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# A MODEL SPACE APPROACH TO SOME CLASSICAL INEQUALITIES FOR RATIONAL FUNCTIONS

ANTON BARANOV AND RACHID ZAROUF

ABSTRACT. We consider the set  $\mathcal{R}_n$  of rational functions of degree at most  $n \geq 1$  with no poles on the unit circle  $\mathbb{T}$  and its subclass  $\mathcal{R}_{n,r}$  consisting of rational functions without poles in the annulus  $\{\xi : r \leq |\xi| \leq \frac{1}{r}\}$ . We discuss an approach based on the model space theory which brings some integral representations for functions in  $\mathcal{R}_n$  and their derivatives. Using this approach we obtain  $L^p$ -analogs of several classical inequalities for rational functions including the inequalities by P. Borwein and T. Erdélyi, the Spijker Lemma and S.M. Nikolskii's inequalities. These inequalities are shown to be asymptotically sharp as  $n$  tends to infinity and the poles of the rational functions approach the unit circle  $\mathbb{T}$ .

## 1. INTRODUCTION

The goal of this paper is to give a unified approach to several classical inequalities for rational functions. This approach is based on integral representations for rational functions and their derivatives. It makes possible to recover several known results and obtain their  $L^p$  analogs where the estimate is given not only in terms of the degree of a rational function but also in terms of the distance from the poles to the boundary.

**1.1. Notations.** Let  $\mathcal{P}_n$  be the space of complex analytic polynomials of degree at most  $n \geq 1$  and let

$$\mathcal{R}_n = \left\{ \frac{P}{Q} : P, Q \in \mathcal{P}_n, Q(\xi) \neq 0 \text{ for } |\xi| = 1 \right\},$$

be the set of rational functions of degree at most  $n$  (where  $\deg \frac{P}{Q} = \max(\deg P, \deg Q)$ ) without poles on the unit circle  $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ . We denote by  $\|f\|_{L^p}$ ,  $1 \leq p \leq \infty$ , the standard norms of the spaces  $L^p(\mathbb{T}, m)$ , where  $m$  stands for the normalized Lebesgue measure on  $\mathbb{T}$ . Denote by  $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$  the unit disc of the complex plane and by  $\overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$  its closure. For a given  $r \in (0, 1)$ , we finally introduce the subset

$$\mathcal{R}_{n,r} = \left\{ \frac{P}{Q} : P, Q \in \mathcal{P}_n, Q(\xi) \neq 0 \text{ for } r \leq |\xi| \leq \frac{1}{r} \right\}$$

of  $\mathcal{R}_n$ , consisting of rational functions of degree at most  $n$  without poles in the annulus  $\{\xi : r \leq |\xi| \leq \frac{1}{r}\}$ .

We also introduce some notations specific to the theory of model subspaces of the Hardy space  $H^p$ ,  $1 \leq p \leq \infty$ . Denote by  $\text{Hol}(\mathbb{D})$  the space of all holomorphic functions on  $\mathbb{D}$ . The Hardy space  $H^p = H^p(\mathbb{D})$ ,  $1 \leq p < \infty$ , is defined as follows:

$$H^p = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{H^p}^p = \sup_{0 \leq \rho < 1} \int_{\mathbb{T}} |f(\rho\xi)|^p dm(\xi) < \infty \right\}.$$

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As usual, we denote by  $H^\infty$  the space of all bounded analytic functions in  $\mathbb{D}$ . For any  $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ , we consider the finite Blaschke product

$$B_\sigma = \prod_{k=1}^n b_{\lambda_k}, \quad b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z},$$

$b_\lambda$  being the elementary Blaschke factor associated to  $\lambda \in \mathbb{D}$ . Define the model subspace  $K_{B_\sigma}$  of the Hardy space  $H^2$  by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

The subspace  $K_{B_\sigma}$  consists of rational functions of the form  $P/Q$ , where  $P \in \mathcal{P}_{n-1}$  and  $Q$  is a polynomial of degree  $n$  with the zeros  $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$  of corresponding multiplicities.

For any Blaschke product  $B$ , the reproducing kernel of the model space  $K_B$  corresponding to a point  $\xi \in \mathbb{D}$  is of the form

$$k_\xi^B(z) = \frac{1 - \overline{B(\xi)}B(z)}{1 - \bar{\xi}z}.$$

**1.2. Some classical inequalities for rational functions.** In this subsection we give a brief review of several well-known inequalities for polynomials and rational functions.

1.2.1. *Pointwise estimates for the derivatives of the functions in  $\mathcal{R}_n$ .* Let us start with the following theorem.

**Theorem.** *For any function  $f \in \mathcal{R}_n$  with the poles  $\{a_k\}$  (counting multiplicities) we have*

$$(1.1) \quad |f'(\xi)| \leq \|f\|_{L^\infty} \left( \sum_{|a_k| < 1} \frac{1 - |a_k|^2}{|a_k - \xi|^2} + \sum_{|a_k| > 1} \frac{|a_k|^2 - 1}{|a_k - \xi|^2} \right), \quad |\xi| = 1.$$

Inequality (1.1) has a long history. It was for the first time explicitly stated and proved (by two different methods) for the case when all poles are outside  $\mathbb{D}$  in a monograph by V.N. Rusak [R, Chapter III, Section 1] in 1979. Also, as Rusak mentions, this inequality is contained (only with a hint of a proof) in the book of V.I. Smirnov and N.A. Lebedev [SL, Chapter V, Section 3, Corollary 3].

At the same time this inequality (for poles both inside  $\mathbb{D}$  and outside  $\overline{\mathbb{D}}$ ) is a very special case of results of M.B. Levin [Le1, Le2] which were obtained already in 1974–1975, but remained unnoticed; these results apply to arbitrary functions admitting pseudocontinuation.

Further extensions of Levin–Rusak inequality were obtained in the 1990s independently by two groups of specialists in polynomial inequalities [BE1, LMR]. In particular, in [BE2] (see also [BE1, Theorem 7.1.7]) P. Borwein and T. Erdélyi obtained the following interesting improvement which shows that the sum in (1.1) may be replaced by the maximum.

**Theorem.** *For any function  $f \in \mathcal{R}_n$  with the poles  $\{a_k\}$  we have*

$$(1.2) \quad |f'(\xi)| \leq \|f\|_{L^\infty} \max \left( \sum_{|a_k| < 1} \frac{1 - |a_k|^2}{|a_k - \xi|^2}, \sum_{|a_k| > 1} \frac{|a_k|^2 - 1}{|a_k - \xi|^2} \right), \quad |\xi| = 1.$$

1.2.2. *Spijker's Lemma.* A well-known result by M.N. Spijker [S] (known as Spijker's Lemma) asserts that the image of  $\mathbb{T}$  under a complex rational map  $f \in \mathcal{R}_n$  has length at most  $2n\pi$ , or, in other words,

$$(1.3) \quad \|f'\|_{L^1} \leq n\|f\|_\infty, \quad f \in \mathcal{R}_n.$$

The inequality is apparently sharp (take  $f(z) = z^n$ ). This result published by Spijker in 1991 ended a long search for the best bound in this inequality (e.g., in 1984 R.J. Leveque and L.N. Trefethen [LT] proved the above inequality with  $2n$  in place of  $n$ ). The importance of the sharp constants in this inequality is related to its role in the *Kreiss Matrix Theorem* [K].

However, it was recently noticed (see [N]) that inequality (1.3) was discovered already in 1978 by E.P. Dolzhenko [D] as a special case of more general results. Unfortunately, this paper (which appeared only in Russian) remained unknown to the specialists. Let us cite the following beautiful theorem from [D] where majorization on the whole circle is replaced by majorization on its subset.

**Theorem.** *If  $E$  is a measurable subset of  $\mathbb{T}$  or of a line in the complex plane and  $f \in \mathcal{R}_n$ ,  $|f(u)| \leq 1$ ,  $u \in E$ , then  $\int_E |f'(u)| |du| \leq 2\pi n$ ; if  $f$  is real valued, then the constant  $2\pi$  can be replaced by 2. The latter estimate is sharp for  $n = 0, 1, 2, \dots$  and for each  $E$  with positive measure.*

We refer to a recent paper by N.K. Nikolski [N] for a detailed account of the history of inequality (1.3) and its relations to the Kreiss Matrix Theorem, as well as for some new developments in the Kreiss theory.

Let us also mention that the Dolzhenko–Spijker inequality (1.3) follows easily from (1.2) (or from (1.1)). This was mentioned by X. Li [Li] (see also [FLM, Remark 7]). Indeed, integrating (1.2) with respect to the normalized Lebesgue measure  $m$ , we obtain (1.3).

1.2.3. *S.M. Nikolskii's inequalities.* By the well-known results of S.M. Nikolskii [SMN] (1951), the essentially sharp inequality

$$(1.4) \quad \|f\|_{L^q} \leq c(p, q) n^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^p}$$

holds for all polynomials  $f$  of degree at most  $n$  and for all  $1 \leq p < q \leq \infty$  with a constant  $c(p, q)$  depending only on  $p$  and  $q$ . Analogous inequalities were proved in [SMN] for trigonometric polynomials of several variables. A few years later (1954) G. Szegö and A. Zygmund [SZ] among other more general results rediscovered (1.4) (for polynomials of one variable only) and extended it to the whole range  $0 < p < q \leq \infty$ .

## 2. MAIN RESULTS

In the present paper we use an approach based on the model space theory to obtain some extensions of inequalities of type (1.2), (1.3), and (1.4) for functions in  $\mathcal{R}_n$  or  $\mathcal{R}_{n,r}$ . The estimates we obtain will depend not only on the degree of a rational function  $n$  but also on the distance  $1 - r$  from the poles to the boundary. We show the sharpness of the obtained inequalities as both  $n \rightarrow \infty$  and  $r \rightarrow 1-$ .

Our main tools are integral representations for rational functions and their derivatives. Integral representations for a derivative of a rational function were introduced and successfully used by R. Jones, X. Li, R.N. Mohapatra and R.S. Rodriguez [JLMR, Lemma 4.3] (for rational functions without poles in  $\overline{\mathbb{D}}$ ) and by X. Li [Li, Lemma 3] in the general case. At the same time such representations are known in the setting of general model spaces where they were also used to produce some estimates for the derivatives [A, B, Dy2].

From now on, for two positive functions  $U$  and  $V$ , we say that  $U$  is dominated by  $V$ , denoted by  $U \lesssim V$ , if there is a constant  $c > 0$  such that  $U \leq cV$ ; we say that  $U$  and  $V$  are comparable, denoted by  $U \asymp V$ , if both  $U \lesssim V$  and  $V \lesssim U$ .

**2.1.  $L^p$ -version of the Borwein–Erdélyi inequality.** We first give an analog of (1.1)–(1.2) in which the  $L^\infty$ -norm at the right is replaced by the  $L^p$ -norm,  $p \geq 1$ .

**Theorem 2.1.** *Let  $1 \leq p \leq \infty$ . Then*

(i) *For any function  $f \in \mathcal{R}_n$  with the set of poles  $a = \{a_k\}$  (repeated counting multiplicities) we have*

$$(2.1) \quad |f'(\xi)| \leq \left( \mathcal{D}_1^{\frac{1}{p}}(a) \sum_{|a_k| < 1} \frac{1 - |a_k|^2}{|a_k - \xi|^2} + \mathcal{D}_2^{\frac{1}{p}}(a) \sum_{|a_k| > 1} \frac{|a_k|^2 - 1}{|a_k - \xi|^2} \right) \|f\|_{L^p}$$

for all  $|\xi| = 1$ , where

$$(2.2) \quad \mathcal{D}_1(a) = \sum_{|a_k| < 1} \frac{1 + |a_k|}{1 - |a_k|}, \quad \mathcal{D}_2(a) = \sum_{|a_k| > 1} \frac{|a_k| + 1}{|a_k| - 1}.$$

(ii) *Moreover, (2.1) is sharp in the following sense: for any  $p \in [1, \infty]$  there exists a constant  $c(p) > 0$  such that for any  $n \geq 2$  and any  $r \in (0, 1)$  there exists  $f \in \mathcal{R}_n$  with the poles  $\{a_k\}$  on the circle  $\{|z| = \frac{1}{r}\}$  such that*

$$\frac{f'(-1)}{\|f\|_{L^p}} \geq c(p) \mathcal{D}_2^{\frac{1}{p}}(a) \sum_{|a_k| > 1} \frac{|a_k|^2 - 1}{|a_k + 1|^2}.$$

Clearly, inequality (1.1) is the limit case of (2.1) when  $p = \infty$ .

**2.2. An  $L^p$ -version of the Dolzhenko–Spijker Lemma.** Next we obtain a version of (1.3) where the  $L^\infty$ -norm at the right is replaced by the  $L^p$ -norm,  $p \geq 1$ . Note that this inequality does not follow from (2.1) by integration on the unit circle  $\mathbb{T}$ , as it was the case for  $p = 1$ .

**Theorem 2.2.** *For every rational function  $f \in \mathcal{R}_n$  having  $n_1$  poles inside  $\mathbb{D}$  and  $n_2$  poles outside of  $\overline{\mathbb{D}}$ , we have*

$$(2.3) \quad \|f'\|_{L^1} \leq \left( n_1^{1-\frac{1}{p}} \mathcal{D}_1^{\frac{1}{p}}(a) + n_2^{1-\frac{1}{p}} \mathcal{D}_2^{\frac{1}{p}}(a) \right) \|f\|_{L^p}$$

where  $a = \{a_k\}$  stand for the poles of  $f$  and  $\mathcal{D}_1(a)$  and  $\mathcal{D}_2(a)$  are defined in (2.2).

This inequality is asymptotically sharp when  $n = n_1 + n_2$  tend to  $\infty$  and  $\text{dist}(\{a_k\}, \mathbb{T}) \rightarrow 0$  (see Theorem 2.3 below).

**2.3.  $L^p - L^q$  Bernstein-type inequality.** Now we consider the following Bernstein-type problem, which could be interpreted as a generalization of (2.1) and (2.3): given  $n \geq 1$ ,  $r \in (0, 1)$  and  $1 \leq p, q \leq \infty$ , let  $\mathcal{C}_{n,r}(L^q, L^p)$  be the best possible constant in the inequality

$$\|f'\|_{L^q} \leq \mathcal{C}_{n,r}(L^q, L^p) \|f\|_{L^p}, \quad f \in \mathcal{R}_{n,r}.$$

One could also introduce  $\mathcal{C}_n(L^q, L^p) = \sup_{r \in (0,1)} \mathcal{C}_{n,r}(L^q, L^p)$ . The Dolzhenko–Spijker Lemma means exactly that  $\mathcal{C}_n(L^1, L^\infty) = n$ . It is however easy to see (take  $f(z) = (1 - rz)^{-1}$  as a test function) that  $\mathcal{C}_n(L^q, L^p) = \infty$  unless  $q = 1$  and  $p = \infty$ . Thus, the dependence on  $r$  (that is, on the distance from the poles to the boundary) appears naturally in the problem.

**Theorem 2.3.** *Let  $n \geq 1$ ,  $r \in (0, 1)$ , and  $1 \leq p, q \leq \infty$ . We have*

$$(2.4) \quad \mathcal{C}_{n,r}(L^q, L^p) \asymp \begin{cases} \left( \frac{n}{1-r} \right)^{1+\frac{1}{p}-\frac{1}{q}}, & q \geq p, \\ \frac{n}{(1-r)^{1+\frac{1}{p}-\frac{1}{q}}}, & q \leq p, \end{cases}$$

with the constants depending only on  $p$  and  $q$ , but not on  $n$  and  $r$ .

Moreover, the constant in upper bound is in both of these two cases  $(1+r)^{1+\frac{1}{p}-\frac{1}{q}}$ : we have

$$(2.5) \quad \mathcal{C}_{n,r}(L^q, L^p) \leq \begin{cases} (1+r)^{1+\frac{1}{p}-\frac{1}{q}} \left(\frac{n}{1-r}\right)^{1+\frac{1}{p}-\frac{1}{q}}, & q \geq p, \\ (1+r)^{1+\frac{1}{p}-\frac{1}{q}} \frac{n}{(1-r)^{1+\frac{1}{p}-\frac{1}{q}}}, & q \leq p. \end{cases}$$

The upper bound for  $\mathcal{C}_{n,r}(L^q, L^p)$  is obviously sharp for the special case  $p = q = \infty$ , since it is reached by the Blaschke product  $b_r^n$ . Moreover, it is proved in [Z] that for any fixed  $r$  in  $(0, 1)$ , there exists a limit

$$\lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n,r}(L^2, L^2)}{n} = \frac{1+r}{1-r},$$

and thus, the bound  $(1+r)^{1+\frac{1}{p}-\frac{1}{q}}$  is again (asymptotically as  $n \rightarrow \infty$ ) sharp for  $p = q = 2$ . It is possible that this constant is sharp in the general case  $1 \leq p, q \leq \infty$ .

In Subsection 4.3 we compare inequality (2.5) with a theorem by K.M. Dyakonov [Dy1, Theorem 11].

**2.4. An extension of S.M. Nikolskii's inequality to rational functions.** Direct analogs (with the dependence on  $n$  only) of (1.4) do not exist for functions in  $\mathcal{R}_n$ . As always, it is easy to check this fact considering the test function  $f(z) = (1-rz)^{-1}$  as  $r$  tends to  $1^-$ . A natural extension of (1.4) for functions in  $\mathcal{R}_{n,r}$  can be stated as follows:

**Theorem 2.4.** *Let  $1 \leq p < q \leq \infty$ ,  $n \geq 1$  and  $r \in (0, 1)$ .*

(i) *We have*

$$(2.6) \quad \|f\|_{L^q} \leq \left(\frac{1+r}{1-r}\right)^{\frac{1}{p}-\frac{1}{q}} \left(n_1^{\frac{1}{p}-\frac{1}{q}} + (n_2+1)^{\frac{1}{p}-\frac{1}{q}}\right) \|f\|_{L^p}, \quad f \in \mathcal{R}_{n,r},$$

where  $n_1$  (respectively,  $n_2$ ) is the number of poles of  $f$  inside  $\mathbb{D}$  (respectively, outside  $\overline{\mathbb{D}}$ ). In particular,

$$(2.7) \quad \|f\|_{L^q} \lesssim \left(\frac{n}{1-r}\right)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p}, \quad f \in \mathcal{R}_{n,r},$$

with a constant depending only on  $p$  and  $q$ .

(ii) *The inequality (2.7) is sharp: for  $1 \leq p < q \leq \infty$  there exists a constant  $c(p, q) > 0$  such that for any  $r \in (0, 1)$  and  $n \geq 2$  there exists  $f \in \mathcal{R}_{n,r}$  with the property*

$$\frac{\|f\|_{L^q}}{\|f\|_{L^p}} \geq c(p, q) \left(\frac{n}{1-r}\right)^{\frac{1}{p}-\frac{1}{q}}.$$

*Remark.* It is possible that for any  $1 \leq p < q \leq \infty$  the upper bound  $(1+r)^{\frac{1}{p}-\frac{1}{q}}$  in (2.6) is asymptotically sharp as  $n$  tends to infinity, for any fixed  $r \in (0, 1)$ . We are able to provide a simple proof of this fact for the special case  $q = \infty$ ,  $2 \leq p < \infty$ . Indeed, using the test function  $f = \frac{1}{1+rz} \sum_{k=0}^{n-1} b_{-r}^k$ ,  $r \in (0, 1)$ , we clearly have  $\|f\|_{L^2}^2 = \frac{n}{1-r^2}$ , since the family  $\{\frac{1}{1+rz} b_{-r}^k\}$  is orthogonal in  $L^2$ . Moreover,  $\|f\|_{L^\infty} = f(-1) = \frac{n}{1-r}$  and thus

$$\|f\|_{L^p}^p \leq \|f\|_{L^\infty}^{p-2} \|f\|_{L^2}^2 = \frac{1}{1+r} \left(\frac{n}{1-r}\right)^{p-1}, \quad 2 \leq p < \infty.$$

As a consequence,

$$\frac{\|f\|_{L^\infty}}{\|f\|_{L^p}} \geq \left(n \frac{1+r}{1-r}\right)^{\frac{1}{p}}, \quad 2 \leq p < \infty,$$

which gives the result since the poles of  $f$  are all outside  $\frac{1}{r}\mathbb{D}$ ,  $n_2 = n = \deg f$ .

## 3. INTEGRAL REPRESENTATIONS FOR RATIONAL FUNCTIONS AND THEIR DERIVATIVES

**3.1. Preliminaries.** In what follows we may assume without loss of generality that the functions  $f = \frac{P}{Q} \in \mathcal{R}_n$  we consider are such that:

- (1)  $\max(\deg P, \deg Q) = n$ , (otherwise  $f \in \mathcal{R}_m$ ,  $m < n$ ),
- (2)  $\deg P \leq \deg Q$ : indeed, if  $n = \deg P > l = \deg Q$  and  $\xi \in \mathbb{T}$ , then we define  $g(\xi) = f(\bar{\xi})$  and supposing that  $P(z) = \sum_{k=0}^n a_k z^k$ ,  $a_n \neq 0$  and  $Q(z) = \sum_{k=0}^l b_k z^k$ ,  $b_l \neq 0$ , we obtain (multiplying by  $\xi^n$ )

$$g(\xi) = \frac{\sum_{k=0}^n a_k \xi^{-k}}{\sum_{k=0}^l b_k \xi^{-k}} = \frac{\sum_{k=0}^n a_k \xi^{n-k}}{\sum_{k=0}^l b_k \xi^{n-k}} = \frac{\sum_{j=0}^n a_{n-j} \xi^j}{\sum_{j=n-l}^n b_{n-j} \xi^j} = \frac{\tilde{P}(\xi)}{\tilde{Q}(\xi)},$$

where  $\tilde{P}(z) = \sum_{j=0}^n a_{n-j} z^j$  and  $\tilde{Q}(z) = \sum_{j=n-l}^n b_{n-j} z^j$  are such that  $\tilde{P}, \tilde{Q} \in \mathcal{P}_n$  and  $\deg \tilde{P} \leq \deg \tilde{Q}$ . Moreover, we clearly have  $|g'(\xi)| = |f'(\bar{\xi})|$  for all  $\xi \in \mathbb{T}$ ,  $\|g\|_{L^p} = \|f\|_{L^p}$ ,  $1 \leq p \leq \infty$ , and  $\|g'\|_{L^q} = \|f'\|_{L^q}$ ,  $1 \leq q \leq \infty$ , since  $g'(\xi) = -\frac{1}{\xi^2} f'(\bar{\xi})$ ,  $\xi \in \mathbb{T}$ , and

- (3) all the poles of  $f$  (i.e., the zeros of  $Q$ ) are pairwise distinct: indeed, we can assume this perturbing slightly the poles of  $f$  and the result will follow by continuity.

From now on, for every function  $f \in \mathcal{R}_n$  we will denote by  $\sigma_1$  and  $\sigma_2$  the sets of poles of  $f$  (repeated counting multiplicities) which are respectively inside  $\mathbb{D}$  or outside  $\overline{\mathbb{D}}$ ,

$$\sigma_1 = \{\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_{n_1}}\}, \quad \sigma_2 = \{1/\overline{\mu_1}, 1/\overline{\mu_2}, \dots, 1/\overline{\mu_{n_2}}\}.$$

Also, we will denote by

$$B_1 = \prod_{j=1}^{n_1} b_{\lambda_j}, \quad \tilde{B}_1 = \prod_{j=1}^{n_1} b_{\overline{\lambda_j}} \quad \text{and} \quad B_2 = \prod_{j=1}^{n_2} b_{\mu_j},$$

the corresponding finite Blaschke products and by  $k_\xi^{B_1}$ ,  $k_\xi^{\tilde{B}_1}$ ,  $k_\xi^{B_2}$  the reproducing kernels at the point  $\xi$  of the corresponding model spaces. Under the assumption (3),  $f$  can be written as

$$(3.1) \quad f(\xi) = a + \sum_{k=1}^{n_1} \frac{c_k}{\xi - \lambda_k} + \sum_{k=1}^{n_2} \frac{d_k}{1 - \overline{\mu_k} \xi}, \quad c_k, d_k \in \mathbb{C},$$

where  $n = n_1 + n_2 = \deg f$  ( $a = 0$  if and only if  $\deg P < \deg Q$ ). We put  $g(\xi) = \sum_{k=1}^{n_1} \frac{c_k}{\xi - \lambda_k}$  and  $h(\xi) = \sum_{k=1}^{n_2} \frac{d_k}{1 - \overline{\mu_k} \xi}$ , so that  $f = a + g + h$ . We denote by  $V_{\sigma_1, \sigma_2}$  the vector space of all functions of the form (3.1).

**3.2. Integral representation for rational functions on the unit circle.** We first obtain an integral representation for a function  $f \in \mathcal{R}_n$ .

**Lemma 3.1.** *Keeping the notations and assumptions 1 and 2 of Subsection 3.1, for any function  $f \in \mathcal{R}_n$  we have*

$$(3.2) \quad f(\xi) = \langle f, \phi_\xi \rangle, \quad |\xi| = 1,$$

where  $\phi_\xi(u) = k_\xi^{zB_2}(u) + \overline{\xi u k_\xi^{\tilde{B}_1}(u)}$ ,  $u, \xi \in \mathbb{T}$ , and  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2}$  stands for the scalar product in  $L^2 = L^2(\mathbb{T}, m)$ .

*Proof.* Without loss of generality, we can suppose that  $f$  satisfies assumption 3 of Subsection 3.1 for the same reason of continuity. Thus,  $f$  satisfies the above formula (3.1):

$$(3.3) \quad f(\xi) = a + h(\xi) + g(\xi).$$



Clearly,  $a + h \in K_{zB_2}$ . Thus, for a fixed  $\xi$  we have

$$a + h(\xi) = \langle a + h, k_\xi^{zB_2} \rangle.$$

Moreover, since  $g \in \overline{H_0^2}$  (where  $H_0^2$  stands for the subspace of  $H^2$  consisting of functions  $f$  such that  $f(0) = 0$ ), we have

$$(3.4) \quad a + h(\xi) = \langle a + g + h, k_\xi^{zB_2} \rangle = \langle f, k_\xi^{zB_2} \rangle, \quad |\xi| \leq 1,$$

Note that by the continuity of the kernel  $k_\xi^{zB_2}$  in  $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ , this formula extends to  $\xi \in \mathbb{T}$ .

To obtain an analogous formula for  $g$ , consider the function

$$(3.5) \quad \varphi(\xi) = \frac{1}{\xi} g\left(\frac{1}{\xi}\right),$$

which belongs to  $K_{B_1}$  and as a consequence we can write, for  $|\xi| < 1$ ,

$$(3.6) \quad \frac{1}{\xi} g\left(\frac{1}{\xi}\right) = \langle \varphi, k_\xi^{B_1} \rangle = \int_{\mathbb{T}} \varphi(u) \frac{1 - \overline{B_1(u)}B_1(\xi)}{1 - \overline{u}\xi} dm(u).$$

Now setting  $w = \frac{1}{\xi}$ ,  $|w| > 1$ , changing the variable  $v = \bar{u}$  and using the fact that  $\varphi(u) = \bar{u}g(\bar{u}) = vg(v)$ ,  $u \in \mathbb{T}$ , we get

$$(3.7) \quad g(w) = \int_{\mathbb{T}} \frac{1}{u} g\left(\frac{1}{u}\right) \frac{1 - \overline{B_1(u)}B_1\left(\frac{1}{w}\right)}{w - \bar{u}} dm(u) = \int_{\mathbb{T}} g(v)v \frac{1 - \overline{B_1(\bar{v})}B_1\left(\frac{1}{w}\right)}{w - v} dm(v).$$

Note also that (3.7) holds for  $|w| > 1$  and, by continuity, also for  $|w| = 1$ . Now, for  $|w| = 1$ ,

$$(3.8) \quad \begin{aligned} g(w) &= \overline{w} \int_{\mathbb{T}} g(v)v \frac{1 - \widetilde{B_1(v)}\overline{B_1(w)}}{1 - v\overline{w}} dm(v) \\ &= \overline{w} \langle g, \overline{zk_w^{\widetilde{B_1}}} \rangle = \langle f, \overline{wzk_w^{\widetilde{B_1}}} \rangle, \end{aligned}$$

where the finite Blaschke product  $\widetilde{B_1}$  is defined in Subsection 3.1 (we have  $B_1(\bar{v}) = \overline{\widetilde{B_1(v)}}$ ,  $|v| = 1$ ), and the last equality is due to the fact that the function  $\overline{zk_w^{\widetilde{B_1}}}$  belongs to  $\overline{H_0^2}$  and thus, is orthogonal to  $a + h$ . Now, combining (3.4) and (3.8), we obtain for any  $\xi \in \mathbb{T}$ :

$$f(\xi) = a + h(\xi) + g(\xi) = \langle f, k_\xi^{zB_2} + \overline{\xi zk_\xi^{\widetilde{B_1}}} \rangle,$$

which completes the proof.  $\square$

**3.3. Integral representation for the derivative of rational functions on the unit circle.** The integral representation in this subsection essentially coincides with the representation due to X. Li [Li, Lemma 3] (for the case of rational functions without poles in  $\overline{\mathbb{D}}$  it was proved by R. Jones, X. Li, R.N. Mohapatra and R.S. Rodriguez [JLMR, Lemma 4.3]). It is possible to reinterpret the proof of Li's Lemma using the theory of model spaces.

**Lemma 3.2.** *Keeping the notations and assumptions 1 and 2 of Subsection 3.1, for any function  $f \in \mathcal{R}_n$ ,*

$$(3.9) \quad f'(\xi) = \langle f, \psi_\xi \rangle, \quad |\xi| = 1,$$

where  $\psi_\xi(u) = u \left( k_\xi^{B_2}(u) \right)^2 - \xi^2 \overline{u} \left( k_\xi^{\widetilde{B_1}}(u) \right)^2$ ,  $u, \xi \in \mathbb{T}$ .



*Proof.* The scheme of the proof repeats the one of Lemma 3.1. We use again the fact that  $f$  can be written as in (3.1). This time, we notice first that  $h \in K_{B_2}$ . Then for a fixed  $\xi$ , we have

$$h'(\xi) = \left\langle h, \frac{z}{(1 - \bar{\xi}z)^2} \right\rangle = \langle h, z (k_\xi^{B_2})^2 \rangle.$$

Here the first equality is the standard Cauchy formula, while the second follows from the fact that  $z(1 - \bar{\xi}z)^{-2} - z(k_\xi^{B_2}(z))^2 \in B_2H^2$  and  $h \perp B_2H^2$ . Thus, for the case when all poles are outside the disc  $\overline{\mathbb{D}}$ , the formula is immediate. Moreover, since  $g \in \overline{H_0^2}$  (where  $\overline{H_0^2}$  is defined above in the proof of Lemma 3.1) we have

$$(3.10) \quad h'(\xi) = \langle g + h, z (k_\xi^{B_2})^2 \rangle = \langle f, z (k_\xi^{B_2})^2 \rangle, \quad |\xi| \leq 1,$$

Again, by the continuity of the kernel in  $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ , this formula extends to  $\xi \in \mathbb{T}$ .

To obtain an analogous formula for  $g'$ , consider the function  $\varphi \in K_{B_1}$  defined by (3.5). Now setting in (3.6)  $w = \frac{1}{\xi}$ ,  $|w| > 1$ , we get

$$g(w) = \int_{\mathbb{T}} \varphi(u) \frac{1 - \overline{B_1(u)}B_1\left(\frac{1}{w}\right)}{w - \bar{u}} dm(u).$$

Then, differentiating with respect to  $w$ , we obtain

$$g'(w) = \int_{\mathbb{T}} \varphi(u) \left( \frac{1 - \overline{B_1(u)}B_1\left(\frac{1}{w}\right)}{w - \bar{u}} \right)' dm(u) = - \int_{\mathbb{T}} \varphi(u) \left( \frac{1 - \overline{B_1(u)}B_1\left(\frac{1}{w}\right)}{w - \bar{u}} \right)^2 dm(u),$$

since, by a direct computation,

$$\left( \frac{1 - \overline{B_1(u)}B_1\left(\frac{1}{w}\right)}{w - \bar{u}} \right)' + \left( \frac{1 - \overline{B_1(u)}B_1\left(\frac{1}{w}\right)}{w - \bar{u}} \right)^2 \in B_1H^2$$

(as a function of  $u$ ), while  $\varphi \perp B_1H^2$ . Changing the variable  $v = \bar{u}$  and using that  $\varphi(u) = \bar{u}g(\bar{u}) = vg(v)$ ,  $u \in \mathbb{T}$ , we get

$$g'(w) = - \int_{\mathbb{T}} g(v)v \left( \frac{1 - \overline{B_1(v)}B_1\left(\frac{1}{w}\right)}{w - v} \right)^2 dm(v).$$

Recalling that the finite Blaschke product  $\widetilde{B}_1$  defined in Subsection 3.1 satisfies  $B_1(\bar{v}) = \overline{\widetilde{B}_1(v)}$ ,  $|v| = 1$ , we have

$$(3.11) \quad \begin{aligned} g'(w) &= - \int_{\mathbb{T}} g(v)v \left( \frac{1 - \widetilde{B}_1(v)\overline{\widetilde{B}_1(w)}}{w - v} \right)^2 dm(v) \\ &= - \int_{\mathbb{T}} f(v)v \left( \frac{1 - \widetilde{B}_1(v)\overline{\widetilde{B}_1(w)}}{w - v} \right)^2 dm(v). \end{aligned}$$

The last equality follows from the fact that  $h \in H^2$ , while  $v \left( \frac{1 - \widetilde{B}_1(v)\overline{\widetilde{B}_1(w)}}{w - v} \right)^2 \in H_0^2$ . Note also that (3.11) holds for  $|w| > 1$  and, by continuity, also for  $|w| = 1$ .

Now, applying formulas (3.10)–(3.11) to  $z = w \in \mathbb{T}$  and recalling that  $f = a + g + h$ ,  $a \in \mathbb{C}$ , we conclude that:

$$f'(\xi) = \int_{\mathbb{T}} (h + g)(u)\overline{\psi_\xi(u)} dm(u) = \langle f, \psi_\xi \rangle,$$

since  $\psi_\xi$  is orthogonal to 1. □

*Remark.* The above integral representation immediately implies inequality (1.1) by Levin and Rusak. Indeed,  $|f'(\xi)| \leq \|\psi_\xi\|_{L^1} \|f\|_{L^\infty} = (|B_1'(\xi)| + |B_2'(\xi)|) \|f\|_{L^\infty}$  (see (4.1) below). It is, however, unclear, whether one can prove the Borwein–Erdélyi inequality (1.2) using the representation (3.9).

## 4. PROOFS OF THE UPPER BOUNDS

In this Section we prove the upper bounds from Section 2, that is, inequalities (2.1), (2.3), (2.5) and (2.6). All these proofs are based on

- (1) integral representations (3.2) or (3.9);
- (2) estimates of  $H^p$ -norms of reproducing kernels  $k_\xi^B$ , where  $B$  is a finite Blaschke product.

In particular, we will often use the fact that for any finite Blaschke product  $B = \prod_{j=1}^d b_{\nu_j}$ ,  $\nu_j \in \mathbb{D}$ , and for any  $\xi \in \mathbb{T}$ ,

$$(4.1) \quad \|k_\xi^B\|_{L^2}^2 = |B'(\xi)| = \sum_{j=1}^d \frac{1 - |\nu_j|^2}{|\xi - \nu_j|^2}.$$

We will use here the assumptions and the notations of Subsection 3.1.

**4.1. Proof of Theorem 2.1, inequality (2.1).** We first assume  $1 \leq p < \infty$  and denote by  $p'$  the conjugate exponent for  $p$ . Applying the Hölder inequality to the identity (3.9), we obtain

$$\begin{aligned} |f'(\xi)| &\leq \|f\|_{L^p} \|\psi_\xi\|_{L^{p'}} \\ &\leq \|f\|_{L^p} \left( \|(k_\xi^{\tilde{B}_1})^2\|_{L^{p'}} + \|(k_\xi^{B_2})^2\|_{L^{p'}} \right) \\ &= \|f\|_{L^p} \left( \|k_\xi^{\tilde{B}_1}\|_{H^{2p'}}^2 + \|k_\xi^{B_2}\|_{H^{2p'}}^2 \right), \quad |\xi| = 1. \end{aligned}$$

Now for any finite Blaschke product  $B = \prod_{j=1}^d b_{\nu_j}$ ,  $\nu_j \in \mathbb{D}$ , we have for  $|\xi| = |u| = 1$

$$(4.2) \quad \|k_\xi^B\|_{H^{2p'}}^{2p'} = \int_{\mathbb{T}} |k_\xi^B(u)|^{2(p'-1)+2} dm(u) \leq \|k_\xi^B\|_{H^2}^2 \max_{|u|=1} |k_\xi^B(u)|^{2(p'-1)}.$$

On one hand, by (4.1),

$$\|k_\xi^B\|_{H^2}^2 = \sum_{k=1}^d \frac{1 - |\nu_k|^2}{|1 - \nu_k \xi|^2}, \quad |\xi| = 1,$$

and on the other hand that for any  $u, \xi \in \mathbb{T}$ ,

$$(4.3) \quad |k_\xi^B(u)| \leq \sum_{j=1}^d \frac{1 - |\nu_j|^2}{|1 - \bar{\nu}_j \xi| |1 - \bar{\nu}_j u|},$$

which, by the Cauchy–Schwarz inequality, gives

$$(4.4) \quad |k_\xi^B(u)|^2 \leq \sum_{j=1}^d \frac{1 - |\nu_j|^2}{|1 - \bar{\nu}_j \xi|^2} \sum_{j=1}^d \frac{1 + |\nu_j|}{1 - |\nu_j|}, \quad u \in \mathbb{T}.$$

Thus combining (4.4) with (4.2), we obtain:

$$(4.5) \quad \|k_\xi^B\|_{H^{2p'}}^{2p'} \leq \left( \sum_{k=1}^d \frac{1 - |\nu_k|^2}{|1 - \nu_k \xi|^2} \right)^{p'} \left( \sum_{j=1}^d \frac{1 + |\nu_j|}{1 - |\nu_j|} \right)^{p'-1}.$$

Applying the last inequality to  $\tilde{B}_1$  and  $B_2$  we get

$$\|k_\xi^{\tilde{B}_1}\|_{H^{2p'}}^2 + \|k_\xi^{B_2}\|_{H^{2p'}}^2 \leq \sum_{k=1}^{n_1} \frac{1 - |\lambda_k|^2}{|1 - \lambda_k \xi|^2} \left( \sum_{k=1}^{n_1} \frac{1 + |\lambda_k|}{1 - |\lambda_k|} \right)^{\frac{1}{p}} + \sum_{k=1}^{n_2} \frac{1 - |\mu_k|^2}{|1 - \mu_k \xi|^2} \left( \sum_{k=1}^{n_2} \frac{1 + |\mu_k|}{1 - |\mu_k|} \right)^{\frac{1}{p}},$$

which, in view of the definition of  $\mathcal{D}_1(a)$  and  $\mathcal{D}_2(a)$ , completes the proof.  $\square$

**4.2. Proof of Theorem 2.2.** We use the integral representation (3.9) and Fubini–Tonelli’s Theorem to get:

$$(4.6) \quad \begin{aligned} \|f'\|_{L^1} &\leq \int_{\mathbb{T}} \left( \int_{\mathbb{T}} |f(\tau)| \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau) \right) dm(\xi) \\ &= \int_{\mathbb{T}} |f(\tau)| \left( |B_2'(\tau)| + |\tilde{B}_1'(\tau)| \right) dm(\tau), \end{aligned}$$

where the last equality comes from (4.1). Now, for any finite Blaschke product  $B = B_{\sigma}$  of degree  $d$ , corresponding to a set  $\sigma = \{\nu_1, \nu_2, \dots, \nu_d\} \subset r\mathbb{D}$ , we have  $\|B'\|_{L^1} = d$  (this follows immediately if we integrate equality (4.1)).

On the other hand,

$$(4.7) \quad \|B'\|_{L^{\infty}} \leq \sum_{k=1}^d \left\| \frac{1 - |\nu_k|^2}{(1 - \bar{\nu}_k z)^2} \right\|_{L^{\infty}} \leq \sum_{k=1}^d \frac{1 + |\nu_k|}{1 - |\nu_k|}.$$

This implies that for any  $1 \leq p' \leq \infty$ ,

$$(4.8) \quad \|B'\|_{L^{p'}} \leq \|B'\|_{L^{\infty}}^{1 - \frac{1}{p'}} \|B'\|_{L^1}^{\frac{1}{p'}} \leq d^{\frac{1}{p'}} \left( \sum_{k=1}^d \frac{1 + |\nu_k|}{1 - |\nu_k|} \right)^{1 - \frac{1}{p'}}.$$

Going back to (4.6) and applying the Hölder inequality, we get

$$\begin{aligned} \|f'\|_{L^1} &= \int_{\mathbb{T}} |f(\tau)| |B_2'(\tau)| dm(\tau) + \int_{\mathbb{T}} |f(\tau)| |\tilde{B}_1'(\tau)| dm(\tau) \\ &\leq \|f\|_{L^p} \left( \|B_2'\|_{L^{p'}} + \|\tilde{B}_1'\|_{L^{p'}} \right), \end{aligned}$$

where  $p'$  is the conjugate of  $p$ . Applying the last inequality combined with (4.8) to  $\tilde{B}_1$  and  $B_2$  we get

$$\|f'\|_{L^1} \leq \|f\|_{L^p} \left( n_1^{1 - \frac{1}{p}} \left( \sum_{k=1}^{n_1} \frac{1 + |\lambda_k|}{1 - |\lambda_k|} \right)^{\frac{1}{p}} + n_2^{1 - \frac{1}{p}} \left( \sum_{k=1}^{n_2} \frac{1 + |\mu_k|}{1 - |\mu_k|} \right)^{\frac{1}{p}} \right),$$

as required.  $\square$

**4.3. Proof of Theorem 2.3, inequality (2.5).** The proof will consist of several steps.

**Step 1. The case  $q = \infty$ ,  $1 \leq p \leq \infty$ .** Clearly, for a function  $f \in \mathcal{R}_{n,r}$  having  $n_1$  poles inside  $\mathbb{D}$  and  $n_2$  poles outside  $\bar{\mathbb{D}}$  we have

$$\mathcal{D}_1 \leq n_1 \frac{1+r}{1-r}, \quad \mathcal{D}_2 \leq n_2 \frac{1+r}{1-r}.$$

Taking the supremum over all  $\xi \in \mathbb{T}$  in (2.1) we obtain

$$(4.9) \quad \|f'\|_{L^{\infty}} \leq \left( \frac{1+r}{1-r} \right)^{1 + \frac{1}{p}} (n_1 + n_2) \|f\|_{L^p}.$$

**Step 2. The case  $q = 1$ ,  $1 \leq p \leq \infty$ .** A direct consequence of the inequality (2.3) is that

$$(4.10) \quad \|f'\|_{L^1} \leq \left( \frac{1+r}{1-r} \right)^{\frac{1}{p}} (n_1 + n_2) \|f\|_{L^p},$$

for any  $f \in \mathcal{R}_{n,r}$  having  $n_1$  poles inside  $\mathbb{D}$  and  $n_2$  poles outside  $\bar{\mathbb{D}}$ .

**Step 3. The case  $p = q$ .** For any  $f \in \mathcal{R}_{n,r}$  (as always of the form (3.1)), we have by (3.9)

$$\begin{aligned} \|f'\|_{L^p}^p &= \int_{\mathbb{T}} |f'(\xi)|^p dm(\xi) \\ &= \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \overline{\tau} f(\tau) u \left( k_{\xi}^{B_2} \right)^2 - \xi^2 \overline{u} \left( k_{\xi}^{\tilde{B}_1} \right)^2 dm(\tau) \right|^p dm(\xi) \\ &\leq \int_{\mathbb{T}} \left( \int_{\mathbb{T}} |f(\tau)| \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau) \right)^p dm(\xi). \end{aligned}$$

Now applying the Hölder inequality ( $p'$  being the conjugate of  $p$ ), we obtain

$$\begin{aligned} \|f'\|_{L^p}^p &\leq \left( \int_{\mathbb{T}} \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau) \right)^{\frac{p}{p'}} \int_{\mathbb{T}} |f(\tau)|^p \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau) \\ &= \left( |B_2'(\xi)| + |\tilde{B}_1'(\xi)| \right)^{\frac{p}{p'}} \int_{\mathbb{T}} |f(\tau)|^p \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau), \end{aligned}$$

where the last equality comes from (4.1). Now, integrating the last inequality on the unit circle with respect to  $\xi$ , we obtain

$$\begin{aligned} \|f'\|_{L^p}^p &\leq \int_{\mathbb{T}} \left( |B_2'(\xi)| + |\tilde{B}_1'(\xi)| \right)^{\frac{p}{p'}} \int_{\mathbb{T}} |f(\tau)|^p \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau) dm(\xi) \\ &\leq \max_{|\xi|=1} \left( |B_2'(\xi)| + |\tilde{B}_1'(\xi)| \right)^{\frac{p}{p'}} \int_{\mathbb{T}} \int_{\mathbb{T}} |f(\tau)|^p \left( |k_{\xi}^{B_2}(\tau)|^2 + |k_{\xi}^{\tilde{B}_1}(\tau)|^2 \right) dm(\tau) dm(\xi) \\ &= \max_{|\xi|=1} \left( |B_2'(\xi)| + |\tilde{B}_1'(\xi)| \right)^{\frac{p}{p'}} \int_{\mathbb{T}} |f(\tau)|^p \left( |B_2'(\tau)| + |\tilde{B}_1'(\tau)| \right) dm(\tau). \end{aligned}$$

Finally, using (4.7) we obtain

$$\begin{aligned} \|f'\|_{L^p}^p &\leq \max_{|\xi|=1} \left( |B_2'(\xi)| + |\tilde{B}_1'(\xi)| \right)^{\frac{p}{p'}+1} \|f\|_{L^p}^p \\ &\leq \left( \sum_{k=1}^{n_1} \frac{1+|\lambda_k|}{1-|\lambda_k|} + \sum_{k=1}^{n_2} \frac{1+|\mu_k|}{1-|\mu_k|} \right)^{\frac{p}{p'}+1} \|f\|_{L^p}^p, \end{aligned}$$

whence

$$(4.11) \quad \|f'\|_{L^p} \leq n \frac{1+r}{1-r} \|f\|_{L^p}.$$

**Step 4. The case  $1 < q < p < \infty$ .** This follows by interpolation between the cases  $q = 1$  and  $q = p$ . For any  $f \in \mathcal{R}_{n,r}$ , we have, by the Hölder inequality with the exponents  $\frac{p-1}{p-q}$  and  $\frac{p-1}{q-1}$ ,

$$\|f'\|_{L^q}^q = \int_{\mathbb{T}} |f'(\tau)|^{\frac{p-q}{p-1} + \frac{p(q-1)}{p-1}} dm(\tau) \leq \|f'\|_{L^1}^{\frac{p-q}{p-1}} \|f'\|_{L^p}^{\frac{p(q-1)}{p-1}}.$$

Now, using both the inequalities (4.10) and (4.11) from Steps 2 and 3, we obtain the required estimate:

$$\begin{aligned} \|f'\|_{L^q}^q &\leq \left( n \left( \frac{1+r}{1-r} \right)^{\frac{1}{p}} \|f\|_{L^p} \right)^{\frac{p-q}{p-1}} \left( n \frac{1+r}{1-r} \|f\|_{L^p} \right)^{\frac{p(q-1)}{p-1}} \\ &\leq n^q \left( \frac{1+r}{1-r} \right)^{\frac{1}{p} \frac{p-q}{p-1} + \frac{p(q-1)}{p-1}} \|f\|_{L^p}^q = n^q \left( \frac{1+r}{1-r} \right)^{q-1 + \frac{q}{p}} \|f\|_{L^p}^q. \end{aligned}$$

**Step 5. The case**  $1 \leq p \leq q \leq \infty$ . We now interpolate between  $q = p$  and  $q = \infty$ :

$$\|f'\|_{L^q}^q \leq \|f'\|_{L^\infty}^{q-p} \|f'\|_{L^p}^p \leq \left( n \frac{1+r}{1-r} \right)^{1+\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p},$$

where we used inequalities (4.9) and (4.11).  $\square$

*Remark.* For the case when  $1 \leq q \leq p \leq \infty$  and the function  $f \in \mathcal{R}_n$  has no poles in  $\overline{\mathbb{D}}$ , a different proof of the upper bound (but without explicit constants) could be given by an application of a result by K.M. Dyakonov [Dy1, Theorem 11]. Namely, applying inequality (11.2) from [Dy1] with  $s = 1$  we obtain that for a rational function  $f$  of degree  $n$  with the poles  $1/\overline{\nu}_1, 1/\overline{\nu}_2, \dots, 1/\overline{\nu}_n \notin \overline{\mathbb{D}}$  we have

$$(4.12) \quad \|f'\|_{L^q} \leq C \|B'\|_{L^\gamma} \|f\|_{L^p},$$

where  $q \leq p$ ,  $\frac{1}{\gamma} = \frac{1}{q} - \frac{1}{p}$ ,  $C$  is a constant depending on  $q$  and  $p$  which is not precised, and  $B = B_\sigma$  is the finite Blaschke product corresponding to  $\sigma = (\nu_1, \nu_2, \dots, \nu_n)$ . It remains to apply inequality (4.8).

**4.4. Proof of the upper bound in Theorem 2.4. Step 1. The special case  $q = \infty$  and  $1 \leq p \leq 2$ .** Following the assumptions of Subsection 3.1 and applying Hölder inequality to (3.2), we obtain (with the notations of Lemma 3.1) that for any  $\xi \in \mathbb{T}$ ,

$$(4.13) \quad |f(\xi)| \leq \|f\|_{L^p} \|\phi_\xi\|_{L^{p'}},$$

where  $p' \geq 2$  stands for the conjugate of  $p$ . Moreover,

$$\|\phi_\xi\|_{L^{p'}} \leq \|k_\xi^{zB_2}\|_{L^{p'}} + \|k_\xi^{\widetilde{B}_1}\|_{L^{p'}}.$$

Now, we prove that for any finite Blaschke product  $B = B_\sigma$  of degree  $d$ , corresponding to a set  $\sigma = \{\nu_1, \nu_2, \dots, \nu_d\} \subset r\mathbb{D}$ , we have

$$(4.14) \quad \|k_\xi^B\|_{L^{p'}} \leq \left( \frac{1+r}{1-r} d \right)^{1+\frac{1}{p}}.$$

Indeed, as a direct consequence of (4.3), we get

$$(4.15) \quad \|k_\xi^B\|_{L^\infty} \leq \sum_{j=1}^d \frac{1+|\nu_j|}{1-|\nu_j|} \leq d \frac{1+r}{1-r}.$$

Moreover, (4.1) clearly gives that for any  $\xi \in \mathbb{T}$ ,

$$(4.16) \quad \|k_\xi^B\|_{L^2}^2 = \sum_{j=1}^d \frac{1-|\nu_j|^2}{|1-\overline{\nu}_j\xi|^2} \leq \sum_{j=1}^d \frac{1+|\nu_j|}{1-|\nu_j|} \leq d \frac{1+r}{1-r}.$$

Thus, combining (4.15) and (4.16), we get that for any  $p' \geq 2$ ,

$$\|k_\xi^B\|_{L^{p'}} \leq \|k_\xi^B\|_{L^\infty}^{1-\frac{2}{p'}} \|k_\xi^B\|_{L^2}^{\frac{2}{p'}} \leq d^{p'-1} \left( \frac{1+r}{1-r} \right)^{p'-1}.$$

Applying this to  $B = \widetilde{B}_1$  and to  $B = zB_2$ , we get

$$(4.17) \quad \|\phi_\xi\|_{L^{p'}} \leq \left( \frac{1+r}{1-r} \right)^{1-\frac{1}{p'}} \left( n_1^{1-\frac{1}{p'}} + (n_2+1)^{1-\frac{1}{p'}} \right),$$

where  $n_1$  (respectively,  $n_2$ ) is the number of poles of  $f$  inside  $\mathbb{D}$  (respectively, outside  $\overline{\mathbb{D}}$ ). Thus, it follows from (4.13) and (4.17) that for any  $1 \leq p \leq 2$ ,

$$(4.18) \quad \|f\|_{L^\infty} \leq \left( \frac{1+r}{1-r} \right)^{\frac{1}{p}} \left( n_1^{\frac{1}{p}} + (n_2+1)^{\frac{1}{p}} \right) \|f\|_{L^p},$$

as required.

**Step 2. The case  $1 \leq p \leq 2$  and  $1 \leq p < q \leq \infty$ .** For any  $f \in \mathcal{R}_{n,r}$ ,

$$\|f\|_{L^q}^q \leq \|f\|_{L^\infty}^{q-p} \|f\|_{L^p}^p,$$

which gives using (4.18),

$$\begin{aligned} \|f\|_{L^q} &\leq \|f\|_{L^\infty}^{1-\frac{p}{q}} \|f\|_{L^p}^{\frac{p}{q}} \\ &\leq \left( \frac{1+r}{1-r} \right)^{\frac{1}{p}-\frac{1}{q}} \left( n_1^{\frac{1}{p}} + (n_2+1)^{\frac{1}{p}} \right)^{1-\frac{p}{q}} \|f\|_{L^p} \\ &\leq \left( \frac{1+r}{1-r} \right)^{\frac{1}{p}-\frac{1}{q}} \left( n_1^{\frac{1}{p}-\frac{1}{q}} + (n_2+1)^{\frac{1}{p}-\frac{1}{q}} \right) \|f\|_{L^p}. \end{aligned}$$

**Step 3. The case  $2 \leq p \leq \infty$  and  $1 \leq p < q \leq \infty$ .** Let  $p'$  and  $q'$  be the conjugates to  $p$  and  $q$ , respectively. Then  $1 \leq q' < p' \leq 2$ . By Step 2 we have for any  $V_{\sigma_1, \sigma_2} \subset \mathcal{R}_{n,r}$  (see Subsection 3.1 for the definition)

$$\|Id\|_{(V_{\sigma_1, \sigma_2}, L^{q'}) \rightarrow (V_{\sigma_1, \sigma_2}, L^{p'})} \leq \left( \frac{1+r}{1-r} \right)^{\frac{1}{q'}-\frac{1}{p'}} \left( n_1^{\frac{1}{q'}-\frac{1}{p'}} + (n_2+1)^{\frac{1}{q'}-\frac{1}{p'}} \right),$$

where  $Id$  is the identity operator. Now denoting by  $Id^*$  the adjoint operator (for the usual Cauchy duality) of  $Id$ , we have  $Id^* = Id$ . Therefore,

$$\begin{aligned} \|Id\|_{(V_{\sigma_1, \sigma_2}, L^p) \rightarrow (V_{\sigma_1, \sigma_2}, L^q)} &= \|Id^*\|_{(V_{\sigma_1, \sigma_2}, L^p) \rightarrow (V_{\sigma_1, \sigma_2}, L^q)} \\ &= \|Id\|_{(V_{\sigma_1, \sigma_2}, L^{q'}) \rightarrow (V_{\sigma_1, \sigma_2}, L^{p'})} \\ &\leq \left( \frac{1+r}{1-r} \right)^{\frac{1}{q'}-\frac{1}{p'}} \left( n_1^{\frac{1}{q'}-\frac{1}{p'}} + (n_2+1)^{\frac{1}{q'}-\frac{1}{p'}} \right) \\ &= \left( \frac{1+r}{1-r} \right)^{\frac{1}{p}-\frac{1}{q}} \left( n_1^{\frac{1}{p}-\frac{1}{q}} + (n_2+1)^{\frac{1}{p}-\frac{1}{q}} \right), \end{aligned}$$

which is the required estimate. □

## 5. PROOFS OF THE LOWER BOUNDS

In this section we prove the asymptotic sharpness of the inequalities in Theorems 2.1, 2.3 and 2.4 as  $n$  tends to infinity and the poles of the rational functions approach the unit circle  $\mathbb{T}$ . From now on, we denote by

$$(5.1) \quad D_n(z) = \sum_{k=0}^{n-1} z^k$$

the Dirichlet kernel of order  $n \geq 1$ . The asymptotic behaviour of  $\|D_n\|_{L^q}$  as  $n$  tends to  $\infty$  is well known: for  $q > 1$ ,

$$(5.2) \quad \lim_{n \rightarrow \infty} \frac{\|D_n\|_{L^q}}{n^{1-\frac{1}{q}}} = \left( \frac{1}{\pi} \int_{\mathbb{R}} \left| \frac{\sin x}{x} \right|^q dx \right)^{\frac{1}{q}}.$$

Put  $I_n = \{z \in \mathbb{T} : z = e^{it}, t \in [\frac{\pi}{4n}, \frac{\pi}{2n}]\}$ . It is well known and easy to see that for  $q > 1$  the integral over the arc  $I_n$  gives a substantial contribution to the norms  $\|D_n\|_{L^q}$  and  $\|D_n D'_n\|_{L^q}$ , namely

$$(5.3) \quad \int_{I_n} |D_n(\xi)|^q dm(\xi) \gtrsim n^{q-1}, \quad \int_{I_{2n}} |D_n(\xi) D'_n(\xi)|^q dm(\xi) \gtrsim n^{3q-1}.$$

Now we introduce the test functions from  $\mathcal{R}_n$  which will be used throughout this section to illustrate the sharpness of the inequalities in Theorems 2.1, 2.3 and 2.4. Put

$$(5.4) \quad f(z) = b'_{-r}(z) \sum_{k=0}^{n-2} b_{-r}^k(z), \quad b_{-r}(z) = \frac{z+r}{1+rz}.$$

Thus,  $f$  is essentially the Dirichlet kernel transplanted from the origin to the point  $\lambda = -r \in (0, 1)$  by the change of variable  $\circ b_{-r}$ . Also, for  $n \geq 4$ , we put

$$(5.5) \quad g(z) = b'_{-r}(z) \left( \sum_{k=0}^N b_{-r}^k(z) \right)^2 \in \mathcal{R}_n,$$

where  $N$  is the integer part of  $\frac{n-2}{2}$ . It is interesting to note that other natural (and simpler) candidates such as  $h(z) = (1-rz)^{-n}$  or  $h(z) = \frac{1}{1-rz} b_r^{n-1}(z)$  will not give the right order of  $n$  in the inequality (2.5) for  $p < q$ .

We will need the following lemma.

**Lemma 5.1.** *We have*

$$(5.6) \quad \|f\|_{L^p} \lesssim \left( \frac{n}{1-r} \right)^{1-\frac{1}{p}}, \quad 1 < p \leq \infty,$$

and

$$(5.7) \quad \|g\|_{L^p} \lesssim \frac{n^{2-\frac{1}{p}}}{(1-r)^{1-\frac{1}{p}}}, \quad 1 \leq p \leq \infty.$$

*Proof.* Let us compute the  $L^p$ -norm of  $f$ . Making use of the change of variable  $\zeta = b_{-r}(\xi)$  we get

$$(5.8) \quad \begin{aligned} \|f\|_{L^p}^p &= \int_{\mathbb{T}} |b'_{-r}(\xi)| |b'_{-r}(\xi)|^{p-1} \left| \sum_{k=0}^{n-2} b_{-r}^k(\xi) \right|^p dm(\xi) \\ &= \int_{\mathbb{T}} |b'_{-r}(b_{-r}(\zeta))|^{p-1} |D_{n-1}(\zeta)|^p dm(\zeta). \end{aligned}$$

By a straightforward computation,  $b'_{-r} \circ b_{-r} = \frac{r^2-1}{(1+rb_{-r})^2} = -\frac{(1+rz)^2}{1-r^2}$ . Thus,

$$\|f\|_{L^p}^p = \frac{1}{(1-r^2)^{p-1}} \int_{\mathbb{T}} |1+r\zeta|^{2(p-1)} |D_{n-1}(\zeta)|^p dm(\zeta) \leq \frac{(1+r)^{p-1}}{(1-r)^{p-1}} \|D_{n-1}\|_{L^p}^p,$$

and the statement follows from (5.2).



Analogously, changing the variable  $\zeta = b_{-r}(\xi)$ , we obtain

$$\begin{aligned} \|g\|_{L^p}^p &= \int_{\mathbb{T}} |b'_{-r}(b_{-r}(\zeta))|^{p-1} |D_{N+1}(\zeta)|^{2p} dm(\zeta) \\ &= \frac{1}{(1-r^2)^{p-1}} \int_{\mathbb{T}} |1+r\zeta|^{2(p-1)} |D_{N+1}(\zeta)|^{2p} dm(\zeta) \\ &\leq \frac{(1+r)^{p-1}}{(1-r)^{p-1}} \|D_{N+1}\|_{L^{2p}}^{2p} \lesssim \frac{(1+r)^{p-1}}{(1-r)^{p-1}} n^{2p-1}. \end{aligned}$$

□

**5.1. Sharpness in Theorem 2.1.** We prove here the statement (ii) of Theorem 2.1. Without loss of generality we assume that  $n \geq 5$  (for small  $n$  the statement is obvious, take the test function  $(1-rz)^{-2}$ ). First we consider the case  $1 < p \leq \infty$ . For  $r \in (0, 1)$  let  $f$  be defined by (5.4). We have

$$f' = b''_{-r} \sum_{k=0}^{n-2} b_{-r}^k + (b'_{-r})^2 \sum_{k=0}^{n-3} (k+1) b_{-r}^k.$$

Moreover,  $b_{-r}(-1) = 1$ ,  $b'_{-r}(-1) = -\frac{1+r}{1-r}$ , and  $b''_{-r}(-1) = \frac{2r(1+r)}{(1-r)^2}$ . This gives

$$f'(-1) = \frac{2r(n-1)(1+r)}{(1-r)^2} + \left(\frac{1+r}{1-r}\right)^2 \sum_{k=0}^{n-3} (k+1) > \left(\frac{1+r}{1-r}\right)^2 \sum_{k=0}^{n-3} (k+1) \gtrsim \left(\frac{n}{1-r}\right)^2.$$

On the other hand, for  $a = \{a_k\}$ ,  $a_k = \frac{1}{r}$ ,  $k = 1, \dots, n$ , we have

$$\max \left( \sum_{|a_k|>1} \frac{|a_k|^2 - 1}{|a_k + 1|^2}, \sum_{|a_k|<1} \frac{1 - |a_k|^2}{|a_k + 1|^2} \right) = n \frac{1+r}{1-r}, \quad \mathcal{D}_2(a) = \sum_{|a_k|>1} \frac{|a_k| + 1}{|a_k| - 1} = \frac{n}{1-r}.$$

Thus, by (5.6),

$$\max \left( \sum_{|a_k|>1} \frac{|a_k|^2 - 1}{|a_k + 1|^2}, \sum_{|a_k|<1} \frac{1 - |a_k|^2}{|a_k + 1|^2} \right) \mathcal{D}_2^{\frac{1}{p}}(a) = \left( n \frac{1+r}{1-r} \right)^{1+\frac{1}{p}} \lesssim \frac{f'(-1)}{\|f\|_{L^p}}.$$

In the case  $p = 1$ , we consider the test function  $g$  defined by (5.5). We have

$$\begin{aligned} (5.9) \quad g' &= b''_{-r} \left( \sum_{k=0}^N b_{-r}^k \right)^2 + 2 (b'_{-r})^2 \left( \sum_{k=0}^{N-1} (k+1) b_{-r}^k \right) \sum_{k=0}^N b_{-r}^k \\ &= -\frac{2}{1+rz} b'_{-r} \left( \sum_{k=0}^N b_{-r}^k \right) \left( r \sum_{k=0}^N b_{-r}^k + \frac{1-r^2}{1+rz} \sum_{k=0}^{N-1} (k+1) b_{-r}^k \right). \end{aligned}$$

As before, this gives

$$g'(-1) = \frac{2r(1+r)}{(1-r)^2} (N+1)^2 + 2 \left( \frac{1+r}{1-r} \right)^2 (N+1) \sum_{k=0}^{N-1} (k+1) \gtrsim \frac{n^3}{(1-r)^2},$$

which gives the required estimate from below for  $g'(-1)/\|g\|_{L^p}$  if we use (5.7). □

**5.2. Sharpness of (2.5).** Here we show the asymptotic sharpness of the constants  $\mathcal{C}_{n,r}(L^q, L^p)$  as  $n \rightarrow \infty$  and  $r \rightarrow 1-$ . Clearly, we need to show the sharpness only for sufficiently large values of  $n$  (for small values of  $n$  one may use the test function  $(1 - rz)^{-2}$ ).

**Step 1. The case  $1 \leq p \leq q \leq \infty$ .** We consider the same test function  $g$  defined in (5.5). Let us estimate from below the norm  $\|g'\|_{L^q}$  using the representation (5.9) for  $g'$ . Taking  $b_{-r}(\xi)$  as the new variable (as in the proof of Lemma 5.1), we get

$$\|g'\|_{L^q}^q = 2^q \int_{\mathbb{T}} \left| \frac{1}{1 + rb_{-r}(\xi)} \right|^q |b'_{-r}(b_{-r}(\xi))|^{q-1} \left| D_{N+1}(\xi) \left( rD_{N+1}(\xi) + \frac{1-r^2}{1+rb_{-r}(\xi)} D'_{N+1}(\xi) \right) \right|^q dm(\xi).$$

Since  $1 + rb_{-r}(z) = \frac{1-r^2}{1+rz}$  and  $b'_{-r} \circ b_{-r}(z) = -\frac{(1+rz)^2}{1-r^2}$ , we have

$$\begin{aligned} \|g'\|_{L^q}^q &= \frac{2^q}{(1-r^2)^{2q-1}} \int_{\mathbb{T}} |1+r\xi|^{3q-2} \left| D_{N+1}(\xi) \left( rD_{N+1}(\xi) + (1+r\xi)D'_{N+1}(\xi) \right) \right|^q dm(\xi) \\ &\geq \frac{2^q}{(1-r^2)^{2q-1}} \int_{I_{2N}} \left| D_{N+1}(\xi) \left( rD_{N+1}(\xi) + (1+r\xi)D'_{N+1}(\xi) \right) \right|^q dm(\xi), \end{aligned}$$

where the arcs  $I_N$  are defined at the beginning of the section. Now, by (5.2) and (5.3),

$$\int_{I_{2N}} \left| D_{N+1}(\xi) \right|^{2q} dm(\xi) \lesssim N^{2-\frac{1}{q}}$$

and

$$\int_{I_{2N}} \left| (1+r\xi)D_{N+1}(\xi)D'_{N+1}(\xi) \right|^q dm(\xi) \geq (1+r^2)^{\frac{q}{2}} \int_{I_N} \left| D_{N+1}(\xi)D'_{N+1}(\xi) \right|^q dm(\xi) \gtrsim N^{3-\frac{1}{q}}.$$

We conclude that

$$(5.10) \quad \|g'\|_{L^q} \gtrsim \left( \frac{1}{1-r^2} \right)^{2-\frac{1}{q}} N^{3-\frac{1}{q}}.$$

As a result, combining (5.10) and (5.7) we obtain

$$\frac{\|g'\|_{L^q}}{\|g\|_{L^p}} \gtrsim \left( \frac{n}{1-r} \right)^{1+\frac{1}{p}-\frac{1}{q}}$$

for sufficiently large values of  $n$ .

**Step 2. The case  $1 \leq q \leq p \leq \infty$ .** We consider now a simpler test function  $h(z) = \frac{1}{1-rz} b_r^{n-1}(z)$ . We have

$$\|h\|_{L^p} = \left\| \frac{1}{1-rz} \right\|_{L^p} \lesssim \frac{1}{(1-r)^{1-\frac{1}{p}}}$$

with a constant depending on  $p$ . Furthermore,

$$h' = (n-1) \frac{1}{1-rz} b'_r b_r^{n-2} + \frac{r}{(1-rz)^2} b_r^{n-1} = b'_r b_r^{n-2} \left( \frac{n-1}{1-rz} - \frac{r}{1-r^2} b_r \right),$$

and, by the change of variable  $\zeta = b_{-r}(\xi)$ ,

$$\begin{aligned} \|h'\|_{L^q}^q &= \int_{\mathbb{T}} |b'_r(\xi)| |b'_r(\xi)|^{q-1} \left| \frac{n-1}{1-r\xi} - \frac{r}{1-r^2} b_r(\xi) \right|^q dm(\xi) \\ &= \int_{\mathbb{T}} |b'_r(b_r(\zeta))|^{q-1} \left| \frac{n-1}{1-rb_r(\zeta)} - \frac{r}{1-r^2} \zeta \right|^q dm(\zeta) \\ &= \frac{1}{(1-r^2)^{2q-1}} \int_{\mathbb{T}} |(1-r\zeta)^2|^{q-1} |(n-1)(1-r\zeta) - r\zeta|^q dm(\zeta). \end{aligned}$$

Now, integrating over the arc  $\frac{\pi}{2} \leq \arg \zeta \leq \frac{3\pi}{2}$ , it is easily seen that for  $n \geq 3$ ,

$$\left( \int_{\mathbb{T}} |(1 - r\zeta)^2|^{q-1} |(n-1)(1 - r\zeta) - r\zeta|^q dm(\zeta) \right)^{1/q} \geq cn,$$

where  $c > 0$  is a numerical constant independent of  $q$ . Thus,  $\|h'\|_{L^q} \gtrsim n(1-r)^{\frac{1}{q}-2}$ ,  $r \in (0, 1)$ , and so

$$\frac{\|h'\|_{L^q}}{\|h\|_{L^p}} \gtrsim \frac{n}{(1-r)^{1+\frac{1}{p}-\frac{1}{q}}}, \quad r \in (0, 1), \quad n \geq 3.$$

□

**5.3. Sharpness of (2.7).** Let  $g$  be the test function defined in (5.5). Recall that, by (5.7),

$$\|g\|_{L^p} \lesssim \frac{n^{2-\frac{1}{p}}}{(1-r)^{1-\frac{1}{p}}}, \quad 1 \leq p \leq \infty.$$

On the other hand, we have (after the change of the variable)

$$\|g\|_{L^q}^q = \frac{1}{(1-r^2)^{q-1}} \int_{\mathbb{T}} |1 + r\xi|^{2q-2} |D_{N+1}(\xi)|^{2q} dm(\xi)$$

and

$$\int_{\mathbb{T}} |1 + r\xi|^{2q-2} |D_{N+1}(\xi)|^{2q} dm(\xi) \geq \int_{I_{N+1}} |D_{N+1}(\xi)|^{2q} dm(\xi) \gtrsim n^{2q-1}.$$

We conclude that for  $1 \leq p < q \leq \infty$ ,

$$\frac{\|g\|_{L^q}}{\|g\|_{L^p}} \gtrsim \frac{n^{\frac{1}{p}-\frac{1}{q}}}{(1-r)^{\frac{1}{p}-\frac{1}{q}}}.$$

□

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