prestige which also nowadays is honouring them. It was then that Vincenzo, having dedicated himself completely to his science, started with all his forces to promote it.

The ‘Analisi Derivata’ (Analysis of Functions) was printed in Pavia in the year 1802. It is in this book that one can find one of the most sublime concepts which was ever conceived by the human mind, that is the principle of derivation. Because of it all the different parts of mathematical sciences are tied and interconnected and it is opened an endless view which allows us to consider as possible their infinite development. Soon he conceived the challenging thought which lead him to re-write the whole body of the doctrine of his science in many volumes, enriched by every novel concept which had been formulated in the modern works.

This endeavour may have frightened everybody except him: he was also pushed by the advice of that Sovereign Investigator of the stars who, being in Milan, wrote to persuade him to start this oeuvre in the year 1800, believing that he was the only one among the Italians who was capable to complete it successfully. The oeuvre of the *Course of Sublime Mathematics* was printed in Florence in four volumes in the years 1804, 1806, 1807, 1808.

One would need a very long time to expound, as it should be done, the merits of this book, but I will want to shortly describe here its contents. The first volume contains the calculus of finite differences. To this calculus, which was originated among the obscure calculations presented by Taylor, which was developed in many memoirs disseminated here and there in the proceedings of many academies, for the first time was given a scientific order and method by the Florentine geometer. He wrote it finding in his ingenious mind all that which was lacking in order to form a perfect theoretical frame, and he infused in it all novel results which he had obtained in his already mentioned works. It was his original contribution the integration of linear equations of second order with variable coefficients, it was his contribution a new formula for the integration of linear equation of all orders with constant coefficients; it was his own the method to complete the integrals to be replaced to the one proposed by D’Alembert, [method] which he successfully introduced also in the differential calculus; besides, [it was his original contribution] the idea of the variable probability and the solution of the related problems, with which he metaphorically could seize the wheel of the fortune and advanced in the field where the genius of Lagrangia had stopped, when in the Proceedings of the Academy of Berlin (1775) he had given the solution of those problems only in the case of constant probability.

While citing the name of Lagrangia I will not neglect to say that Brunacci was the first in Italy to see that admirable light which the theory of analytical functions can spread among the mysterious smog which was obscuring the infinitesimal analysis. He immediately conceived the idea of introducing it also among us: but oh! how difficult was that endeavour! The Lagrangian notation, completely new, produced a kind of revulsion: not all minds were firm enough to be able to maintain – in the middle of a revolution – their contact with the spirit of the calculus: he himself told me many times about the great obstacles which he needed courageously to confront in pursuing his effort. He finally managed to reach his aim, by reconciling the Lagrangian ideas with the Leibnitz notation, together with the brackets introduced by Fontaine.

In this way are written the other three volumes of the said course, where, however, the author did not neglect to introduce with great skill whenever possible the notation of the geometer of Turin in order to make it familiar to us.We will only add that in the remaining part of his great oeuvre one can find the rich results of mathematical analysis gathered from the most recent memoirs of the most celebrated geometers and especially from the immense body of works of the great Euler which he called his delight and from which he admitted to have learnt that lucid order which makes his own works so brilliant.

Oh! How many times I heard him talking about Euler with a great enthusiasm and to urge me, and many of his other students, to study the work of the only author who is suitable to educate a geometer! The great men, even when are quoting the results of other authors are able to give to the subject their own mark. This statement is true for that book where Brunacci infused many of his ideas, not only those which we mentioned but also many others which equally would merit to be mentioned, and in particular in those various problems of every kind of applied mathematics and in the calculus of variations which is reduced to the differential calculus and is there exposed with a great detail, and finally in the mixed calculus, of which he was the first to give the true principles and to expound in orderly way the doctrine.

However a triumph which Brunacci obtained in front of all his rivals attracts my attention. The theory of the hydraulic water hammer, which seemed to be rebellious to the lordship of mathematical analysis, and which was demanded with a golden prize – without success – by the Academy of Berlin to the greatest geometers of Europe in the year 1810 and then again in the year 1812 doubling the prize, since 1810 was discovered by Brunacci who should have had received the promised reward if an accident – which I do not want to recall here – had not defrauded him of the deserved glory; this theory was published in the treatise of the hydraulic water hammer of which were printed two editions; in this Treatise the mentioned theory, reduced to formulae and problems, is expounded in the most efficacious way.

It is custom of the brave to prepare himself to the new victories and not to be proud of the past ones. Therefore a new arena was chosen by our athlete where he managed to defeat strong rivals. If he competes for discovering the nature in hydraulic problems, Brunacci is awarded by the Società Italiana: if he needs to reach the highest abstraction in order to find the best metaphysics for the Calculus, Brunacci is awarded by the Accademia di Padova. The Proceedings of the Illustrious Società Italiana carry the name of Brunacci as author of many of the best memoirs: too long would be to cite all of them. I will only mention that one on particular solutions for the finite difference equations, which our author treats in a way which is similar to the one used by Lagrangia for differential equations, and where he discovered some very elegant theorems valid for finite difference equations which are not true for differential equations. And I will mention the other one on shock waves in fluids which embellishes the last volume printed by said cited Società [Italiana], memoir where the analytical spirit really is triumphant.

Also the Istituto Nazionale Italiano was immediately honoured in its first volume with a memoir by Brunacci on the theory of maxima and high minima; subject which was remarkably advanced later in another memoir. The Società Italiana and the Istituto oh! how greatly will grieve the loss of a man who honoured them with many and valuable works!

Also the Academies of Berlin, Munich, Turin and Lucca, and the others to which he belonged, will perceive the great emptiness which is now left in them.

I simply quickly cite the textbook on the *Elements of Algebra and Geometry* written by our author for the high school in few days, of which one has to praise the order and the distribution of subjects and which was published in many editions.

I will mention as meriting great praise the *Compendium of Sublime Calculus* which was issued in two volumes in the year 1811, where it is gathered everything which is sufficient to educate thoroughly a young geometer. In writing it the author greatly improved and carefully modified many parts of the complete course, and added many new results and arguments.

It is not licit for me neglecting to indicate another subject in which – with honoured efforts – our professor distinguished himself. The Journal of Physical Chemistry of Pavia was illustrated in many of his pages by his erudite pen; I will content myself to indicate here three memoirs where he examines the doctrine of capillary attraction of Mr Laplace, comparing it with that of Pessutti and where with his usual frankness, which is originated by his being persuaded of how well founded was his case, he proves with his firm reasonings, whatever it is said by the French geometers, some propositions which are of great praise for the mentioned Italian geometer.

One could think that a man who wrote so much in his short life actually should have been remained closed all the time alone in his office. On the contrary: he not only was a great theoretician but also he was excellent in all practical hydrometric and geodetic operations. He was professor also in these disciplines and with great dedication he worked heavily along the banks of Ticino river in order to educate the best engineers.

He was a really skilled experimentalist and he often investigated natural phenomena, getting favourable answers. I know very well how much interest pushed him to these experimental activities, as it is proven by the Hydrometry Laboratory of the University which he founded and improved (sometimes at his own expenses) with high quality instruments.

Also in these more practical activities his capabilities won him a universal esteem, so that he was called everywhere on the river banks sometimes in order to monitor their construction or for preventing their collapse or sometimes on the navigation canals, among which the famous one in Pavia started under his direction, confided to him by the past government. The same government nominated him inspector of waters and streets, inspector general of the public instruction and knight.

His character was strong in his resolutions, [it was] constant and resolute in his sentiments, vigorous in the spirit, ready to well reason and ponder, [it was] active and ready to engage in the [needed] efforts but above all he was friendly and urbane: [his character] made him the centre of social life and the joy of friendship. Particularly with his students he was renouncing to all the superiority of the 'maestro' and assumed the attitude of the father: I must avoid this memory, woe is me!, because it too strongly makes tears to come to my eyes. Those who need the evidence of my last statement has simply to see how his students wanted to honour that great man and to manifest their sorrow: they carried on their shoulders his mortal remains, they decorated in an extraordinary way his funeral parlour and now are praising the departed's merits with their tears and their silent grief which are more eloquent than all spoken lamentation.

Everybody who is now promising to contribute to exact sciences in Lombardy is a student of Brunacci, and indeed among his disciples there are those who, as their mentor himself often said, is now an eagle who can fly with his own wings. Such [an eagle] is the professor in Bologna, author of the essay on poligonometry, such is the other one who is the author of the treatise on the contours of the shadows and whose noteworthy voice is entitled to succeed to that of his maestro on the banks of Ticino, such is a third disciple who has already shown that Italy can hope to have soon a geometer who will emulate the great genius who wrote the theory of celestial bodies.

What a great misfortune was to see the departure of a man in the age of his maturity who already had greatly contributed to science and who was bound to contribute even more copiously to it! I know very well, as I had many times the privilege of his confidences about the subject of his studies, how many precious works can be found in his manuscripts. Among them, some excellent documents which he wanted to gather to form a commentary to the analytical mechanics, many very beautiful discourses read on occasion of the defense of theses, some sequels of memoirs containing the description and the calculation of many machines inspired by the Hydraulic Architecture authored by Belidor, gathering which he intended to complete an oeuvre which would have been of great utility.

May these last achievements of such an inventive and ingenious geometer be delivered up to a capable and educated scholar, who could enlighten them as they deserve, for the advancement of SCIENCES, for the glory of the AUTHOR and for the prestige of ITALY.

Milan, 18 June 1818.

Appendix C Peridynamics: A new/old model for deformable bodies

The celebrated and fundamental textbook by Lagrange [145] is, with few and biased exceptions, generally regarded as a milestone in mechanical sciences and, unanimously, as novel in its content and style of presentation.

Indeed Lagrange himself, differently from what was done by his epigones, puts his work in the correct perspective by giving the due credit to all his predecessors. Indeed the *Mécanique analytique* starts with an interesting historical introduction, which can be considered the initiation of the modern history of mechanics. Unfortunately this aspect of the Lagrangian lesson is not very often followed in modern science either.

A very new continuum mechanical theory has been recently announced and developed: peridynamics. Actually, the ideas underlying peridynamics are very interesting and most likely they deserve the full attention of experts in continuum, fracture and damage mechanics.

Indeed starting from a balance law of the form (N3) for instance in [146–148] (but many other similar treatments are available in the literature) one finds a formulation of continuum mechanics which relaxes the standard one transmitted by the apologists of the Cauchy format and seems suitable (see the few comments below) for describing many and interesting phenomena, for example in crack formation and growth.

However even those scientists whose mother tongue is Italian actually seem unaware of the contribution of Gabrio Piola in this field: this loss of memory and this lack of credit to the major sources of our knowledge, even in those cases in which their value is still topical, is very dangerous, as proven in detail by the analysis developed in Russo [2, 3].

Unfortunately this tendency towards a mindless 'modernism' seems to become more and more aggravated.

In [148] the analysis started by Piola is continued, seemingly as if the author, Silling, were one of his closer pupils: arguments are very similar and also a variational formulation of the presented theories is found and discussed. In [62] and in [71] the following is stated in the abstracts:

> The peridynamic model is a framework for continuum mechanics based on the idea that pairs of particles exert forces on each other across a finite distance. The equation of motion in the peridynamic model is an integro-differential equation. In this paper, a notion of a peridynamic stress tensor derived from nonlocal interactions is defined.

> The peridynamic model of solid mechanics is a nonlocal theory containing a length scale. It is based on direct interactions between points in a continuum separated from each other by a finite distance. The maximum interaction distance provides a length scale for the material model. This paper addresses the question of whether the peridynamic model for an elastic material reproduces the classical local model as this length scale goes to zero. We show that if the motion, constitutive model, and any nonhomogeneities are sufficiently smooth, then the peridynamic stress tensor converges in this limit to a Piola–Kirchhoff stress tensor that is a function only of the local deformation gradient tensor, as in the classical theory. This limiting Piola–Kirchhoff stress tensor field is differentiable, and its divergence represents the force density due to internal forces.

The reader is invited to compare these statements with those which can be found in Appendix A.

It is very interesting to see how fruitful the ideas formulated 167 years ago by Piola can be. It is enough to read the abstract of [149]:

The paper presents an overview of peridynamics, a continuum theory that employs a nonlocal model of force interaction. Specifically, the stress/strain relationship of classical elasticity is replaced by an integral operator that sums internal forces separated by a finite distance. This integral operator is not a function of the deformation gradient, allowing for a more general notion of deformation than in classical elasticity that is well aligned with the kinematic assumptions of molecular dynamics. Peridynamics' effectiveness has been demonstrated in several applications, including fracture and failure of composites, nanofiber networks, and polycrystal fracture. These suggest that peridynamics is a viable multiscale material model for length scales ranging from molecular dynamics to those of classical elasticity.

Also see the abstract of the paper by Parks et al. [150]:

Peridynamics (PD) is a continuum theory that employs a nonlocal model to describe material properties. In this context, nonlocal means that continuum points separated by a finite distance may exert force upon each other. A meshless method results when PD is discretized with material behaviour approximated as a collection of interacting particles. This paper describes how PD can be implemented within a molecular dynamics (MD) framework, and provides details of an efficient implementation. This adds a computational mechanics capability to an MD code enabling simulations at mesoscopic or even macroscopic length and time scales.

It is remarkable how strictly related non-local continuum theories are with the discrete theories of particles bound to the nodes of a lattice. How deep the insight of Piola was can be understood by looking at the literature about the subject, which includes for instance [59–64, 71, 146–149].

Appendix D On an expression for ∇F deduced in Piola [7] on pp. 158–159

In this appendix we deduce, by means of Levi-Civita tensor calculus, the expression for the second gradient of placement that is needed to transform equation (N4) into equation (N5) and that is obtained by [7]; see Appendix A for the appropriate translation. The original calculations are rather lengthy and cumbersome; it is however the opinion of the authors that Piola had caught their 'tensorial', or at least their algebraic, structure. Indeed the notation he used made the identification of the tensorial objects involved rather easy.

We start from the following identification between modern and Piola's notation:

$$F^i_\alpha \leftrightharpoons \begin{pmatrix} \frac{dx}{da} & \frac{dy}{da} & \frac{dz}{da} \\ \frac{dx}{db} & \frac{dy}{db} & \frac{dz}{db} \\ \frac{dx}{dc} & \frac{dy}{dc} & \frac{dz}{dc} \end{pmatrix}, \quad \det F \leftrightharpoons H, \quad (\det F)\left(F^{-1}\right)^\beta_j \leftrightharpoons \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}, \tag{N7}$$

so that we can state that equation (28) on p. 26 of [7] is equivalent to

$$(\det F)\, F^{-1} F = (\det F)\, I,$$

where I is the identity matrix. Moreover equation (6) on p. 57 is equivalent to

$$\begin{pmatrix} t_1 & t_4 & t_5 \\ t_4 & t_2 & t_6 \\ t_5 & t_6 & t_3 \end{pmatrix} \leftrightharpoons C = F^{\mathrm{T}} F \quad C_{\alpha\beta} = F^i_\alpha F_{i\beta}. \tag{N8}$$

The equation on pp. 158–159 in [7] is written in tensorial form as follows:

$$\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\beta} := D_{\alpha\beta\eta}\left(F^{-1}\right)^{i\eta}, \quad F_{i\eta}\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\beta} = \frac{\partial F^i_\alpha}{\partial X^\beta} F_{i\eta} = D_{\alpha\beta\eta}. \tag{N9}$$

Now by recalling that

$$\frac{\partial F_{i\beta}}{\partial X^\gamma} = \frac{\partial F_{i\gamma}}{\partial X^\beta}$$

we have the symmetry of D with respect to the first two indices,

$$D_{\alpha\beta\eta} = D_{\beta\alpha\eta} \tag{N10}$$

and, because of this expression, we can perform the following simple calculations (usual symmetrization, $A_{(ab)} = A_{ab} + A_{ba}$, and skew-symmetrization, $A_{[ab]} = A_{ab} - A_{ba}$, conventional symbols are used):

$$\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} = \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} + F^i_\alpha \frac{\partial F_{i\beta}}{\partial X^\gamma}$$

$$\begin{aligned} 2\frac{\partial C_{\alpha[\beta}}{\partial X^{\gamma]}} &= \frac{\partial C_{\alpha\beta}}{\partial X^\gamma} - \frac{\partial C_{\alpha\gamma}}{\partial X^\beta} = \\ &= \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} + F^i_\alpha \frac{\partial F_{i\beta}}{\partial X^\gamma} - \frac{\partial F^i_\alpha}{\partial X^\beta} F_{i\gamma} - F^i_\alpha \frac{\partial F_{i\gamma}}{\partial X^\beta} = \\ &= \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} - \frac{\partial F^i_\alpha}{\partial X^\beta} F_{i\gamma} = D_{\alpha\gamma\beta} - D_{\alpha\beta\gamma} = 2D_{\alpha[\gamma\beta]} \end{aligned}$$

$$\begin{aligned} \frac{\partial C_{\alpha\beta}}{\partial X^\gamma} = \frac{\partial \left(F^i_\alpha F_{i\beta}\right)}{\partial X^\gamma} = \frac{\partial F^i_\alpha}{\partial X^\gamma} F_{i\beta} + F^i_\alpha \frac{\partial F_{i\beta}}{\partial X^\gamma} = \\ = D_{\gamma\alpha\beta} + D_{\gamma\beta\alpha} = 2D_{\gamma(\alpha\beta)}. \end{aligned}$$

By decomposing D into its skew and symmetric parts (with respect to the second and third index, see also (N10)) one gets

$$D_{\gamma\alpha\beta} = D_{\gamma(\alpha\beta)} + D_{\gamma[\alpha\beta]} = \frac{1}{2}\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma[\beta}}{\partial X^{\alpha]}} = \frac{1}{2}\left(\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma\beta}}{\partial X^\alpha} - \frac{\partial C_{\gamma\alpha}}{\partial X^\beta}\right). \qquad \text{(N11)}$$

The third-order tensor $D_{\gamma\alpha\beta}$ which we have introduced allows us to reproduce in the compact form (N9) the formula which occupies nearly two pages of Piola's work. Moreover, we have obtained the formula N11 with an easy calculation process which is much less involved than the one first conceived by Piola.

From (N11) we have

$$F_{i\beta}\frac{\partial^2 \chi^i}{\partial X^\alpha \partial X^\gamma} = \frac{1}{2}\left(\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma\beta}}{\partial X^\alpha} - \frac{\partial C_{\gamma\alpha}}{\partial X^\beta}\right) \qquad \text{(N12)}$$

which is equivalent to

$$\frac{\partial^2 \chi^j}{\partial X^\alpha \partial X^\gamma} = \frac{1}{2}\left(F^{-1}\right)^{j\beta}\left(\frac{\partial C_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial C_{\gamma\beta}}{\partial X^\alpha} - \frac{\partial C_{\gamma\alpha}}{\partial X^\beta}\right). \qquad \text{(N13)}$$

To compare the two formalisms let us state the identification of the left-hand side of one line, that is, of the 11th one divided by $2H$ of the formula appearing on p. 158 in [7]; in other words,

$$\frac{\partial^2 \chi^2}{\partial X^1 \partial X^2} \leftrightarrows \frac{d^2 y}{dadb}.$$

Thus, from (N11) with $\alpha = 1, j = \gamma = 2$, by recalling the symmetry of the tensor C and the identifications (N7) and (N8),

$$
\begin{aligned}
\frac{\partial^2\chi^2}{\partial X^1\partial X^2} &= \frac{1}{2}\left(F^{-1}\right)^{2\beta}\left(\frac{\partial C_{1\beta}}{\partial X^2}+\frac{\partial C_{2\beta}}{\partial X^1}-\frac{\partial C_{21}}{\partial X^\beta}\right)= \\
&= \frac{1}{2}\left(F^{-1}\right)^{21}\left(\frac{\partial C_{11}}{\partial X^2}+\frac{\partial C_{21}}{\partial X^1}-\frac{\partial C_{21}}{\partial X^1}\right)+ \\
&+\frac{1}{2}\left(F^{-1}\right)^{22}\left(\frac{\partial C_{12}}{\partial X^2}+\frac{\partial C_{22}}{\partial X^1}-\frac{\partial C_{21}}{\partial X^2}\right)+ \\
&+\frac{1}{2}\left(F^{-1}\right)^{23}\left(\frac{\partial C_{13}}{\partial X^2}+\frac{\partial C_{23}}{\partial X^1}-\frac{\partial C_{21}}{\partial X^3}\right)= \\
&= \frac{1}{2}\left(F^{-1}\right)^{21}\frac{\partial C_{11}}{\partial X^2}+\frac{1}{2}\left(F^{-1}\right)^{22}\frac{\partial C_{22}}{\partial X^1}+ \\
&+\frac{1}{2}\left(F^{-1}\right)^{23}\left(\frac{\partial C_{13}}{\partial X^2}+\frac{\partial C_{23}}{\partial X^1}-\frac{\partial C_{21}}{\partial X^3}\right)\leftrightarrows \\
&\leftrightarrows \frac{l_2}{2H}\frac{dt_1}{db}+\frac{m_2}{2H}\frac{dt_2}{da}+\frac{n_2}{2H}\left(\frac{dt_5}{db}+\frac{dt_6}{da}-\frac{dt_4}{dc}\right)
\end{aligned}
$$

which is, multiplying both members by $2H$, the 11th equality on p. 158 in [7]

$$
2H\,\frac{d^2y}{dadb}=l_2\frac{dt_1}{db}+m_2\frac{dt_2}{da}+n_2\left(\frac{dt_5}{db}+\frac{dt_6}{da}-\frac{dt_4}{dc}\right).
$$

Piola continued the calculations by considering the third-order derivatives. However, the obtained expressions are too long to be reproduced in printed form. So he states:

> The trinomials with third order derivatives are of three kinds: there are those constituted by derivatives of first and third order, and one can count 30 of them; there are those constituted by derivatives of second and third order, and they are 60 in number; and there are those which contain only third order derivatives and they are 55 in number. I am not writing them, as everybody who is given the needed patience can easily calculate them by himself, as it can be also done for those trinomials containing derivatives of higher order.

As we can use Levi-Civita tensor calculus it is easier for us to find the needed patience, at least for calculating the trinomials constituted by first- and third-order derivatives. Indeed from

$$
F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta}=\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta}+\frac{\partial C_{\beta\gamma}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right) \tag{N14}
$$

by differentiating (N14) we get

$$
\frac{\partial}{\partial X^\eta}\left(F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta}\right)=\frac{\partial}{\partial X^\eta}\left(\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta}+\frac{\partial C_{\beta\gamma}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right)\right)
$$

and rearranging the terms,

$$
F_{i\gamma}\frac{\partial^3\chi^i}{\partial X^\alpha\partial X^\beta\partial X^\eta}=-\frac{\partial^2\chi_i}{\partial X^\gamma\partial X^\eta}\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta}+\frac{1}{2}\left(\frac{\partial^2 C_{\alpha\gamma}}{\partial X^\eta\partial X^\beta}+\frac{\partial^2 C_{\beta\gamma}}{\partial X^\eta\partial X^\alpha}-\frac{\partial^2 C_{\beta\alpha}}{\partial X^\eta\partial X^\gamma}\right). \tag{N15}
$$

By replacing the following equality due to (N9)

$$
\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta}=\frac{1}{2}\left(F^{-1}\right)^{i\delta}\left(\frac{\partial C_{\alpha\delta}}{\partial X^\beta}+\frac{\partial C_{\beta\delta}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\delta}\right) \tag{N16}
$$

in the identity (N15) one gets

$$
\begin{aligned}
F_{i\gamma}\frac{\partial^3\chi^i}{\partial X^\alpha\partial X^\beta\partial X^\eta} &= -\frac{1}{2}\left(F^{-1}\right)_i^{\nu}\left(\frac{\partial C_{\gamma\nu}}{\partial X^\eta}+\frac{\partial C_{\eta\nu}}{\partial X^\gamma}-\frac{\partial C_{\eta\gamma}}{\partial X^\nu}\right)\frac{1}{2}\left(F^{-1}\right)^{i\delta}\left(\frac{\partial C_{\alpha\delta}}{\partial X^\beta}+\frac{\partial C_{\beta\delta}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\delta}\right) \\
&+\frac{1}{2}\left(\frac{\partial^2 C_{\alpha\gamma}}{\partial X^\eta\partial X^\beta}+\frac{\partial^2 C_{\beta\gamma}}{\partial X^\eta\partial X^\alpha}-\frac{\partial^2 C_{\beta\alpha}}{\partial X^\eta\partial X^\gamma}\right),
\end{aligned}
$$

which can easily be rewritten in the form

$$F_{i\gamma}\frac{\partial^3\chi^i}{\partial X^\alpha\partial X^\beta\partial X^\eta}=-\frac{1}{4}\left(C^{-1}\right)^{\nu\delta}\left(\frac{\partial C_{\gamma\nu}}{\partial X^\eta}+\frac{\partial C_{\eta\nu}}{\partial X^\gamma}-\frac{\partial C_{\eta\gamma}}{\partial X^\nu}\right)\left(\frac{\partial C_{\alpha\delta}}{\partial X^\beta}+\frac{\partial C_{\beta\delta}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\delta}\right)$$
$$+\frac{1}{2}\left(\frac{\partial^2 C_{\alpha\gamma}}{\partial X^\eta\partial X^\beta}+\frac{\partial^2 C_{\beta\gamma}}{\partial X^\eta\partial X^\alpha}-\frac{\partial^2 C_{\beta\alpha}}{\partial X^\eta\partial X^\gamma}\right)$$

which has the structure sought after by Piola.

With easy calculation, from the last equation we get

$$\frac{1}{2}\frac{\partial^3\rho^2(\bar{X},X)}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma}=\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta}+\frac{\partial C_{\beta\gamma}}{\partial X^\alpha}+\frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right)+\left(\left(\chi_i(\bar{X})-\chi_i(X)\right)\frac{\partial^3\chi_i(\bar{X})}{\partial\bar{X}_\alpha\partial\bar{X}_\beta\partial\bar{X}_\gamma}\right). \tag{N17}$$

In Appendix E an induction argument will be presented which allows us to prove the conjecture put forward by Piola at the beginning of sect. 74 on p. 156.

Appendix E ‘After these calculations I abandoned myself to the analogy’

We are not enthusiastic about the work of Piola to the extent that we cannot clearly see the limits of his mathematical proofs.

Indeed the important property which he discusses in sect. 74 is obtained by means of a proof ‘by analogy’ which is not considered acceptable nowadays. Although there are examples of mathematical induction which are very ancient (see the discussion in [3] and references therein) only after Boole and Dedekind did it become a universally known and (nearly universally) accepted method.

Actually Piola states here that, because of objectivity, the expression of virtual work must depend only on deformation measure $C_{\gamma\beta}$ and its derivatives. However, as we have already pointed out, his proof is based, for higher derivatives, on an argument which the majority of contemporary mathematicians would consider no more than a (maybe well grounded) conjecture. Indeed at the beginning of p. 157 of [7] one reads ‘after these calculations I abandoned myself to the analogy: and this will sooner or later be unavoidable, because our series is infinite and it will be impossible to check all its terms.’

We reproduce here an inductive argument which indeed follows the original spirit of Piola.

Let us start by proving the following.

Lemma 3 ***Representation of placement higher-order derivatives***. *For every n there exists a family of (polynomial) functions $M_{\gamma\alpha_1\ldots\alpha_n}$ of the tensor variables $C,\nabla C,\ldots\nabla^{n-1}C$ such that*

$$\left(\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}^\gamma}\frac{\partial^n\chi^i(\bar{X})}{\partial\bar{X}^{\alpha_1}\ldots\partial\bar{X}^{\alpha_n}}\right)=M_{\gamma\alpha_1\ldots\alpha_n}\left(C,\ldots,\nabla^{n-1}C\right). \tag{N18}$$

As in Appendix D we have proven such a lemma for $n=2$, that is,

$$F_{i\gamma}\frac{\partial^2\chi^i}{\partial X^\alpha\partial X^\beta}=\frac{1}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^\beta}+\frac{\partial C_{\beta\gamma}}{\partial X^\alpha}-\frac{\partial C_{\beta\alpha}}{\partial X^\gamma}\right). \tag{N19}$$

In order to prove (N18) for every n it is sufficient to prove that if it is valid for all $N\le n$ then it is also valid for $N=n+1$.

Let us start by remarking that (N18) implies that

$$\frac{\partial^n\chi^i(\bar{X})}{\partial\bar{X}^{\alpha_1}\ldots\partial\bar{X}^{\alpha_n}}=\left(F^{-1}\right)^{i\eta}M_{\eta\alpha_1\ldots\alpha_n}\left(C,\ldots,\nabla^{n-1}C\right). \tag{N20}$$

Let us then differentiate (N18) assumed valid for $N=n$ to get

$$\frac{\partial}{\partial\bar{X}^{\alpha_{n+1}}}\left(\frac{\partial\chi_i(\bar{X})}{\partial\bar{X}^\gamma}\frac{\partial^n\chi^i(\bar{X})}{\partial\bar{X}^{\alpha_1}\ldots\partial\bar{X}^{\alpha_n}}\right)=\frac{\partial}{\partial\bar{X}^{\alpha_{n+1}}}\left(M_{\gamma\alpha_1\ldots\alpha_n}\left(C,\ldots,\nabla^{n-1}C\right)\right)$$

which implies

$$\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^\gamma} \frac{\partial^{n+1} \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n} \partial \bar{X}^{\alpha_{n+1}}} = \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}} \left(M_{\gamma\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-1} C \right) \right) - \frac{\partial^2 \chi_i(\bar{X})}{\partial \bar{X}^\gamma \partial \bar{X}^{\alpha_{n+1}}} \frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n}}.$$

Now by replacing equation (N20) two times (for $n = 2$ and for $N = n$) we get

$$\begin{aligned} \frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^\gamma} \frac{\partial^{n+1} \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n} \partial \bar{X}^{\alpha_{n+1}}} &= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}} \left(M_{\gamma\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-1} C \right) \right) \\ &- \left(\left(F^{-1}\right)_i^\eta M_{\eta\alpha_{n+1}} \left(C, \dots, \nabla^{n-1} C \right) \right) \left(\left(F^{-1}\right)^{i\beta} M_{\beta\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-1} C \right) \right) \\ &= M_{\gamma\alpha_1 \dots \alpha_n \alpha_{n+1}} \left(C, \dots, \nabla^{n-1} C \right), \end{aligned}$$

where we have introduced the definition

$$\begin{aligned} M_{\gamma\alpha_1 \dots \alpha_n \alpha_{n+1}} \left(C, \dots, \nabla^{n-1} C \right) := &\frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}} \left(M_{\gamma\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-1} C \right) \right) \\ &- \left(C^{-1}\right)^{\beta\eta} M_{\eta\alpha_{n+1}} \left(C, \dots, \nabla^{n-1} C \right) M_{\beta\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-1} C \right). \end{aligned}$$

The proof by induction of the lemma is thus complete. To prove that the generic nth-order derivative of ρ^2 can also be expressed, when $\bar{X} = X$, in terms of the $(n-2)$th-order derivatives of $C_{\gamma\beta}$, we can again use a simple recursion argument based on the previous lemma.

Indeed the following lemma is true.

Lemma 4 ***Representation of the derivatives of the distance function*** ρ*. For every n there exists a family of (polynomial) functions $L_{\alpha_1 \dots \alpha_n}$ of the variables $C, \dots, \nabla^{n-2} C$ such that*

$$\frac{\partial^n \rho^2(\bar{X}, X)}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n}} = L_{\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-2} C \right) + \left(\left(\chi_i(\bar{X}) - \chi_i(X) \right) \frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n}} \right). \tag{N21}$$

To prove the lemma we assume by inductive hypothesis that it is true for $N = n$ and prove that it is true for $N = n + 1$. As we have proven formula (N17), that is, the previous lemma for $n = 3$, then the lemma follows by the mathematical induction principle.

Therefore by differentiating equation (N21) one gets

$$\begin{aligned} \frac{\partial^{n+1} \rho^2(\bar{X}, X)}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_{n+1}}} &= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}} \left(L_{\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-2} C \right) \right) + \left(\frac{\partial \chi_i(\bar{X})}{\partial \bar{X}^{\alpha_{n+1}}} \frac{\partial^n \chi^i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n}} \right) \\ &+ \left(\sum_{i=1}^3 \left(\chi_i(\bar{X}) - \chi_i(X) \right) \frac{\partial^{n+1} \chi_i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n} \partial \bar{X}^{\alpha_{n+1}}} \right) \end{aligned}$$

which by replacing equation (N18) becomes

$$\begin{aligned} \frac{\partial^{n+1} \rho^2(\bar{X}, X)}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_{n+1}}} &= \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}} \left(L_{\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-2} C \right) \right) + M_{\alpha_{n+1}\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-1} C \right) \\ &+ \left(\sum_{i=1}^3 \left(\chi_i(\bar{X}) - \chi_i(X) \right) \frac{\partial^{n+1} \chi_i(\bar{X})}{\partial \bar{X}^{\alpha_1} \dots \partial \bar{X}^{\alpha_n} \partial \bar{X}^{\alpha_{n+1}}} \right), \end{aligned}$$

which proves the lemma once one has introduced the following recursive definition:

$$L_{\alpha_1 \dots \alpha_n \alpha_{n+1}} \left(C, \dots, \nabla^{n-2} C \right) := \frac{\partial}{\partial \bar{X}^{\alpha_{n+1}}} \left(L_{\alpha_1 \dots \alpha_n} \left(C, \dots, \nabla^{n-2} C \right) \right) + M_{\alpha_{n+1}\alpha_1 \dots \alpha_n} \left(C, \dots y, \nabla^{n-1} C \right).$$

Appendix F An Italian mathematical genealogy

In [2] it is discussed that a loss of knowledge most likely occurred at the end of the Punic wars. More generally, all the processes of erasure and removal of previously well established scientific knowledge are related to the simultaneous occurrence of two circumstances: the loss of continuity in the chain between maestro and student in the academic institutions and the loss of the awareness (in the whole society) of the strict connection which exists between science (in all its most abstract expressions,

The final result of the simultaneous occurrence of these two circumstances is that the societies in which they occur do not invest resources in the storage and transmission of theoretical knowledge and that, as a consequence, the contact between maestro and pupil, established when a living scientist teaches to his students the content of the most difficult and important textbooks, is broken. As a final result, in those societies, at first the theoretical knowledge and subsequently, after a while, the technological capabilities also are lost.

We want to underline in this appendix that there is a direct genealogy starting from Gabrio Piola and leading to the founders of absolute tensor calculus. The Italian school of the 19th century was started under the momentum impressed by the Napoleonic reforms of the political organization of the Italian nation: in this context the reader should see the eulogy in Appendix B where Piola, talking about the textbook of mathematical analysis written by his maestro Vincenzo Brunacci, writes '...was also pushed by the advice of that Sovereign who was an investigator of the stars who, being in Milan, wrote to persuade him to start this oeuvre in the year 1800, believing that he was the only one among the Italians who was capable to complete it successfully.'

Gabrio Piola never accepted a university chair; however, his pupil Francesco Brioschi was the founder of the Politecnico di Milano. Brioschi mentored Enrico Betti and Eugenio Beltrami. Ulisse Dini was a pupil of Enrico Betti, also being his successor as the chair of Mathematical Analysis and Geometry at the Università di Pisa. Gregorio Ricci-Curbastro was a pupil of Ulisse Dini, Eugenio Beltrami and Enrico Betti. Tullio Levi-Civita was a pupil of Gregorio Ricci-Curbastro.

The strength of the Italian school of mathematical physics, mathematical analysis and differential geometry has been weakened by two processes, one which it shares with all other national schools and in general with all groups of scientists, the other which is more peculiar to the Italian nation.

1. It happens very often that some theories need to be rediscovered and reformulated several times in different circles before becoming a universally recognized part of knowledge. For instance, the basic ideas of functional analysis and its founding concept of functional (which goes back to the calculus of variations and which can be defined with the sentence 'a function whose argument is a function') were already treated in the papers by Erik Ivar Fredholm and in Hadamard's 1910 textbook, and had previously been introduced in 1887 by the Italian mathematician and physicist Vito Volterra. The theory of nonlinear functionals was continued by students of Hadamard, in particular Fréchet and Lévy. Hadamard also founded the modern school of linear functional analysis, further developed by Riesz and the group of Polish mathematicians around Stefan Banach. However, Heisenberg and Dirac did need to rediscover many parts of a theory already known and they developed that theory until the moment in which von Neumann could recognize that quantum mechanics had actually been formulated in terms of what he called Hilbert spaces. Simple laziness or the difficulty of understanding the formalism introduced by other authors, lack of time or of economical means: all of these may lead some very brilliant scientists to ignore results obtained by other scientists, which are nevertheless relevant to their work. Many of the mathematicians listed in the previous genealogy rediscovered many times the results which their predecessors had already obtained because of the aforementioned first process. Such a process could be called removal and/or ignorance of the results which appear not to be relevant. This first removal process is indeed observed very often in the history of science applied to different groups, independent of their nationality, and the case of the rediscovery of functional analysis is a striking example of its occurrence.
2. Napoleon favoured the birth of an Italian mathematical school, and among many other actions he pushed Vincenzo Brunacci to write the first Italian textbook in mathematical analysis. However, he could not enforce in the Italian school the habit, always followed by the French school, which leads all French scientists to recognize, to develop and to glorify the contributions of their compatriots. Instead, the Italian scientists always preferred to follow the tradition of their predecessors, that is, the scientists of Greek language who developed Hellenistic science (see [3]). Hellenic tradition is based on the intentional removal and contempt of the contribution due to compatriots and on the continuous preference for the approval of foreign scientists. The described process leads the members of a national

group to consider other national groups always to be stronger, more qualified and more productive, while actively acting to impeach the cultural, political and academic growth of compatriot scientists.

The momentum given to the Italian school by Napoleon eventually led to the birth of tensor calculus, but was exhausted by the typical Italian negative attitude towards compatriots, which was exemplified by the removal of Levi-Civita, due to Mussolini's racial persecutions, from his chair in Rome, which was immediately occupied by Signorini. Finally, however, it at least has to be recognized that:

- The strict relationship between differential geometry and continuum mechanics was discovered and developed by the Italian school started by Piola and culminating in Levi-Civita;
- The great advancement of Riemannian geometry produced by the recognition of the unicity of the parallel transport compatible with a Riemannian metric (the so-called Levi-Civita theorem) has deep roots already in Piola's works (recall the well known concept of Piola's transformation);
- Ulisse Dini's theorem mathematically clarifies the concept of constraint intensively used in the works of Piola. Indeed the crucial concept of independent constraints (defined as those having non-singular Jacobians) was clarified by Dini several decades after Piola had proven its importance in continuum mechanics.