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FULL ABSTRACTION FOR FAIR TESTING IN CCS

TOM HIRSCHOWITZ

ABSTRACT. In previous work with Pous, we defined a semantics for CCS which may both be viewed as an innocent presheaf semantics and as a concurrent game semantics. It is here proved that a behavioural equivalence induced by this semantics on CCS processes is fully abstract for fair testing equivalence.

The proof relies on a new algebraic notion called *playground*, which represents the ‘rule of the game’. From any playground, two languages, equipped with labelled transition systems, are derived, as well as a strong, functional bisimulation between them.

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1. INTRODUCTION

Motivation. In spite of active research on its mathematical formalisation [44, 43, 17, 16, 52, 2, 24], programming language theory remains largely conducted in an informal way. We are far from having a satisfactory body of general results, leading to efficient proofs of, e.g., compiler correctness. More specifically, even though there is a well-established theory of behavioural equivalences over labelled transition systems (LTSS) [48], which have been put to efficient use in proofs of concrete results, there appears to be few effective tools to generate the desired LTSS from elementary data, and reason about translations between them. Actually, solutions to the PoplMark challenge [4], as well as the remarkable achievement by Leroy and colleagues [34, 36, 35] of a certified compiler, seem to have convinced the community that no mathematical progress was to be expected: one has to go through lengthy proofs by induction, and there is no hope for abstract results.

The author still believes there is room for improvement, and the present paper is an attempt in this direction. Before conceiving a general theory of programming language semantics, it seems reasonable to try and reconcile various existing approaches to denotational semantics. We here consider two successful such semantics:

- *Innocent* game semantics was introduced by Hyland and Ong [26] to construct fully abstract models for functional languages, in particular PCF, where programs are interpreted as strategies in a game.
- *Presheaf* models [29, 11] were introduced by Joyal et al. as a semantics for process algebras, in particular Milner’s CCS [40].

Is it possible to reconcile these apparently very different approaches? Beyond the technical matters explained below, we feel that the present work, building on previous work with Pous [25, 23] (HP), hints at a positive answer. To summarise:

- on the one hand, we generalise innocent game semantics to (1) take seriously the possibility of games with more than two players and (2) consider strategies which may accept plays in more than one way;
- on the other hand, we refine presheaf models to take parallel composition more seriously.

This leads to a model of CCS which may both be seen as a concurrent game semantics, and as an innocent presheaf model.

Concurrent games vs. innocent presheaves. To see that presheaf models are a concurrent, non-innocent variant of game semantics, recall that the base category, say \mathbb{C} , for such a presheaf model typically has as objects sequences of labels, or configurations in event structures, morphisms being given by prefix inclusion. Such

objects may be understood as plays in some game. Now, in standard game semantics, a strategy is a prefix-closed (non-empty) set of plays. Unfolding the definition, this is the same as a functor $\mathbb{C}^{op} \rightarrow 2$, where 2 is the poset category $0 \leq 1$: the functor maps a play to 1 when it is accepted by the strategy, and to 0 otherwise. It is known since Harmer and McCusker [21] that this notion of strategy does not easily adapt to non-determinism or concurrency, Presheaf semantics only slightly generalises it by allowing strategies to accept a play in several ways. A strategy S now maps each play p to a *set* $S(p)$. The play is accepted when $S(p)$ is non-empty, and, because there are then no functions $S(p) \rightarrow \emptyset$, being accepted remains a prefix-closed property of plays. The passage from 2 to more general sets allows to express branching-time semantics.

This links presheaf models with game models, but would be of little interest without the issue of *innocence*. Game models, indeed, do not always accept *any* prefix-closed set of plays S as a strategy: they demand that any choice of move in S depends only on its *view*. E.g., consider the CCS process $P = (a|(b \oplus c))$, where \oplus denotes internal choice, and a candidate strategy accepting the plays $\epsilon, (a), (b), (c), (ab)$, but not (ac) . This strategy refuses to choose c after a has been played. Informally, there are two players here, one playing a and the other playing $b \oplus c$; the latter should have no means to know whether a has been played or not. We want to rule out this strategy on the grounds that it is not innocent.

Our technical solution for doing so is to refine the notion of play, making the number of involved players more explicit. Plays still form a category, but they admit a subcategory of *views*, which represent a single player's possible perceptions of the game. This leads us to two equivalent categories of strategies. In the first, strategies are presheaves on views. In the second category, strategies are certain presheaves on arbitrary plays, satisfying an innocence condition. Parallel composition, in the game semantical sense, is best understood in the former category: it merely amounts to copairing. Parallel composition, in the CCS sense, which in standard presheaf models is a complex operation based on some labelling of transitions or events, is here just a move in the game. The full category of plays is necessary for understanding the global behaviour of strategies. It is in particular needed to define our semantic variant of fair testing equivalence, described below. One may think of presheaves on views as a syntax, and of innocent presheaves on plays as a semantics. The combinatorics of passing from local (views) to global (arbitrary plays) are dealt with by right Kan extension.

Main results. Once our semantics is defined, we should demonstrate how close it is to operational semantics. For this, we provide two results. The most important, in the author's view, is full abstraction w.r.t. *fair testing semantics*, hence the title. But the second result might be considered more convincing by many, since it establishes that our semantics is fully abstract w.r.t. weak bisimilarity. The reason it is here considered less important is that it relies on something external to the model itself, namely an LTS for strategies, constructed in an *ad hoc* way. Fair testing equivalence, being an internal notion of behavioural equivalence, appears more legitimate.

Now, why consider fair testing among the many testing equivalences? First of all, let us mention that we could probably generalise our result to any testing equivalence, in a sense to be made precise: it is enough for us that the testing predicate \perp be invariant under weak bisimilarity. But this paper is already quite complicated,

and pushes generalisation rather far in other respects (see below). We thus chose to remain concrete regarding the considered equivalence. It was then natural to choose fair testing, as it both is one of the most prominent testing equivalences, and one of the finest. It was introduced independently by Natarajan and Cleaveland [42], and by Brinksma et al. [8, 46] (under the name of *should* testing in the latter paper), with the aim of reconciling the good properties of observation congruence [41] w.r.t. divergence, and the good properties of previous testing equivalences [12] w.r.t. choice. Typically, $a.b + a.c$ and $a.(b \oplus c)$ (where $+$ denotes guarded choice and \oplus denotes internal choice) are not observation congruent, which is perceived as excessive discriminating power of observation congruence. Conversely, $(! \tau) \mid a$ and a are not must testing equivalent, which is perceived as excessive discriminating power of must testing equivalence. Fair testing rectifies both defects, and has been the subject of further investigation, of which an excellent survey can be found in Cacciagrano et al. [10].

Overview. We now give a bit more detail on the contents. In HP, a game is defined, and CCS processes are interpreted as strategies in this game. A semantic form of fair testing equivalence, denoted by \sim_f is defined on strategies. In this paper, we prove that the translation of HP from CCS to strategies, here decomposed as $\llbracket - \rrbracket \circ \theta$ (see below), is such that $P \sim_{f,s} Q$ iff $\llbracket \theta(P) \rrbracket \sim_f \llbracket \theta(Q) \rrbracket$, where $\sim_{f,s}$ is standard fair testing equivalence (Corollary 7).

In order to derive our main result, our first step is to define an LTS for strategies, over the standard alphabet \mathbb{A} for CCS, and prove that the graph of $\llbracket - \rrbracket \circ \theta$ is included in weak bisimilarity (Corollary 6). The proof of this result rests upon a general theory developed in Sections 4, 6, and 5. There, we define an algebraic gadget called *playground*, and show that any such playground \mathbb{D} gives rise to two LTSS $\mathcal{T}_{\mathbb{D}}$ and $\mathcal{S}_{\mathbb{D}}$, whose elements are respectively called *process terms* and *strategies*, and a strong bisimulation $\llbracket - \rrbracket : \mathcal{T}_{\mathbb{D}} \rightarrow \mathcal{S}_{\mathbb{D}}$ between them. We construct in Section 8 a playground \mathbb{D}^{CCS} for CCS, such that the elements of $\mathcal{S}_{\mathbb{D}^{CCS}}$ are the strategies of HP. Process terms in $\mathcal{T}_{\mathbb{D}^{CCS}}$ form a language into which CCS easily embeds, weakly bisimilarly. Up to a few changes of base, we recover the translation of HP as the composite

$$CCS \xrightarrow{\theta} \mathcal{T}_{\mathbb{D}^{CCS}} \xrightarrow{\llbracket - \rrbracket} \mathcal{S}_{\mathbb{D}^{CCS}},$$

and deduce that its graph is included in weak bisimilarity.

Our main result is then derived in Section 7, using a general setting for comparing LTSS w.r.t. fair testing equivalence. We first define a notion of *effective graph with complementarity*, which is slightly more general than Hennessy and De Nicola's original setting for testing equivalences [12]. The extra generality is useful to define fair testing equivalence directly on strategies. Inspired by the notion of *failure* [46], we then define what it means for such an effective graph with complementarity G to have *enough A-trees*. Namely, G should come equipped with a structure of LTS over a graph A , such that, among other conditions, for any t in a certain class of tree-like LTSS over A , there exists $x_t \in G$ weakly bisimilar to t . Finally, we prove that if two effective graphs with complementarity G and H have enough A -trees, and if a relation $R : G \rightarrow H$ is included in weak bisimilarity over A , then R preserves and reflects fair testing equivalence. We then prove that CCS , $\mathcal{T}_{\mathbb{D}^{CCS}}$, and $\mathcal{S}_{\mathbb{D}^{CCS}}$ have enough \mathbb{A} -trees. Since we have seen that both θ and $\llbracket - \rrbracket$ are included in weak bisimilarity over \mathbb{A} , this entails the result.

Related work. Other general frameworks aiming at an effective, general theory of programming languages include mathematical operational semantics [52], Kleene coalgebra [7, 6], the Tile model [17, 9], relative monads [3], and cartesian closed 2-categories [24]. All these frameworks, to the author’s knowledge, with the notable exception of Kleene coalgebra, attempt to organise the traditional techniques of syntax with variable binding and reduction rules into some algebraic, coalgebraic, or categorical structure. Our approach, like Kleene coalgebra, sees syntax and its associated LTS as derived notions. In an attempt to compare the two, our approach resembles Kleene coalgebra without quantitative aspects, while Kleene coalgebra resembles our approach without innocence.

Beyond general frameworks, Rideau and Winskel’s [47] recent work also contains a notion of innocent, non-deterministic strategy, which builds upon Melliès’s earlier work [38]. Neither our nor their notion of innocence has yet been shown to entirely coincide with, or specialise to, innocence in the sense of Hyland and Ong, and both are rather aimed at generalising the latter. The precise links between the three settings remain to be better investigated. Furthermore, Melliès [39], although in a deterministic and linear setting, incorporates some ‘concurrency’ into plays by presenting them as string diagrams. Our innocentisation procedure also bears some similarity with Harmer et al.’s [22] presentation of innocence based on a distributive law.

Perspectives. We plan to adapt our semantics to more complicated calculi like π , the Join and Ambients calculi, functional calculi, possibly with extra features (e.g., references, data abstraction, encryption), with a view to eventually generalising it. Preliminary investigations already led to a playground for π , whose adequacy remains to be established. More speculative directions include

- defining a notion of morphisms for playgrounds, which should induce translation functions between strategies, and find sufficient conditions for such morphisms to preserve, resp. reflect testing equivalences;
- generalising playgrounds to apply them beyond programming language semantics; in particular, preliminary work shows that playgrounds easily account for cellular automata; this raises the question of how morphisms playgrounds would compare with various notions of simulations between cellular automata [13];
- trying and recast the issue of deriving transition systems (LTSS) from reductions [51, 32, 50, 45] in terms of the tools developed in Section 7.

As a final remark, we mentioned that the theory developed in Section 7 would generalise to other testing equivalences. It might be interesting to investigate to which extent this holds, and whether it has useful applications.

2. PRELIMINARIES

Sets, categories, presheaves. **Set** is the category of sets; **set** is a skeleton of the category of finite sets, e.g., the category of finite ordinals and arbitrary maps between them; **ford** is the category of finite ordinals and monotone maps between them. For any category \mathbb{C} , $\widehat{\mathbb{C}} = [\mathbb{C}^{op}, \mathbf{Set}]$ denotes the category of presheaves on \mathbb{C} , while $\widehat{\mathbb{C}}^f = [\mathbb{C}^{op}, \mathbf{set}]$ and $\widehat{\mathbb{C}} = [\mathbb{C}^{op}, \mathbf{ford}]$ respectively denote the categories of presheaves of finite sets and of finite ordinals. One should distinguish, e.g., ‘presheaf of finite sets’ $\mathbb{C}^{op} \rightarrow \mathbf{set}$ from ‘finite presheaf of sets’ $F: \mathbb{C}^{op} \rightarrow \mathbf{Set}$. The latter means that

the disjoint union $\sum_{c \in \text{ob}(\mathbb{C})} F(c)$ is finite. Throughout the paper, any finite ordinal n is seen as $\{1, \dots, n\}$ (rather than $\{0, \dots, n-1\}$).

For any functor $F: \mathbb{C} \rightarrow \mathbb{D}$ and object $D \in \mathbb{D}$, let F_D denote the comma category on the left below, and $F(D)$ denote the pullback category on the right:

$$\begin{array}{ccc} F_D & \longrightarrow & 1 \\ \downarrow & \nearrow & \downarrow r_{D'} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{D} \end{array} \qquad \begin{array}{ccc} F(D) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow r_{D'} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{D}. \end{array}$$

When F is clear from context, we simply write \mathbb{C}_D , resp. $\mathbb{C}(D)$.

Transition systems. The notion of LTS that we'll use here is a little more general than the usual one, but this does not change much. In particular, all our results may be reflected back to usual LTSS. Let \mathbf{Gph} be the category of reflexive graphs, which has as objects diagrams $s, t: E \rightrightarrows V$ in \mathbf{Set} , equipped with a further arrow $id: V \rightarrow E$ such that $s \circ id = t \circ id = id_V$. Morphisms are those morphisms between underlying graphs which preserve identity arrows.

Definition 1. For any $A \in \mathbf{Gph}$, let \mathbf{LTS}_A , the category of LTSS over A , be the slice category \mathbf{Gph}/A .

Notation. A is called the *alphabet*, which goes slightly beyond the usual notion of an alphabet. The latter would here come in the form of the graph with one vertex, an identity edge, plus an edge for each letter. By convention, and mainly to ease graphical intuitions in Sections 4, 6, and 5, for any LTS $p: G \rightarrow A$, we understand an edge $e: x' \rightarrow x$ in G as a transition from x to x' . Of course, to recover a more standard notation, one may replace all graphs with their opposites. When e does not matter, but $p(e)$ does, we denote such a transition by $x \xleftarrow{p(e)} x'$, omitting the subscript A when clear from context.

For any reflexive graph A , we denote by A^* the graph with the same vertices and arbitrary paths as edges. A^* is reflexive, with identity edges given by empty paths. It, however, admits a useful quotient, which equates (id) and $()$, i.e., the singleton, identity path and the empty one. Formally, the forgetful functor $U: \mathbf{Cat} \rightarrow \mathbf{Gph}$ has a left adjoint, which we denote by fc . In $\text{fc}(A)$, morphisms are paths in A , considered equivalent modulo removal of identity edges. Any path ρ has a normal form, obtained by removing all identity edges and denoted by $\tilde{\rho}$. We denote by $x \xleftarrow{a} x'$ any path $\rho: x' \rightarrow^* x$ in G , mapped by p to a path in A whose composite in $\text{fc}(A)$ is the singleton (a) . Concretely, if a is an identity, then $p(\rho)$ only consists of identity edges; otherwise, $p(\rho)$ consists of a , possibly surrounded by identity edges. In the former case, ρ may well be empty, and we further abbreviate the notation to $x \Leftarrow x'$.

Constructions. Any morphism $f: A \rightarrow B$ induces by pullback a change-of-base functor $f^*: \mathbf{Gph}/B \rightarrow \mathbf{Gph}/A$, which has a left adjoint $f_!$ given by composition with f .

Any $f: G \rightarrow G'$ over A is a *functional bisimulation* iff for all $x \in G$, $y \in G'$, and $e' \in G'(y, f(x))$, there exist $x' \in G$ and $e \in G(x', x)$ such that $f(e) = e'$.

Proposition 1. For any morphism of graphs $f: A \rightarrow B$, both functors $f^*: \mathbf{Gph}/B \rightarrow \mathbf{Gph}/A$ and $f_!: \mathbf{Gph}/A \rightarrow \mathbf{Gph}/B$, i.e., pullback along and post-composition with f , preserve functional bisimulations.

Proof. First recall Joyal's observation that, for any alphabet A , a morphism of graphs $G \rightarrow G'$ is a functional bisimulation iff for any edge $e: a \rightarrow b$ in A and commuting square as the exterior of

$$\begin{array}{ccc} \{b\} & \longrightarrow & G \\ \downarrow & \dashrightarrow & \downarrow \\ e & \longrightarrow & G', \end{array}$$

there exists a dashed arrow making both triangles commute.

The case of $f_!$ is an easy diagram chase.

For f^* , by the pullback lemma, the square

$$\begin{array}{ccc} f^*(G) & \longrightarrow & G \\ \downarrow \lrcorner & & \downarrow \\ f^*(G') & \longrightarrow & G', \end{array}$$

is a pullback. We check that $f^*(G) \rightarrow f^*(G')$ is again a bisimulation. Indeed, consider any $e: a \rightarrow b$ in A and square

$$\begin{array}{ccc} \{b\} & \longrightarrow & f^*(G) \\ \downarrow & & \downarrow \\ e & \longrightarrow & f^*(G'). \end{array}$$

Pasting this with the above pullback square, we obtain the solid part of the following diagram of graphs over B :

$$\begin{array}{ccccc} \{f(b)\} & \longrightarrow & f^*(G) & \longrightarrow & G \\ \downarrow & \dashrightarrow & \downarrow \lrcorner & & \downarrow \\ f(e) & \dashrightarrow & f^*(G') & \longrightarrow & G'. \end{array}$$

Because $G \rightarrow G'$ is a bisimulation, we obtain the dashed arrow making the triangles commute. But then by universal property of pullback, we obtain the dotted arrow, making the corresponding bottom triangle commute. Finally, the top triangle commutes after postcomposition with $f^*(G) \rightarrow G$, and after composition with $f^*(G) \rightarrow f^*(G')$, hence commutes by uniqueness in the universal property of pullback. \square

A *simulation* is a relation $R: \text{ob}(G) \dashrightarrow \text{ob}(G')$ over A between vertices, such that for all $e \in G(x', x)$, if $R(x, y)$ then there exist y' and $e' \in G'(y', y)$ such that $R(x', y')$ and e and e' are mapped to the same edge in A . A *bisimulation* is a simulation whose converse also is a simulation. Such a relation R is a *weak simulation* iff for all $e \in G(x', x)$, if $R(x, y)$ then there exist y' and a path $p: y' \rightarrow^* y$ in G' such that $R(x', y')$ and e and p are mapped to the same morphism in the free category on A (where identity edges are units for composition). A *weak bisimulation* is a weak simulation whose converse also is a weak simulation.

For any $x \in G$ and $y \in G'$, with $G, G' \in \mathbf{Gph}/A$, x and y are *strongly*, resp. *weakly bisimilar*, notations $x \sim_A y$ and $x \simeq_A y$, iff there exists a strong, resp. weak, bisimulation $G \dashrightarrow G'$ relating them. Both are equivalence relations.

CCS. Our (infinite) CCS terms are coinductively generated by the typed grammar

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P|Q} \quad \frac{\Gamma, a \vdash P}{\Gamma \vdash \nu a.P} \quad \frac{\dots \quad \Gamma \vdash P_i \quad \dots}{\Gamma \vdash \sum_{i \in n} \alpha_i.P_i} \quad (n \in \mathbb{N}),$$

where α_i is either a , \bar{a} , for $a \in \Gamma$, or \heartsuit . The latter is a ‘tick’ move used in the definition of fair testing equivalence. As a syntactic facility, we here understand Γ as ranging over \mathbb{N} , i.e., the free names of a process always are $1 \dots n$ for some n . E.g., Γ, a denotes just $n + 1$, and $a \in \Gamma$ means $a \in \{1, \dots, \Gamma\}$.

Definition 2. Let \mathbb{A} be the reflexive graph with vertices given by finite ordinals, edges $\Gamma \rightarrow \Gamma'$ given by \emptyset if $\Gamma \neq \Gamma'$, and by $\Gamma + \Gamma + \{\tau, \heartsuit\}$ otherwise, $\tau: \Gamma \rightarrow \Gamma$ being the identity edge on Γ . Elements of the first summand are denoted by $a \in \Gamma$, while elements of the second summand are denoted by \bar{a} .

We view terms as a graph *CCS* over \mathbb{A} with the usual transition rules. The graph \mathbb{A} only has ‘endo’-edges, which allows to only compare terms with the same set of free channels. Some LTSS below do use more general graphs.

Pseudo double categories. Finally, we briefly recall from Grandis and Pare [19, 20], Leinster [33], or Garner [18] the notion of pseudo double category. This is a weakening of Ehresmann’s double categories [14, 15], with one dimension strict and the other weak (i.e., bicategory-like). We need to consider proper pseudo double categories, notably we use cospans in examples, but there is no subtlety in the way we handle pseudo-ness, so we often lazily treat them just as double categories.

A pseudo double category \mathbb{D} consists of a set $\text{ob}(\mathbb{D})$ of *objects*, shared by two categories \mathbb{D}_h and \mathbb{D}_v . \mathbb{D}_h is called the *horizontal* category of \mathbb{D} , and \mathbb{D}_v is the *vertical* category. \mathbb{D} is furthermore equipped with a set of *double cells*. A double cell α has vertical domain and codomain, denoted by $\text{dom}_v(\alpha)$ and $\text{cod}_v(\alpha)$, and similarly horizontal ones, denoted by $\text{dom}_h(\alpha)$ and $\text{cod}_h(\alpha)$. We picture this as:

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ u \downarrow & \nearrow \alpha & \downarrow u' \\ Y & \xrightarrow{h'} & Y' \end{array}$$

where $u = \text{dom}_h(\alpha)$, $u' = \text{cod}_h(\alpha)$, $h = \text{dom}_v(\alpha)$, and $h' = \text{cod}_v(\alpha)$. A pseudo double category is furthermore equipped with operations for composing double cells: \circ composes them along a common vertical morphism, \bullet composes along horizontal morphisms. In this paper, \bullet may only be associative up to coherent isomorphism. The full axiomatisation is given in the above listed references, and we omit it here. Finally, these data must satisfy the *interchange law*, which says that the two ways of parsing

$$\begin{array}{ccccc} X & \xrightarrow{h} & X' & \xrightarrow{k} & X'' \\ u \downarrow & \nearrow \alpha & u' \downarrow & \nearrow \alpha' & \downarrow u'' \\ Y & \xrightarrow{h'} & Y' & \xrightarrow{k'} & Y'' \\ v \downarrow & \nearrow \beta & v' \downarrow & \nearrow \beta' & \downarrow v'' \\ Z & \xrightarrow{h''} & Z' & \xrightarrow{k''} & Z'' \end{array}$$

namely $(\beta' \circ \beta) \bullet (\alpha' \circ \alpha)$ and $(\beta' \bullet \alpha') \circ (\beta \bullet \alpha)$, coincide.

For any (pseudo) double category \mathbb{D} , we denote by \mathbb{D}_H the category with vertical morphisms as objects and double cells as morphisms, and by \mathbb{D}_V the category with horizontal morphisms as objects and double cells as morphisms. \mathbb{D}_H is called the *1-horizontal* category, whilst \mathbb{D}_V is the *1-vertical* category.

We introduce a bit more notation.

Definition 3. *A double cell is special when its vertical domain and codomain are (horizontal) identities.*

For any object $X \in \text{ob}(\mathbb{D})$, $\mathbb{D}_H(X)$ denotes the category with

- objects all vertical morphisms to X , and

$$\bullet \text{ morphisms } u \rightarrow v \text{ all double cells } \begin{array}{ccc} Y & \xrightarrow{h} & Y' \\ u \downarrow & \xrightarrow{\alpha} & \downarrow u' \\ X & \xrightarrow{k} & X \end{array} \text{ with } \text{cod}_v(\alpha) = id_X.$$

3. SUMMARY OF PREVIOUS WORK

In this section and the next, we recall some material from HP.

3.1. Recalling the game.

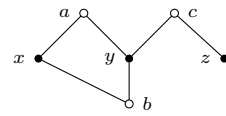
3.1.1. *Positions, Moves, and Plays.* In this section, we start by explaining positions, plays, and views, trying to emphasise how the definition is driven by that of CCS. Here, positions, moves, and plays will be just string diagrams. Formally, they are represented as presheaves over a certain category \mathbb{C} , as developed in HP. In this section, the reader may understand a ‘game’ to be a category, whose objects are ‘positions’ and whose morphisms are ‘plays’. There is a class of morphisms called ‘moves’, and plays should admit at least one decomposition into moves.

Positions. First, we introduce positions, which are essentially multi-hole evaluation contexts in CCS, up to structural congruence. Without going through the formal definition, consider evaluation contexts generated from the grammar

$$\frac{}{\Gamma; x: n \vdash x(a_1, \dots, a_n)} \quad \frac{\Gamma; \Delta_1 \vdash e_1 \quad \Gamma; \Delta_2 \vdash e_2}{\Gamma; \Delta_1, \Delta_2 \vdash e_1|e_2},$$

where, in the first rule, $\forall i \in n, a_i \in \Gamma$, and in the second $\text{dom}(\Delta_1) \cap \text{dom}(\Delta_2) = \emptyset$. Here, x ranges over a fixed set of *variables*, and Δ, \dots range over finite maps from variables to natural numbers. Evaluation contexts are furthermore considered equivalent up to associativity and commutativity of $|$.

Positions are essentially a combinatorial, direct representation of such contexts, namely a kind of hypergraphs. For example, the context $x(a, b) | y(a, b, c) | z(c)$ is represented by the graph on the right, where x, y, z are the bullets and a, b, c are the circles. Variables will represent players in the game. (Observe in passing that two α -equivalent contexts will be represented by isomorphic graphs.) An important detail is that the channels adjacent to a player are linearly ordered. The graphical representation is thus ambiguous, since it does not indicate which name comes first in the ordering. The formal representation rectifies this, and we will live with the graphical ambiguities, since they should not hinder understanding.



Morphisms of positions are just embeddings, in the straightforward sense.

Proposition 2. *Positions and morphisms between them form a category \mathbb{D}_h^{CCS} .*

Local moves. The moves of the game will be defined in two stages. We first define *local moves*, in which all involved players are active, and then *global moves*, which may involve passive players. Local moves are themselves defined in three stages:

- *basic* local moves will be in 1-1 correspondence with (premise, conclusion) pairs in the natural deduction presentation of CCS recalled above;
- the next stage will group together pairs with a common conclusion (separating cases in guarded sum);
- and the last stage will deal with synchronisation.

First, we have basic moves for the left and right premises of a parallel composition, which are pictured like this in the case of a process with two channels:



The left and right orientations serve to differentiate the left premise from the right (see HP for a formal definition). These moves are called *left* and *right fork*. The game interpretation is that they are moves from the bottom position $\circ \bullet \circ$ to the top position (which here is the same). In both shown moves, the considered player knows two channels. These are just instances for $n = 2$ of left and right forks with n channels, which we respectively denote by π_n^l and π_n^r .

The next two basic moves are output and input:



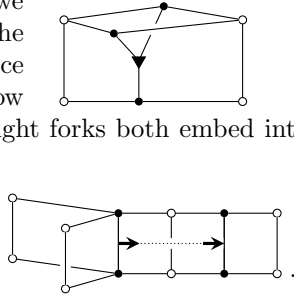
These are the instances for $n = 3, i = 2, m = 2, j = 1$ of, respectively, output on the i th of n channels ($o_{n,i}$), and input on the j th of m channels ($l_{m,j}$).

The next basic moves are *tick*, a special move used for defining fair testing equivalence, and *channel creation*, which starts from a player with n channels, and leads to a player with $n + 1$ channels. Here are the instances \heartsuit_2 and ν_2 :

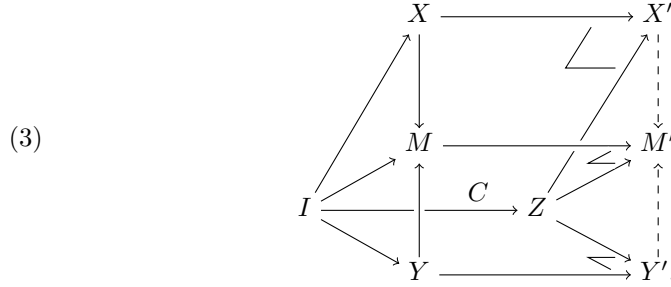
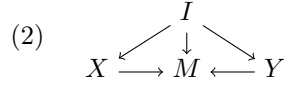
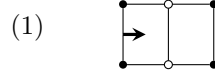


Let us now consider second-stage moves. There, we have one move (scheme), called *fork*, which looks as on the right. As for left and right forks, this is the $n = 2$ instance of a general pattern, denoted by π_n . We will define below a category whose *objects* are plays, in which left and right forks both embed into this full fork.

The final layer of moves consists of just synchronisation, which looks as on the right. As for forks, input and output embed into it, and there are instances $\tau_{n,i,m,j}$ for all sensible n, i, m, j .



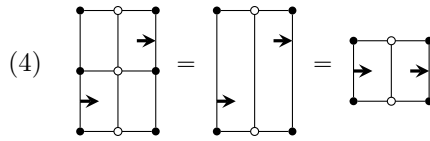
Global moves and plays. The local moves introduced above are in fact just ‘seeds’ for the actual moves of our game. We have the informal definition: a *global move* is a local move whose initial position has been embedded into a larger position. For example, $o_{1,1}$ embeds into the global move (1). A bit more formally, all kinds of string diagrams above are represented as finite presheaves on a category \mathbb{C} . An *interface* is a diagram consisting only of channels. Local moves may be equipped with a canonical interface, consisting of the channels of their initial position. If $X \rightarrow M \leftarrow Y$ is a local move (say from Y to X), and I is its canonical interface, we obtain a commuting diagram (2) in $\widehat{\mathbb{C}}$. A global move is formally defined to be a cospan obtained by pushing such a local move with interface along any morphism $I \rightarrow Z$, for any position Z , as the dashed cospan in



Definition 4. A play is a composite of global moves, i.e., a diagram obtained from a compatible sequence of global moves by glueing the final position of each move to the initial position of the next. (Formally, we use pushouts in $\widehat{\mathbb{C}}$, i.e., composition of cospans.)

Proposition 3. Plays form a bicategory \mathbb{D}_v^{CCS} .

This definition features some concurrency. For instance, composing the above global move with a similar global move by the other player, we obtain the play on the right. Note in passing that this embeds into a synchronisation, but is not one, since the input and output moves are not related. This play should be understood as each player communicating with the outside world.



Definition 5. A move is full iff it is neither a left nor a right fork. We call \mathbb{F} the graph of global, full moves.

Intuitively, a move is full when its final position contains all possible premises of the involved natural deduction rules.

3.2. Behaviours and strategies.

3.2.1. *Behaviours.* Recall from HP the category \mathbb{E}

- whose objects are maps $U \leftarrow X$ in $\widehat{\mathbb{C}}$, such that there exists a play $Y \rightarrow U \leftarrow X$, i.e., objects are plays, where we forget the final position;
- and whose morphisms $(U \leftarrow X) \rightarrow (U' \leftarrow X')$ are commuting diagrams as on the right with all arrows monic.

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}$$

Morphisms $U \rightarrow U'$ in \mathbb{E} represent extensions of U , both spatially (i.e., embedding into a larger position) and dynamically (i.e., adding more moves).

We may relativise this category \mathbb{E} to a particular position X , yielding a category $\mathbb{E}(X)$ of plays on X . Consider the functor $\text{cod}: \mathbb{E} \rightarrow \mathbb{D}_h^{\text{CCS}}$ mapping any play $U \leftarrow X$ to its initial position X , and consider the pullback category $\mathbb{E}(X)$ as defined in Section 2. The objects of $\mathbb{E}(X)$ are just plays $(U \leftarrow X)$ on X , and morphisms are morphisms of plays whose lower border is id_X . This allows the definition of a category of ‘naive’ strategies, called behaviours.

Definition 6. *The category \mathbb{B}_X of behaviours on X is the category $\widehat{\mathbb{E}(X)}^f$ of presheaves of finite sets on $\mathbb{E}(X)$.*

Behaviours suffer from the deficiency of allowing unwanted cooperation between players. HP (Example 12) exhibits a behaviour where players choose with whom they synchronise, which clearly is not allowed in CCS.

3.2.2. Strategies. To rectify this, we consider the full subcategory of \mathbb{E} consisting of *views*, i.e., compositions of basic local moves. Calling this category \mathbb{E}^{\vee} , we relativise views to a position X by considering the comma category \mathbb{E}_X^{\vee} as defined in Section 2. Its objects are pairs of a view $V \leftarrow [n]$ on a single n -ary player connected to n distinct channels (which we denote by $[n]$), and an embedding $[n] \hookrightarrow X$, i.e., a player of X .

Definition 7. *The category \mathbb{S}_X of strategies on X is the category $\widehat{\mathbb{E}_X^{\vee}}$ of presheaves of finite ordinals on \mathbb{E}^{\vee} .*

This rules out undesired behaviours. Recall from HP how to map strategies to behaviours: let first \mathbb{E}_X be the category obtained by taking a comma category instead of a pullback in the definition of $\mathbb{E}(X)$. Then, embedding $\widehat{\mathbb{E}_X^{\vee}}$ into $\widehat{\mathbb{E}_X^{\vee}}^f$ via $\text{ford} \hookrightarrow \text{set}$, followed by right Kan extension to \mathbb{E}_X^{op} followed by restriction to $\mathbb{E}(X)^{\text{op}}$ yields a functor $\overline{(-)}: \mathbb{S}_X \rightarrow \mathbb{B}_X$, which is full, faithful, and essentially injective on objects. The image of a strategy S may be computed as S'' in

$$\begin{array}{ccccc} (\mathbb{E}_X^{\vee})^{\text{op}} & \hookrightarrow & \mathbb{E}_X^{\text{op}} & \longleftarrow & \mathbb{E}(X)^{\text{op}} \\ S \downarrow & & S' \downarrow & \swarrow S'' & \\ \text{ford} & \hookrightarrow & \text{set} & & \end{array}$$

where S' is here obtained by right Kan extension (the embedding $(\mathbb{E}_X^{\vee})^{\text{op}} \hookrightarrow \mathbb{E}_X^{\text{op}}$ being full and faithful, we may choose the diagram to strictly commute).

By the standard formula for right Kan extensions as ends [37] we have for any $S: \overline{S}(U) = \int_{v \in \mathbb{E}_X^{\vee}} S(v)^{\mathbb{E}_X(v, U)}$. If S is boolean, i.e., takes values in $\{\emptyset, 1\}$, then the involved end may be viewed as a conjunction, saying that U is accepted by S'' whenever all its views are accepted by S . Equivalently, $\overline{S}(U)$ is a limit of $(\mathbb{E}_X^{\vee}/U)^{\text{op}} \xrightarrow{\text{dom}} (\mathbb{E}_X^{\vee})^{\text{op}} \xrightarrow{S} \text{ford} \hookrightarrow \text{set}$.

Finally, $\overline{(-)}$ admits a left adjoint, which we might call ‘innocentisation’, because it maps naive strategies (behaviours) to innocent ones.

3.2.3. Decomposition: a syntax for strategies. Our definition of strategies is rather semantic in flavour. Indeed, presheaves are akin to domain theory. However, they also lend themselves well to a syntactic description.

First, it is shown in HP that strategies on an arbitrary position X are in 1-1 correspondence with families of strategies indexed by the players of X . Recall that $[n]$ is the position consisting of one n -ary player. A player of X is the same as a morphism $[n] \rightarrow X$ (for some n) in \mathbb{D}_h^{CCS} . Thus, we define the set $\text{Pl}(X) = \sum_{n \in \mathbb{N}} \mathbb{D}_h^{CCS}([n], X)$ of players of X .

Proposition 4. *We have $\mathbb{S}_X \cong \prod_{(n,x) \in \text{Pl}(X)} \mathbb{S}_{[n]}$. For any $S \in \mathbb{S}_X$, we denote by S_x the component corresponding to $x \in \text{Pl}(X)$ under this isomorphism.*

So, strategies on arbitrary positions may be entirely described by strategies on ‘typical’ players $[n]$. As an important particular case, we may let two strategies interact along an interface (recall from Section 3.1.1 that this means a position consisting only of channels). This will be the basis of our semantic definition of fair testing equivalence. We proceed as follows. Consider any pushout Z of $X \leftarrow I \rightarrow Y$ where I is an interface. We have

Corollary 1. $\mathbb{S}_Z \cong \mathbb{S}_X \times \mathbb{S}_Y$.

Proof. We have $\mathbb{E}_Z^\vee \cong \mathbb{E}_X^\vee + \mathbb{E}_Y^\vee$, and conclude by universal property of coproduct. \square

We denote by $[S, T]$ the image of $(S, T) \in \mathbb{S}_X \times \mathbb{S}_Y$ under this isomorphism.

Having shown how strategies may be decomposed into strategies on ‘typical’ players $[n]$, we now explain that strategies on such players may be further decomposed. For any strategy S on $[n]$ and basic move $b: [n'] \rightarrow [n]$, let the *residual* $S \cdot b$ of S after b be the strategy playing like S after b , i.e., for all $v \in \mathbb{E}_{[n']}^\vee$, $(S \cdot b)(v) = S(b \bullet v)$. S is almost determined by its residuals. The only information missing from the $S \cdot b$ ’s to reconstruct S is the set of initial states and how they relate to the initial states of each $(S \cdot b)$. This may be taken into account as follows.

For any initial state $\sigma \in S(id)$, let $S_{|\sigma}$ be the restriction of S determined by

$$S_{|\sigma}(v) = \{\sigma' \in S(v) \mid S(!)(\sigma') = \sigma\},$$

where $S(!)$ denotes the image by S of the unique morphism $!: id \rightarrow v$. S is determined by its set $S(id)$ of initial states, plus the function $(\sigma, b) \mapsto (S_{|\sigma} \cdot b)$ mapping any $\sigma \in S(id)$ and isomorphism class b of basic moves to $S_{|\sigma} \cdot b$. In other words, we have for all n :

Theorem 1. $\mathbb{S}_{[n]} \cong (\prod_{b: [n'] \rightarrow [n]} \mathbb{S}_{[n']})^*$.

Given an element (D_1, \dots, D_m) of the right-hand side, the corresponding strategy maps the identity view id to m , and any non-identity view $b \bullet v$ on $[n]$ to the sum $\sum_{i \in m} D_i(b)(v)$.

A closely related result is that strategies on a player $[n]$ are in bijection with infinite terms in the following typed grammar, with judgements $n \vdash_D D$ and $n \vdash S$, where D is called a *definite prestrategy* and S is a *strategy*:

$$\frac{\dots n_b \vdash S_b \dots (\forall b: [n_b] \rightarrow [n] \in [\mathbb{B}]_n)}{n \vdash_D \langle (S_b)_{b \in [\mathbb{B}]_n} \rangle} \quad \frac{\dots n \vdash_D D_i \dots (\forall i \in m)}{n \vdash \bigoplus_{i \in m} D_i} \quad (m \in \mathbb{N}),$$

where $[\mathbb{B}]_n$ denotes the set of all isomorphism classes of basic moves from $[n]$. This achieves the promised syntactic description of strategies. We may readily define the translation of CCS processes, coinductively, as follows. For processes with channels in Γ , we define

$$\begin{aligned} (\sum_{i \in n} \alpha_i.P_i) &= \langle b \mapsto \oplus_{\{i \in n | b = \{\alpha_i\}\}} \langle P_i \rangle \rangle & \langle a \rangle &= \iota_{\Gamma, a} \\ \langle \nu a.P \rangle &= \langle \nu_{\Gamma} \mapsto \langle P \rangle, - \mapsto \emptyset \rangle & \langle \bar{a} \rangle &= o_{\Gamma, a} \\ \langle P \mid Q \rangle &= \langle \pi_{\Gamma}^l \mapsto \langle P \rangle, \pi_{\Gamma}^r \mapsto \langle Q \rangle, - \mapsto \emptyset \rangle & \langle \heartsuit \rangle &= \heartsuit_{\Gamma}. \end{aligned}$$

For example, $a.P + a.Q + \bar{b}.R$ is mapped to

$$\langle \iota_{\Gamma, a} \mapsto (\langle P \rangle \oplus \langle Q \rangle), o_{\Gamma, b} \mapsto \langle R \rangle, - \mapsto \emptyset \rangle.$$

3.3. Semantic fair testing. The tools developed in the previous section allow the following semantic analogue of fair testing equivalence.

Definition 8. Closed-world moves are (the global variants of) ν, \heartsuit, π_n , and $\tau_{n,i,m,j}$. A play is closed-world when it is a composite of closed-world moves.

Let a closed-world play be *successful* when it contains a \heartsuit move. Let then \perp_Z denote the set of behaviours $B \in \mathbf{B}_Z$ such that for any unsuccessful, closed-world play admits a successful extension. Formally, for any unsuccessful, closed-world play $U \leftarrow Z$ and $\sigma \in B(U)$, there exists $f: U \rightarrow U'$ with closed-world U' and $\sigma' \in B(U')$ such that $B(f)(\sigma') = \sigma$ and U' is successful. Finally, let us say that a triple (I, h, S) , for any $h: I \rightarrow X$ where I is an interface, and strategy $S \in \mathbf{S}_X$, passes the test consisting of a morphism $k: I \rightarrow Y$ of positions and a strategy $T \in \mathbf{S}_Y$ iff $[\overline{S}, \overline{T}] \in \perp_Z$, where Z is the pushout of h and k . Let S^\perp denote the set of all such (k, T) .

Definition 9. For any $h: I \rightarrow X$, $h': I \rightarrow X'$, $S \in \mathbf{S}_X$, and $S' \in \mathbf{S}_{X'}$, $(I, h, S) \sim_f (I, h', S')$ iff $(I, h, S)^\perp = (I, h', S')^\perp$.

This yields an equivalence relation, analogous to standard fair testing equivalence, which we hence also call fair testing equivalence.

This raises the question of whether the translation $\langle - \rangle$ preserves or reflects fair testing equivalence. The rest of the paper is devoted to proving that it does both. As announced in the introduction, this is done by organising the game into a *playground*, as defined in the next section.

4. PLAYGROUNDS: FIRST AXIOMS AND DEFINITIONS

4.1. Motivation: a pseudo double category. In order to motivate the notion of playground, we organise the game described above into a (pseudo) double category. We have seen that positions are the objects of the category \mathbb{D}_h^{CCS} , whose morphisms are embeddings of positions. But positions are also the objects of the category \mathbb{D}_v^{CCS} , whose morphisms are plays. It should seem natural to define a pseudo double category structure with double cells given by commuting diagrams as on the right in $\widehat{\mathbb{C}}$. Here, Y is the initial position and X is the final one; all arrows are mono.

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ s \downarrow & & \downarrow s' \\ U & \xrightarrow{k} & V \\ t \uparrow & & \uparrow t' \\ Y & \xrightarrow{l} & Y' \end{array}$$

Proposition 5. This forms a pseudo double category \mathbb{D}^{CCS} . Furthermore the functor $\text{cod}_v: \mathbb{D}_H^{CCS} \rightarrow \mathbb{D}_h^{CCS}$ is a Grothendieck fibration [27].

In the sequel, we will refer to dom_v and cod_v simply as dom and cod , reserving subscripts for dom_h and cod_h .

The only non-trivial point is that cod is a fibration, which is proved among other facts in Section 8. ‘Pseudo’ here means that both vertical compositions are only associative and unital up to canonical isomorphism (since they are defined by pushout).

Proposition 5 was the starting point of the notion of playground: which axioms can we demand of a pseudo double category in order to be able to define the constructions of HP? We follow the constructions in this section, considering an arbitrary pseudo double category \mathbb{D} , on which we impose axioms along the way. Objects and vertical morphisms will respectively be called *positions* and *plays*. The pseudo double category \mathbb{D}^{CCS} does satisfy the axioms, albeit in a non-trivial way. This is stated and proved in Section 8, but we use the result in advance in examples to illustrate our constructions.

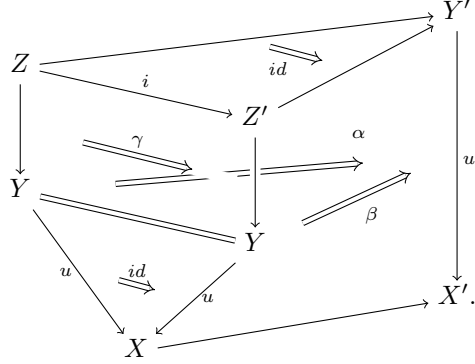
For the reader’s convenience, we here record the axioms imposed on \mathbb{D} in the next sections to obtain Theorem 3:

- (P1), page 16,
- (P2)—(P5), page 17,
- (P6), page 17,
- (P7), page 18,
- (P8), page 18,
- (P9), page 25,
- (P10), page 30.

4.2. Behaviours. The easiest construction of HP to carry over to the abstract setting of playgrounds is that of behaviours. First, let us stress that, in the case of \mathbb{D}^{CCS} , \mathbb{D}_H^{CCS} is very different from the category of plays called \mathbb{E} in HP. Indeed, any morphism $\alpha: u \rightarrow u'$ in \mathbb{D}_H^{CCS} in particular induces an embedding of the final position $\text{dom}(u)$ of u into that of u' . In \mathbb{E} , instead, a morphism $u \rightarrow u'$ may involve prolongating u . For instance, the play (1) embeds into (4) in the sense of \mathbb{E} , but not in the sense of \mathbb{D}_H^{CCS} .

So our first step is to construct an analogue of \mathbb{E} from any playground \mathbb{D} . Let it have as objects all plays, and as morphisms $u \rightarrow u'$ all pairs (w, α) as on the right. Actually, this definition is slightly wrong, in that a morphism carries some information about how w embeds into u' , while we are only interested in how u embeds into u' . Thus, we instead define morphisms $u \rightarrow u'$ to be pairs (w, α) as in (5), quotiented by the equivalence relation generated by pairs (w, α) and (w', β) such that there exists morphisms i and γ satisfying $\alpha = \beta \circ (u \bullet \gamma)$, as in

$$(5) \quad \begin{array}{ccc} Z & \xrightarrow{h} & Y' \\ w \downarrow & \alpha \Rightarrow & \downarrow u' \\ Y & & \\ u \downarrow & & \\ X & \xrightarrow{k} & X' \end{array}$$



In order to define composition in this category, we state the following axiom.

Axiom. (P1) (Fibration) *The vertical codomain functor $\text{cod} : \mathbb{D}_H \rightarrow \mathbb{D}_h$ is a fibration.*

Composition may now be defined by pullback (i.e., cartesian lifting in the fibration $\text{cod} : \mathbb{D}_H \rightarrow \mathbb{D}_h$) and pasting:

$$\begin{array}{ccccc}
 Z'' & \longrightarrow & Z' & \longrightarrow & V \\
 w'' \downarrow \lrcorner & & w' \downarrow & & \downarrow \\
 Z & \longrightarrow & Y' & & \downarrow \\
 w \downarrow & & \downarrow & \xrightarrow{\beta} & \downarrow \\
 Y & \xrightarrow{\alpha} & u' & & u \\
 u \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & X' & \longrightarrow & U.
 \end{array}$$

Quotienting makes composition functional and associative, and furthermore it is compatible with the above equivalence. Identities are obvious.

Proposition 6. *This forms a category \mathbb{E} .*

The quotient we make is designed to retain from a morphism $(w, \alpha) : u \rightarrow u'$ just the way u embeds into u' , so the way w does is irrelevant.

Following HP, we state:

Definition 10. *The category \mathbb{B}_X of behaviours on X is $\widehat{\mathbb{E}(X)}^f$, i.e., the category of presheaves of finite sets on $\mathbb{E}(X)$.*

This construction also has a bit of structure. First, observe that the map $X \mapsto \mathbb{E}(X)$ extends to a functor $\mathbb{E}(-) : \mathbb{D}_v \rightarrow \mathbf{Cat}$ by vertical post-composition. Post-composing the opposite of this functor by $\widehat{(-)}^f : \mathbf{Cat}^{op} \rightarrow \mathbf{Cat}$, we obtain a functor $\mathbb{B}_- : \mathbb{D}_v^{op} \rightarrow \mathbf{Cat}$, satisfying $\mathbb{B}_u(B)(u') = B(u \bullet u')$.

4.3. More axioms. We now turn to generalising further constructions of HP to the general setting of playgrounds. We mentioned in the introduction that strategies on a position X should be defined as presheaves on the category of views on X . We will further want to generalise the decomposition theorems for strategies of HP, which crucially rely on a property of views stated (below in Section 4.4) as Proposition 13.

In order to define strategies, while retaining this property, we require more axioms on \mathbb{D} . Roughly, the axioms equip in particular \mathbb{D} with a notion of *player* for a position X . Each position has a set of players, each player having a certain ‘type’. Furthermore, in Section 4.4, \mathbb{D} is equipped with a notion of ‘view’, mimicking that of game semantics; and views have a type, too. Proposition 13 states that views on a position X form a coproduct, over all players x in X , of views over the type of x .

We first state a series of simple axioms, and then, building on these, two more complicated axioms.

Axiom. \mathbb{D} is equipped with

- a full subcategory $\mathbb{I} \hookrightarrow \mathbb{D}_h$ of objects called individuals,
- a replete class \mathbb{M} of vertical morphisms called moves, with replete subclasses \mathbb{B} and \mathbb{F} , respectively called basic and full moves,

satisfying the following conditions:

(P2) \mathbb{I} is discrete.

(P3) (Individuality) Basic moves have individuals as both domain and codomain.

(P4) (Atomicity) Up to a special isomorphism in \mathbb{D}_H , all plays u admit decompositions into moves. All such decompositions have the same number $|u|$ of moves, which is called their length.

For any vertical $u: X \rightarrow Y$ of length 0, there exists a unique isomorphism $\alpha^u: u \rightarrow id_Y$ with $\text{cod}(\alpha^u) = id_Y$, and a unique isomorphism $\alpha_u: id_X \rightarrow u$ with $\text{dom}(\alpha_u) = id_X$, which furthermore satisfy $\text{cod}(\alpha_u) = \text{dom}(\alpha^u): X \rightarrow Y$.

Furthermore, for any cell $\alpha: v \rightarrow u$, (1) if $|u| = 0$ then also $|v| = 0$, and (2) if v is a view, then $|v| \leq |u|$. ((2) is actually a consequence of (1) if all other axioms are satisfied.)

(P5) (Fibration, continued) Restrictions of moves (resp. full moves) to individuals either are moves (resp. full moves), or have length 0.

Replete means stable under isomorphism (here in \mathbb{D}_H). A *player* in a position (i.e., object) X , is a pair (d, x) , where $d \in \mathbb{I}$ and $x: d \rightarrow X$. Let $\text{Pl}(X)$ be the set of players of X .

Example 1. In \mathbb{D}^{CCS} , we take individuals to consist of positions, denoted by $[n]$, consisting for some n of a single n -ary player, connected to n distinct channels. Actually, for each isomorphism class of such positions we pick one representative: this makes \mathbb{I} discrete. Furthermore, we take basic moves to be local basic moves.

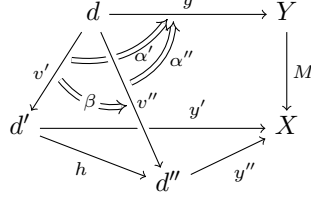
Here is a further, crucial axiom. Let \mathbb{B}_0 be the full subcategory of \mathbb{D}_H consisting of basic moves and morphisms of length 0.

Axiom. (P6) (Views) For any move $M: Y \rightarrow X$ in \mathbb{D}_v , the domain functor $\text{dom}: \mathbb{B}_0/M \rightarrow \mathbb{I}/Y$ is an equivalence of categories.

In elementary terms, this means that, for any $y: d \rightarrow Y$ in \mathbb{D}_h with $d \in \mathbb{I}$, there exists a cell

$$\begin{array}{ccc} d & \xrightarrow{y} & Y \\ \downarrow v^{y,M} & \begin{array}{c} \dashv \dashv \dashv \dashv \dashv \\ \alpha^{y,M} \\ \dashv \dashv \dashv \dashv \dashv \end{array} & \downarrow M \\ d^{y,M} & \dashrightarrow & X, \\ & y^M & \end{array}$$

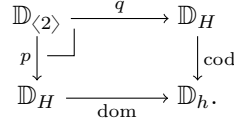
with $v^{y,M} \in \mathbb{B}_0$, which is unique up to canonical isomorphism of such. An isomorphism between two such triples, say (d', v', y') and (d', v'', y'') is a diagram



such that $\alpha'' \circ \beta = \alpha'$.

Example 2. *This axiom is obviously satisfied by \mathbb{D}^{CCS} .*

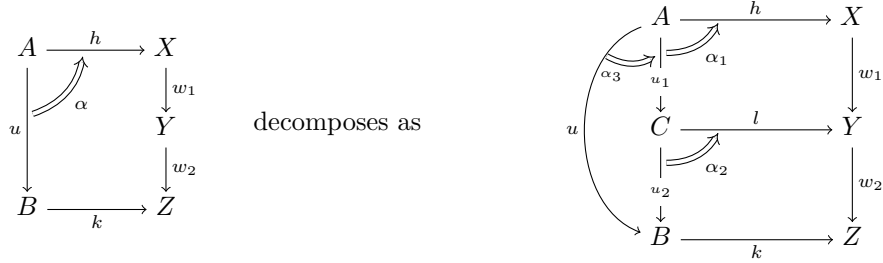
We then have two decomposition axioms. Consider first the category $\mathbb{D}_{\langle 2 \rangle}$ obtained by pullback



For any composable $X \xrightarrow{u_1} Y \xrightarrow{u_2} Z$ in \mathbb{D}_v , we may consider the slice category $\mathbb{D}_{\langle 2 \rangle}/(u_2, u_1)$, which admits a functor to $\mathbb{D}_H/(u_2 \bullet u_1)$ given by vertical composition.

Axiom. (P7) (Left decomposition) *For any such composable u_1 and u_2 , the functor $\mathbb{D}_{\langle 2 \rangle}/(u_2, u_1) \rightarrow \mathbb{D}_H/(u_2 \bullet u_1)$ is an equivalence.*

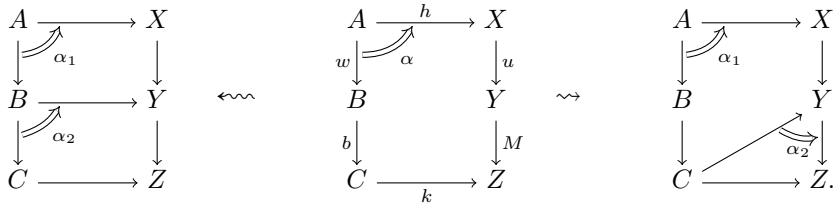
This says in particular that any double cell



with α_3 an isomorphism, in an essentially unique way.

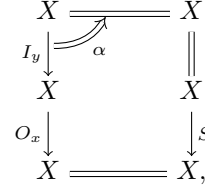
Here is our second decomposition axiom. It is kind of symmetric to the first, but not quite.

Axiom. (P8) (Right decomposition) *Any double cell as in the center below, where b is a basic move and M is a move, decomposes in exactly one of the forms on the left and right:*



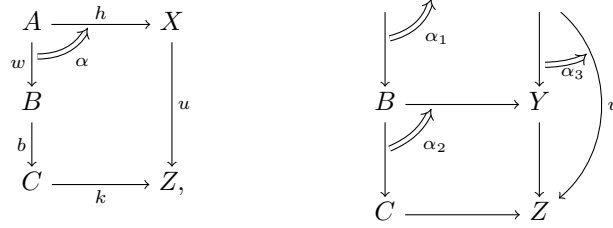
Example 3.

That this axiom is satisfied by \mathbb{D}^{CCS} is not obvious and is proved in Section 8. However, let us show that it does not satisfy the more general version where b is not required to be basic. Indeed, let X consist of two players x and y sharing a channel a . Let $I_y: X \rightarrow X$ be the play where y inputs on a , $O_x: X \rightarrow X$ be the play where x outputs on a , let $S: X \rightarrow X$ be the play where both players synchronise on a . We obtain a double cell as on the right, which does not decompose as in (P8). The problem here is that, on the left-hand side, the upper input by y has to be mapped to the same part of the right-hand side as the lower output on x , which prevents any suitable decomposition.



In the next section, we define and study views. But before that, let us draw a few consequences from our axioms on basic moves.

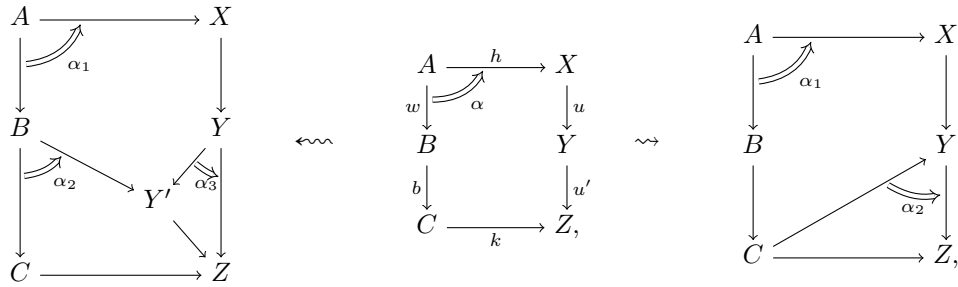
Proposition 7. Any double cell as on the left



where b is a basic move, decomposes in at least one way as on the right, with α_3 an isomorphism in \mathbb{D}_H .

Proof. By induction on $|u|$. It cannot be 0 by atomicity, so we apply Axiom (P8) until we fall in the left-hand case. \square

Proposition 8. Any double cell as in the center below, where b is a basic move, decomposes in at least one of the forms on the left and right:



with α_3 an isomorphism in \mathbb{D}_H .

Proof. We proceed by induction on $|u'|$. If $|u'| = 0$, then w.l.o.g. u' is an identity (this is actually a bit tricky, but it works, using atomicity), and we fall in the right-hand case, with $\alpha_1 = \alpha$ and $\alpha_2 = id_k$.

Otherwise, we decompose u' as $M \bullet u'_1$, apply Axiom (P8), and easily conclude by induction hypothesis. \square

We now continue by defining and studying views.

4.4. Views.

Definition 11. A view in \mathbb{D} is a play which is isomorphic via a special isomorphism in \mathbb{D}_H to a possibly empty (vertical) composite of basic moves. I.e., if

$$d_n \xrightarrow{b_n} d_{n-1} \dots d_1 \xrightarrow{b_1} d_0$$

are all basic moves, then the composite is a view. Let \mathbb{V} be the full subcategory of \mathbb{D}_H consisting of views.

The definition includes the ‘identity’ view given by the empty path on d . In \mathbb{D}^{ccs} , this of course coincides with views as defined in HP.

We start with an analogue of (P6).

Proposition 9. For any $y: d \rightarrow Y$ in \mathbb{D}_h with $d \in \mathbb{I}$, for any $u: Y \rightarrow X$ in \mathbb{D}_v , there exists a cell

$$\begin{array}{ccc} d & \xrightarrow{y} & Y \\ v^{y,u} \downarrow \lrcorner & \alpha^{y,u} = \rhd & \downarrow u \\ d^{y,u} & \dashrightarrow_{y^u} & X, \end{array}$$

with $v^{y,u}$ a view, which is unique up to canonical isomorphism of such.

Proof. We find $v^{y,u}$ by repeated application of (P6). For essential uniqueness, observe that any morphism $\alpha': v' \rightarrow u$ with $\text{dom}(\alpha') = y$ is canonically isomorphic to a vertical composition of morphisms $b \rightarrow M$ from a basic move to a move and morphisms $id \rightarrow M$ to a move. Any two such vertical compositions must be canonically isomorphic by repeated application of essential uniqueness in (P6). \square

This provides a better understanding of \mathbb{V}_H .

Proposition 10. \mathbb{V}_H is an equivalence relation, compatible with length.

Proof. Consider any morphism $\alpha: v \rightarrow v'$. By Proposition 9, α is the unique isomorphism between itself and $id_{v'}$, hence it is in particular invertible. Furthermore, this directly entails that $|v| = |v'|$. \square

Actually, we even have:

Proposition 11. For any $\alpha: v \rightarrow v'$ with $v \cong (b_1 \bullet \dots \bullet b_n)$ and $v' \cong (b'_1 \bullet \dots \bullet b'_{n'})$, we have $n = n'$ and α is the vertical composition of the unique morphisms $b_i \cong b'_i$, for $i \in n$.

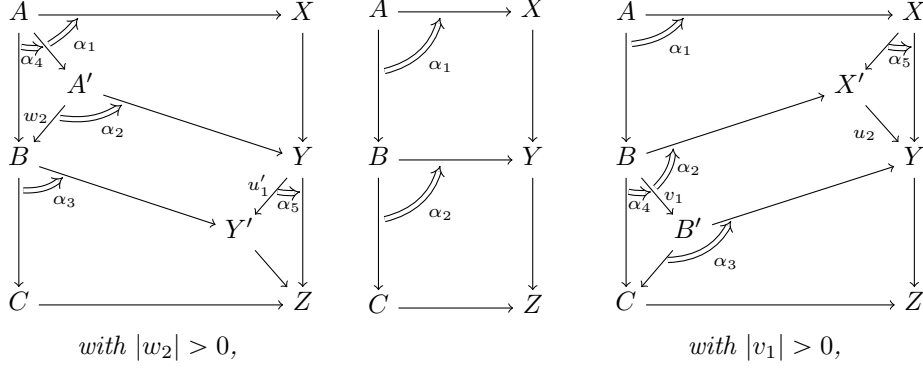
Proof. By repeated application of (P8). \square

We continue with an analogue of (P8), replacing M with an arbitrary play u' .

Proposition 12. Any double cell

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ w \downarrow \lrcorner \alpha & & \downarrow u \\ B & & Y \\ v \downarrow & & \downarrow u' \\ C & \xrightarrow{k} & Z, \end{array}$$

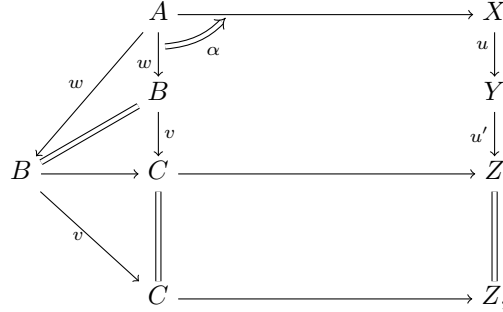
where v is a view, decomposes in exactly one of the following forms:



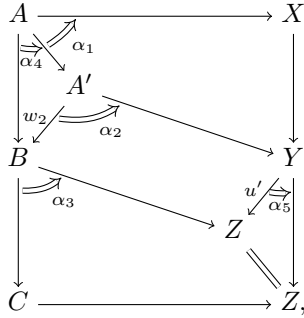
with in the left and right cases α_4 and α_5 iso in \mathbb{D}_H .

A possible reading of this is that in the left and middle cases, the whole of v embeds into u' . In the left case, a non-trivial part of w embeds into the remaining part of u' . In the right case, a non-trivial part of v embeds into u .

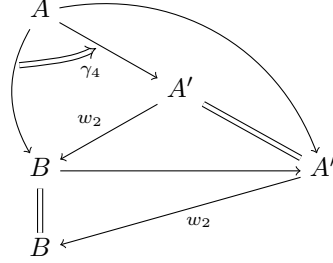
Proof. We proceed by induction on v . If v has length 0, then we cannot be in the right-hand case. Furthermore, α decomposes as



where all empty cells are either identities or obtained by atomicity. The composite is equal to α by atomicity and Proposition 10. Applying (P7) to the upper cell $w \rightarrow (u' \bullet u)$, α further decomposes as



with γ_4 and γ_5 isos. If $|w_2| \neq 0$, then we are in the left-hand case of the proposition, and the middle case is impossible by essential uniqueness in (P7). Otherwise, we may decompose γ_4 as



by atomicity (again, empty cells are either identity or given by atomicity), so we are in the middle case of the proposition.

Assume now that $|v| > 0$. We proceed by induction on $|u'|$. If $|u'| = 0$, one easily sees that we are in the right-hand case, and that both other cases are impossible. If now $|u'| > 0$, we decompose u' as $M \bullet u'_1$ and apply Axiom (P8), and, in each case, easily conclude by induction hypothesis. \square

Lastly, we need a few more definitions before Proposition 13.

Definition 12. Let \mathbb{E}^\vee be the full subcategory of \mathbb{E} consisting of views.

Following HP, we now define relativisation, i.e., we construct categories of plays and views over a given position X . As for \mathbb{D}_H , taking vertical codomain yields a functor $\text{cod}: \mathbb{E} \rightarrow \mathbb{D}_h$ mapping (5) to k , and we consider, for any X , the comma category \mathbb{E}_X as defined in Section 2. Similarly, consider \mathbb{E}_X^\vee . Concretely, an object of \mathbb{E}_X^\vee is a pair of a view $v: d' \rightarrow d$, and a player $x: d \rightarrow X$ of X . A morphism $(v_1, x_1) \rightarrow (v_2, x_2)$ is a morphism $(w, \alpha): v_1 \rightarrow v_2$ in \mathbb{E}^\vee , such that $x_2 \circ \text{cod}(\alpha) = x_1$. Because \mathbb{I} is discrete, $\text{cod}(\alpha) = id$, so our morphism is just a morphism $v_1 \rightarrow v_2$ in \mathbb{E}^\vee with identity lower border.

Recall now from Section 2 the pullback category $\mathbb{E}(X)$, which is the full subcategory of \mathbb{E}_X consisting of pairs (u, x) where $x = id_X$.

Proposition 13. $\mathbb{E}_d^\vee \cong \mathbb{E}^\vee(d)$, $\mathbb{E}_X^\vee \cong \sum_{(d,x) \in \text{Pl}(X)} \mathbb{E}_d^\vee$, and $\mathbb{E}^\vee(d)$ is a preorder.

Proof. First, because \mathbb{I} is discrete, $\mathbb{D}_h(d, d) = \{id_d\}$, so $\mathbb{E}_d^\vee \cong \mathbb{E}^\vee(d)$. Furthermore, the functor $\mathbb{E}_X^\vee \rightarrow \sum_{(d,x) \in \text{Pl}(X)} \mathbb{E}_d^\vee$ mapping any (v, x) to $((d, x), (v, id_d))$, with $v: d' \rightarrow d$ a view and $x: d \rightarrow X$ a player, has as inverse the functor mapping any $((d, x), (v, id_d))$ to (v, x) . Finally, consider any morphisms $v_1 \rightarrow v_2$ in $\mathbb{E}^\vee(d)$, say

$$\begin{array}{ccc}
 X_1 & \xrightarrow{h_1} & d_2 \\
 w_1 \downarrow & & \downarrow \\
 d_1 & \xrightarrow{\alpha_1} & v_2 \\
 v_1 \downarrow & & \downarrow \\
 d & \xlongequal{\quad} & d
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 X_2 & \xrightarrow{h_2} & d_2 \\
 w_2 \downarrow & & \downarrow \\
 d_1 & \xrightarrow{\alpha_2} & v_2 \\
 v_1 \downarrow & & \downarrow \\
 d & \xlongequal{\quad} & d
 \end{array}$$

Fixing decompositions of v_1 and v_2 into basic moves, by repeated application of (P8), in which we have to be in the middle case, we obtain that α_1 and α_2 respectively decompose as

$$\begin{array}{ccc}
X_1 & \xrightarrow{h_1} & d_2 \\
w_1 \downarrow & \xrightarrow{\alpha_1^1} & \downarrow v_2^1 \\
d_1 & \xrightarrow{\quad} & d' \\
v_1 \downarrow & \xrightarrow{\alpha_1^2} & \downarrow v_2^2 \\
d & \xlongequal{\quad} & d
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_2 & \xrightarrow{h_2} & d_2 \\
w_2 \downarrow & \xrightarrow{\alpha_2^1} & \downarrow v_2^1 \\
d_1 & \xrightarrow{\quad} & d' \\
v_1 \downarrow & \xrightarrow{\alpha_2^2} & \downarrow v_2^2 \\
d & \xlongequal{\quad} & d.
\end{array}$$

By 10, $\alpha_1^2 = \alpha_2^2$. Furthermore, we conclude by (P1) that both morphisms are equal in $\mathbb{E}^\vee(d)$ to $\alpha_1^2 \bullet id_{v_2^1}$. \square

4.5. Strategies.

Definition 13. *The category \mathbb{S}_X of strategies on X is the category $\widehat{\mathbb{E}}_X^\vee$ of presheaves of finite ordinals on \mathbb{E}_X^\vee .*

Remark 1. *We could here replace finite ordinals with a wider category and still get a valid semantics. But then to show the correspondence with the syntax we would work with the subcategory of presheaves of finite ordinals.*

Example 4. *On \mathbb{D}^{CCS} , \mathbb{E}_X^\vee as defined here yields a category equivalent to the definition in HP, so the categories of strategies are also equivalent (even isomorphic because ford contains no non-trivial automorphism).*

The rest of this section and the next two develop some structure on strategies, which is needed for constructing the LTS in Section 5.3. We start in this section by extending the assignment $X \mapsto \mathbb{S}_X$ to a pseudo double functor $\mathbb{D}^{op} \rightarrow \mathbb{QCat}$, where \mathbb{QCat} is Ehresmann's double category of *quintets* on the 2-category \mathbb{Cat} .

Actually, already the assignment $X \mapsto \mathbb{E}_X^\vee$ extends to a pseudo double functor. Indeed, define the action of a horizontal map $h: X \rightarrow X'$ to map any object (v, x) of \mathbb{E}_X^\vee to $(v, h \circ x)$ (and any morphism to itself viewed as a morphism in $\mathbb{E}_{X'}^\vee$). (This functor is induced by universal property of comma category.) This defines a functor $\mathbb{E}_-^\vee: \mathbb{D}_h \rightarrow \mathbb{Cat}$.

There is also a functor $\mathbb{D}_v \rightarrow \mathbb{Cat}$, which is a bit harder to construct. For any $u: Y \rightarrow X$ in \mathbb{D}_v and $y: d \rightarrow Y$, each cell $\alpha^{y,u}$ from Proposition 9 induces a functor $v_1^{y,u}: \mathbb{E}^\vee(d) \rightarrow \mathbb{E}^\vee(d^{y,u})$ mapping any $v: d' \rightarrow d$ to $v^{y,u} \bullet v$. Composing with the coproduct injection $inj_{d^{y,u}, y^u}: \mathbb{E}^\vee(d^{y,u}) \hookrightarrow \sum_{(d',x) \in \text{PI}(X)} \mathbb{E}^\vee(d')$, because $\mathbb{E}_X^\vee \cong \sum_{(d',x) \in \text{PI}(X)} \mathbb{E}^\vee(d')$, we obtain functors

$$\mathbb{E}^\vee(d) \xrightarrow{v_1^{y,u}} \mathbb{E}^\vee(d^{y,u}) \xleftarrow{inj_{d^{y,u}, y^u}} \mathbb{E}_X^\vee,$$

whose copairing defines a functor $u_1: \mathbb{E}_Y^\vee \rightarrow \mathbb{E}_X^\vee$.

Now, for any cell as on the left below, we obtain by Proposition 9 a canonical natural isomorphism as on the right

$$\begin{array}{ccc}
Y & \xrightarrow{k} & Y' \\
u \downarrow & \xrightarrow{\alpha} & \downarrow u' \\
X & \xrightarrow{h} & X'
\end{array}
\quad
\begin{array}{ccc}
\mathbb{E}_Y^\vee & \xrightarrow{\mathbb{E}_k^\vee} & \mathbb{E}_{Y'}^\vee \\
u_1 \downarrow & \xrightarrow{\cong} & \downarrow u'_1 \\
\mathbb{E}_X^\vee & \xrightarrow{\mathbb{E}_h^\vee} & \mathbb{E}_{X'}^\vee.
\end{array}$$

Recall Ehresmann's double category of *quintets* \mathbb{QC} on any 2-category \mathbb{C} . \mathbb{QC} has small categories as objects, functors as both horizontal and vertical morphisms, and natural transformations as double cells. By canonicity of the double cell above, we have

Proposition 14. *This assignment defines a pseudo double functor $\mathbb{E}_-^{\vee} : \mathbb{D} \rightarrow \mathbb{Q}\text{Cat}$.*

Definition 14. *Let the opposite \mathbb{D}^{op} of a pseudo double category be the pseudo double category obtained by reversing both vertical and horizontal arrows, and the double cells.*

(In the literature, this would probably rather be denoted by \mathbb{D}^{coop} . We stick to our notation for conciseness.)

We obtain:

Definition 15. *Let $S : \mathbb{D}^{op} \rightarrow \mathbb{Q}\text{Cat}$ be the composite $\mathbb{D}^{op} \xrightarrow{(\mathbb{E}_-^{\vee})^{op}} \mathbb{Q}\text{Cat}^{op} \xrightarrow{\hat{\cdot}} \mathbb{Q}\text{Cat}$.*

As a shorthand, we denote $S(u)(S)$ by $S \cdot u$. Concretely, for any horizontal $h : Z \rightarrow X$, $S \cdot h$ is determined by

$$(S \cdot h)(v, z) = S(v, h \circ z).$$

For any vertical $u : Y \rightarrow X$, $S \cdot u$ is determined by

$$(S \cdot u)(v, y) = S(v^{y,u} \bullet v).$$

We now reproduce in the abstract setting the constructions of HP recalled in Section 3.2, and furthermore define the LTS for strategies. First, we define behaviours in the general setting, and relate them to innocent strategies. Then, we state decomposition theorems, which lead to a syntax for strategies. Finally, we define our LTS.

5. PLAYGROUNDS: CONSTRUCTIONS ON STRATEGIES

5.1. Strategies and behaviours. We now consider *extension* of strategies, which maps strategies to behaviours, and generalise the construction to arbitrary playgrounds. First, let $k_X : \widehat{\mathbb{E}}_X^{\vee} \rightarrow \widehat{\mathbb{E}}_X^{\vee f}$ denote postcomposition with $\text{ford} \hookrightarrow \text{set}$. Because views form a full subcategory of \mathbb{D}_H , all embeddings $i_X : \mathbb{E}_X^{\vee} \hookrightarrow \mathbb{E}_X$ also are. This entails that right Kan extension from one category to the other is almost a full embedding.

Lemma 1. *For all X , right Kan extension $(i_X^{op})_{\star} : \widehat{\mathbb{E}}_X^{\vee f} \hookrightarrow \widehat{\mathbb{E}}_X^f$ along i_X^{op} is well-defined, full, and faithful.*

Recall that any full and faithful functor $F : \mathbb{C} \rightarrow \mathbb{D}$ reflect isomorphisms, in the sense that if $F(C) \cong F(C')$ then $C \cong C'$, for all $C, C' \in \mathbb{C}$.

Proof. One easily shows that, when defined, right extension along a full and faithful functor is full and faithful.

It remains to show that the considered right extensions exist. It is well-known [37] that for any $S \in \widehat{\mathbb{E}}_X^{\vee f}$, its right Kan extension, if it exists, maps any u to the limit of the functor $(\mathbb{E}_X^{\vee}/u)^{op} \rightarrow (\mathbb{E}_X^{\vee})^{op} \xrightarrow{S} \text{set}$.

Since finite limits exist in set (though not in ford , which explains why we use set instead of ford for extending strategies), it is enough to prove that each \mathbb{E}_X^{\vee}/u is essentially finite, i.e., equivalent to a finite category. This is proved in the next lemma. \square

Lemma 2. *For any play $u : Y \rightarrow X$, the category \mathbb{E}_X^{\vee}/u is essentially finite.*

For this lemma to hold, we need more axioms.

Axiom. (P9) (*Finiteness*) For any position X , there are only finitely many players, i.e., the category \mathbb{I}/X is finite. Furthermore, up to isomorphism in $\mathbb{D}_H(X)$, there are only finitely many moves with initial position X .

Proof of Lemma 2. Let us fix a play u . By (P9), there is a finite number of players in X , so it is enough to prove that for a fixed player $x: d \rightarrow X$, there are, up to isomorphism, finitely many morphisms from a view v on d to u . Furthermore, one shows by induction, using Axiom (P8) and letting $n = |u|$, that for any such morphism, $m = |v|$ may not exceed n , hence by (P9) again, using Proposition 10, there is only a finite number of possible views v , up to isomorphism. Hence, it is enough to show that for fixed x and v , there is up to isomorphism only finitely many morphisms $v \rightarrow u$ with lower border x in \mathbb{E}_X^V/u .

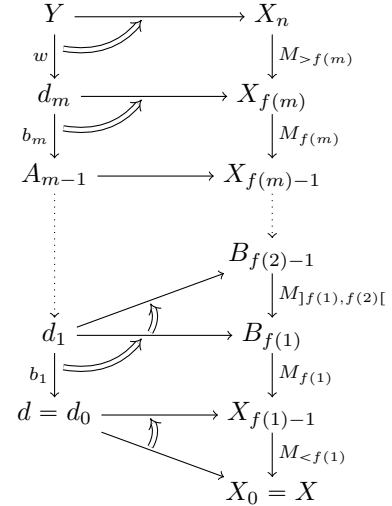
Let us now fix a decomposition of v , say as $b_1 \bullet \dots \bullet b_m$, and, using (P4), a decomposition of u into moves, say as $M_1 \bullet \dots \bullet M_n$. We construct a function $\phi: \mathbb{E}_X^V(v, u) \rightarrow n^m$. We prove that ϕ has finite fibres, hence, since n^m is finite, this entails the result.

We start by defining our function on representatives of morphisms $v \rightarrow u$. Consider any morphism (w, α) . We proceed by lexicographic induction on the pair (m, n) . If $m = 0$ then our map $\phi(w, \alpha): m \rightarrow n$ is the unique map $0 \rightarrow n$. Otherwise, we apply (P8) with $v = b_1$, $w = (b_2 \bullet \dots \bullet b_{m-1} \bullet w)$, $u = M_1$ and $u' = (M_2 \bullet \dots \bullet M_{n-1})$.

- If we are in the left-hand case, α decomposes as $\alpha_1 \bullet \alpha_2$, with $\alpha_1: b_1 \rightarrow M_1$ and $\alpha_2: (b_2 \bullet \dots \bullet b_m \bullet w) \rightarrow (M_2 \bullet \dots \bullet M_n)$. By induction hypothesis, we obtain a map $\phi(w, \alpha_2): m-1 \rightarrow n-1$. We then let our map $\phi(w, \alpha): m \rightarrow n$ map 1 to 1, and $p+1$ to $\phi(w, \alpha_2)(p)+1$ for any $p \in (m-1)$.
- If we are in the right-hand case, we obtain a map $\phi(w, \alpha_2): m \rightarrow n-1$, and return the map $p \mapsto f(p)+1$.

The map $f = \phi(w, \alpha)$ so defined is obviously strictly monotone, and by construction, we obtain double cells as on the right, whose vertical composition is equal (up to the isomorphisms $v \cong (b_1 \bullet \dots \bullet b_m)$ and $(M_1 \bullet \dots \bullet M_n) \cong u$) to α . It is easily shown that two equivalent pairs (w, α) are mapped to the same $\phi(w, \alpha)$, hence we have indeed defined the desired map.

We now show that ϕ has finite fibres. For fixed $f: m \rightarrow n$, the morphism $v \rightarrow u$ denoted by any ladder as above is determined by the player $d_m \rightarrow X_{f(m)}$ (by Proposition 9), and there are finitely many such players by (P9), hence the result. \square



This achieves the proof of Lemma 1, i.e., that right Kan extension along $i_X^{op}: (\mathbb{E}_X^V)^{op} \hookrightarrow \mathbb{E}_X^{op}$ yields a full and faithful functor. We now return to constructing the functor from strategies to behaviours. Consider the embedding $j_X: \mathbb{E}(X) \hookrightarrow \mathbb{E}_X$ mapping any u to (u, id_X) . Restriction along $(j_X)^{op}$ defines a functor $((j_X)^{op})^*: \widehat{\mathbb{E}_X^f} \rightarrow \widehat{\mathbb{E}(X)^f}$.

Definition 16. For any X , let the extension functor $\text{ext}_X: S_X \rightarrow \widehat{\mathbb{E}(X)}^f$ be the composite $((j_X)^{op})^*(i_X^{op})_*k_X$.

Notation: when X is clear from context, we abbreviate $\text{ext}_X(S)$ as \bar{S} .

We call a behaviour on X *innocent* when it is in the essential image of ext_X , and we sometimes abbreviate $\text{ext}_X(S)$ as \bar{S} .

5.2. A syntax for strategies. In this section, we prove in the abstract setting of playgrounds the decomposition results of HP, which lead in particular to proving that strategies form a terminal coalgebra for a certain polynomial functor. This is equivalent to saying that they are essentially infinite terms in a typed grammar. We use this in the next section to give to define and study transitions in $\mathbb{S}_{\mathbb{D}}$.

Decomposition. First, we have *spatial* decomposition:

Proposition 15. We have $S_X \cong \prod_{(d,x) \in \text{Pl}(X)} S_d$.

Proof. We have:

$$\begin{aligned} \text{Cat}((\mathbb{E}_X^{\vee})^{op}, \text{ford}) &\cong \text{Cat}(\sum_{(d,x) \in \text{Pl}(X)} \mathbb{E}^{\vee}(d)^{op}, \text{ford}) \quad (\text{by Proposition 13}) \\ &\cong \prod_{(d,x) \in \text{Pl}(X)} \text{Cat}(\mathbb{E}^{\vee}(d)^{op}, \text{ford}) \\ &\cong \prod_{(d,x) \in \text{Pl}(X)} S_d. \quad \square \end{aligned}$$

Temporal decomposition is harder, but the proof goes through essentially as in the concrete case.

Definition 17. For any $\sigma \in S(id_d)$, let the restriction $S|_{\sigma} \in S_d$ of S to σ be defined by the fact that $S|_{\sigma}(v) = \{\sigma' \in S(v) \mid \sigma' \cdot v = \sigma\}$.

(Here, we freely use the fact that $\mathbb{E}_d^{\vee} \cong \mathbb{E}^{\vee}(d)$.)

Let, for any category \mathbb{C} , $\text{Fam}_o(\mathbb{C})$ denote the familiar family construction, but replacing arbitrary sets with finite ordinals. Objects are thus maps $p: X \rightarrow \text{ob}(\mathbb{C})$ in Set , with $X \in \text{ford}$; maps to $q: Y \rightarrow \text{ob}(\mathbb{C})$ are pairs (f, g) , with f in ford , making the following diagram commute:

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \xrightarrow{f} & Y \\ p \downarrow & & g \downarrow & & \downarrow q \\ \text{ob}(\mathbb{C}) & \xleftarrow{\text{dom}} & \text{mor}(\mathbb{C}) & \xrightarrow{\text{cod}} & \text{ob}(\mathbb{C}). \end{array}$$

Definition 18. Define, for all $d \in \mathbb{I}$ $\partial_d: \text{ob}(S_d) \rightarrow \text{ob}(\text{Fam}_o(\prod_{b \in [\mathbb{B}]_d} S_{\text{dom}(b)}))$ to map any S to the $S(id_d)$ -indexed family defined for each $\sigma \in S(id_d)$ by $b \mapsto (S|_{\sigma}) \cdot b$.

This map actually extends to a functor, but we do not need it here.

The \mathbb{I} -indexed family formed by codomains of this map is actually the image of S under a *polynomial* endofunctor [30] of Set/\mathbb{I} . Recall that, given any locally cartesian closed category \mathbb{C} , e.g., Set , and two objects C and D in \mathbb{C} , a functor $F: \mathbb{C}/C \rightarrow \mathbb{C}/D$ is polynomial when there exists a diagram

$$C \xleftarrow{s} A \xrightarrow{p} O \xrightarrow{t} D$$

in \mathbb{C} such that F is isomorphic to $t_! p_* s^*$, where s^* denotes pullback along s , p_* denotes push forward along p (which is right adjoint to p^*), and $t_!$ denotes composition with t (which is left adjoint to t^*). In sets, when F is polynomial we have

for any family $U \in \text{Set}/C$, and $d \in D$,

$$(F(U))_d = \sum_{o \in t^{-1}(d)} \prod_{a \in p^{-1}(o)} U_{s(a)}.$$

Let $G_{\mathbb{D}}: \text{Set}/\mathbb{I} \rightarrow \text{Set}/\mathbb{I}$ be the functor mapping any family U to

$$(G_{\mathbb{D}}(U))_d = \left(\prod_{b \in [\mathbb{B}]_d} U_{\text{dom}(b)} \right)^*.$$

This functor is polynomial, as $(G_{\mathbb{D}}(U))_d = \sum_{n \in \mathbb{N}} \left(\prod_{i \in n, b \in [\mathbb{B}]_d} U_{\text{dom}(b)} \right)$.

Theorem 2. *The map $\partial: \mathbb{S} \rightarrow G_{\mathbb{D}}(\mathbb{S})$ makes \mathbb{S} into a terminal $G_{\mathbb{D}}$ -coalgebra.*

This intuitively means that strategies, on individuals, are infinite terms for the following typed grammar with judgements $d \vdash_D D$ and $d \vdash S$, where D is a *definite prestrategy* and S is a *strategy*.

$$\frac{\dots d' \vdash S_b \dots (\forall b: d' \rightarrow d \in [\mathbb{B}]_d)}{d \vdash_D \langle (S_b)_{b \in [\mathbb{B}]_d} \rangle} \quad \frac{\dots d \vdash_D D_i \dots (\forall i \in n)}{d \vdash \bigoplus_{i \in n} D_i} \quad (n \in \mathbb{N})$$

The rest of this section is a proof of Theorem 2.

Consider any $G_{\mathbb{D}}$ -coalgebra $a: U \rightarrow G_{\mathbb{D}}U$.

We define by induction on N a sequence of maps $f_N: U \rightarrow \mathbb{S}$, such that for any $d \in \mathbb{I}$, $u \in U_d$, view v of length less than N , and any $N' > N$, $f_{N'}(u)(v) = f_N(u)(v)$, and similarly the action of $f_N(u)$ on morphisms is the same as that of $f_{N'}(u)$.

To start the induction, take $f_0(u)$ to be the strategy mapping id_d to $\pi(a(u))$, i.e., the length of $a(u) \in \sum_{n \in \text{ford}} \left(\prod_b U_{\text{dom}(b)} \right)^n$, and all other views to 0.

Furthermore, given f_N , define f_{N+1} to be

$$U \xrightarrow{a} G_{\mathbb{D}}U \xrightarrow{G_{\mathbb{D}}(f_N)} G_{\mathbb{D}}(\mathbb{S}) \xrightarrow{\cong} \mathbb{S},$$

where the equivalence is by temporal decomposition. In other words, f_N is

$$U \xrightarrow{a} G_{\mathbb{D}}U \xrightarrow{G_{\mathbb{D}}(a)} \dots \rightarrow G_{\mathbb{D}}^{N-1}U \xrightarrow{G_{\mathbb{D}}^{N-1}a} G_{\mathbb{D}}^N U \xrightarrow{G_{\mathbb{D}}^N f_0} G_{\mathbb{D}}^N \mathbb{S} \cong \mathbb{S}.$$

Unfolding the definitions yields:

Lemma 3. *Consider any $u \in U_d$, and $a(u) = (z_1, \dots, z_k)$. For any basic move $b: d' \rightarrow d$ and view $v: d'' \rightarrow d'$ of length at most N , we have $f_{N+1}(u)(b \bullet v) = \sum_{i \in k} f_N(z_i(b))(v)$.*

For any $u \in U_d$, we have a sequence $f_0(u) \hookrightarrow f_1(u) \hookrightarrow \dots \hookrightarrow f_N(u) \hookrightarrow f_{N+1}(u) \hookrightarrow \dots$ which is pointwise stationary. This sequence thus has a colimit in $\mathbb{S}_d = \widehat{\mathbb{E}}_d^{\vee}$, the presheaf mapping any view v of length N to $f_N(v)$ (or equivalently $f_{N'}(v)$ for any $N' \geq N$), which allows us to define:

Definition 19. *Let $f: U \rightarrow \mathbb{S}$ map any $u \in U_d$ to the colimit of the $f_N(u)$'s.*

Lemma 4. *The following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{a} & G_{\mathbb{D}}U \\ f \downarrow & & \downarrow G_{\mathbb{D}}(f) \\ \mathbb{S} & \xleftarrow{\cong} & G_{\mathbb{D}}(\mathbb{S}). \end{array}$$

Proof. We proceed by induction on views. The result is then a consequence of Lemma 3. \square

Corollary 2. *The map f is a map of $G_{\mathbb{D}}$ -coalgebras.*

Lemma 5. *The map f is the unique map $U \rightarrow \mathcal{S}$ of $G_{\mathbb{D}}$ -coalgebras.*

Proof. Consider any such map g of coalgebras. It must be such that $g(u)(id_d) = \pi(a(u))$, and furthermore, using the notations of Lemma 3,

$$g(u)(b \bullet v) = \sum_{i \in \pi(a(u))} g(z_i(b))(v),$$

which imposes by induction that $f = g$. \square

The last two results directly entail Theorem 2.

5.3. The labelled transition system for strategies. In this section, we go beyond HP, and define an LTS for strategies, for an arbitrary playground \mathbb{D} . For this, we need to define the restriction of a strategy to a ‘family of local initial states’.

Definition 20. *A strategy $S \in \mathcal{S}_X$ is definite when $\bar{S}(id_X) \cong 1$, or equivalently when for all players $(d, x) \in \text{Pl}(X)$, we have $S(id_d, x) = 1$.*

We use the following notation for cartesian lifting (by (P1)) of a play u along a horizontal morphism h :

$$\begin{array}{ccc} D_{k,u} & \xrightarrow{h_{k,u}} & X' \\ u|_k \downarrow & \xrightarrow{\alpha_{k,u}} & \downarrow u \\ Y & \xrightarrow{k} & X. \end{array}$$

Definition 21. *A quasi-move is a vertical morphism which locally either is a move or has length 0. More precisely, a play $u: X \rightarrow Y$ is a quasi-move iff for all players $y: d \rightarrow Y$, $u|_y$ either is a move or has length 0.*

A quasi-move is full when, it locally either is a full move or has length 0. Let \mathbb{Q} denote the subgraph of \mathbb{D}_v consisting of full quasi-moves.

Observe that a quasi-move on an individual either is a move or has length 0.

Furthermore, consider, for any $S \in \mathcal{S}_X$, any $\sigma \in \prod_{d,x \in \text{Pl}(X)} S(id_d, x)$. We slightly extend the notion of restriction.

Definition 22. *Let the restriction $S|_{\sigma} \in \mathcal{S}_X$ of S to σ be defined by the fact that for any player $x: d \rightarrow X$, $S|_{\sigma}(v, x) = \{\sigma' \in S(v, x) \mid \sigma' \cdot v = \sigma(d, x)\}$.*

For any full quasi-move $M: X' \rightarrow X$, observe that for any player $x': d' \rightarrow X'$, $v^{x', M}$ has length at most 1 (consider $M|_{(x')^M}$), and let

$$\text{Pl}_M(X') = \{(d', x') \in \text{Pl}(X') \mid |v^{x', M}| \neq 0\}.$$

Lemma 6. *For any definite $S \in \mathcal{S}_X$, and full quasi-move $M: X' \rightarrow X$,*

$$\bar{S}(M) \cong \prod_{\{(d', x') \in \text{Pl}_M(X')\}} S(v^{x', M}, x'^M).$$

Proof. Recall that $\overline{S}(M)$ is a limit of

$$(\mathbb{E}_X^{\vee}/M)^{op} \xrightarrow{\text{dom}} (\mathbb{E}_X^{\vee})^{op} \xrightarrow{S} \text{ford} \hookrightarrow \text{set}.$$

Consider now the poset P with underlying set $\text{Pl}(X) + \text{Pl}_M(X')$ and ordering given by $(d, x) < (d', x')$ iff $x = (x')^M$. Consider the functor $p: P \rightarrow \text{ob}(\mathbb{E}_X^{\vee}/M)$ mapping any $(d, x) \in \text{Pl}(X)$ to the unique morphism $id_d \rightarrow M$ with lower border x , and any $(d', x') \in \text{Pl}_M(X')$ to $\alpha^{x', M}$. Since P is a poset, p is faithful. It is furthermore easily checked to be full. Finally, by Proposition 12 and (P4), for any $\alpha: v \rightarrow M$ in \mathbb{E}_X^{\vee}/M ,

- either $|v| = 0$ and there is a unique player $x: d \rightarrow X$ such that α is the (unique) morphism $id_d \rightarrow M$ with lower border x ,
- or $|v| = 1$ and there exists a unique player $(d', x') \in X'$ such that $v = v^{x', M}$.

This entails that p is essentially surjective on objects, hence an equivalence. Thus, $\overline{S}(M)$ is also a limit of

$$P^{op} \simeq (\mathbb{E}_X^{\vee}/M)^{op} \xrightarrow{\text{dom}} (\mathbb{E}_X^{\vee})^{op} \xrightarrow{S} \text{ford} \hookrightarrow \text{set}.$$

But now, this functor maps any $(d, x) \in \text{Pl}(X)$ to a singleton, hence $\overline{S}(M)$ is also a limit of

$$\text{Pl}_M(X') \hookrightarrow P^{op} \simeq (\mathbb{E}_X^{\vee}/M)^{op} \xrightarrow{\text{dom}} (\mathbb{E}_X^{\vee})^{op} \xrightarrow{S} \text{ford} \hookrightarrow \text{set},$$

i.e., isomorphic to $\prod_{(d', x') \in \text{Pl}_M(X')} S(v^{x', M}, (x')^M)$, as desired. \square

Using the lemma, if S is definite, any $\sigma \in \overline{S}(M)$ uniquely corresponds to a $\sigma' \in \prod_{\{(d', x') \in \text{Pl}(X')\}} S(v^{x', M}, (x')^M)$. Indeed, if $|v^{x', M}| = 0$, then $\sigma'(d', x')$ is the unique element of $S(id_{d'}, (x')^M)$, and otherwise $(d', x') \in \text{Pl}_M(X')$ and we apply the lemma.

Definition 23. Let $\psi_M: \overline{S}(M) \rightarrow \prod_{\{(d', x') \in \text{Pl}(X')\}} S(v^{x', M}, (x')^M)$ denote the bijection so defined.

We now define our LTS for strategies over \mathbb{Q}^{op} .

Definition 24. The underlying graph $\mathbb{S}_{\mathbb{D}}$ for our first LTS is the graph with as vertices all pairs (X, S) where X is a position and $S \in \mathbb{S}_X$ is a definite strategy, and whose edges $(X', S') \rightarrow (X, S)$ are all full quasi-moves $M: X' \rightarrow X$ such that there exists a state $\sigma \in \overline{S}(M)$ with

$$S' = (S \cdot M)|_{\psi(\sigma)},$$

i.e.,

$$S'_{d', x'} = (S_{(x')^M} \cdot v^{x', M})|_{\psi(\sigma)(d', x')}$$

for all $(d', x') \in \text{Pl}(X')$.

The assignment $(X, S) \mapsto X$ defines a morphism $p_{\mathbb{S}}: \mathbb{S}_{\mathbb{D}} \rightarrow \mathbb{Q}$ of graphs, which is our LTS.

Essentially, to construct a transition from (X, S) to (X', S') , pick a full quasi-move $M: X' \rightarrow X$, and a state $\sigma \in \overline{S}(M)$. This σ induces a family of local states $\psi_M(\sigma) \in \prod_{\{(d', x') \in \text{Pl}(X')\}} S(v^{x', M}, (x')^M)$. All players (d', x') of X' have a view through M , $v^{x', M}$: just check that $S'_{d', x'}$ is the restriction of $(S \cdot v^{x', M})$ to $\psi_M(\sigma)(d', x')$.

We now give characterisations of transitions.

Proposition 16. *If $S = \langle (S_b)_{b \in [\mathbb{B}]_d} \rangle$ is a definite strategy on $d \in \mathbb{I}$, and if for all $b \in [\mathbb{B}]_d$, $S_b = \bigoplus_{i \in n_b} D_i^b$ for definite D_i^b , then we have $(d, S) \xleftarrow{M} (X', S')$ iff*

- for all $(d', x') \in \text{Pl}_M(X')$, there exists $i_{x'} \in n_{v_{x'}, M}$ such that $S'_{x'} = D_{i_{x'}}^{v_{x'}, M}$,
- and for all $(d', x') \in \text{Pl}(X') \setminus \text{Pl}_M(X')$, $S'_{x'} = S$.

Recall the notation from just above Definition 21, that the cartesian lifting, w.r.t. cod, of a play $u: X' \rightarrow X$ along any horizontal $h: Y \rightarrow X$ is denoted by $u|_h: D_{h,u} \rightarrow Y$. We have:

Proposition 17. *We have $(X, S) \xleftarrow{M} (X', S')$ iff for all $(d, x) \in \text{Pl}(X)$,*

$$(d, S_x) \xleftarrow{M_x} (D_{x,M}, S'_{|D_{x,M}}).$$

Proposition 18. *Let, for all $(d, x) \in \text{Pl}(X)$, $S_x = \langle (S_b^x)_{b \in [\mathbb{B}]_d} \rangle$ and for all $b \in [\mathbb{B}]_d$, $S_b^x = \bigoplus_{i \in n_b^x} D_i^{x,b}$ for definite $D_i^{x,b}$.*

Then, we have $(X, S) \xleftarrow{M} (X', S')$ iff

- for all $(d', x') \in \text{Pl}_M(X')$, there exists $i_{x'} \in n_{v_{x'}, M}^{(x')^M}$ such that $S'_{x'} = D_{i_{x'}}^{(x')^M, v_{x'}, M}$,
- and for all $(d', x') \in \text{Pl}(X') \setminus \text{Pl}_M(X')$, $S'_{x'} = S_{(x')^M}$.

6. PLAYGROUNDS: PROCESS TERMS AND A STRONG BISIMULATION

6.1. Process terms. In the previous section, starting from a playground \mathbb{D} , we have constructed an LTS $\mathcal{S}_{\mathbb{D}}$ of strategies. We now construct a syntactic LTS $\mathcal{J}_{\mathbb{D}}$.

Definition 25. *For any X , let $[\mathbb{F}]_X$ be the set of isomorphism classes of full moves to X , in $\mathbb{D}_H(X)$.*

For any $[M] \in [\mathbb{F}]_X$, let

$$\chi(M) = \{[b] \in [\mathbb{B}] \mid \exists \alpha \in \mathbb{D}_H(b, M)\}.$$

Let $[\mathbb{F}^1]_d$ denote the subset of $[\mathbb{F}]_d$ consisting of (isomorphism classes of) full moves $M: X' \rightarrow d$ such that $\text{Pl}_M(X')$ is a singleton (and hence so is $\chi(M)$). Let $[\mathbb{F}^+]_d$ denote the complement subset.

The map χ is easily checked to be well-defined.

We state one more axiom to demand that basic sub-moves of a full move $[M] \in [\mathbb{F}]_d$ may not be sub-moves of other full moves.

Axiom. (P10) *(Basic vs. full) For any $d \in \mathbb{I}$ and $M, M' \in [\mathbb{F}]_d$, if $[M] \neq [M']$, then $\chi[M] \cap \chi[M'] = \emptyset$.*

Let *process terms* be infinite terms in the typed grammar:

$$\frac{\dots d_i \vdash T_i \dots \ (\forall i \in n)}{d \vdash \sum_{i \in n} M_i.T_i} \quad (n \in \mathbb{N}; \forall i \in n, M_i \in [\mathbb{F}^1]_d, \text{ and } \chi[M_i] = \{b_i: d_i \rightarrow d\})$$

$$\frac{\dots d' \vdash T_b \dots \ (\forall (b: d' \rightarrow d) \in \chi[M])}{d \vdash M \langle (T_b)_{b \in \chi[M]} \rangle} \quad (M \in [\mathbb{F}^+]_d).$$

The first rule is a guarded sum, in a sense analogous to guarded sum in CCS. It should be noted that guards have to be full moves with only one non-trivial view. There is good reason for that, since allowing general moves as guards would break

bisimilarity between process terms and strategies. To understand this, consider a hypothetic guarded sum $R = (P|Q) + (P'|Q')$. Since this has no interaction before the choice is made, R behaves, in CCS, just like an internal choice $(P|Q) \oplus (P'|Q')$. However, our translation to strategies does not translate guarded sum as an internal choice, with right, since other guarded sums, e.g., $a.P + b.Q$ should certainly not be translated this way. Instead, R would be translated as something equivalent to $(P|Q) \oplus (P'|Q) \oplus (P|Q') \oplus (P'|Q')$, which is clearly not bisimilar to R in general.

Also, we could easily include internal choice in the grammar, since strategies do model it, directly. We refrain from doing so for simplicity.

Remark 2. *Again, process terms form a terminal coalgebra for a polynomial functor. There does not appear to be a standard presentation for terminal coalgebras of such polynomial functors. Although this question lies beyond the scope of this paper, we mention that Adámek and Porst [1] use an elementary presentation of infinite trees to characterise the terminal coalgebra for polynomial endofunctors on $\mathbf{Set}/1$. It might be efficient to use Kock's presentation of trees as polynomial endofunctors [30] to extend the result to arbitrary \mathbf{Set}/X .*

Definition 26. *Let $\mathbb{T}_{\mathbb{D}}$ be the set of process terms.*

Example 5. *For \mathbb{D}^{CCS} , the obtained syntax is equivalent to*

$$\frac{\dots \Gamma \cdot \alpha_i \vdash P_i \dots}{\Gamma \vdash \sum_i \alpha_i.P_i} \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P|Q}.$$

where

- Γ ranges over natural numbers;
- $\alpha ::= a \mid \bar{a} \mid \heartsuit \mid \nu$ (for $a \in \Gamma$);
- $\Gamma \cdot \alpha$ denotes $(\Gamma + 1)$ if $\alpha = \nu$ and just Γ otherwise.

This grammar obviously contains CCS, and we let $\theta: \text{ob}(CCS) \hookrightarrow \mathbb{T}$ be the injection.

6.2. The labelled transition system for process terms. We now define the LTS $\mathbb{T}_{\mathbb{D}}$. The states, i.e., the vertices of the graph underlying this LTS, are pairs (X, T) of a position X and a family T of process terms, indexed by the players of X , i.e., $T \in \prod_{d, x \in \text{Pl}(X)} (\mathbb{T}_{\mathbb{D}})_d$, where $(\mathbb{T}_{\mathbb{D}})_d$ is the set of process terms of type d .

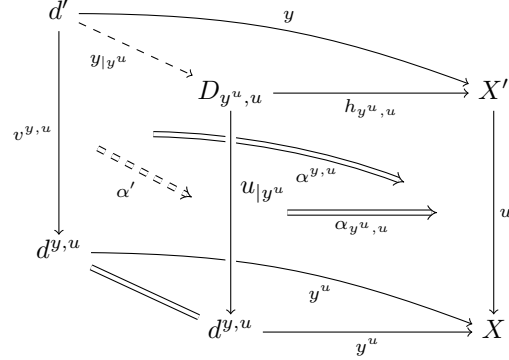
To define edges, we need a lemma. For any play $u: X' \rightarrow X$ and $x: d \rightarrow X$, we again use the notation just before Definition 21 for the cartesian lifting of u along x . We fix a global choice of such liftings. Consider now the map

$$r^u : \begin{array}{ccc} \sum_{d, x \in \text{Pl}(X)} \text{Pl}(D_{x,u}) & \rightarrow & \text{Pl}(X') \\ ((d, x), (d', x')) & \mapsto & h_{x,u} \circ x' \end{array}$$

sending any $(d, x) \in \text{Pl}(X)$ and $x': d' \rightarrow D_{x,u}$ to $d' \xrightarrow{x'} D_{x,u} \xrightarrow{h_{x,u}} X'$.

Lemma 7. *The map r^u is a bijection.*

Proof. Consider the map, say i^u , in the other direction mapping any $y: d' \rightarrow X'$ to $((d^{y,u}, y^u), (d', y|_{y^u}))$, where $y|_{y^u}$ is the (domain in \mathbb{D}_V of the) unique morphism making the diagram on the right commute (by (P1)). Clearly $r^u \circ i^u = id$, so in particular r^u is surjective. Furthermore, consider $a_1 = ((d_1, x_1), (d'_1, x'_1))$ and $a_2 = ((d_2, x_2), (d'_2, x'_2))$ such that $r^u(a_1) = r^u(a_2) = (d', y)$. Both views $v^{x'_1, u|_{x_1}}$ and $v^{x'_2, u|_{x_2}}$ must be canonically isomorphic to $v^{y,u}$, hence $d^{x'_1, u|_{x_1}} = d^{x'_2, u|_{x_2}} =$



$d_1 = d_2$ (dy discreteness of \mathbb{I}) and $x_1 = x_2$, so $\alpha_{y^u,u} = \alpha_{x_1,u} = \alpha_{x_2,u}$. Thus, by cartesianness of $\alpha_{y^u,u}$, $x'_1 = x'_2$, hence r^u is injective. \square

Note in passing that the inverse of r^u is i^u .

Let us return to the definition of our LTS. We first say that for any full quasi-move $M: D \rightarrow d$, a process term $d \vdash T$ has an M -transition to (D, T') , for $T' \in \prod_{(d', x') \in \text{Pl}(D)} \mathbb{T}_{d'}$, when

- (i) either $[M'] = [M] \in [\mathbb{F}^+]$, $T = M' \langle T'' \rangle$, and, for all $(d', x') \in \text{Pl}(D)$,
 - if $v^{x', M}$ is a basic move, then $T'_{d', x'} = T''_{v^{x', M}}$,
 - otherwise $|v^{x', M}| = 0$ (hence $d' = d$), and $T'_{d', x'} = T$,
- (ii) or $[M] \in [\mathbb{F}^1]$, $T = \sum_{i \in n} M_i \cdot T_i$, $M_{i_0} \cong M$ for some $i_0 \in n$, and for all players $x': d' \rightarrow D$
 - if $[v^{x', M}] \in \chi(M)$, then $T'_{d', x'} = T_{i_0}$,
 - and otherwise ($|v^{x', M}| = 0$), $T'_{d', x'} = T$,
- (iii) or $|M| = 0$ and for all $(d', x') \in \text{Pl}(D)$, $T'_{d', x'} = T$ (which, again, makes sense by (P4)).

We denote such a transition by $T \xleftarrow{M} (D, T')$.

Definition 27. Let $\mathcal{T}_{\mathbb{D}}$ be the graph with pairs (X, T) as vertices, and as edges $(X, T) \leftarrow (X', T')$ full quasi-moves $M: X' \rightarrow X$ such that for all $(d, x) \in \text{Pl}(X)$, $T_{d,x} \xleftarrow{M|_x} (D_{x,M}, (T' \circ (h_{x,M})!))$. Here, we let $(h_{x,M})!$ denote composition with $h_{x,M}: D_{x,M} \rightarrow X'$, viewed as a map $\text{Pl}(D_{x,M}) \rightarrow \text{Pl}(X')$.

$\mathcal{T}_{\mathbb{D}}$ is viewed as an LTS over \mathbb{Q} , by mapping $(X, T) \xleftarrow{M} (X', T')$ to $X \xleftarrow{M} X'$.

Example 6. For \mathbb{D}^{CCS} , the obtained LTS differs subtly, but significantly from the usual LTS for CCS. In order to explain them clearly, let us introduce some notation. Recall evaluation contexts from Section 3.1.1. Leaving the details aside, states in $\mathcal{T}_{\mathbb{D}^{\text{CCS}}}$ are pairs (X, T) of an evaluation context X , plus, for each occurrence $x(a_1, \dots, a_n)$ of an n -ary variable in X , a process term over n in the grammar of Example 5. Instead of separately writing the evaluation context and the map from its variables to process terms, we inline the process terms, between brackets in the context, thus avoiding variables. Moves are either put in context similarly, or located implicitly. E.g., for a state (X, T) where X contains two players respectively mapped by T to process terms P and Q , we would write $[P][Q]$. There is some ambiguity in this notation, e.g., in case some channels are absent from P : are they absent from the arity of P , or only unused? Since we use this

notation mostly for clarifying examples, we will avoid such ambiguities. Finally, we sometimes use brackets to denote the fact that some holes are filled with the given state. E.g., $X[[P]][[Q]]$ denotes a state X , where a hole has been replaced by a parallel composition of two holes, respectively filled with P and Q .

Returning to our examination of $\mathcal{T}_{\mathbb{D}CCS}$, of course, a first difference is the fact that labels may contain several moves, as quasi-moves only locally have length 1.

A second difference is the presence of heating rules for parallel composition and name creation, in a sense close to the chemical abstract machine [5]. For example, we have transitions $X[P|Q] \xleftarrow{\pi} X[[P]][[Q]]$, for any sensible P and Q .

There is a third important difference, related to name creation. For instance, using the above notation, we have transitions

$$[\nu a.a.P] \xleftarrow{\nu} \nu a.[a.P] \xleftarrow{L_a} \nu a.[P].$$

The second transition cannot occur in a closed-world setting, since the environment cannot know a .

A final difference with the expected transition rules is that labels contain too much information to be relevant for behavioural equivalences. E.g., they contain the whole evaluation context in which the transition takes place, as well as which players are involved. All defects will be corrected below.

6.3. Translation and a first correctness result. Our translation from terms to strategies is defined coinductively by

$$(6) \quad \begin{aligned} \llbracket \sum_{i \in n} M_i.T_i \rrbracket &= \langle b \mapsto \bigoplus_{\{i \in n \mid b \in \chi[M_i]\}} \llbracket T_i \rrbracket \rangle \\ \llbracket M \langle (T_b)_{b \in \chi[M]} \rangle \rrbracket &= \left\langle b' \mapsto \begin{cases} \llbracket T_{b'} \rrbracket & \text{if } b' \in \chi[M] \\ \emptyset & \text{otherwise} \end{cases} \right\rangle. \end{aligned}$$

Let us extend the map $\llbracket - \rrbracket : \mathbb{T} \rightarrow \mathbb{S}$ to a map $\llbracket - \rrbracket : \text{ob}(\mathcal{T}_{\mathbb{D}}) \rightarrow \text{ob}(\mathcal{S}_{\mathbb{D}})$, defined by $\llbracket X, T \rrbracket = (X, (\llbracket T_{(d,x)} \rrbracket)_{(d,x) \in \text{Pl}(X)})$.

Theorem 3. *The map $\llbracket - \rrbracket : \text{ob}(\mathcal{T}_{\mathbb{D}}) \rightarrow \text{ob}(\mathcal{S}_{\mathbb{D}})$ is a functional, strong bisimulation.*

Proof. The theorem follows from Proposition 17 and the next lemma. \square

Lemma 8. *For any full quasi-move $M : X' \rightarrow d$, for any $T \in \mathbb{T}_d$ and $S' \in \mathbb{S}'_{X'}$, we have*

$$(d, \llbracket T \rrbracket) \xleftarrow{M} (X', S') \quad \text{iff} \quad \exists T', (T \xleftarrow{M} (X', T')) \wedge ((X', S') = \llbracket X', T' \rrbracket).$$

Note the implicit typing: $T' \in \prod_{(d',x') \in \text{Pl}(X')} \mathbb{T}_{d'}$. Also the second condition on the right equivalently means $\forall x' : d' \rightarrow X', S'_{x'} = \llbracket T'_{x'} \rrbracket$.

Proof. If $|M| = 0$, then both sides are equivalent to the fact that for all $x' : d' \rightarrow X'$, $S'_{x'} = \llbracket T \rrbracket$.

Otherwise, we proceed by case analysis on T .

If $T = M' \langle (T''_b)_{b \in \chi(M')} \rangle$, then by (P10) both sides are equivalent to $M \cong M'$, plus

- for all $(d', x') \in \text{Pl}_M(X')$, $S'_{x'} = \llbracket T''_{v_{x',M}} \rrbracket$, and
- for all $(d', x') \in \text{Pl}(X') \setminus \text{Pl}_M(X')$, $S'_{x'} = \llbracket T \rrbracket$.

Indeed, for any $b \in \chi(M)$, $\llbracket T \rrbracket \cdot b = \llbracket T''_b \rrbracket$ is definite. We then put $T'_{d',x'} = T''_{v_{x',M}}$ in the first case and $T'_{d',x'} = T$ in the second case.

If $T = \sum_{i \in n} M_i.T_i$, then both sides are equivalent to the existence of $i_0 \in n$ such that $M \cong M_{i_0}$ and

- for the unique $(d', x') \in \text{Pl}_M(X')$, $S'_{x'} = \llbracket T_{i_0} \rrbracket$, and
- for all $(d', x') \in \text{Pl}(X') \setminus \text{Pl}_M(X')$, $S'_{x'} = \llbracket T \rrbracket$.

This uses (P10), since the left-hand side unfolds to the existence of $x': d' \rightarrow X'$ such that $v^{x', M} \in \chi(M)$ and $\llbracket T \rrbracket \cdot v^{x', M} \neq 0$, i.e., $v^{x', M} \in \chi(M_{i_0})$ for some $i_0 \in n$, by definition of $\llbracket T \rrbracket$. \square

To conclude this section, we state a simple corollary.

Definition 28. Let \mathbb{Q}^I denote the graph with morphisms $k: I \rightarrow X$ as vertices, where I is an interface, and whose edges $k \rightarrow h$, for $k: J \rightarrow Y$, are full quasi-moves $M: Y \rightarrow X$.

Let $U: \mathbb{Q}^I \rightarrow \mathbb{Q}$ be the obvious forgetful morphism. Let \mathcal{S}^I and \mathcal{T}^I be the pullbacks of \mathcal{S} and \mathcal{T} along U .

Corollary 3. The map $\text{ob}(\mathcal{S}^I) \rightarrow \text{ob}(\mathcal{T}^I)$ obtained by pullback, which we also denote by $\llbracket - \rrbracket$, is a strong bisimulation.

Proof. By Proposition 1. \square

7. GRAPHS AND FAIR MORPHISMS

In this section, we derive our main result. For this, we develop a notion of *graphs with complementarity*, which aims at being a theory of alphabets over which fair testing makes sense. Although the theory would apply with any predicate \perp compatible with \simeq_Σ equivalence classes (see below), the question of whether such a generalisation would have useful examples is deferred for now.

For any graph with complementarity A and relation $R: G \leftrightarrow H$ over A , we exhibit sufficient conditions for R to be *fair*, i.e., to preserve and reflect fair testing equivalence. We then relate this theory to our semantics, and show that it entails our main result. For now, this section lies outside the scope of playground theory. Some aspects of it could be formalised there, but we leave the complete formalisation for further work. Because the only playground involved is \mathbb{D}^{CCS} , we often omit sub or superscripts, e.g., in \mathbb{D} , $\mathcal{S}_{\mathbb{D}}$, etc.

Before we start, let us define \mathbb{W}_{CCS} to be the set of *closed-world* quasi-moves. Let $\mathbb{D}^{\mathbb{W}}$ be the subcategory of \mathbb{D}_v generated by \mathbb{W}_{CCS} , and Σ be the free reflexive graph on an endo-edge \heartsuit . Finally, let $\ell_{\mathbb{D}}: \mathbb{D}^{\mathbb{W}} \rightarrow \Sigma$ be the functor determined by the mapping $\mathbb{W}_{CCS} \rightarrow \Sigma$ sending all closed-world quasi-moves to *id* except \heartsuit moves, which are sent to \heartsuit .

7.1. Graphs with complementarity. A relation $A \leftrightarrow B$ between two reflexive graphs A and B is a subgraph $R \hookrightarrow A \times B$. Such a relation R is *total* when, for all vertices, resp. edges, $x \in A$, there exists a vertex, resp. an edge $y \in B$, such that $(x, y) \in R$. It is *functional* when y is furthermore unique.

Definition 29. A graph with complementarity is a reflexive graph A , equipped with an identity-on-vertices subgraph $A^{\mathbb{W}}$, a subgraph $A^{\circ} \hookrightarrow A^2$, a total relation $\triangleright^A: A^{\circ} \leftrightarrow A^{\mathbb{W}}$, and a map $\ell^A: A^{\mathbb{W}} \rightarrow \Sigma$, such that, writing $a \circ a'$ for $(a, a') \in A^{\circ}$,

- A° is symmetric, i.e., for any two vertices or edges x, y of A , if $(x, y) \in A^{\circ}$, then $(y, x) \in A^{\circ}$;
- the composite $A^{\circ} \leftrightarrow A^{\mathbb{W}} \rightarrow \Sigma$ is functional and symmetric; we denote it by $(a, b) \mapsto (a \Downarrow b)$;

A morphism of graphs with complementarity is a morphism $f: A \rightarrow B$ of reflexive graphs such that

$$f(A^{\mathbb{W}}) \subseteq B^{\mathbb{W}} \quad f^2(A^{\circ}) \subseteq B^{\circ} \quad \ell^B \circ f^{\mathbb{W}} = \ell^A$$

$$((a_1, a_2) \triangleright^A a_3) \Rightarrow ((f(a_1), f(a_2)) \triangleright^A f(a_3)),$$

where $f^{\mathbb{W}}$ is the obvious map $A^{\mathbb{W}} \rightarrow B^{\mathbb{W}}$.

Example 7. \mathbb{Q}^I is a graph with complementarity. For any $h: I \rightarrow X$ and $k: J \rightarrow Y$, let $h \circ k$ iff $I = J$. Let $c: I \rightarrow Z$ be such that $(h, k) \triangleright^{\mathbb{Q}^I} c$ iff $Z = h +_I k$ and c is the corresponding map $I \rightarrow Z$. Similarly, for any $M_h: h' \rightarrow h$, $M_k: k' \rightarrow k$, and $M_c: c' \rightarrow c$, let $(M_h, M_k) \triangleright^{\mathbb{Q}^I} M_c$ iff there exists a diagram

$$(7) \quad \begin{array}{ccccc} I' & \xrightarrow{\quad} & Y' & & \\ & \searrow^{h'} & \downarrow & \searrow & \\ & & X' & \xrightarrow{\quad} & Z' \\ & & \downarrow & & \downarrow \\ I & \xrightarrow{M_h} & Y & \xrightarrow{\quad} & Z \\ & \searrow^h & \downarrow & \searrow & \\ & & X & \xrightarrow{\quad} & Z \end{array}$$

where M_c is a closed-world move and double cells with a ‘double pullback’ mark are cartesian. (One easily shows that the upper square is also a pushout and that I' is an interface.)

Let $(\mathbb{Q}^I)^{\mathbb{W}}$ consist of all closed-world moves, let $(\mathbb{Q}^I)^{\circ}$ consist of all pairs (M_h, M_k) for which there exists a diagram of the shape (7) (which a posteriori justifies the name $\triangleright^{\mathbb{Q}^I}$), and let $\ell^{\mathbb{Q}^I}$ be the composite $(\mathbb{Q}^I)^{\mathbb{W}} \hookrightarrow (\mathbb{D})^{\mathbb{W}} \xrightarrow{\ell_{\mathbb{D}}} \Sigma$. It thus maps tick moves to \heartsuit and all other closed-world moves to id . The relation $\triangleright^{\mathbb{Q}^I}$ is total by construction; the composite $(\mathbb{Q}^I)^{\circ} \xrightarrow{\triangleright^{\mathbb{Q}^I}} (\mathbb{Q}^I)^{\mathbb{W}} \rightarrow \Sigma$ is indeed functional; and all other axioms are easily verified.

Example 8. Let \mathcal{L} be the identity-on-vertices subgraph of \mathbb{Q}^I whose edges $k \rightarrow h$ are given by diagrams

$$(8) \quad \begin{array}{ccc} I & \xrightarrow{k} & Y \\ \parallel & \nearrow^{\alpha} & \downarrow M \\ I & \xrightarrow{h} & X \end{array}$$

in \mathbb{D}_H , where M is either a full move or an identity, such that if M is an input or an output, then the corresponding channel is in the image of I . (Equivalently, for the last condition, the involved edge M in \mathbb{Q}^I is part of a diagram (7).) \mathcal{L} forms a reflexive graph with identities given by the case where $M = id$.

Let \mathcal{L}° , $\mathcal{L}^{\mathbb{W}}$, and $\triangleright^{\mathcal{L}}$ be given by restricting $(\mathbb{Q}^I)^{\circ}$, $(\mathbb{Q}^I)^{\mathbb{W}}$, and $\triangleright^{\mathbb{Q}^I}$ to \mathcal{L} .

When passing from \mathbb{Q}^I to \mathcal{L} , we have removed many edges from each $h: I \rightarrow X$, but the axioms still hold.

There is an obvious morphism $\chi: \mathcal{L} \rightarrow \mathbb{Q}^I$ of graphs with complementarity.

Example 9. Recall the alphabet \mathbb{A} for CCS. It also forms a graph with complementarity, as follows. Let $\mathbb{A}^{\mathbb{W}}$ consist of all vertices and of all \heartsuit and id edges. Let \mathbb{A}° consist, on vertices, of the diagonal, i.e., all pairs (n, n) . On edges, let $e \circ e'$ when $\text{dom}(e) \circ \text{dom}(e')$ and:

- either of e and e' is in $\mathbb{A}^{\mathbb{W}}$, the other being an identity,
- or either of e and e' is an input on, say, $i \in \text{dom}(e)$, the other being an output on i .

Define now $\triangleright^{\mathbb{A}}$ by mapping all coherent pairs $e \circ e'$ to id, except when one is a \heartsuit , in which case the pair is mapped to $\heartsuit \in \Sigma$. The axioms are easily satisfied.

Let $\xi: \mathcal{L} \rightarrow \mathbb{A}$ map any vertex $h: I \rightarrow X$ to $n = I(\star)$, and any edge (8) to

- id_n if M is an identity, a synchronisation, a fork, or a channel creation,
- \heartsuit_n if M is a tick move,
- i if M is an input on $h_{\star}(i)$,
- \bar{i} if M is an output on $h_{\star}(i)$.

This map ξ is a morphism of graphs with complementarity.

Example 10. \mathcal{S}^I and \mathcal{T}^I are graphs with complementarity over \mathbb{Q}^I , and CCS is a graph with complementarity over \mathbb{A} , by the following general argument. Consider any A with complementarity, and faithful $p: G \rightarrow A$. Let $G^{\circ} = G^2 \times_{A^2} A^{\circ}$ and $G^{\mathbb{W}} = G \times_A A^{\mathbb{W}}$.

Assume given a choice, for all $x, y \in G$ and $a \in A$ such that $(p(x), p(y)) \triangleright^A a$, of a vertex $[x, y]_a \in G$ such that

- $p([x, y]_a) = a$
- and for all $e_x \circ e_y$ with $e_x: x' \rightarrow x$, $e_y: y' \rightarrow y$, and $e_a: a' \rightarrow a$, if $(p(e_x), p(e_y)) \triangleright^A e_a$, then there exists a (unique by faithfulness) $[e_x, e_y]_{e_a}: [x', y']_{a'} \rightarrow [x, y]_a$ such that $p([e_x, e_y]_{e_a}) = e_a$.

Let ℓ^G be the composite $G^{\mathbb{W}} \rightarrow A^{\mathbb{W}} \rightarrow \Sigma$, and let $\triangleright^G: G^{\circ} \rightarrow G^{\mathbb{W}}$ be defined by $(x, y) \triangleright^G [x, y]_a$ for all a such that $(p(x), p(y)) \triangleright^A a$, and similarly for edges.

Then, G , equipped with $G^{\mathbb{W}}$, G° , \triangleright^G , and ℓ^G , forms a graph with complementarity.

First, symmetry of G° is by construction, and totality of \triangleright^G is by hypothesis.

Let us now show that $G^{\circ} \rightarrow G^{\mathbb{W}} \rightarrow A^{\mathbb{W}} \rightarrow \Sigma$ is functional and symmetric. It is total since \triangleright^G is. Furthermore, For any $e_x \circ e_y$ related to two edges by \triangleright^G , then these two edges must have the shape $[e_x, e_y]_{e_i}$, for some e_i such that $(p(e_x), p(e_y)) \triangleright^A e_i$, with $i \in \{1, 2\}$. These edges are mapped by $G^{\mathbb{W}} \rightarrow A^{\mathbb{W}}$ to e_1 and e_2 respectively, and hence $\ell^A(e_1)$ and $\ell^A(e_2)$ must coincide as images of $(p(e_x), p(e_y))$ under $A^{\circ} \rightarrow A^{\mathbb{W}} \rightarrow \Sigma$. Finally, the composite is symmetric because the one for A is.

This shows that G is a graph with complementarity. It follows straightforwardly that p is a morphism of such.

\mathcal{S}^I and \mathcal{T}^I clearly satisfy the hypotheses, hence form graphs with complementarity over \mathbb{Q}^I in a canonical way. Similarly, CCS is a graph with complementarity over \mathbb{A} .

Example 11. Let $\mathcal{S}^{\mathcal{L}} = \xi^*(\mathcal{S}^I)$ and $\mathcal{T}^{\mathcal{L}} = \xi^*(\mathcal{T}^I)$ be the pullbacks of \mathcal{S}^I and \mathcal{T}^I along ξ . They form graphs with complementarity over \mathcal{L} .

7.2. Effective graphs. We now introduce the notion of *effective* graph, which is appropriate for defining fair testing. We could actually introduce fair testing for

arbitrary graphs with complementarity, but the extra generality would make little sense.

For any graph with complementarity G , G° forms an LTS over Σ , through $G^\circ \xrightarrow{\downarrow} \Sigma$.

Definition 30. G is effective iff \triangleright^G is a strong bisimulation over Σ .

Concretely, if $x' \xrightarrow{e_x} x$, $y' \xrightarrow{e_y} y$, and $(x, y) \triangleright^G z$, with $e_x \Downarrow e_y = \sigma$, then by totality of \triangleright^G there exists $e: z' \rightarrow z$ such that $(e_x, e_y) \triangleright^G e$. Then, $\ell^G(e) = \sigma$ by definition, hence $z' \xrightarrow{\sigma} z$. This holds for any graph with complementarity without further hypotheses.

Conversely, if $(x, y) \triangleright^G z$ and $e: z' \rightarrow z$ in $G^{\mathbb{W}}$, the hypothesis yields the existence of $e_x: x' \rightarrow x$ and $e_y: y' \rightarrow y$ such that $(e_x, e_y) \triangleright^G e$.

Example 12. \mathcal{S}^I and \mathcal{T}^I , as well as CCS, are effective.

We now define fair testing in any effective graph, and compare with \sim_f and $\sim_{f,s}$.

Lemma 9. For any $x, y, z, t \in G$, if $(x, y) \triangleright^G z$ and $(x, y) \triangleright^G t$, then $z \sim_\Sigma t$.

Proof. We have $z \sim_\Sigma (x, y) \sim_\Sigma t$. □

In any such effective graph, we denote by $[x, y]$ any z such that $(x, y) \triangleright^G z$. By the lemma, the choice of z does not matter as long as we only consider properties invariant under \sim_Σ . Here, we only need the standard predicate for fair testing.

For any graph G over Σ , let \perp^G denote the set of all $x \in G$ such that for all $x \Leftarrow x'$ there exists $x' \overset{\heartsuit}{\Leftarrow} x''$. When G is a graph with complementarity, we often denote $\perp^{G^{\mathbb{W}}}$ by \perp^G . There is no confusion because G is not even a graph over Σ .

In any effective graph with complementarity G , let, for any $x \in G$, $x^\circ = \{y \mid x \circ y\}$, and let $x \bowtie y$ iff $x^\circ = y^\circ$.

Definition 31. For any $x, y \in G$, let $x \sim^G y$ iff $x \bowtie y$ and for all $z \in x^\circ$, $[x, z] \in \perp^{G^{\mathbb{W}}}$ iff $[y, z] \in \perp^{G^{\mathbb{W}}}$.

We have a first, easy characterisation of fair testing.

Proposition 19. For any $x \circ y$, $[x, y] \in \perp^G$ iff $(x, y) \in \perp^{G^\circ}$.

Proof. Indeed, $\triangleright^G: G^\circ \dashrightarrow G^{\mathbb{W}}$ only relates strongly bisimilar vertices, which entails the result. □

We now prove that the general definition of fair testing equivalence instantiates correctly for \mathcal{S}^I and CCS. We wish to compare $\perp^{\mathcal{S}^I}$, as defined in this section, and \perp , as defined in the previous one. As an intermediate step, we consider the following, bare \perp , which lives over \mathbb{Q} (as opposed to \mathbb{Q}^I), but is defined in terms of LTSS (as opposed to succesful states of strategies).

Let $\mathcal{S}^{\mathbb{W}}$ be the restriction of \mathcal{S} to closed-world transitions; this is an LTS over Σ via $\ell_{\mathbb{D}}$. Let \perp be the set of strategies $S \in \mathcal{S}$ such that for all $S \Leftarrow S'$ there exists $S' \overset{\heartsuit}{\Leftarrow} S''$. (Here, we omit underlying positions.)

Lemma 10. We have $\perp = \perp$.

We first observe:

Lemma 11. For any two closed-world plays W, W' over X , and $\alpha: W \rightarrow W'$ in \mathbb{D}_H , if $\text{cod}(\alpha) = \text{id}_X$, then α is an isomorphism, and it is unique.

Proof of Lemma 10. Let $S \in \mathbb{S}_X$ and assume $S \in \perp_X$. Let $S \Leftarrow S'$ (over Σ). This means that there exists a path p

$$X = X_0 \xleftarrow{M_1} X_1 \xleftarrow{M_2} \dots X_n = X',$$

such that, omitting positions,

$$S = S_0 \xleftarrow{M_1} S_1 \xleftarrow{M_2} \dots S_n = S',$$

and p is mapped by $\ell_{\mathbb{D}}$ to the path of length n consisting only of *id* edges. This implies by induction the existence of $\sigma \in \overline{\mathbb{S}}(W)$, where $W = M_1 \bullet \dots \bullet M_n$ is closed-world and unsuccessful, such that $S' = (S \cdot W)|_{\psi(\sigma)}$. Because $S \in \perp_X$, there exists a successful W' and $f: W \rightarrow W'$ in $\mathbb{E}^{\mathbb{W}}(X)$ and $\sigma' \in \overline{\mathbb{S}}(W')$ such that $\sigma' \cdot f = \sigma$. By Lemma 11, W' is an extension of W with closed-world moves, say $W' = W \bullet M_{n+1} \bullet \dots \bullet M_{n+m}$. By induction on m , we obtain a path

$$S' = S_n \xleftarrow{M_{n+1}} S_{n+1} \xleftarrow{M_{n+2}} \dots S_{n+m},$$

where $S_{n+m} = (S' \cdot W')|_{\psi(\sigma')}$. Because W' is successful and $\ell_{\mathbb{D}}$ is a functor, there exists i such that $\ell_{\mathbb{D}}(M_{n+i}) = \heartsuit$, hence $S' \Leftarrow S_{n+m}$. Thus, $S \in \perp$.

Conversely, assume $S \in \mathbb{S}_X$ is in \perp . Let W be an unsuccessful, closed-world play over X and $\sigma \in \overline{\mathbb{S}}(W)$. Picking a decomposition $W = M_1 \bullet \dots \bullet M_n$ of W , we obtain a path p

$$S = S_0 \xleftarrow{M_1} S_1 \dots \xleftarrow{M_n} S_n = S'$$

in \mathbb{S} , which by functorality of $\ell_{\mathbb{D}}$ and the fact that W is unsuccessful, is mapped to a path $S \Leftarrow S'$. Because $S \in \perp$, there exists $S' \Leftarrow S'' \Leftarrow S'''$, with underlying path

$$S' = S_n \xleftarrow{M_{n+1}} S_{n+1} \dots \xleftarrow{M_{n+m}} S_{n+m} = S'' \xleftarrow{M_{n+m+1}} S'''$$

in $\mathbb{S}^{\mathbb{W}}$, such that $\ell_{\mathbb{D}}(M_{n+i}) = id$ for all $i \in m$ and $\ell_{\mathbb{D}}(M_{n+m+1}) = \heartsuit$. But by definition this means that $S''' = (S' \cdot W')|_{\psi(\sigma')}$ for some

$$\sigma' \in \overline{\mathbb{S}}(W') = \overline{(S \cdot W)|_{\psi(\sigma)}}(W') = \{\sigma'' \in \overline{\mathbb{S}}(W \bullet W') \mid \sigma'' \cdot f = \sigma\},$$

where $W' = M_{n+1} \bullet \dots \bullet M_{n+m+1}$ and $f: W \rightarrow (W \bullet W')$ is the extension. By construction, $\sigma' \cdot f = \sigma$. Hence, $S \in \perp_X$. \square

We furthermore easily obtain that, for any sensible $h: I \rightarrow X$ and $S \in \mathbb{S}_X$, $(I, h, S) \in \perp^{S^f}$ iff $(X, S) \in \perp$. This entails:

Corollary 4. *For any $h: I \rightarrow X$, $h': I \rightarrow X'$, $S \in \mathbb{S}_X$, and $S' \in \mathbb{S}_{X'}$, $(I, h, S) \sim_f (I, h', S')$ iff $(I, h, S) \sim^{S^f} (I, h', S')$.*

Proposition 20. *For any two CCS processes P and Q over n , $P \sim_{f,s} Q$ iff $P \sim^{CCS} Q$.*

7.3. Adequacy. We now define the notions of *blind composition* of two LTSS over a given alphabet A , and of *adequacy* of an effective G over A , which yield a handy characterisation of fair testing.

Definition 32. *Let, for any G and H over A , $G \diamond_A H = (G \times H) \times_{A^2} A^\circ$ be their blind composition over A . This forms an LTS over Σ via $G \diamond_A H \rightarrow A^\circ \rightarrow \Sigma$.*

In order to state the main property of blind composition, let us introduce further notation. For any graph with complementarity A and paths $\rho_a: a' \rightarrow a$ and $\rho_b: b' \rightarrow b$ in A^* , let $\rho_a \circ \rho_b$ iff there exists a path $\rho \in (A^\circ)^*$, whose projections $\rho_1: a' \rightarrow a$ and $\rho_2: b' \rightarrow a$ in A^* are equivalent to ρ_a and ρ_b under the quotient map $A^* \rightarrow \text{fc}(A)$, i.e., are such that $\widetilde{\rho}_a = \widetilde{\rho}_1$ and $\widetilde{\rho}_b = \widetilde{\rho}_2$. Intuitively, ρ_a and ρ_b are coherent if, up to insertion of identities at appropriate places, they consist of pairwise coherent edges. We call $(\text{fc}(A))^\circ$ the reflexive graph with pairs of coherent vertices of A as vertices, and such coherent pairs of (equivalence classes of) paths as edges $(a', b') \rightarrow (a, b)$. For two such paths $\rho_a \circ \rho_b$, let $(\rho_a, \rho_b) \triangleright^{\text{fc}(A)} \rho'$ iff there exists ρ_3 , of the same length n as ρ_1 and ρ_2 , such that $\widetilde{\rho}' = \widetilde{\rho}_3$ and $(\rho_1^i, \rho_2^i) \triangleright^A \rho_3^i$, for all $i \in n$. Note that any such ρ' has to be a path in A^{W} by construction. We may now state the main property of blind composition:

Proposition 21. *For any graphs with complementarity G and H over A , paths ρ and ρ' in A , all transition sequences $x \xleftarrow{G}^{\rho_x} x'$ and $y \xleftarrow{H}^{\rho_y} y'$, if $(\rho_x, \rho_y) \triangleright^{\text{fc}(A)} \rho$, then $(x, y) \xleftarrow{G \diamond_A H}^{\rho} (x', y')$.*

Proof. Let $p: G \rightarrow A$ and $q: H \rightarrow A$ be the given projections. Let also $(\rho_1^i, \rho_2^i) \triangleright^A \rho_3^i$ for all $i \in n$ witness the fact that $(\rho_x, \rho_y) \triangleright^{\text{fc}(A)} \rho$. It is enough to prove $(x, y) \xleftarrow{G \diamond_A H}^{\rho_3} (x', y')$, which is in fact a trivial induction on the length of ρ_3 using the definition of $G \diamond_A H$. \square

Let us now introduce the notion of adequacy of $G \rightarrow A$, which holds when fair testing may be checked on the blind ‘square’ of G over A , i.e., its blind product with itself.

Definition 33. *Let $p: G \rightarrow A$ be an effective graph over a graph A with complementarity. G is adequate (over A) iff the graph of $\text{ob } G^\circ \hookrightarrow \text{ob}(G \diamond_A G)$ is included in \simeq_Σ .*

Concretely, any transition $(e_1, e_2) \in G^\circ$ is matched, without any hypothesis on G , by (e_1, e_2) itself. Conversely, having a transition $(x_1, x_2) \xleftarrow{e_1, e_2} (x'_1, x'_2)$ in $G \diamond_A G$ means that we have edges $e_i: x'_i \rightarrow x_i$, for $i \in 2$, such that $p(e_1) \Downarrow p(e_2) = \sigma$. Adequacy demands that there exists a path $(\rho_1, \rho_2): (x''_1, x''_2) \rightarrow^* (x_1, x_2)$ in G° , such that the composite of $\rho_1 \Downarrow \rho_2$ in $\text{fc}(\Sigma)$ is the singleton path (σ) , and $(x''_1, x''_2) \simeq_\Sigma (x'_1, x'_2)$, where the left-hand side is in G° while the right-hand side is in $G \diamond_A G$.

Proposition 22. *CCS, $\mathcal{S}^\mathcal{L}$, and $\mathcal{T}^\mathcal{L}$ are adequate over \mathbb{A} .*

Proof. For *CCS*, the projection $\text{CCS} \rightarrow \mathbb{A}$ is actually a strong bisimulation over \mathbb{A} .

For the both other graphs $p: G \rightarrow \mathbb{A}$ over \mathbb{A} , the graph of $\text{ob } G^\circ \hookrightarrow \text{ob}(G \diamond_A G)$ is a weak bisimulation over \mathbb{A} , because for all e and e' in G , if $p(e) \circ p(e')$, then either $e \circ e'$, or both interleavings are coherent, i.e., $(e, \text{id}) \circ (\text{id}, e')$ and $(\text{id}, e) \circ (e', \text{id})$, pointwise. \square

The only subtle point is that this only holds because \mathcal{L} is restricted to edges of the shape (8). E.g., this does not hold for \mathcal{S}^I over \mathbb{A} . To define \mathcal{S}^I as a graph over \mathbb{A} , extend ξ to $\chi: \mathbb{Q}^I \rightarrow \mathbb{A}$ in the obvious way. Consider now the moves $o_{2,1}, \iota_{2,1}: [2] \rightarrow [2]$, let $I = 2 \cdot \star$, and let h be one of the two embeddings $I \rightarrow [2]$, say the one which is an inclusion at \star , h' being the other. We have transitions $(I, h, [2]) \xleftarrow{\bar{1}} (I, h, [2])$ and

$(I, h, [2]) \stackrel{1}{\leftarrow} (I, h', [2])$, and $(I, h, [2]) \supset (I, h, [2])$. However, the two transitions are not coherent for any attempt to construct a diagram (7) (with here $k = k' = h$) fails to make the upper square commute. This is the very reason we use \mathcal{L} instead of \mathbb{Q}^I .

Proposition 23. *For any adequate $p: G \rightarrow A$ and $x \supset y$ in G , we have $[x, y] \in \perp^{G^{\mathbb{W}}}$ iff $(x, y) \in \perp^{G \diamond_A G}$.*

Proof. We have $[x, y] \simeq_{\Sigma} ((x, y) \in G^{\supset}) \simeq_{\Sigma} ((x, y) \in G \diamond_A G)$. \square

Proposition 24. *For any H over A and adequate G over A , $x_1, x_2 \in G$, and y in H , if $x_2 \simeq_A y$, then*

$$[x_1, x_2] \in \perp^G \text{ iff } (x_1, y) \in \perp^{G \diamond_A H}.$$

Proof. By effectiveness, it is enough to prove that the right-hand side is equivalent to $(x_1, x_2) \in \perp^{G \diamond_A G}$, which is straightforward by hypothesis. \square

7.4. Trees. Let us return to our main goal and see what we have. We know that the graph morphism $\mathcal{T}^{\mathcal{L}} \rightarrow \mathcal{S}^{\mathcal{L}}$ is included in weak bisimilarity over \mathbb{A} . We will further show below that the map $\text{ob}(CCS) \hookrightarrow \text{ob}(\mathcal{T}^{\mathcal{L}})$ also is. We here examine the question of when this is enough for the composite to be *fair*, i.e., to preserve and reflect fair testing equivalence.

Our tool to answer this question will be the notion of A -tree, for any graph with complementarity A , which is directly inspired by the work of Brinksma et al. on *failures* [46].

Let the set \mathcal{H}^A of A -trees consist of possibly infinite terms in the grammar

$$\frac{\dots \quad v_i \vdash t_i \quad \dots \quad (\forall i \in n)}{v \vdash \sum_{i \in n} a_i \cdot t_i} \quad (n \in \mathbb{N})$$

where for all $i \in n$, $a_i: v_i \rightarrow v$ in A is not silent, i.e., $a_i \notin A^{\mathbb{W}}$ or $\ell^A(a_i) \neq id$. A -trees form a reflexive graph over A with edges determined by

$$(v \vdash \sum_{i \in n} a_i \cdot t_i) \xleftarrow{a_i} (v_i \vdash t_i).$$

An effective graph $p: G \rightarrow A$ over A has enough A -trees iff for all $x \in G$, $v \in A$ such that $p(x) \supset v$, and A -trees $v \vdash t$, there exists $x_t \in G$ such that $x \supset x_t$ and x_t is weakly bisimilar to t (over A).

For such graphs, we may define a new equivalence, called A -tree equivalence, as follows.

Definition 34. *For any effective $p: G \rightarrow A$ with enough A -trees, let \sim_A^G be the relation defined by $x \sim_A^G y$ iff $x \bowtie y$ and for all $v \in A$ such that $p(x) \supset v$ and A -trees $t \in \mathcal{H}_v^A$,*

$$(x, t) \in \perp^{G \diamond_A \mathcal{H}^A} \text{ iff } (y, t) \in \perp^{G \diamond_A \mathcal{H}^A}.$$

A graph with complementarity A has enough ticks iff for all $a \in A$, there exists an edge $\heartsuit_a: a' \rightarrow a$ such that $\ell^A(\heartsuit_a) = \heartsuit$. Furthermore, A is inertly silent iff for all $e: b \rightarrow a$ in $A^{\mathbb{W}}$ such that $\ell^A(e) = id$, we have $a = b$.

Definition 35. *A graph with complementarity A is a nice alphabet iff it has enough ticks, and is finitely branching and inertly silent.*

The main property of A -trees is:

Proposition 25. *Consider any adequate $p: G \rightarrow A$, where G has enough A -trees and A is a nice alphabet. Then, $\sim^G = \sim_A^G$.*

We start with some preparation. Let a path in A be *pure* iff it contains no silent edge, and *\heartsuit -free* iff no edge is mapped to $(\ell^A)^{-1}(\heartsuit)$. Let the set \mathcal{F}_a of *failures* over $a \in A$ consist of all pairs (p, L) , where $p: a' \rightarrow a$ is any pure, \heartsuit -free path in A and $L \subseteq A^*$ is a set of pure paths such that for all $q \in L$, $\text{cod}(q) = a'$.

We define a map $\mathbf{fl}: \mathcal{F}_a \rightarrow \mathcal{H}_a^A$ to A -trees over a , for all a , by induction on p , followed by coinduction on L :

$$\begin{aligned} (e \circ p, L) &\mapsto e.(\mathbf{fl}(p, L)) + \heartsuit_a.0 \\ (\epsilon, L) &\mapsto \mathbf{fl}(L) \\ L &\mapsto \sum_{\{e \in A(-, a) \mid L \cdot e \neq \emptyset\}} e.\mathbf{fl}(L \cdot e) \end{aligned}$$

where $L \cdot e$ is the set of paths p such that $(e \circ p) \in L$. Note in particular that if $L = \emptyset$ or $\{\epsilon\}$, then $\mathbf{fl}(L) = 0$. The sum is finite at each stage because A is finitely branching, and we use the fact that A has enough ticks.

Proof of Proposition 25. It is straightforward to show that $\sim^G \subseteq \sim_A^G$, by Proposition 7.3. For the converse, assume $x \bowtie y$ and $x \approx^G y$. This means that there exists z such that $x \circ z$ and $y \circ z$, and, w.l.o.g., $(y, z) \in \perp^{G \diamond_A G}$ and $(x, z) \notin \perp^{G \diamond_A G}$.

By the latter, we obtain a transition sequence $(x, z) \Leftarrow (x', z')$, such that for no (x'', z'') we have $(x', z') \overset{\heartsuit}{\Leftarrow} (x'', z'')$. Let ρ be the ‘purification’ of the image of the given path $(x, z) \Leftarrow (x', z')$ under the second projection $G \diamond_A G \rightarrow A^\circ \xrightarrow{\pi'} A$, where purification means removing the silent edges (which is possible thanks to A being inertly silent). Further let $L \subseteq A^*$ be the set of purifications of images of paths $z' \leftarrow^* z''$ under p . Let

$$t = \mathbf{fl}(\rho, L).$$

We show $(x, t) \notin \perp^{G \diamond_A \mathcal{H}^A}$ and $(y, t) \in \perp^{G \diamond_A \mathcal{H}^A}$.

For the first point, $t \overset{p}{\Leftarrow} t'$, with $t' = \mathbf{fl}(\epsilon, L)$, hence $(x, t) \Leftarrow (x', t')$. Now, assume $(x', t') \overset{\heartsuit}{\Leftarrow} b''$. By definition of (x', t') , we split this into $x' \overset{p}{\Leftarrow} x''$ and $t' \overset{q}{\Leftarrow} t''$, with $b'' = (x'', t'')$. But then $z' \overset{q}{\Leftarrow} z''$ by construction of t , and hence $(x', z') \overset{\heartsuit}{\Leftarrow} (x'', z'')$, contradicting $(x, z) \notin \perp^{G \diamond_A G}$.

Let us now show $(y, t) \in \perp^{G \diamond_A \mathcal{H}^A}$. For any $(y, t) \Leftarrow (y', t')$, we have accordingly $t \overset{p'}{\Leftarrow} t'$. Let $\rho'_1 = \rho' \wedge \rho$ be the greatest common prefix of ρ' and ρ .

- If ρ'_1 is a strict prefix, then by construction $t' \overset{\heartsuit}{\Leftarrow} 0$ and we are done, by Propopsition 21, since $(id, \heartsuit) \heartsuit \diamond_A$.
- If $\rho'_1 = \rho$, then let ρ'_2 be the unique path such that $\rho' = \rho'_1 \circ \rho'_2$. We have $\rho'_2 \in L$, hence by construction of L there exists z'' such that $z \overset{\rho'_1}{\Leftarrow} z' \overset{\rho'_2}{\Leftarrow} z''$, and thus $(y, z) \Leftarrow (y', z'')$, by Propopsition 21. By $(y, z) \in \perp^{G \diamond_A G}$, there exists $(y', z'') \overset{\heartsuit}{\Leftarrow} (y'', z''')$, which projects to $y' \overset{\rho_y}{\Leftarrow} y''$ and $z'' \overset{\rho_z}{\Leftarrow} z'''$. But then $t' \overset{p'}{\Leftarrow} t''$, hence $(y', t') \overset{\heartsuit}{\Leftarrow} (y'', t'')$, by Propopsition 21 again, which concludes the proof. \square

Corollary 5. *For any adequate G and H over a nice alphabet A and relation $R: G \dashrightarrow H$ over A such that $R \subseteq \simeq_A$, if G and H have enough A -trees and R preserves and reflects \bowtie , then for any xRx' and yRy' , we have $x \sim^G y$ iff $x' \sim^H y'$.*

Proof. We have

$$\begin{aligned}
& x \sim^G y \\
& \quad \Downarrow (\text{by Proposition 25}) \\
& x \bowtie y \text{ and } \forall v \in p(x)^\circ. \forall t \in \mathcal{H}_v^A. (x, t) \in \perp^{G \circ_A \mathcal{H}^A} \Leftrightarrow (y, t) \in \perp^{G \circ_A \mathcal{H}^A} \\
& \quad \Downarrow (\text{by weak bisimilarity over } A) \\
& x' \bowtie y' \text{ and } \forall v \in p(x)^\circ. \forall t \in \mathcal{H}_v^A. (x', t) \in \perp^{H \circ_A \mathcal{H}^A} \Leftrightarrow (y', t) \in \perp^{H \circ_A \mathcal{H}^A} \\
& \quad \Downarrow (\text{by Proposition 25 again}) \\
& x' \sim^H y'.
\end{aligned}$$

□

Lemma 12. *The graph of $\theta: \text{ob } CCS \rightarrow \text{ob } \mathcal{T}^\mathcal{L}$ is included in weak bisimilarity over \mathbb{A} .*

Proof. Let, for any $h: I \rightarrow X$ and family $P \in \prod_{n \in \mathbb{N}} \prod_{x \in X[n]} CCS_n$, $h[P]$ be

$$I(\star) \vdash \nu(X(\star) \setminus h(I(\star))). (\mid_n \mid_{x \in X[n]} P_x [l \mapsto x \cdot s_l]).$$

This is not a valid CCS process in general, because ν may bind names which are less than names in the interface, which is prohibited by our syntax. This may be overcome by letting γ_h be the unique non-decreasing isomorphism $(X(\star) \setminus h(I(\star))) \rightarrow (X(\star) - I(\star))$ (which exists thanks to h being monic). We may then let $h[P]$ be

$$I(\star) \vdash \nu^{X(\star) - I(\star)}. \left(\mid_n \mid_{x \in X[n]} P_x \left[\begin{array}{ll} l \mapsto \epsilon a. (h_\star(a) = x \cdot s_l) & \text{if } x \cdot s_l \in h_\star(I(\star)) \\ l \mapsto \gamma_h(x \cdot s_l) & \text{otherwise} \end{array} \right] \right),$$

where ϵ is Hilbert's definite description operator, i.e., $\epsilon a. A(a)$ denotes the unique a such that $A(a)$ holds, and $\nu^n.P$ denotes $\nu \dots \nu.P$, n times.

Let $\mathcal{J}: \text{ob } CCS \dashrightarrow \text{ob } \mathcal{T}^\mathcal{L}$ consist of all pairs $(h[P], (I, h, \theta(P)))$, for any $P \in \prod_{n \in \mathbb{N}} \prod_{x \in X[n]} CCS_n$.

Let \mathcal{R} be the composite relation

$$\text{ob } CCS \xrightarrow{\equiv} \text{ob } CCS \xrightarrow{\mathcal{J}} \text{ob } \mathcal{T}^\mathcal{L}.$$

We show that \mathcal{R} is an expansion [49], which implies that it is a weak bisimulation. Hence, since the graph of θ is included in \mathcal{R} , this entails the desired result. Let $x \xrightarrow{\hat{\alpha}} x'$ iff

- either α is an identity and $x \xrightarrow{\tau}^* x'$ in zero or one step,
- or α is not an identity and $x \xleftarrow{\alpha} x'$.

Recall that \mathcal{R} being an expansion means that for all $P\mathcal{R}T$,

- if $P \xrightarrow{\alpha} P'$, then that there exists T' such that $P'\mathcal{R}T'$ and $T \xleftarrow{\alpha} T'$; and
- $T \xrightarrow{\alpha} T'$ implies that there exists P' such that $P \xrightarrow{\hat{\alpha}} P'\mathcal{R}T'$.

First, one easily shows that transitions in CCS are dealt with by ‘heating’ the right-hand side until it may match the given transition.

Conversely, we show below in (1) that for any transition $(I, h, \theta(P)) \xrightarrow{M} (I, k, T')$, for $M: k \rightarrow h$ in \mathcal{L} , where M is either a fork or a channel creation, then $T' = \theta(P')$, for some $P' \in \prod_{n \in \mathbb{N}} \prod_{y \in Y[n]} CCS_n$, and $h[P] \equiv k[P']$.

Thus, any such transition, which is silent is matched by the empty transition sequence, as in

$$\begin{array}{ccc} Q & \equiv & h[P] \quad \mathcal{J} \quad (I, h, \theta(P)) \\ \parallel & & \parallel \quad \uparrow_M \\ Q & \equiv & k[P'] \quad \mathcal{J} \quad (I, k, T') \end{array}$$

Similarly, for any transition $(I, h, \theta(P)) \xrightarrow{M} (I, k, T')$ not falling in the previous cases, we prove below in (2) that there exists $P' \in \prod_{n \in \mathbb{N}} \prod_{y \in Y[n]} CCS_n$ and Q' such that $h[P] \xleftarrow{\xi(M)} Q' \equiv k[P']$. Thus, any such transition is matched as in

$$\begin{array}{ccc} Q & \equiv & h[P] \quad \mathcal{J} \quad (I, h, \theta(P)) \\ \xi(M) \uparrow & \nearrow \xi(M) & \uparrow_M \\ Q' & \equiv & k[P'] \quad \mathcal{J} \quad (I, k, T'). \end{array}$$

Since \mathcal{R} contains the graph of θ , this entails the result.

(1) Let us now consider the case of a transition $(I, h, \theta(P)) \xrightarrow{M} (I, k, T')$, for $M: k \rightarrow h$ in \mathcal{L} , where M is either a fork or a channel creation. Consider first the case where M is a fork. Let x_1, \dots, x_n be the players of X , let m_1, \dots, m_n be their respective arities, and let $i_0 \in n$ be the forking player. Let, for any $i \in n + 1$,

$$\mu(i) = \begin{cases} i & \text{if } i < i_0 \\ i_0 & \text{if } i = i_0 \text{ or } i = i_0 + 1 \\ i - 1 & \text{if } i > i_0 + 1 \end{cases}$$

and

$$P'_i = \begin{cases} P_i & \text{if } i < i_0 \\ P_{i_0}^1 & \text{if } i = i_0 \\ P_{i_0}^2 & \text{if } i = i_0 + 1 \\ P_{i-1} & \text{if } i > i_0 + 1, \end{cases}$$

For all $j \in n + 1$, we have that y_j is an avatar of $x_{\mu(j)}$, and $P'_j = P_{\mu(j)}$ if $\mu(j) \neq i_0$, while $P'_{i_0} = P'_{i_0} \mid P'_{i_0+1}$.

Thanks to the restriction of edges

$$\begin{array}{ccc} & I & \\ h \swarrow & \downarrow u & \searrow k \\ X & \xrightarrow{t} M \xleftarrow{s} & Y \end{array}$$

in \mathcal{L} , for any $j \in n + 1$, if $\mu(j) = i \neq i_0$, $l \in m_i$ and $a, b \in I(\star)$, we have that $h_\star(a) = x_i \cdot s_l$ and $k_\star(b) = y_j \cdot s_l$, then, since $sy_j = tx_i$, both squares

$$\begin{array}{ccc} \star & \xrightarrow{a,b} & I \\ s_l \downarrow & & \downarrow u \\ [m_i] & \xrightarrow{sy_j (=tx_i)} & M \end{array}$$

commute, hence $a = b$ by monicity of u . Furthermore, if $\mu(j) = i_0$, and $a, b \in I(\star)$, we have that if $h_\star(a) = x_i \cdot s_l$ and $k_\star(b) = y_j \cdot s_l$, then, since $sy_j s_l = tx_i s_l$, we again have $a = b$.

So, for all $j \in n + 1$ and $l \in m_i$, for $i = \mu(j)$, we have $x_i \cdot s_l \in h_\star(I(\star))$ iff $y_j \cdot s_l \in k_\star(I(\star))$, in which case

$$\epsilon a.(h_\star(a) = x_i \cdot s_l) = \epsilon b.(k_\star(b) = y_j \cdot s_l).$$

We have a commuting diagram

$$\begin{array}{ccc} & \delta_M & \\ & \curvearrowright & \\ X(\star) \setminus h(I(\star)) & \xrightarrow{\cong} & M(\star) \setminus u(I(\star)) \xleftarrow{\cong} Y(\star) \setminus k(I(\star)) \\ \gamma_h \downarrow & & \downarrow \gamma_k \\ X(\star) - I(\star) & \xrightarrow{\delta'_M} & Y(\star) - I(\star), \end{array}$$

of bijections, where δ_M and δ'_M are obtained by composition, which is such that for all $j \in n + 1$ and $i = \mu(j)$, $l \in m_i$, if $x_i \cdot s_l \notin h(I(\star))$, then $\delta_M(x_i \cdot s_l) = y_j \cdot s_l$. We have

$$h[P] = \nu^{X(\star) - I(\star)}. \left(\prod_{i \in n} P_i \left[\begin{array}{ll} l \mapsto \epsilon a.(h_\star(a) = x_i \cdot s_l) & \text{if } x_i \cdot s_l \in h_\star(I(\star)) \\ l \mapsto \gamma_h(x_i \cdot s_l) & \text{otherwise} \end{array} \right] \right)$$

and

$$k[P'] = \nu^{Y(\star) - I(\star)}. \left(\prod_{j \in n+1} P'_j \left[\begin{array}{ll} l \mapsto \epsilon b.(k_\star(b) = y_j \cdot s_l) & \text{if } y_j \cdot s_l \in k_\star(I(\star)) \\ l \mapsto \gamma_k(y_j \cdot s_l) & \text{otherwise} \end{array} \right] \right).$$

Via the renaming δ'_M , we have

$$\begin{aligned} h[P] &\equiv \nu^{Y(\star) - I(\star)}. \left(\prod_{j \in n+1, j \neq i_0+1} P^{\mu(j)} \left[\begin{array}{ll} l \mapsto \epsilon b.(k_\star(b) = y_j \cdot s_l) & \text{if } y_j \cdot s_l \in k_\star(I(\star)) \\ l \mapsto \gamma_k(\delta_M(x_i \cdot s_l)) & \text{otherwise} \end{array} \right] \right) \\ &\equiv \nu^{Y(\star) - I(\star)}. \left(\prod_{j \in n+1, j \neq i_0+1} P^{\mu(j)} \left[\begin{array}{ll} l \mapsto \epsilon b.(k_\star(b) = y_j \cdot s_l) & \text{if } y_j \cdot s_l \in k_\star(I(\star)) \\ l \mapsto \gamma_k(y_j \cdot s_l) & \text{otherwise} \end{array} \right] \right) \\ &\equiv \nu^{Y(\star) - I(\star)}. \left(\prod_{j \in n+1} P'_j \left[\begin{array}{ll} l \mapsto \epsilon b.(k_\star(b) = y_j \cdot s_l) & \text{if } y_j \cdot s_l \in k_\star(I(\star)) \\ l \mapsto \gamma_k(y_j \cdot s_l) & \text{otherwise} \end{array} \right] \right) \\ &\equiv k[P']. \end{aligned}$$

The case of a channel creation move is similar.

(2) Consider now any transition $(I, h, \theta(P)) \xleftarrow{M} (I, k, T')$, where M is an input or an output on some channel $c \in h_\star(I(\star))$, or a synchronisation, or a tick. Then, proceeding as for the forking move above, we may take $\mu = id$, and still obtain δ_M and δ'_M . In all cases, we have $T'_i = \theta(P'_i)$, for some family P' of CCS processes. E.g., if M is an input on c by x_{i_0} , then $P'_i = P_i$ for all $i \neq i_0$, and $P_{i_0} \equiv c.P'_{i_0} + P''$.

We have $h[P] \xleftarrow{\xi(M)} Q$, where

$$Q = \nu^{X(\star) - I(\star)}. \left(\prod_{i \in n} P'_i \left[\begin{array}{ll} l \mapsto \epsilon a.(h_\star(a) = x_i \cdot s_l) & \text{if } x_i \cdot s_l \in h_\star(I(\star)) \\ l \mapsto \gamma_h(x_i \cdot s_l) & \text{otherwise} \end{array} \right] \right),$$

which via the renaming δ'_M , is structurally congruent to

$$\begin{aligned}
& \nu^{Y(\star)-I(\star)}. \left(\bigsqcup_{i \in n} P'_i \left[\begin{array}{ll} l \mapsto \epsilon b.(k_\star(b) = y_i \cdot s_l) & \text{if } y_i \cdot s_l \in k_\star(I(\star)) \\ l \mapsto \gamma_k(\delta_M(x_i \cdot s_l)) & \text{otherwise} \end{array} \right] \right) \\
& \equiv \nu^{Y(\star)-I(\star)}. \left(\bigsqcup_{i \in n} P'_i \left[\begin{array}{ll} l \mapsto \epsilon b.(k_\star(b) = y_i \cdot s_l) & \text{if } y_i \cdot s_l \in k_\star(I(\star)) \\ l \mapsto \gamma_k(y_i \cdot s_l) & \text{otherwise} \end{array} \right] \right) \\
& \equiv k[P'],
\end{aligned}$$

which concludes the proof. \square

This leads to our first full abstraction result:

Corollary 6. *The composite $\text{ob}(CCS) \hookrightarrow \text{ob}(\mathcal{T}^\mathcal{L}) \rightarrow \text{ob}(\mathcal{S}^\mathcal{L})$ is included in weak bisimilarity.*

Proof. By the previous lemma and Proposition 1. \square

Lemma 13. *For any $x, y \in \mathcal{S}^I$, we have $x \sim^{\mathcal{S}^I} y$ iff $x \sim^{\mathcal{S}^\mathcal{L}} y$.*

Proof. First, the projection $p_\chi: \mathcal{S}^\mathcal{L} \rightarrow \mathcal{S}^I$ restricts to a map $(\mathcal{S}^\mathcal{L})^\mathbb{W} \rightarrow (\mathcal{S}^I)^\mathbb{W}$ over Σ , which is included in strong bisimilarity. Indeed, consider the relation \mathcal{R} defined for all $h: I \rightarrow X$, $k: J \rightarrow Y$, $S \in \mathbf{S}_X$, and $T \in \mathbf{S}_Y$, by $(I, h, S)\mathcal{R}(J, k, T)$ iff there exists an isomorphism $\gamma: X \rightarrow Y$ in \mathbb{D}_h such that $S = T \cdot \gamma$. \mathcal{R} is a strong bisimulation over Σ .

Hence, for any $x \in \mathcal{S}^\mathcal{L}$, $x \simeq_\Sigma p_\chi(x)$, hence $x \in \perp^{\mathcal{S}^\mathcal{L}}$ iff $p_\chi(x) \in \perp^{\mathcal{S}^I}$, which directly entails the result. \square

Corollary 7. *The composite $\text{ob } CCS \xrightarrow{\theta} \text{ob } \mathcal{T}^\mathcal{L} \xrightarrow{\llbracket - \rrbracket} \text{ob } \mathcal{S}^\mathcal{L}$ is fair, and we have for all CCS processes P and Q over any common n :*

$$P \sim_{f,s} Q \quad \text{iff} \quad \llbracket \theta(P) \rrbracket \sim_f \llbracket \theta(Q) \rrbracket.$$

Proof. We have:

$$\begin{aligned}
& P \sim_{f,s} Q \\
& \quad \Downarrow \\
& P \sim^{CCS} Q \\
& \quad \Downarrow \text{(by Lemma 12)} \\
& \theta(P) \sim^{\mathcal{T}^\mathcal{L}} \theta(Q) \\
& \quad \Downarrow \text{(by Proposition 1)} \\
& \llbracket \theta(P) \rrbracket \sim^{\mathcal{S}^\mathcal{L}} \llbracket \theta(Q) \rrbracket \\
& \quad \Downarrow \text{(by Lemma 13)} \\
& \llbracket \theta(P) \rrbracket \sim^{\mathcal{S}^I} \llbracket \theta(Q) \rrbracket \\
& \quad \Downarrow \text{(by Lemma 10)} \\
& \llbracket \theta(P) \rrbracket \sim_f \llbracket \theta(Q) \rrbracket,
\end{aligned}$$

as desired. \square

8. CCS AS A PLAYGROUND

At last, we prove that \mathbb{D}^{CCS} forms a playground. We rewind to the beginning of Section 4.1, to state things a bit more formally.

8.1. A pseudo-double category. Recall from HP the embeddings of categories $\mathbb{C}_1 \hookrightarrow \mathbb{C}_2 \hookrightarrow \mathbb{C}_3 \hookrightarrow \mathbb{C}_4 = \mathbb{C}$, and how plays were constructed as denumerable compositions in $\mathbf{Cospan}(\widehat{\mathbb{C}})$ of global moves. There is a notion of dimension in \mathbb{C} : \star is the sole object of dimension 0, all $[n]$'s have dimension 1, all $o_{n,i}$, $\iota_{n,i}$, π_n^l , π_n^r , \heartsuit_n , and ν_n have dimension 2, all π_n have dimension 3, and all $\tau_{n,i,m,j}$ have dimension 4. By extension, a presheaf F has dimension i if F is empty over objects of dimension strictly greater than i . We call *interfaces* the presheaves of dimension 0 (i.e., empty beyond dimension 0), *positions* the finite presheaves of dimension 1.

We start by constructing the base double category for our playground, and then prove the required axioms. We form a first pseudo double category $\mathbb{D}^{CCS,0}$ with:

- as objects all positions,
- horizontal category $\mathbb{D}_h^{CCS,0}$ the full subcategory of $\widehat{\mathbb{C}}^f$ consisting of positions;
- vertical (bi)category $\mathbb{D}_v^{CCS,0}$ the sub-bicategory of $\mathbf{Cospan}(\widehat{\mathbb{C}}^f)$ consisting of positions and cospans of monic arrows between them;
- and all commuting diagrams

$$(9) \quad \begin{array}{ccc} X & \xleftarrow{h} & X' \\ s \downarrow & & \downarrow s' \\ U & \xleftarrow{k} & V \\ t \uparrow & & \uparrow t' \\ Y & \xleftarrow{l} & Y' \end{array} \text{ as double cells} \quad \begin{array}{ccc} X & \xrightarrow{h} & X' \\ U \downarrow & \xrightarrow{(h,k,l)} & \downarrow V \\ Y & \xrightarrow{l} & Y' \end{array}$$

with all \hookrightarrow arrows monic.

Horizontal composition of double cells is induced by composition in $\widehat{\mathbb{C}}^f$. Vertical composition of double cells is induced by pushout in $\widehat{\mathbb{C}}^f$. It is of course the vertical direction here which is pseudo.

Definition 36. Let \mathbb{D}^{CCS} be the pseudo-double category obtained by restricting $\mathbb{D}^{CCS,0}$ to vertical morphisms which are finite composites of global moves [25].

Since \mathbb{D}^{CCS} is again the only involved (candidate) playground in this section, we often omit the superscript. E.g., \mathbb{D}^0 denotes $\mathbb{D}^{CCS,0}$.

The rest of Section 8 is devoted to proving:

Theorem 4. \mathbb{D} , equipped with

- as individuals, all positions of the shape $\mathbb{C}(-, [n])$, i.e., all strictly representable presheaves,
- global moves as moves, local basic moves as basic moves, and full, global moves as full moves.

forms a playground.

We start with a combinatorial correctness criterion for characterising plays $U: X \rightarrow Y$ among general cospans $X \hookrightarrow U \hookrightarrow Y$, which we then put to use in proving the theorem. Our convention for plays $X \hookrightarrow U \hookrightarrow Y$ is that the (candidate) final position is always on the left.

8.2. Correctness. We prove a few properties of plays, which we then find are sufficient for a cospan to be a play.

Given a play $X \hookrightarrow U \hookrightarrow Y$, we start by forgetting the cospan structure and exhibiting some properties of U alone.

Definition 37. A core of a presheaf $U \in \widehat{\mathbb{C}}^f$ is an element of dimension > 1 which is not the image (under the action of some morphism of \mathbb{C}) of any element of higher dimension.

Here is a first easy property of plays. Before the next definition, observe that for all local moves $Y \hookrightarrow M \hookrightarrow X$, M is a representable presheaf.

Definition 38. A presheaf U is locally 1-injective iff for any local move $Y \hookrightarrow M \hookrightarrow X$ with interface I and core $\mu \in U(M)$, if two elements of M are identified by the Yoneda morphism $\mu: M \rightarrow U$, then they are in (the image of) $I(\star)$.

The name ‘locally 1-injective’ is designed to evoke the fact that $M \rightarrow U$ is injective above dimension 0.

Proposition 26. Any play U is locally 1-injective.

Proof. Choose a decomposition of U into global moves; μ corresponds to precisely one such global move, say M' , obtained, by definition, from some local move M as a pushout (3). By construction of pushouts in presheaf categories, M' is obtained from M by identifying some channels according to $I \rightarrow Z$. \square

We now extract from any presheaf a graph, which represents its candidate causal structure. Observe that, in \mathbb{C} , for any object μ of dimension > 1 (i.e., a move), all morphisms from a player, i.e., an object of the shape $[n]$, to μ have exactly one of the shapes $f \circ s \circ f'$ and $f \circ t \circ f'$. In the former case, the given player belongs in the final position of μ and we say that it is a *source* of μ ; in the latter, it belongs in the initial position and we call it a *target*. We extend these notions to arbitrary presheaves.

Definition 39. In any U , the sources of a core μ are the players x with a morphism, in $\text{el}(U)$, of the shape $x \xrightarrow{f \circ s \circ f'} \mu$ to μ ; its targets are the players y with a morphism of the shape $y \xrightarrow{f \circ t \circ f'} \mu$.

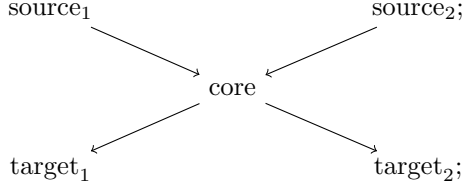
Example 13. In the representable π_n , there is one target, $l \circ t$ (or equivalently $r \circ t$), and two sources, $s_1 = l \circ s$ and $s_2 = r \circ s$, respectively the left and right players obtained by forking. Another example is $\tau_{n,i,m,j}$, which has two targets, the sender $\epsilon \circ t$ and the receiver $\rho \circ t$, and two sources $\epsilon \circ s$ and $\rho \circ s$.

Definition 40. A channel $a \in M(\star)$ is created by a local move $Y \xrightarrow{s} M \xleftarrow{t} X$ iff $a \in Y(\star) \setminus X(\star)$.

Recall that in \mathbb{C} , the channels known to a player $[n]$ are represented by morphisms $s_1, \dots, s_n: \star \rightarrow [n]$, so that in a presheaf $U \in \widehat{\mathbb{C}}^f$, the channels known to $x \in U[n]$ are $x \cdot s_1, \dots$, and $x \cdot s_n$.

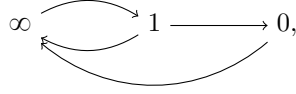
Given a presheaf U , we construct its *causal (simple) graph* G_U as follows:

- its vertices are all channels, players, and cores in U ;
- there is an edge to each core from its sources and one from each core to its targets, as in



- there is an edge $x \rightarrow x \cdot s_i$ for all $x \in U[n]$ and $i \in n$;
- there is an edge $a \rightarrow \mu$ for each channel a created by μ .

This graph is actually a binary relation, since there is at most one edge between any two vertices. It is also a colored graph, in the sense that it comes equipped with a morphism to the graph L :



mapping cores to ∞ , players to 1, and channels to 0. (Observe in particular that there are no edges from channels to players nor from cores to channels.)

Consider any simple graph G , equipped with a morphism $l: G \rightarrow L$. We call vertices of G channels, players, or cores, according to their label.

The assignment $U \mapsto G_U$ actually extends to a functor $G_-: \widehat{\mathbb{C}} \rightarrow \mathbf{Gph}/L$, as follows. Let, first, for any move $x \in U$, the *core associated to x* $\text{core}(x)$ be the unique core reachable from x in $\text{el}(U)$, i.e., the unique core μ for which there exists f in \mathbb{C} such that $\mu \cdot f = x$. Now, for any $\alpha: U \rightarrow U'$ in $\widehat{\mathbb{C}}$, let $G_\alpha: G_U \rightarrow G_{U'}$ map any core x in G_U to $\text{core}(\alpha(x)) \in G_{U'}$, and any non-core vertex $x \in G_U$ to $\alpha(x) \in G_{U'}$. By naturality, this indeed defines a unique morphism of simple graphs.

Proposition 27. $G_-: \widehat{\mathbb{C}} \rightarrow \mathbf{Gph}/L$ is a functor.

Definition 41. Seen as an object of \mathbf{Gph}/L , G is *source-linear* iff for any cores μ, μ' , and other vertex (necessarily a player or a channel) x , $\mu \leftarrow x \rightarrow \mu'$ in G , then $\mu = \mu'$. G is *target-linear* iff for any cores μ, μ' and player x , if $\mu \rightarrow x \leftarrow \mu'$ in G , then $\mu = \mu'$. G is *linear* iff it is both source-linear and target-linear.

Proposition 28. For any play $Y \xrightarrow{s} U \xleftarrow{t} X$, G_U is linear.

Proof. By induction on any decomposition of U into global moves. □

Proposition 29. For any play as above, G_U is acyclic (in the directed sense).

Proof. Again by induction on any decomposition of U . □

Definition 42. A player x in U is *final* iff it is not the target of any move, i.e., for no move $\mu \in U$, $x = \mu \cdot t$.

Lemma 14. A player is final in U iff it has no edge from any core in G_U .

Definition 43. A player is *initial* in U when it is not the source of any move, i.e., for no move $\mu \in U$, $x = \mu \cdot s$. A channel is *initial* when it is not created by any move.

Lemma 15. A player is initial in U iff it has no edge to any core in G_U .

Now, here is the expected characterisation:

Theorem 5. *A cospan $Y \xrightarrow{s} U \xleftarrow{t} X$ is a play iff*

- (1) U is locally 1-injective,
- (2) X contains precisely the initial players and channels in U ,
- (3) Y contains all channels, plus precisely the final players in U ,
- (4) and G_U is linear and acyclic.

Of course, we have almost proved the ‘only if’ direction, and the rest is easy, so only the ‘if’ direction remains to prove. The rest of this section is devoted to this. First, let us familiarise ourselves with removing elements from a presheaf. For two morphisms of presheaves $U \xrightarrow{f} V \xleftarrow{g} W$, we denote by $U \setminus W$ the topos-theoretic difference $U \cap \neg W$ of (the images of) f and g in the lattice $\text{Sub}(V)$ of subobjects of V . This differs in general from what we denote $U - W$, which is the set of elements in V which are in the image of U but not that of W , i.e., $\sum_{c \in \mathbb{C}} U(c) \setminus W(c)$. More generally, for any morphism of presheaves $f: U \rightarrow V$ and set W , let $U - W = \sum_{c \in \mathbb{C}} \text{Im}(U(c)) \setminus W$. $U - W$ is generally just a set, not a presheaf; i.e., its elements are not necessarily stable under the action of morphisms in \mathbb{C} . Proposition 30 below exhibits a case where they are, which is useful to us.

Definition 44. *For any local move $Y \hookrightarrow M \hookleftarrow X$, let the past $\text{past}(M) = M - Y$ of M be the set of its elements not in the image of Y . For any such M , presheaf U , and core $\mu \in U(M)$, let $\text{past}(\mu) = \text{Im}(\text{past}(M))$ consist of all images of $\text{past}(M)$.*

To explain the statement a bit more, by Yoneda, we see μ as a map $M \rightarrow U$, so we have a set-function

$$\text{past}(M) \hookrightarrow \text{el}(M) \rightarrow \text{el}(U).$$

Observe that $\text{past}(\mu)$ is always a set of players and moves only, since channels present in X always are in Y too.

Given a core $\mu \in U$, an important operation for us will consist of considering

$$U \setminus \mu = \bigcup \{V \hookrightarrow U \mid \text{el}(V) \cap \text{past}(\mu) = \emptyset\}.$$

$U \setminus \mu$ is thus the largest subpresheaf of U not containing any element of the past of μ . The good property of this operation is:

Proposition 30. *If μ is a maximal core in G_U (i.e., there is no path to any further core) and G_U is target-linear, then $U \setminus \mu = U - \text{past}(\mu)$, i.e., $(U \setminus \mu)(c) = U(c) \setminus \text{past}(\mu)$ for all c .*

Proof. The direction $(U \setminus \mu)(c) \subseteq U(c) \setminus \text{past}(\mu)$ is by definition of \setminus . Conversely, it is enough to show that $c \mapsto U(c) \setminus \text{past}(\mu)$ forms a subpresheaf of U , i.e., that for any $f: c \rightarrow c'$ in \mathbb{C} , and $x \in U(c') \setminus \text{past}(\mu)$, $x \cdot f \notin \text{past}(\mu)$. Assume on the contrary that $x' = x \cdot f \in \text{past}(\mu)$. Then, of course f cannot be the identity. Furthermore, x' is either a player or a move; so, up to pre-composition of f with a further morphism, we may assume that x' is a player. But then, since f is non-identity, x must be a move, with x' being one of its sources or targets. Now, up to post-composition of f with a further morphism, we may assume that x is a core. So, there is either an edge $x \rightarrow x'$ or an edge $x' \rightarrow x$ in G_U . However, $x \neq \mu$, so $x \rightarrow x'$ is impossible by target-linearity of G_U , and $x' \rightarrow x$ is impossible by maximality of μ . \square

Proof of Theorem 5. We proceed by induction on the number of moves in U . If it is zero, then U is a position; by ((ii)), t is an iso, and by ((iii)) so is s , hence the

cospan is a play. For the induction step, we first decompose U into

$$Y \xrightarrow{s_2} U' \xleftarrow{t_2} Z \xrightarrow{s_1} M' \xleftarrow{t_1} X,$$

and then show that M' is a global move and U' satisfies the conditions of the theorem.

So, first, pick a maximal core μ in G_U , i.e., one with no path to any other core. Let

$$\begin{array}{ccccc} & & I_0 & & \\ & \swarrow & \downarrow & \searrow & \\ Y_0 & \longrightarrow & M_0 & \longleftarrow & X_0 \end{array}$$

be the local move with interface corresponding to μ , so we have the Yoneda morphism $\mu: M_0 \rightarrow U$.

Let $U' = (U \setminus \mu)$, and $X_1 = X - \text{Pl}(X_0)$. X_1 is a subpresheaf of X , since it contains all names. The square

$$\begin{array}{ccc} I_0 & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

is a pushout, since it just adds the missing players to X_1 . Define now Z , M' , s_1 , and t_1 by the pushouts

$$\begin{array}{ccccccc} & & Y_0 & \longrightarrow & Z & & \\ & & \downarrow & & \downarrow & \swarrow & \\ & & M_0 & \longrightarrow & M' & \longleftarrow & U \\ I_0 & \longrightarrow & \downarrow & \xrightarrow{C} & X_1 & \longrightarrow & \downarrow \\ & & X_0 & \longrightarrow & X & \xrightarrow{t_1} & \\ & & & & & \downarrow & \\ & & & & & s_1 & \\ & & & & & \downarrow & \\ & & & & & X & \end{array}$$

and the induced arrows. We further obtain arrows to U by universal property of pushout, which are monic because $X \hookrightarrow U$ is, using ((i)). We observe that $U = M' \cup U'$, i.e., the square

$$\begin{array}{ccc} Z & \hookrightarrow & U' \\ \downarrow & & \downarrow \\ M' & \hookrightarrow & U \end{array}$$

is a pushout, so U is indeed a composite as claimed, with $Z \hookrightarrow M' \hookrightarrow X$ a global move by construction. So, it remains to prove that $Y \hookrightarrow U' \hookrightarrow Z$ satisfies the conditions. First, as a subpresheaf of U , U' is locally 1-injective and has a linear and acyclic causal graph, so satisfies ((i)) and ((iv)). U' furthermore satisfies ((ii)) by construction of Z and source-linearity of G_U , and ((iii)) because removing $\text{past}(\mu)$ cannot make any non-final player final. \square

8.3. **CCS as a pre-playground.** We now start proving:

Theorem 6. \mathbb{D} forms a playground.

Axioms (P2)–(P4) are easy, as well as (P6), (P9) and (P10). Furthermore, once (P1) is clear, (P5) is also easy. This leaves (P1) and the decomposition axioms.

For (P1), i.e., the fact that $\text{cod}: \mathbb{D}_H \rightarrow \mathbb{D}_h$ is a fibration, we introduce the notion of ‘history’ for plays. For a presheaf $U \in \widehat{\mathcal{C}}^f$, let $\downarrow U_j$ be its restriction to dimension 3, i.e., U emptied over $\tau_{n,i,m,j}$ ’s, and let $\text{El}(U) = \sum_{c \in \text{ob}(\mathcal{C})} \downarrow U_j(c)$ be the set of elements of $\downarrow U_j$. We have a category $\text{El}(\widehat{\mathcal{C}}^f)$, whose objects are those of $\widehat{\mathcal{C}}^f$, and whose morphisms $U \rightarrow U'$ are set-functions $\text{El}(U) \rightarrow \text{El}(U')$. We denote such morphisms with special arrows $U \twoheadrightarrow U'$. There is a forgetful functor $\text{El}: \widehat{\mathcal{C}}^f \rightarrow \text{El}(\widehat{\mathcal{C}}^f)$, which we implicitly use in casting arrows $U \rightarrow U'$ to arrows $U \twoheadrightarrow U'$.

Definition 45. Consider any local move $X \hookrightarrow M \leftarrow Y$ which is not a synchronisation, where Y is the initial position and X is the final position. Then Y is a representable position, say $[n]$, and we let the history of M be the map $p_M: \text{El}(M) \rightarrow \text{El}(Y)$ sending

- all channels in $\text{El}(M) \cap \text{El}(Y)$ to themselves,
- all other elements to $\text{id}_{[n]}$.

The history $p_{M'}$ of a global move M' is the map obtained by pushout of the history of its generating local move M , as in

$$\begin{array}{ccccc}
 & & M & \xrightarrow{\quad} & M' \\
 & \nearrow & \downarrow & \xrightarrow{C} & \nearrow \\
 I & \xrightarrow{\quad} & Z & & \\
 & \searrow & \downarrow & \xrightarrow{\quad} & \searrow \\
 & & Y & \xrightarrow{\quad} & Y'
 \end{array}$$

This defines the history of global moves. We have:

Proposition 31. For any global move $X \hookrightarrow M \leftarrow Y$, we have $p_M \circ t = \text{id}$.

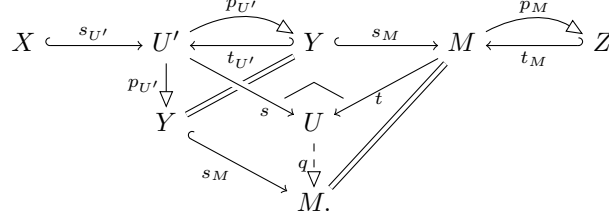
We graphically represent histories by arrows between the presheaves, as p in

$$(10) \quad X \xleftarrow{s} U \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{t} \end{array} Y.$$

We now define the history of sequences of global moves, which we here call *sequential plays*. We denote such a sequence $X_n \xrightarrow{M_n} X_{n-1} \dots X_1 \xrightarrow{M_1} X_0$ by (M_n, \dots, M_1) .

Definition 46. Define now the history of a sequential play $X \rightarrow (M_n, \dots, M_1) \leftarrow Y$, letting $U = M_1 \bullet \dots \bullet M_n$ be the corresponding play, to be the map $U \twoheadrightarrow Y$ defined by induction on n as follows:

- if $|U| = 0$, then t is an isomorphism, and the history is the inverse of the corresponding bijection on elements;
- if $|U| = 1$, then U is a global move M and its history is that of M ;
- if $|U| > 1$, then $U = (U', M)$ for some global move M and sequential play U' ; letting $p_{U'}$ be the history of U' obtained by induction hypothesis, we let $p_U = p_M \circ q$, where q is defined by universal property of pushout in



Proposition 32. *For any sequential plays $U_1, U_2: X \rightarrow Y$ with isomorphic compositions, we have $p_{U_1} = p_{U_2}$.*

Proof. For any presheaf U such that G_U is source-linear and acyclic, consider the function $h_U: \text{El}(U) \rightarrow \text{El}(U)$ mapping

- initial players and channels to themselves,
- non-initial players and channels to the (unique by source-linearity of G_U) core that created them,
- elements of dimension 2 to their image under t ,
- elements of higher dimensions to the image of one of their images in dimension 2 (which all map to the same element by a simple case analysis).

Observe that this map is ultimately idempotent because it is strictly increasing w.r.t. G_U , and let H_U be the corresponding idempotent function.

It is easy to see that if $X \hookrightarrow U \hookrightarrow Y$ is a global move, then $\text{Im}(H_U) = Y$ and $p_U = H_U$.

Furthermore, for all composable plays $X \xrightarrow{U'} Y \xrightarrow{U} Z$, we have $H_{U \bullet U'} = H_U \circ H_{U'}^U$, where $H_{U'}^U: \text{El}(U \bullet U') \rightarrow \text{El}(U)$ is the extension of $H_{U'}$ to $\text{El}(U \bullet U')$ which is the identity on $\text{El}(U) \setminus \text{El}(U')$. Because $\text{Im}(H_{U'}) = Y$, this indeed goes to $\text{El}(U)$.

When U is a global move, this is actually equivalent to the diagrammatic definition of $p_{M \bullet U'}$, which entails by induction that for any play U , $p_U = H_U$, which does not depend on the decomposition of U into moves. \square

Just as for moves, the target map is a section of the history:

Proposition 33. *For any play $X \hookrightarrow U \xleftarrow{t} Y$, we have $p_U \circ t = \text{id}_Y$.*

Proposition 34. *Any double cell (h, k, l) as above is compatible with histories $p: U \twoheadrightarrow Y$ and $p': U' \twoheadrightarrow Y'$, in the sense that the square*

$$\begin{array}{ccc} U & \xrightarrow{k} & V \\ p \downarrow & & \downarrow p' \\ Y & \xrightarrow{l} & Y' \end{array}$$

commutes.

The important point for us is:

Proposition 35. *The vertical codomain functor $\text{cod}: \mathbb{D}_H \rightarrow \mathbb{D}_h$ is a fibration.*

Proof. We first consider the restriction of cod to the full subcategory of \mathbb{D}_H consisting of moves and isomorphisms. Given a move $X \xrightarrow{s} M \xleftarrow{t} Y$ and a morphism $l: Y' \rightarrow Y$ in \mathbb{D}_h , consider the pullback (in sets) and the induced arrow t' :

$$\begin{array}{ccccc}
 Y' & \xrightarrow{l} & Y & & \\
 & \searrow^{t'} & & \searrow^t & \\
 & & U_0 & \xrightarrow{k_0} & M \\
 & & \downarrow p' & \lrcorner & \downarrow p \\
 & & Y' & \xrightarrow{l} & Y.
 \end{array}$$

Now, consider U_0 as a presheaf over \mathbb{C}_3 by giving each element the type of its image under k_0 , and checking that U_0 , viewed as an $\text{ob}(\mathbb{C}_3)$ -indexed family of subsets of M , is stable under the action of morphisms in \mathbb{C}_3 . This, in passing, equips k_0 and t' with the structure of maps in $\widehat{\mathbb{C}}^f$.

Furthermore, let the (n, i, m, j) -horn (see, e.g., Joyal and Tierney [28] for the origin of our terminology) $\tau_{n,i,m,j}^-$ be the representable presheaf on $\tau_{n,i,m,j}$, minus the element $id_{\tau_{n,i,m,j}}$, and consider the family A of commuting squares

$$\begin{array}{ccc}
 \tau_{n,i,m,j}^- & \xrightarrow{w} & U_0 \\
 i \downarrow & & \downarrow k_0 \\
 \tau_{n,i,m,j} & \xrightarrow{w'} & M,
 \end{array}$$

where i is the inclusion. Define then U and k by pushout as in

$$\begin{array}{ccc}
 \sum_{a \in A} \tau_{n_a, i_a, m_a, j_a}^- & \xrightarrow{[w_a]_{a \in A}} & U_0 \\
 \sum_{a \in A} i_a \downarrow & & \downarrow k_0 \\
 \sum_{a \in A} \tau_{n_a, i_a, m_a, j_a} & \xrightarrow{[w'_a]_{a \in A}} & M. \\
 & \searrow^{[w'_a]_{a \in A}} & \\
 & & U \xrightarrow{k} M.
 \end{array}$$

Informally, U is U_0 , where we add all the $\tau_{n,i,m,j}$'s that exist in M and whose horn is in U_0 . We have by construction $\text{El}(U) = \text{El}(U_0)$, so p' is indeed a left inverse to $t': \text{El}(Y') \rightarrow \text{El}(U)$.

Finally, define X' , h , and s' by the pullback

$$\begin{array}{ccc}
 X' & \xrightarrow{h} & X \\
 s' \lrcorner & & \lrcorner s \\
 U & \xrightarrow{k} & M.
 \end{array}$$

This altogether yields a vertical morphism

$$X' \xrightarrow{s'} U \begin{array}{c} \xrightarrow{p'} \\ \xrightarrow{t'} \end{array} Y',$$

in \mathbb{D}_v^0 . A tedious case analysis (made less tedious by $l: Y' \hookrightarrow Y$ being monic) shows that, because M is a move, U is either a move or an isomorphism. So it is in \mathbb{D}_v . U is our candidate cartesian lifting of M along l . More generally, for any play $X \xrightarrow{s} U \xrightarrow{t} Y$, choose a decomposition into moves. We obtain a candidate

cartesian lifting $X' \xrightarrow{s'} U' \xleftarrow{t'} Y'$ for U , with morphism (h, k, l) to U , along any $l: Y' \hookrightarrow Y$ by taking the successive candidates for each move in the obvious way, and composing them.

To show that this indeed yields a cartesian lifting, consider any vertical morphism $X'' \xrightarrow{s''} U'' \xleftarrow{t''} Y''$ and diagram

$$\begin{array}{ccc} X'' & \xrightarrow{h''} & X \\ s'' \downarrow & & \downarrow s \\ U'' & \xrightarrow{k''} & U \\ t'' \uparrow & & \uparrow t \\ Y'' & \xrightarrow{l''} & Y, \end{array}$$

together with a map $l': Y'' \rightarrow Y'$ such that $l \circ l' = l''$. By Proposition 34, letting p'' be the history of U'' , the diagram

$$\begin{array}{ccc} U'' & \xrightarrow{k''} & U \\ p'' \downarrow & & \downarrow p \\ Y'' & \xrightarrow{l''} & Y \end{array}$$

commutes, so by universal property of pullback, we obtain a map $k'_0: \text{El}(U'') \rightarrow \text{El}(U')$, such that $k_0 \circ k'_0 = k''_0$, where k''_0 is the restriction of k'' to dimensions < 4 . Furthermore, the expected map $k': U'' \rightarrow U'$, is given by universal property of pushout in

$$\begin{array}{ccccc} & & \sum_{a \in A} \tau_{n_a, i_a, m_a, j_a}^- & \xrightarrow{\quad} & U'_0 \\ & \nearrow & \downarrow & & \downarrow \\ \sum_{b \in B} \tau_{n_b, i_b, m_b, j_b}^- & \xrightarrow{\quad} & U''_0 & \xrightarrow{\quad} & U'_0 \\ & \searrow & \downarrow & & \downarrow \\ & & \sum_{a \in A} \tau_{n_a, i_a, m_a, j_a} & \xrightarrow{\quad} & U' \\ & \nearrow & \downarrow & & \downarrow \\ \sum_{b \in B} \tau_{n_b, i_b, m_b, j_b} & \xrightarrow{\quad} & U'' & \xrightarrow{\quad} & U' \\ & & \downarrow & \nearrow & \\ & & & & k' \end{array}$$

where B is the family of all commuting squares

$$\begin{array}{ccc} \tau_{n, i, m, j}^- & \xrightarrow{w} & U''_0 \\ i \downarrow & & \downarrow k_0 \\ \tau_{n, i, m, j} & \xrightarrow{w'} & U'' \end{array}$$

Finally, the desired map $h': X'' \rightarrow X'$ follows from universal property of X' as a pullback, and the square

$$\begin{array}{ccc} U'' & \xrightarrow{k'} & U' \\ t'' \uparrow & & \uparrow t' \\ Y'' & \xrightarrow{l'} & Y' \end{array}$$

commutes by uniqueness in the universal property of U' as a pullback. \square

8.4. Towards CCS as a playground. In this section, we prove an intermediate result for proving the decomposition axioms.

Consider a double cell α of the shape

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ w \downarrow & \nearrow \alpha & \downarrow u \\ B & & Y \\ v \downarrow & & \downarrow k \\ C & \xrightarrow{k} & Y \end{array}$$

where v is a view. Let now D_α denote the category with

- objects all tuples $T = (Z, l, u_1, u_2, \alpha_1, \alpha_2, \alpha_3)$ such that

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ w \downarrow & \nearrow \alpha_2 & \downarrow u_2 \\ B & \xrightarrow{l} & Z \\ v \downarrow & \nearrow \alpha_1 & \downarrow u_1 \\ C & \xrightarrow{k} & Y \end{array} \quad \begin{array}{c} \left. \begin{array}{c} \alpha_3 \\ u \end{array} \right\} \\ \left. \begin{array}{c} \alpha_1 \\ u_1 \end{array} \right\} \end{array}$$

equals α and α_3 is an isomorphism;

- with morphisms $T \rightarrow T'$ given by tuples $(U, f, \beta, \gamma, \delta)$ (where f is vertical) such that

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ w \downarrow & & \downarrow u \\ B & \xrightarrow{l} & Z \\ v \downarrow & & \downarrow k \\ C & \xrightarrow{k} & Y \end{array} \quad \begin{array}{c} \left. \begin{array}{c} \alpha_2 \\ u_2 \\ \alpha_3 \end{array} \right\} \\ \left. \begin{array}{c} \alpha_1 \\ u_1 \end{array} \right\} \end{array} \quad \begin{array}{c} \left. \begin{array}{c} \delta \\ \beta \\ \gamma \end{array} \right\} \\ \left. \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \right\} \end{array}$$

commutes, i.e., $\gamma \circ (\alpha_1 \bullet \delta) = \alpha'_1$, $\beta \circ \alpha_2 = \delta \bullet \alpha'_2$, and $\alpha'_3 \circ (\gamma \bullet u'_2) \circ (u_1 \bullet \beta) = \alpha_3$, and β and γ are isomorphisms;

- composition and identities are obvious.

So, objects of D_α are decompositions of u permitting corresponding decompositions of α . The rest of this section is a proof of:

Lemma 16. D_α has a weak initial object, i.e., an object T such that for any object T' there is a morphism $T \rightarrow T'$.

We start by some properties of \mathbb{D} .

Definition 47. A filiform play is any play U such that the restriction of G_U to cores and players is a filiform graph, i.e., a graph of the shape $\cdot \rightarrow \cdot \rightarrow \dots$

Typically, any view is filiform.

Lemma 17. *Any epimorphic (in \mathbb{D}_H , hence isomorphic) double cell*

$$(11) \quad \begin{array}{ccc} A & \xrightarrow{h} & X \\ w \downarrow & \nearrow \alpha & \downarrow u \\ B & & Y \\ v \downarrow & & \downarrow k \\ C & \xrightarrow{k} & Y \end{array}$$

where v is filiform decomposes as

$$\begin{array}{ccccc} A & \xrightarrow{h} & X & & \\ w \downarrow & \nearrow \alpha_2 & \downarrow u_2 & \nearrow \alpha_3 & \\ B & \xrightarrow{\quad} & Z & & \\ v \downarrow & \nearrow \alpha_1 & \downarrow u_1 & & \\ C & \xrightarrow{k} & Y & & \end{array} \quad \begin{array}{c} \uparrow \\ u \\ \downarrow \end{array}$$

with α_3 an isomorphism, α_1 and α_2 epimorphic, uniquely up to isomorphism. In this case, u_1 is filiform.

Proof. B has just one player, say b . Let $b' = \alpha(b)$. Because α is epi, α induces a morphism $G_\alpha: G_{v \bullet w} \rightarrow G_u$ of graphs, which is also epi. So, G_u may be decomposed as a pushout

$$\begin{array}{ccc} b' & \hookrightarrow & G_1 \\ \downarrow & \lrcorner & \downarrow \\ G_2 & \hookrightarrow & G_u \end{array}$$

with $G_1 = \text{Im}_{G_\alpha}(G_v)$ and $G_2 = \text{Im}_{G_\alpha}(G_w)$. From this one deduces a decomposition of u and α . \square

Lemma 18. *For any sensible α and β , if $\alpha \bullet \beta$ is epi, then so are α and β .*

Proof. Easy. \square

Lemma 19. *For any α , D_α has a weak initial object.*

Proof. The double cell α induces morphisms of graphs $G_v \rightarrow G_u \leftarrow G_w$, by Proposition 27. Let

$$u_1 = \bigcap \{u' \subseteq u \mid (Y \subseteq u') \wedge (\text{Im}_\alpha(G_v) \subseteq G_{u'})\}.$$

Thus, $v \rightarrow u$ factors as $v \rightarrow u_1 \rightarrow u$. Let Z be the position containing all channels of u_1 , and all final players of u_1 . Further let $\uparrow Z$ denote the full subgraph of G_u containing all vertices x with a path to some vertex of Z . Let then

$$u_2 = \bigcap \{u'' \subseteq u \mid G_{u''} \supseteq \uparrow Z\}.$$

The union $u_1 \cup u_2$ is u , i.e., the square

$$\begin{array}{ccc}
Z & \longrightarrow & u_1 \\
\downarrow & \lrcorner & \downarrow \\
u_2 & \longrightarrow & u
\end{array}$$

is a pushout, i.e., $u_2 \bullet u_1 \cong u$ in $\text{Cospan}(\widehat{\mathbb{C}}^f)$. So it only remains to prove that $Z \rightarrow u_1 \leftarrow X$ and $Y \rightarrow u_2 \leftarrow Z$ are plays, for which we use Theorem 5. First, u_1 and u_2 , as subpresheaves of u , both are locally 1-injective. Furthermore, G_{u_1} and G_{u_2} , as subgraphs of a linear and acyclic graph, are also linear and acyclic. Now, by definition of Z , $Z \rightarrow u_1$ contains all channels and the final players of u_1 . Further, since $X \subseteq u_1$, being initial in u implies being initial in u_1 , so $Z \rightarrow u_1 \leftarrow X$ indeed is a play. Symmetrically, no player of u_1 not in Z is final, so $Y \subseteq u_2$, and hence $Y \rightarrow u_2$ indeed contains all channels and final players. Finally, the players and channels of Z are precisely the initial players and channels of u_2 .

It remains to show that the induced decomposition of α is weakly initial. But any decomposition, inducing a decomposition $u'_1 \bullet u'_2$ of u , should satisfy $Y \subseteq u'_1$, $\text{Im}_\alpha(G_v) \subseteq G_{u'_1}$, and $G_{u'_2} \subseteq \uparrow Z$, so, ignoring isomorphisms for readability, $u_1 \subseteq u'_1$ and $u'_2 \subseteq u_2$, as desired. \square

This ends the proof of Lemma 16.

8.5. CCS as a playground. We are now ready to prove the decomposition axioms, which entail Theorem 8.3. They are proved in Lemmas 22 and 21 below.

Let us start with the following easy lemma.

Lemma 20. *If $u = u_2 \bullet u_1$, then, in G_u*

- *no player of u_1 is reachable from any core of u_2 ;*
- *no core of u_1 is reachable from any element of u_2 .*

Proof. For the first point, cores of u_2 only reach initial channels of u_1 .

For the second point, we further observe that channel and players of u_2 only reach initial players and channels of u_1 , hence no core. \square

The easiest decomposition axiom is (P8).

Lemma 21. \mathbb{D} *satisfies (P8).*

Proof. Although the statement is complicated, this is rather easy: α restricts to a map of presheaves $f: b \rightarrow (M \bullet u)$, on which we proceed by case analysis.

If $\text{Im}(f) \subseteq M$, then by Lemma 16, then by correctness we are in the left-hand case. Otherwise, assume that a move $\mu' \in M$ is in the image of α , say of a move $\mu \in w$. We have a path $\mu \rightarrow b$ in $b \bullet w$, hence a path $\text{core}(\mu') \rightarrow \alpha(b)$ in $M \bullet u$, contradicting Lemma 20. \square

Let us now attack the last axiom.

Lemma 22. \mathbb{D} *satisfies (P7).*

We need a few lemmas.

Lemma 23. *For any plays $A \xrightarrow{u_1} B \xrightarrow{u_2} C$, for any player or channel $x \in u_2$ and core $\mu \in u_1$, there is no edge $x \rightarrow \mu$ in $u_2 \bullet u_1$.*

Proof. The existence of $e: x \rightarrow \mu$ implies $x \in B$, hence x initial in u_1 , which contradicts the very existence of e . \square

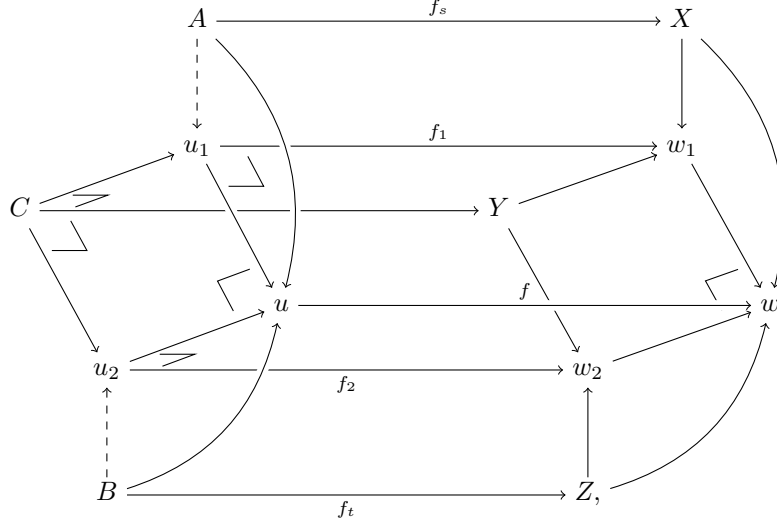


FIGURE 1. Proof of Lemma 22

Lemma 24. *Morphisms of plays preserve finality.*

Proof. If a player is final in the domain, then it is in the final position, hence has an image in the final position of the codomain, hence is final there. \square

Lemma 25. *For any map $\alpha: u \rightarrow w$ in \mathbb{D}_H , for any player x in u and edge $e': \mu' \rightarrow \alpha(x)$ from a core in w , there exists a core $\mu \in u$ and an edge $e: \mu \rightarrow x$ in u such that $G_\alpha(e) = e'$.*

Proof. Let first $X \rightarrow u \leftarrow Y$ and $X' \rightarrow w \leftarrow Y'$ be the considered morphisms.

Then, observe that x is not final in u , for otherwise it would be in X , hence $\alpha(x)$ would be in X' and final, contradicting the existence of e' .

So there exists $e: \mu \rightarrow x$ in u . But now, by target-linearity, $G_\alpha(\mu) = \mu'$, which entails the result. \square

Lemma 26. *In any double cell (9), both squares are pullbacks.*

Proof. X must consist precisely of all final players and channels of G_U , which must also be final in G_V , so finality in G_U implies finality in G_V . Conversely, any player or channel mapped to a final one in G_V has to be final. So X is a pullback of U and X' . The lower square being a pullback follows from similar reasoning. \square

Proof of Lemma 22. Consider any α , and construct C, u_1, u_2 , and the morphisms in Figure 1, as follows. First, let u_1 be the pullback $u \times_w w_1$, and then $C = u_1 \times_{w_1} Y$. Let then $u_2 = u \times_w w_2$, and the arrow $C \rightarrow u_2$ be induced by universal property of pullback. By the pullback lemma, $C = u_2 \times_{w_2} Y$. Because presheaf categories are adhesive [31], \widehat{C}^f is, and, $Y \rightarrow w_1$ being monic, we have a Van Kampen square. Thus, by the main axiom for adhesive categories, u is a pushout $u_1 +_C u_2$, i.e., $u \cong u_2 \bullet u_1$ in $\text{Cospan}(\widehat{C}^f)$. Letting α_i be the arrow $u_i \rightarrow w_i$, for $i = 1, 2$, this yields the desired decomposition of α .

We still need to show that $A \rightarrow u_1 \leftarrow C$ and $C \rightarrow u_2 \leftarrow B$ are plays, and that the obtained decomposition is unique. Uniqueness follows from adhesivity of \widehat{C} and Lemma 26. Indeed, any decomposition looks like Figure 1, except that u_1 , u_2 , and C are not *a priori* obtained by pullback. But by Lemma 26, both back faces have to be pullbacks, hence so are the front faces by adhesivity.

Let us finally show that u_1 and u_2 are plays. It is easy to see that non-linearity or non-acyclicity of G_{u_1} (resp. G_{u_2}) would entail non-linearity or non-acyclicity of u or w_1 (resp. or w_2). Local 1-injectivity is also easy.

Let us now prove the missing conditions for $A \rightarrow u_1 \leftarrow C$.

a) Any player x of u_1 in the image of A is final, for otherwise its image in w_1 would be in the image of X and non-final.

b) Conversely, if a player $x \in u_1$ is final but not in A , then its image in u must be non-final by Theorem 5, because $u_1 \rightarrow u$ is monic. But then there is a core μ of u_2 with a path $\mu \rightarrow x$ in G_u , whose images in w yield a path from a core of w_2 to a player of w_1 , contradicting Lemma 20. So A contains precisely the final players of u_1 .

c) Now, if a channel $x \in u_1$ is not in A , then its image in u must be in A , hence $u_1 \rightarrow u$ cannot be mono, so neither can $w_1 \rightarrow w$, so neither can $Y \rightarrow w_2$, contradiction.

d) Finally, by construction, C contains precisely the initial players and channels of u_1 .

Now, for $C \rightarrow u_2 \leftarrow B$.

a) By universal property of pullback, C contains all channels of u_2 .

b) For players, clearly, for any player x in C , x is final in u_2 . Indeed, otherwise, there would be a path $\mu \rightarrow x$ from a core μ in u_2 , yielding a path $f_2(\mu) \rightarrow f_2(x)$ in w_2 . But since x is in C , $f_2(x) \in Y$, which hence contains a non-final player, contradiction.

c) Conversely, if x is final in u_2 , then $x' = f_2(x)$ is final in w_2 . Indeed, otherwise, there would be an edge $\mu' \rightarrow x'$ from a core in w_2 , so, by Lemma 25, an edge $\mu \rightarrow x$ in u with $f(\mu) = \mu'$. But then, $\mu \in u_2$, so x cannot be final. This shows that x' is final in w_2 . But then $x' \in Y$, so, because $C = u_2 \times_w Y$, $x \in C$.

d) Consider now any player or channel x initial in u_2 . First, x is also initial in u : otherwise, there would be an edge $x \rightarrow \mu$ to a core in u , with $\mu \in u_1$, hence an edge $f(x) \rightarrow f(\mu)$ in w from a channel of w_2 to a core of w_1 , which is impossible by Lemma 23. So x is initial in u , hence $x \in B$.

e) Now, for any player or channel $x \in B$, x is initial in u , hence x is *a fortiori* initial in u_2 . \square

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