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GLOBAL WELL-POSEDNESS FOR A NONLINEAR WAVE EQUATION COUPLED TO THE DIRAC SEA

JULIEN SABIN

ABSTRACT. We prove the global well-posedness and we study the linear response for a system of two coupled equations composed of a Dirac equation for an infinite rank operator and a nonlinear wave or Klein-Gordon equation.

1. INTRODUCTION

The purpose of this article is to study the time evolution of a classical scalar field propagating in *relativistic* vacuum. The free evolution of a scalar field W of mass m in the *non-relativistic* vacuum is governed by a nonlinear wave (m = 0) or Klein-Gordon (m > 0) equation of the form

$$(\partial_t^2 - \Delta_x + m^2)W = -W^3, \quad t \in \mathbb{R}, \, x \in \mathbb{R}^3.$$
(1)

The role of the cubic nonlinearity is to take into account some internal phenomena of the field and it is not essential for our analysis. When relativistic effects are taken into account, the structure of the vacuum changes. According to Dirac's picture [2], the relativistic vacuum is composed of infinitely many virtual particles, whose distribution is uniform in space and thus unobservable. These virtual particles can however react to the presence of the field W, resulting in a new distribution of particles not uniform anymore: we say that the vacuum becomes *polarized*. The goal of this article is to study mathematically how the scalar field reacts when coupled to the Dirac sea, from a time-dependent point of view.

A rigorous description of an infinite number of Dirac particles is not an easy task, due to the ill-posedness of many-body relativistic quantum mechanics. As a consequence, we use the Hartree-Fock approximation of Quantum Electrodynamics developed by Hainzl, Lewin, and Séré [7], which provides a well-defined functional setting to describe systems with an infinite number of particles. In this framework, the state of the vacuum is given by a self-adjoint operator $\gamma = \gamma(t)$ on a Hilbert space \mathfrak{H} satisfing the constraint $0 \leq \gamma \leq 1$. The Hilbert space \mathfrak{H} we consider is

$$\mathfrak{H} = \mathfrak{H}_{\Lambda} := \{ f \in L^2(\mathbb{R}^3, \mathbb{C}^4), \ \widehat{f}(k) = 0 \text{ for a.e. } |k| \ge \Lambda \},$$

where $\Lambda > 0$ is an ultraviolet cut-off, taken to avoid well-known divergences in relativistic theories. Typically, γ is an orthogonal projection of infinite rank, and if $(\varphi_i)_i$ is an orthonormal basis of Ran (γ) , then the $(\varphi_i)_i$ are interpreted as the wavefunctions

of the particles present in the vacuum, which here are taken to be fermions. The free evolution of the vacuum is governed by the following von Neumann-type equation

$$i\partial_t \gamma = \left[D^0, \gamma \right], \tag{2}$$

where D^0 denotes the Dirac operator on \mathfrak{H}_{Λ} given by

$$D^0 := \sum_{k=1}^3 \alpha_k (-i\partial_{x_k}) + \beta_k$$

and $(\alpha_1, \alpha_2, \alpha_3, \beta)$ are the usual Dirac matrices [19]. Notice that D^0 stabilizes \mathfrak{H}_{Λ} since it is a Fourier multiplier. The vacuum in the absence of fields (the *free* vacuum) is described by the negative spectral projection of the Dirac operator $\gamma(t) = P_-^0 :=$ $\chi_{(-\infty,0)}(D^0)$, which is formally the state of lowest possible energy and is a stationary solution to (2). When the vacuum reacts to the presence of the scalar field, one has to couple the evolution (1) of the field to the evolution (2) of the vacuum. The dynamics of the system is then governed by two coupled Partial Differential Equations

$$\begin{cases} (\partial_t^2 - \Delta_x + m^2)W = f(\gamma, W), \\ i\partial_t \gamma = \prod_{\Lambda} \left[D^0 + g(W), \gamma \right] \Pi_{\Lambda}, \end{cases}$$
(3)

where Π_{Λ} is the orthogonal projection from $L^2(\mathbb{R}^3, \mathbb{C}^4)$ onto \mathfrak{H}_{Λ} , needed since the term g(W) may not stabilize the space \mathfrak{H}_{Λ} . The coupling between the Dirac field γ and the scalar field W is expressed through the functions f and g. These functions must satisfy the crucial property that $f(P^0_{-}, 0) = 0$ and g(0) = 0, meaning that the free vacuum without any field $(\gamma(t), W(t)) \equiv (P^0_{-}, 0)$ is a stationary solution to (3). We could study the general equation (3) under appropriate assumptions on f and g, but we will for shortness restrict ourselves to two physical situations.

Case 1: Coulomb case. In the first case, W = V is a classical Coulomb potential that polarizes the Dirac sea γ composed of an infinity of relativistic electrons. The parameters are given by

$$m = 0, \qquad f(\gamma, V) = 4\pi e \rho_{\gamma - 1/2}, \qquad g(V) = eV,$$
 (P1)

where e > 0 is the absolute value of the charge of an electron, and $\rho_{\gamma-1/2}$ is the charge density associated to the state γ , defined formally by

$$\forall x \in \mathbb{R}^3, \ \rho_{\gamma-1/2}(x) := \operatorname{Tr}_{\mathbb{C}^4}\left(\gamma - \frac{1}{2}\right)(x, x) \in \mathbb{R}$$

The substraction of 1/2 in $\gamma - 1/2$ is to ensure charge conjugation symmetry [10]. One can also easily verify that $\rho_{P_{-}^{0}-1/2} \equiv 0$, so that $f(P_{-}^{0}, 0) = 0$ as desired, and we also have $\rho_{\gamma-1/2} = \rho_{\gamma-P_{-}^{0}}$. For this set of parameters, the equation is thus

$$\begin{cases} (\partial_t^2 - \Delta_x)V = 4\pi e \rho_{\gamma - 1/2}, \\ i \partial_t \gamma = \prod_{\Lambda} \left[D^0 + eV, \gamma \right] \prod_{\Lambda}. \end{cases}$$
(4)

This model describes the polarization of the Dirac sea γ by the *repulsive retarded* potential V generated by its own charge distribution $\rho_{\gamma-1/2}$. This picture is however physically incomplete, because this time-dependent charge distribution should also produce a magnetic field. Including a magnetic field in the Hartree-Fock approximation of QED leads to a much more complicated model [4], and therefore (4) is only a toy model for studying retardation effects in the polarized Dirac sea.

Case 2: Meson case. The second case concerns a model of nuclear physics, where W = U represents an attractive classical scalar meson field and γ is a quantum Dirac nucleon field. The new set of parameters is

$$m > 0, \qquad f(\gamma, U) = -U^3 - 4\pi e \rho_{\beta(\gamma - P_{-}^0)}, \qquad g(U) = e\beta U,$$
 (P2)

where we recall that β is the fourth Dirac matrix. The equation is thus

$$\begin{cases} (\partial_t^2 - \Delta_x + m^2)U = -U^3 - 4\pi e \rho_{\beta(\gamma - P_-^0)}, \\ i\partial_t \gamma = \prod_{\Lambda} \left[D^0 + e\beta U, \gamma \right] \prod_{\Lambda}. \end{cases}$$
(5)

Modelling the meson field by a classical field and adding a nonlinear term U^3 are wellknown procedures in the so-called *relativistic mean-field theory* [15, 13, 16, 17]. Here, e > 0 should be interpreted as the square root of the coupling constant $\alpha = e^2 > 0$ between the two fields.

In both cases, the initial condition for γ is chosen to be of the form $\gamma_{|t=0} = P_{-}^{0} + Q_{0}$, where Q_{0} is a Hilbert-Schmidt (hence compact) operator. We show that for all times $t \ge 0$, $\gamma(t)$ remains of this form, i.e. $\gamma(t) = P_{-}^{0} + Q(t)$ with Q(t) Hilbert-Schmidt. We thus study compact perturbations of the stationary state P_{-}^{0} , exactly as in [7, 8, 11]. Hence, the operator $\gamma(t)$ is infinite-rank for all t and really describes a vacuum composed of an infinity of particles. We can rewrite the system (3) in the variables (Q, W):

$$\begin{cases} (\partial_t^2 - \Delta_x + m^2)W = f(P_-^0 + Q, W), \\ i\partial_t Q = \prod_{\Lambda} \left[D^0 + g(W), P_-^0 + Q \right] \prod_{\Lambda}. \end{cases}$$
(6)

The advantage of this formulation is that it has a much nicer functional setting, since Q is a Hilbert-Schmidt operator on \mathfrak{H}_{Λ} . In particular, the quantities $\rho_{\gamma-1/2} = \rho_Q$ and $\rho_{\beta(\gamma-P_{-}^0)} = \rho_{\beta Q}$, which were present in (P1) and (P2), are well-defined square integrable functions [9, Lemma 1].

The main difference between the Coulomb case and the meson case is the sign of the coupling term $\pm 4\pi e\rho$ in (P1) and (P2), reminiscent of the fact that the Coulomb potential is repulsive and the meson potential is attractive. While our mathematical treatment of these two types of interactions is essentially the same thanks to the presence of the cut-off Λ , it is worth mentioning that in a model without cut-off, the

long time behaviour of the repulsive case looks more mathematically involved than the attractive case, even for simpler models (see Remark 4.2 below).

When V is instantaneous (i.e. when $\partial_t^2 - \Delta_x$ is replaced by $-\Delta_x$ in (4)), global wellposedness for the equation on Q has been proved in [11] via conservation of an energy, after the pioneering works on the so-called *Bogoliubov-Dirac-Fock model* in [7, 8]. In [11], it is remarkable that the conservation of energy leads to global well-posedness since usual energies for Dirac equations are unbounded and cannot control any norm of the solution. The coupling of a wave or Klein-Gordon equation with a Schrödinger or Dirac equation for one particle has been extensively studied. We mention [1, 3] and references therein for the study of the Cauchy problem in the energy space and scattering for the Wave-Schrödinger system, and [14] for the local existence theory in the energy space for the much more involved Maxwell-Dirac system, which includes magnetic fields as well. We are not aware of any rigorous work on the coupling between infinitely many Dirac particles and the nonlinear Klein-Gordon equation.

In this article, we prove the global existence and uniqueness of solutions to (6), for $(Q, W, \partial_t W)$ in the space $\mathfrak{S}_2(\mathfrak{H}_\Lambda) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here, $\mathfrak{S}_2(\mathfrak{H}_\Lambda)$ denotes the space of Hilbert-Schmidt operators on \mathfrak{H}_Λ . Our two main results are Theorem 1 and Theorem 2 below. Furthermore, in the Coulomb case, we study the linear response associated to the evolution problem, and compare it with the results of [5] in the timeindependent case. Finally, we explain the difficulties of deriving a "time-dependent charge renormalization" formula, which we have not succeeded in proving yet. To our knowledge, this is the first result on global well-posedness for coupled wave-Dirac equations with an infinite number of Dirac particles. Sections 2 and 3 are devoted to the proof of global well-posedness. In Section 4, we study the linear response theory for our system, and discuss the charge renormalization question.

2. Local existence theory

For initial conditions $(Q_0, W_0, W_1) \in \mathfrak{S}_2(\mathfrak{H}_\Lambda) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, we prove local existence and uniqueness of solutions

$$(Q, W, \partial_t W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2 \times H^1 \times L^2) \cap \mathcal{C}^1([-T, T], \mathfrak{S}_2 \times L^2 \times H^{-1})$$

to the system (6) for some small T > 0, for both the Coulomb and meson cases, such that $(Q, W, \partial_t W)_{|t=0} = (Q_0, W_0, W_1)$. This is equivalent to the existence of $(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2 \times H^1)$ solution to the following Duhamel integral equation:

$$\begin{cases} Q(t) = e^{-itD^{0}}Q_{0}e^{itD^{0}} - i\int_{0}^{t} e^{-i(t-t')D^{0}}\Pi_{\Lambda}[g(W(t')), Q(t') + P_{-}^{0}]\Pi_{\Lambda}e^{i(t-t')D^{0}} dt', \\ W(t) = \cos(tK_{m})W_{0} + \frac{\sin(tK_{m})}{K_{m}}W_{1} + \int_{0}^{t} \frac{\sin((t-t')K_{m})}{K_{m}}f(Q(t'), W(t')) dt', \end{cases}$$

$$\tag{7}$$

where $K_m := \sqrt{-\Delta + m^2}$ and we have written f(Q, W) instead of $f(P_-^0 + Q, W)$. We prove local existence of solutions to (7) by a standard fixed-point argument. In particular, the strategy to handle the nonlinearity U^3 in the meson case (P2) is wellknown, and seems to go back to [12]. Hence, the difficulty here is not the nonlinearity U^3 but rather the coupling between the two equations.

Proposition 1 (Local Existence). Let $\Lambda > 0$, m, f, g satisfying either (P1) or (P2), and R > 0. Then, there exists $T = T(\Lambda, m, e, R) > 0$ such that for any $(Q_0, W_0, W_1) \in \mathfrak{S}_2(\mathfrak{H}_\Lambda) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfying $||(Q_0, W_0, W_1)||_{\mathfrak{S}_2 \times H^1 \times H^{L^2}} \leq R$, there exists $(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2 \times H^1)$ solution to (7).

Proof. Let T > 0 to be chosen later. For any $(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2 \times H^1)$, we introduce $\Phi(Q, W) = (\Phi_1(Q, W), \Phi_2(Q))$, defined for any $t \in [-T, T]$ by

$$\Phi_{1}(Q,W)(t) = e^{-itD^{0}}Q_{0}e^{itD^{0}} - i\int_{0}^{t} e^{-i(t-t')D^{0}}\Pi_{\Lambda}[g(W(t')),Q(t') + P_{-}^{0}]\Pi_{\Lambda}e^{i(t-t')D^{0}} dt',$$

$$\Phi_{2}(Q)(t) = \cos(tK_{m})W_{0} + \frac{\sin(tK_{m})}{K_{m}}W_{1} + \int_{0}^{t} \frac{\sin((t-t')K_{m})}{K_{m}}f(Q(t'),W(t')) dt'.$$
(8)

We apply a fixed-point argument to Φ . We first need to show that Φ stabilizes the space $\mathcal{C}^0([-T,T],\mathfrak{S}_2 \times H^1)$.

Lemma 2. If $(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2 \times H^1)$, then $\Phi(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2 \times H^1)$, and we have, for some universal constant C > 0,

$$\begin{split} \|\Phi(Q,W)\|_{\mathcal{C}^{0}([-T,T],\mathfrak{S}_{2}\times H^{1})} &\leq \|(Q_{0},W_{0},W_{1})\|_{\mathfrak{S}_{2}\times H^{1}\times L^{2}} \\ &+ Ce\Lambda^{3/2}T(1+T)(1+\|Q\|_{\mathcal{C}^{0}(\mathfrak{S}_{2})})(1+\|W\|_{\mathcal{C}^{0}(H^{1})}). \end{split}$$
(9)

Proof of Lemma 2. We first prove that

$$t \mapsto \left(e^{-itD^0}Q_0e^{itD^0}, \cos(tK_m)W_0 + \frac{\sin(tK_m)}{K_m}W_1\right)$$

belongs to $\mathcal{C}^{0}([-T,T], \mathfrak{S}_{2} \times H^{1})$. Let us show that $t \in \mathbb{R} \mapsto e^{-itD^{0}}Q_{0}e^{itD^{0}} \in \mathfrak{S}_{2}$ is continuous. Since $(e^{-itD^{0}} \cdot e^{itD^{0}})_{t \in \mathbb{R}}$ is a semi-group of unitary operators on \mathfrak{S}_{2} , it is sufficient to show the continuity at t = 0. We know that for any $\varphi \in \mathfrak{H}_{\Lambda}$, $e^{-itD^{0}}\varphi \to \varphi$ as $t \to 0$ in \mathfrak{H}_{Λ} . Using this fact, we obtain the continuity at t = 0 of $t \in \mathbb{R} \mapsto e^{-itD^{0}}Q_{0}e^{itD^{0}} \in \mathfrak{S}_{2}$ if Q_{0} is finite-rank. Then, continuity for any $Q_{0} \in \mathfrak{S}_{2}$ is obtained by density of finite rank operators in \mathfrak{S}_{2} . The continuity of $t \in \mathbb{R} \mapsto \cos(tK_{m})W_{0} + \frac{\sin(tK_{m})}{K_{m}}W_{1} \in H^{1}$ is a well-known fact about the free wave equation: for instance one can write

$$\|\cos(tK_m)W_0 - \cos(t'K_m)W_0\|_{H^1}^2$$

= $\int_{\mathbb{R}^3} (\cos(tK_m(p)) - \cos(t'K_m(p)))^2 (1+|p|^2)|\widehat{W}_0(p)|^2 dp$

with $K_m(p) := \sqrt{|p|^2 + m^2}$, to see that it goes to 0 as $t \to t'$ by Lebesgue's dominated convergence theorem. The proof works the same for $t \mapsto \frac{\sin(tK_m)}{K_m}W_1$, and we obtain the estimates for all $t \in [-T, T]$

$$\begin{aligned} \|e^{-itD^0}Q_0e^{itD^0}\|_{\mathfrak{S}_2} &= \|Q_0\|_{\mathfrak{S}_2}, \quad \|\cos(tK_m)W_0\|_{H^1} \leqslant \|W_0\|_{H^1}, \\ \left\|\frac{\sin(tK_m)}{K_m}W_1\right\|_{H^1} \leqslant (1+|t|)\|W_1\|_{L^2}. \end{aligned}$$

We now turn to the terms with the time-integrals in (8). We first show that $t' \mapsto e^{it'D^0} \prod_{\Lambda} [g(W(t')), Q(t') + P_-^0] \prod_{\Lambda} e^{-it'D^0} \in L^{\infty}([-T, T], \mathfrak{S}_2)$, which is enough to prove that $\Phi_1(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2)$. Notice that for any $U, V \in H^1$, we have

$$\|\Pi_{\Lambda} V \Pi_{\Lambda}\|_{L^2 \to L^2} \leqslant \|\Pi_{\Lambda} V \Pi_{\Lambda}\|_{\mathfrak{S}_2} \leqslant \|V \Pi_{\Lambda}\|_{\mathfrak{S}_2} = C\Lambda^{3/2} \|V\|_{L^2} \leqslant C\Lambda^{3/2} \|V\|_{H^1},$$

 $\|\Pi_{\Lambda}\beta U\Pi_{\Lambda}\|_{L^{2}\to L^{2}} \leq \|\Pi_{\Lambda}\beta U\Pi_{\Lambda}\|_{\mathfrak{S}_{2}} \leq \|U\Pi_{\Lambda}\|_{\mathfrak{S}_{2}} = C\Lambda^{3/2}\|U\|_{L^{2}} \leq C\Lambda^{3/2}\|U\|_{H^{1}},$ where we have used that for all $f, g \in L^{2}(\mathbb{R}^{3}),$

$$\|f(x)g(-i\nabla)\|_{\mathfrak{S}^2}^2 = \operatorname{Tr}(f(x)^2g(-i\nabla)^2) = (2\pi)^{-3} \|f\|_{L^2}^2 \|g\|_{L^2}^2.$$

Hence, we have for g satisfying either (P1) of (P2),

$$\|\Pi_{\Lambda}g(W(t'))\Pi_{\Lambda}\|_{L^{2}\to L^{2}} \leqslant Ce\Lambda^{3/2}\|W(t')\|_{H^{1}},$$

 $\|[\Pi_{\Lambda}g(W(t'))\Pi_{\Lambda}, P^{0}_{-}]\|_{\mathfrak{S}_{2}} \leq 2\|\Pi_{\Lambda}g(W(t'))\Pi_{\Lambda}\|_{\mathfrak{S}_{2}} \leq Ce\Lambda^{3/2}\|W(t')\|_{H^{1}}.$ (10) Therefore, for all $t' \in [-T, T]$,

$$\|e^{it'D^{0}}\Pi_{\Lambda}[g(W(t')),Q(t')+P_{-}^{0}]\Pi_{\Lambda}e^{-it'D^{0}}\|_{\mathfrak{S}_{2}} \leq Ce\Lambda^{3/2}\|W\|_{\mathcal{C}^{0}([-T,T],H^{1})}(1+\|Q\|_{\mathcal{C}^{0}([-T,T],\mathfrak{S}_{2})}),$$

and this shows that

$$t' \mapsto e^{it'D^0} \Pi_{\Lambda}[W(t'), Q(t') + P_{-}^0] \Pi_{\Lambda} e^{-it'D^0} \in L^{\infty}([-T, T], \mathfrak{S}_2) \subset L^1([-T, T], \mathfrak{S}_2),$$

which, in turn, implies that $\Phi_1(Q, W) \in \mathcal{C}^0([-T, T], \mathfrak{S}_2)$. Furthermore, we also have

$$\|\Phi_1(Q,W)\|_{\mathcal{C}^0([-T,T],\mathfrak{S}_2)} \leq \|Q_0\|_{\mathfrak{S}_2} + Ce\Lambda^{3/2}T\|W\|_{\mathcal{C}^0([-T,T],H^1)}(1+\|Q\|_{\mathcal{C}^0([-T,T],\mathfrak{S}_2)}).$$

Finally, for any $t \in [-T,T]$ and for any $\rho \in \mathcal{C}^0([-T,T],L^2(\mathbb{R}^3))$, we have

$$\int_0^t \frac{\sin((t-t')K_m)}{K_m} \rho(t') \, \mathrm{d}t' = t \int_0^1 \frac{\sin(t(1-u)K_m)}{K_m} \rho(tu) \, \mathrm{d}u$$

Since for any $u \in [0, 1], t \mapsto \frac{\sin(t(1-u)K_m)}{K_m}\rho(tu) \in \mathcal{C}^0([-T, T], H^1)$ with

$$\left\|\frac{\sin(t(1-u)K_m)}{K_m}\rho(tu)\right\|_{H^1} \leqslant C(1+T)\|\rho\|_{\mathcal{C}^0([-T,T],L^2)},\tag{11}$$

we infer again from Lebesgue's dominated convergence theorem that

$$t \mapsto \int_0^t \frac{\sin((t-t')K_m)}{K_m} \rho(t') \, \mathrm{d}t' \in \mathcal{C}^0([-T,T], H^1),$$

for any $\rho \in \mathcal{C}^0([-T,T], L^2)$. Notice that, in the bound (11), the constant C(1+T)on the right side can be replaced by C = C(m) if m > 0. It thus remains to prove that $t \mapsto f(W(t), Q(t))$ belongs to $\mathcal{C}^0([-T,T], L^2)$. The map $Q \in \mathfrak{S}_2 \mapsto \rho_Q \in L^2$ is linear and continuous with the estimate

$$\|\rho_Q\|_{L^2} \leqslant C\Lambda^{3/2} \|Q\|_{\mathfrak{S}_2}.$$
 (12)

This can be proved by duality:

$$\forall W \in L^2(\mathbb{R}^3), \ \int_{\mathbb{R}^3} \rho_Q(x) W(x) \, \mathrm{d}x = \mathrm{Tr} \, WQ,$$

and we have the estimate valid for any $W \in L^2(\mathbb{R}^3)$:

$$|\operatorname{Tr} WQ| = |\operatorname{Tr} W\Pi_{\Lambda} Q| \leqslant C\Lambda^{3/2} ||W||_{L^2} ||Q||_{\mathfrak{S}_2}.$$
(13)

Together with the inequality

$$||W(t')^{3}||_{L^{2}} = ||W(t')||_{L^{6}}^{3} \leqslant C ||W(t')||_{H^{1}}^{3},$$
(14)

this proves that the maps

$$t' \mapsto 4\pi e \rho_Q(t')$$
$$t' \mapsto -4\pi e \rho_{\beta Q}(t') - W(t')^3$$

both belong to $\mathcal{C}^0([-T,T], L^2)$ and we have $\Phi_2(Q) \in \mathcal{C}^0([-T,T], H^1)$, with the estimate

$$\|\Phi_{2}(Q)\|_{\mathcal{C}^{0}([-T,T],H^{1})} \leq (1+T)\|(W_{0},W_{1})\|_{H^{1}\times L^{2}} + CT(1+T)(e\Lambda^{3/2}\|Q\|_{\mathcal{C}^{0}([-T,T],\mathfrak{S}_{2})} + \|W\|_{\mathcal{C}^{0}(\mathbb{R},H^{1})}^{3}).$$

This concludes the proof of Lemma 2.

By Lemma 2, we see that for any T satisfying

$$T(1+T) \leqslant \frac{1}{CR(1+e\Lambda^{3/2}+R)},$$

 Φ maps $B(2R) = \{(Q, W), \|(Q, W)\|_{\mathcal{C}^0([-T,T],\mathfrak{S}_{2}\times H^1} \leq 2R\}$ into itself. We are now in position to use the fixed point theorem of Banach-Picard to prove that Φ has a unique fixed point on B(2R). To do so, it remains to prove that Φ is a contraction on B(2R), for T small enough. Let $(Q, W), (Q', W') \in B(2R)$. We start with the second term: for any $t \in [-T, T]$, we have

$$\begin{split} \|\Phi_{2}(Q)(t) - \Phi_{2}(Q')(t)\|_{H^{1}} \\ &\leqslant C(1+T) \int_{0}^{T} \|f(Q(t'), W(t')) - f(Q'(t'), W'(t'))\|_{L^{2}} \, \mathrm{d}t' \\ &\leqslant CT(1+T)(e\Lambda^{3/2} \|Q - Q'\|_{\mathcal{C}^{0}([-T,T],\mathfrak{S}_{2})} + R^{2} \|W - W'\|_{\mathcal{C}^{0}([-T,T],H^{1})}). \end{split}$$

In the previous inequality, we have used the fact that

$$||W^{3} - W'^{3}||_{L^{2}} \leq ||W - W'||_{L^{6}} ||W^{2} + WW' + W'^{2}||_{L^{3}} \leq CR^{2} ||W - W'||_{H^{1}}$$

For the term involving Φ_1 , we have for any $t \in [-T, T]$

$$\begin{split} \Phi_1(Q,W)(t) &- \Phi_1(Q',W')(t) \\ &= -i \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda}[g(W(t')),Q(t') - Q'(t')] \Pi_{\Lambda} e^{i(t-t')D^0} \,\mathrm{d}t' \\ &- i \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda}[g(W(t')) - g(W'(t')),Q'(t') + P_-^0] \Pi_{\Lambda} e^{i(t-t')D^0} \,\mathrm{d}t' \end{split}$$

Hence, by the estimates we already mentioned, we have

$$\|\Phi_1(Q,W)(t) - \Phi_1(Q',W')(t)\|_{\mathfrak{S}_2} \leq Ce\Lambda^{3/2}T(1+R)\|(Q,W) - (Q',W')\|_{\mathcal{C}^0([-T,T],\mathfrak{S}_2 \times H^1)}.$$

If we choose T small enough such that

If we choose T small enough such that

$$T(1+T) \leq \min\left(\frac{1}{CR(1+e\Lambda^{3/2}+R)}, \frac{1}{2C(e\Lambda^{3/2}+R^2)}, \frac{1}{2Ce\Lambda^{3/2}(1+R)}\right)$$

then, for any $(Q, W), (Q', W') \in B(2R)$, we have

$$\|\Phi(Q,W) - \Phi(Q',W')\|_{\mathcal{C}^{0}([-T,T],\mathfrak{S}_{2}\times H^{1})} \leqslant \frac{1}{2}\|(Q,W) - (Q',W')\|_{\mathcal{C}^{0}([-T,T],\mathfrak{S}_{2}\times H^{1})},$$

meaning that we can apply the fixed point theorem of Banach-Picard. This ends the proof of Proposition 1. $\hfill \Box$

Remark 2.1. With the same estimates, we also have Lipschitz continuity of the solution map as a function of the initial data.

We not only have uniqueness of solutions in B(2R), but also global uniqueness:

Proposition 3 (Uniqueness). If $I = [T_-, T_+]$ is a time interval containing 0, and if $(Q, W), (Q', W') \in C^0(I, \mathfrak{S}_2 \times H^1)$ are two solutions to (7), then $(Q, W) \equiv (Q', W')$ on I.

Proof. The proof relies on a Grönwall-type argument. For any $t \in I$ we have

$$Q(t) - Q'(t) = -i \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda}[g(W(t')), Q(t') - Q'(t')] \Pi_{\Lambda} e^{i(t-t')D^0} dt'$$

$$- i \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda}[g(W(t')) - g(W'(t')), Q'(t') + P_-^0] \Pi_{\Lambda} e^{i(t-t')D^0} dt',$$

$$W(t) - W'(t) = \int_0^t \frac{\sin((t-t')K_m)}{K_m} (f(Q(t'), W(t')) - f(Q'(t'), W'(t'))) dt'.$$

Hence, there exists $C = C(\Lambda, m, e, T_{\pm}, \|W\|_{\mathcal{C}^{0}(H^{1})}, \|W'\|_{\mathcal{C}^{0}(H^{1})}, \|Q'\|_{\mathcal{C}^{0}(\mathfrak{S}_{2})})$ such that for any $t \in I$,

$$\|Q(t) - Q'(t)\|_{\mathfrak{S}_2} + \|W(t) - W'(t)\|_{H^1} \leq C \int_0^t (\|Q(t') - Q'(t')\|_{\mathfrak{S}_2} + \|W(t') - W'(t')\|_{H^1}) \,\mathrm{d}s$$

which implies $Q \equiv Q'$ and $W \equiv W'$ on I.

Proposition 1 and Proposition 3 imply the existence of a unique maximal solution to (7) belonging to $\mathcal{C}^0((T^*_-, T^*_+), \mathfrak{S}_2 \times H^1), -\infty \leq T^*_- < 0 < T^*_+ \leq +\infty$, with

$$(T_{-}^{*}, T_{+}^{*}) := \bigcup \left\{ I \text{ open interval containing } 0, \exists (Q, W) \in \mathcal{C}^{0}(I, \mathfrak{S}_{2} \times H^{1}) \text{ satisfying } (7) \right\}.$$

We have the following blowup criterion:

Proposition 4 (Blow-Up Criterion). Let $(Q, W) \in C^0((T_-^*, T_+^*), \mathfrak{S}_2 \times H^1)$ be the unique maximal solution to (6) with initial conditions $(Q_0, W_0, W_1) \in \mathfrak{S}_2 \times H^1 \times L^2$. Then we have the following criterion:

$$T_{+}^{*} < +\infty \Longrightarrow \lim_{\substack{t \to T_{+}^{*} \\ t \in (T_{-}^{*}, T_{+}^{*})}} \|(Q(t), W(t), \partial_{t} W(t))\|_{\mathfrak{S}_{2} \times H^{1} \times L^{2}} = +\infty,$$
(15)

$$T_{-}^{*} < +\infty \Longrightarrow \lim_{\substack{t \to T_{-}^{*} \\ t \in (T_{-}^{*}, T_{+}^{*})}} \| (Q(t), W(t), \partial_{t} W(t)) \|_{\mathfrak{S}_{2} \times H^{1} \times L^{2}} = +\infty.$$
(16)

Proof. We prove it for T_+^* , the proof for T_-^* being the same. Assume that $T_+^* < +\infty$ and that there exists a sequence of times (t_n) such that $t_n \in (T_-^*, T_+^*)$ for all n and $t_n \to T_+^*$ as $n \to \infty$, with $\|Q(t_n), W(t_n), \partial_t W(t_n)\|_{\mathfrak{S}_2 \times H^1 \times L^2}$ remaining bounded: there exists R > 0 such that for all n,

$$\|Q(t_n), W(t_n), \partial_t W(t_n)\|_{\mathfrak{S}_2 \times H^1 \times L^2} \leqslant R.$$

By Proposition 1, from any time t_n , we can extend the solution to $t_n + T(\Lambda, m, e, R)$, $T(\Lambda, m, e, R) > 0$ being independent of n, hence $t_n \leq T^*_+ - T(\Lambda, m, e, R)$. This contradicts the fact that $T^*_+ < +\infty$ since $t_n \to T^*_+$.

3. GLOBAL EXISTENCE

Global existence of solutions is obtained by a Grönwall-type argument, based on a Lyapunov functional which depends on whether we work with the Coulomb case (P1) or the meson case (P2).

3.1. Meson case. We introduce the following functional

$$F(Q,U) := \frac{1}{2} \operatorname{Tr} Q(t)^2 + \frac{1}{2} \int_{\mathbb{R}^3} \left((\partial_t U)^2 + |\nabla U|^2 + m^2 U^2 \right) \, \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} U^4 \, \mathrm{d}x.$$

If (Q, U) is a solution to (6) for the parameters (P2), then for any $T_{-}^* < t < T_{+}^*$, the map $t' \mapsto F(Q(t'), U(t')) \in \mathcal{C}^1([\min(0, t), \max(0, t)], \mathbb{R})$ and we have

$$\frac{d}{dt}F(Q(t'), U(t')) = -ie\operatorname{Tr} Q(t')[\beta U(t'), P_{-}^{0}] - 4\pi e \int_{\mathbb{R}^{3}} \rho_{\beta Q}(t')\partial_{t}U(t') \,\mathrm{d}x.$$

As a consequence, by (10) and (12) we have for all $t' \in [\min(0, t), \max(0, t)]$

$$\frac{d}{dt}F(Q(t'), U(t')) \leqslant Ce\Lambda^{3/2} \|Q(t')\|_{\mathfrak{S}_2}(\|U(t')\|_{H^1} + \|\partial_t U(t')\|_{L^2})$$
$$\leqslant C_{\Lambda,m,e}F(Q(t'), U(t')).$$

Hence, by Grönwall's inequality, for all $t \in (T_{-}^{*}, T_{+}^{*})$ we have

$$F(Q(t), U(t)) \leqslant F(Q(0), U(0))e^{C_{m,\Lambda}|t|}$$

Combining this inequality with the blow-up criterion (15), we deduce the

Theorem 1. Let m > 0, e > 0 and $\Lambda > 0$. Then, for any $(Q_0, U_0, U_1) \in \mathfrak{S}_2 \times H^1 \times L^2$, there exists a unique $(Q, U) \in \mathcal{C}^0(\mathbb{R}, \mathfrak{S}_2 \times H^1) \cap \mathcal{C}^1(\mathbb{R}, \mathfrak{S}_2 \times L^2)$ such that $\partial_t U \in \mathcal{C}^1(\mathbb{R}, H^{-1})$, and satisfying for all $t \in \mathbb{R}$

$$\begin{cases} (\partial_t^2 - \Delta_x + m^2)U(t) = -4\pi e \rho_{\beta Q(t)} - U(t)^3 & \text{in } H^{-1}(\mathbb{R}^3), \\ i\partial_t Q(t) = \Pi_\Lambda \left[D^0 + \beta U(t), P^0_- + Q(t) \right] \Pi_\Lambda & \text{in } \mathfrak{S}_2(\mathfrak{H}_\Lambda), \end{cases}$$
(17)

with initial conditions $(Q, U, \partial_t U)_{|t=0} = (Q_0, U_0, U_1).$

Remark 4.1. The conserved energy of the system is

$$\mathcal{E}_t(Q,U) = \text{Tr}_0(D^0Q) + e \int \rho_Q U \,\mathrm{d}x + \frac{1}{2} \int \left((\partial_t U)^2 + |\nabla_x U|^2 + m^2 U^2 \right) \,\mathrm{d}x + \frac{1}{4} \int U^4 \,\mathrm{d}x,$$

where $\operatorname{Tr}_0(D^0Q) := \operatorname{Tr}(|D^0|(Q_{++} - Q_{--})) \geq \operatorname{Tr} Q^2$, and with the notation $Q_{++} := (1 - P_-^0)Q(1 - P_-^0), Q_{--} := P_-^0QP_-^0$. The term $\operatorname{Tr}_0(D^0Q)$ represents the kinetic energy of Q [7]. For it to be well defined and positive, it is enough to know that $P_-^0 + Q_0$ is an orthogonal projection [11]. Then, estimating

$$\left| e \int \rho_Q U \, \mathrm{d}x \right| \leqslant C e \Lambda^3 \operatorname{Tr} Q^2 + C e \int U^2 \, \mathrm{d}x,$$

we deduce that for e small enough (depending on Λ), the conservation of energy also leads to global well-posedness, if $P_{-}^{0} + Q_{0}$ is an orthogonal projection. These are stronger assumptions than those of Theorem 1, but they imply furthermore that the solution belongs to $L^{\infty}(\mathbb{R}, \mathfrak{S}_{2} \times H^{1})$.

3.2. Coulomb case. In this case, we consider the functional

$$G(Q, V) = \frac{1}{2} \operatorname{Tr} Q^2 + \frac{1}{2} \int_{\mathbb{R}^3} \left((\partial_t V)^2 + |\nabla V|^2 \right) \, \mathrm{d}x.$$

It verifies

$$\frac{d}{dt}G(Q(t), V(t)) = -ie\operatorname{Tr}(Q(t)[V(t), P_{-}^{0}]) + 4\pi e \int_{\mathbb{R}^{3}} \rho_{Q}(t)\partial_{t}V(t) \,\mathrm{d}x.$$

for any solution (Q, V) of (6) with parameters (P1). By the same estimates as in the previous section, together with the inequality

$$\|[V, P_{-}^{0}]\|_{\mathfrak{S}_{2}(\mathfrak{H}_{\Lambda})} \leqslant C_{\Lambda} \|\nabla V\|_{L^{2}}$$

which has been proved in [11, Lemma 3.1], we also obtain

$$\frac{d}{dt}G(Q(t), V(t)) \leqslant C_{\Lambda, e}G(Q(t), V(t)).$$

We again deduce

$$\forall t \in (T_{-}^*, T_{+}^*), \quad G(Q(t), V(t)) \leqslant e^{C_{\Lambda, e}|t|}.$$

To use the blow-up criterion (15), it remains to obtain a control on $||V(t)||_{L^2}$, but since we have

$$\|V(t)\|_{L^{2}} \leq \|V(0)\|_{L^{2}} + \left|\int_{0}^{t} \|\partial_{t}V(s)\|_{L^{2}} \,\mathrm{d}s\right| \leq \|V(0)\|_{L^{2}} + \left|\int_{0}^{t} \sqrt{2G(Q(s), V(s))} \,\mathrm{d}s\right|,$$

we have proved the

Theorem 2. Let e > 0 and $\Lambda > 0$. Then, for any $(Q_0, V_0, V_1) \in \mathfrak{S}_2 \times H^1 \times L^2$, there exists a unique $(Q, V) \in \mathcal{C}^0(\mathbb{R}, \mathfrak{S}_2 \times H^1) \cap \mathcal{C}^1(\mathbb{R}, \mathfrak{S}_2 \times L^2)$ such that $\partial_t V \in \mathcal{C}^1(\mathbb{R}, H^{-1})$, and satisfying for all $t \in \mathbb{R}$

$$\begin{cases} (\partial_t^2 - \Delta_x)V(t) = 4\pi e\rho_{Q(t)} & \text{in } H^{-1}(\mathbb{R}^3), \\ i\partial_t Q(t) = \Pi_\Lambda \left[D^0 + eV(t), P_-^0 + Q(t) \right] \Pi_\Lambda & \text{in } \mathfrak{S}_2(\mathfrak{H}_\Lambda), \end{cases}$$
(18)

with initial conditions $(Q, V, \partial_t V)_{|t=0} = (Q_0, V_0, V_1).$

Remark 4.2. The main tool to prove global well-posedness is the cut-off Λ . Without the cut-off, it is not clear at all that global well-posedness holds, even for the apparently simpler *repulsive* Schrödinger-Klein-Gordon system

$$\begin{cases} (\partial_t^2 - \Delta + m^2)W = e|u|^2, \\ (i\partial_t + \Delta)u = eWu, \end{cases}$$

which conserved energy has no particular sign. When $e|u|^2$ is replaced by $-e|u|^2$ in the equation above (*attractive* case), global well-posedness in the energy space was proved in [1]. It heavily relies on the fact that, in this case, the energy is positive and controls the energy norm.

4. Linear response in the Coulomb case

In this section, we study the linear response of Duhamel's equation

$$Q(t) = e^{itD^0} Q_0 e^{itD^0} - i \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda} [eV(t'), Q(t') + P_-^0] \Pi_{\Lambda} e^{i(t-t')D^0} dt',$$

which corresponds to the first equation of (6) in the Coulomb case. More precisely, we compute the density ρ_{Q_1} of the operator Q_1 defined by the formula

$$Q_1(t) = -ie \int_0^t e^{-i(t-t')D^0} [V(t'), P_-^0] e^{i(t-t')D^0} dt', \, \forall t \in \mathbb{R}.$$

This density ρ_{Q_1} is relevant since it corresponds to the divergent part (with respect to the cut-off Λ) of the operator Q(t), in analogy with the time-independent version of this equation studied in [5]. As a consequence, controlling this term is a key first step in the process of understanding the limit $\Lambda \to +\infty$ of Equation (6). In Section 4.1, we compute explicitly ρ_{Q_1} and estimate it. In Section 4.2, we discuss the difficulties of studying the limit $\Lambda \to +\infty$ and its link with charge renormalization.

4.1. **Pointwise estimate.** The integral kernel of $Q_1(t)$ can be computed explicitly in Fourier space,

$$\widehat{Q_{1}}(t;p,q) = \frac{-ie}{(2\pi)^{\frac{3}{2}}} \int_{0}^{t} \widehat{V}(t',p-q) e^{-i(t-t')D^{0}(p)} (\widehat{P_{-}^{0}}(q) - \widehat{P_{-}^{0}}(p)) e^{i(t-t')D^{0}(q)} dt'$$

$$= \frac{-ie}{(2\pi)^{\frac{3}{2}}} \int_{0}^{t} \widehat{V}(t',p-q) \left(e^{-i(t-t')(E(p)+E(q))} \widehat{P_{+}^{0}}(p) \widehat{P_{-}^{0}}(q) - e^{i(t-t')(E(p)+E(q))} \widehat{P_{-}^{0}}(p) \widehat{P_{$$

where $P^{0}_{+} := 1 - P^{0}_{-}$ and

$$E(p) := \sqrt{1+|p|^2}$$

for all $p \in \mathbb{R}^3$. Hence, the (space) Fourier transform of the density $\rho_{Q_1}(t)$ can be written as

$$\begin{split} \widehat{\rho_{Q_1}}(t,k) &= \frac{-e}{4\pi^3} \int_0^t \widehat{V}(t',k) \times \\ &\times \int_{\substack{|p+k/2| \leq \Lambda \\ |p-k/2| \leq \Lambda}} \sin((t-t')(E(p+k/2) + E(p-k/2))) \left(1 - \frac{1 + (p+k/2) \cdot (p-k/2)}{E(p+k/2)E(p-k/2)}\right) \, \mathrm{d}p \, \mathrm{d}t', \end{split}$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{R}^3$. Integrating by parts, one finds the formula

$$\widehat{\rho_{Q_1}}(t,k) = -\frac{e}{4\pi} |k|^2 \widehat{V}(t,k) B_{\Lambda}(0,k) + \frac{e}{4\pi} |k|^2 \widehat{V}(0,k) B_{\Lambda}(t,k) + \frac{e}{4\pi} \int_0^t |k|^2 (\partial_t \widehat{V})(t',k) B_{\Lambda}(t-t',k) \, \mathrm{d}t', \quad (19)$$

where

$$B_{\Lambda}(t,k) = \frac{1}{\pi^2 |k|^2} \times \int_{\substack{|p+k/2| \leq \Lambda \\ |p-k/2| \leq \Lambda}} \frac{\cos(t(E(p+k/2) + E(p-k/2)))}{E(p+k/2) + E(p-k/2)} \left(1 - \frac{1 + (p+k/2) \cdot (p-k/2)}{E(p+k/2)E(p-k/2)}\right) dp.$$

This formula is a time-dependent generalization of the function B^0_{Λ} defined in [5, Eq. (80)], which was rigorously studied in [5, Appendix A] but already present in [7]. The relation between these two functions is just

$$\forall k \in \mathbb{R}^3, \ B^0_{\Lambda}(k) = B_{\Lambda}(0,k)$$

It is proved in [5] that $B^0_{\Lambda}(0)$ is logarithmically divergent: $B^0_{\Lambda}(0) \sim \frac{2}{3\pi} \log \Lambda$ as $\Lambda \to +\infty$, which leads to a *charge renormalization* formula [5, Eq. (28)]. In our case, this logarithmic divergence actually disappears as soon as $t \neq 0$ and it a posteriori justifies the decomposition in (19):

Lemma 5. There exists a universal constant C > 0 such that we have

$$|B_{\Lambda}(t,k)| \leqslant C(1+|\log(|t|E(k))|),$$
(20)

for all $\Lambda > 0$, for all $t \neq 0$, and for all $k \in \mathbb{R}^3$.

Proof. Notice that $B_{\Lambda}(t,k)$ is an even function of t, so we may assume t > 0 in the following. Using the same change of variables as in [5, Appendix A], one finds the formula

$$B_{\Lambda}(t,k) = \frac{|k|}{2\pi} \int_{0}^{Z_{\Lambda}(|k|)} \frac{\cos(t(2E(\Lambda) - |k|z))}{E(\Lambda) - |k|z/2} (z - z^{2}/3) dz + \frac{1}{\pi} \int_{0}^{Z_{\Lambda}(|k|)} \cos\left(\frac{2t}{\sqrt{1 - z^{2}}} \sqrt{1 + \frac{|k|^{2}}{4}(1 - z^{2})}\right) \frac{z^{2} - z^{4}/3}{(1 - z^{2})(1 + |k|^{2}(1 - z^{2})/4)} dz,$$

with

$$Z_{\Lambda}(r) = \frac{E(\Lambda) - E(\Lambda - r)}{r}.$$

The function $B_{\Lambda}(t, \cdot)$ is supported in the ball $B(0, 2\Lambda)$, and for all $0 \leq r \leq 2\Lambda$, one has $0 \leq Z_{\Lambda}(r) \leq 1$. Furthermore, for all $0 \leq z \leq Z_{\Lambda}(r)$, one has $E(\Lambda) - |k|z/2 \geq E(\Lambda)/2$. Hence, we obtain

$$\left|\frac{|k|}{2\pi} \int_0^{Z_{\Lambda}(|k|)} \frac{\cos(t(2E(\Lambda) - |k|z))}{E(\Lambda) - |k|z/2} (z - z^2/3) \,\mathrm{d}z\right| \leqslant C \frac{|k|}{E(\Lambda)} Z_{\Lambda}(|k|) \leqslant C.$$

To estimate the second term of $B_{\Lambda}(t,k)$, which we denote by I, we make the change of variables $u = \frac{1}{\sqrt{1-z^2}}$, which leads to

$$I = \frac{1}{3\pi} \int_{1}^{R} \cos\left(2tu\sqrt{1 + \frac{|k|^{2}}{4u^{2}}}\right) \sqrt{u^{2} - 1} \frac{1 + 2u^{2}}{u^{2}} \frac{\mathrm{d}u}{u^{2} + |k|^{2}/4},$$

with $R = (1 - Z_{\Lambda}(|k|))^{-1/2}$. First, we have

$$\left|\frac{1}{3\pi} \int_{1}^{R} \cos\left(2tu\sqrt{1+\frac{|k|^{2}}{4u^{2}}}\right) \frac{\sqrt{u^{2}-1}}{u^{2}} \frac{\mathrm{d}u}{u^{2}+|k|^{2}/4}\right| \leqslant C \int_{1}^{\infty} \frac{\mathrm{d}u}{u(u^{2}+|k|^{2}/4)} \\ \leqslant C \min\left(1,\frac{1}{|k|}\right).$$

Then, using that $|\sqrt{u^2 - 1} - u| \leq 1/(2u)$ for all $u \ge 1$, we obtain

$$\begin{aligned} \left| \frac{2}{3\pi} \int_{1}^{R} \cos\left(2tu\sqrt{1+\frac{|k|^{2}}{4u^{2}}}\right) (\sqrt{u^{2}-1}-u) \frac{\mathrm{d}u}{u^{2}+|k|^{2}/4} \right| &\leq C \int_{1}^{\infty} \frac{\mathrm{d}u}{u(u^{2}+|k|^{2}/4)} \\ &= C \min\left(1,\frac{1}{|k|}\right). \end{aligned}$$

Hence,

$$|I| \leqslant C + \left| \frac{2}{3\pi} \int_{1}^{R} \cos\left(2tu\sqrt{1 + \frac{|k|^{2}}{4u^{2}}}\right) \frac{u\,\mathrm{d}u}{u^{2} + |k|^{2}/4} \right|.$$

Performing the change of variables $v = \sqrt{u^2 + |k|^2/4}$, we obtain

$$\int_{1}^{R} \cos\left(2tu\sqrt{1+\frac{|k|^{2}}{4u^{2}}}\right) \frac{u\,\mathrm{d}u}{u^{2}+|k|^{2}/4} = \int_{E(k/2)}^{\sqrt{R^{2}+|k|^{2}/4}} \frac{\cos(2tv)}{v}\,\mathrm{d}v$$
$$= \int_{tE(k/2)}^{t\sqrt{R^{2}+|k|^{2}/4}} \frac{\cos(2v)}{v}\,\mathrm{d}v.$$

Assume first $tE(k/2) \leq 1$. We distinguish two cases: if $t\sqrt{R^2 + |k|^2/4} \leq 1$, then

$$\left| \int_{tE(k/2)}^{t\sqrt{R^2 + |k|^2/4}} \frac{\cos(2v)}{v} \, \mathrm{d}v \right| \leqslant \int_{tE(k/2)}^{1} \frac{\mathrm{d}v}{v} = |\log(tE(k/2))|.$$

Else, if $t\sqrt{R^2+|k|^2/4} \ge 1$, we have

$$\left| \int_{tE(k/2)}^{t\sqrt{R^2 + |k|^2/4}} \frac{\cos(2v)}{v} \, \mathrm{d}v \right| \leq \int_{tE(k/2)}^{1} \frac{\mathrm{d}v}{v} + \left| \int_{1}^{t\sqrt{R^2 + |k|^2/4}} \frac{\cos(2v)}{v} \, \mathrm{d}v \right|$$
$$\leq C + |\log(tE(k/2))|.$$

On the other hand, if $tE(k/2) \ge 1$, then

$$\left| \int_{tE(k/2)}^{t\sqrt{R^2 + |k|^2/4}} \frac{\cos(2v)}{v} \,\mathrm{d}v \right| \leqslant C.$$

In any case, we get

$$\left| \int_{tE(k/2)}^{t\sqrt{R^2 + |k|^2/4}} \frac{\cos(2v)}{v} \, \mathrm{d}v \right| \le C(1 + |\log(tE(k))|).$$

Remark 5.1. With the same changes of variables, one can easily show that for each $t \neq 0$ and $k \in \mathbb{R}^3$ fixed, $B_{\Lambda}(t,k)$ has a limit as $\Lambda \to +\infty$, which is

$$\lim_{\Lambda \to +\infty} B_{\Lambda}(t,k) = \frac{2}{3\pi} \int_{|t|E(k/2)}^{\infty} \frac{\cos(2v)}{v} dv + \frac{1}{3\pi} \int_{1}^{\infty} \cos\left(2tu\sqrt{1+\frac{|k|^2}{4u^2}}\right) \left[\frac{\sqrt{u^2-1}}{u^2} + 2(\sqrt{u^2-1}-u)\right] \frac{du}{u^2+|k|^2/4}.$$

This has to be compared with the situation at t = 0, where only $\lim_{\Lambda \to +\infty} (B_{\Lambda}(0, 0) - B_{\Lambda}(0, k))$ exists.

4.2. Discussion: time-dependent charge renormalization. In Equation (19), we have isolated the logarithmic divergent term $B_{\Lambda}(0, k)$ which is independent of t. The other term is actually uniformly bounded with respect to Λ , which suggests that the limit $\Lambda \to +\infty$ of $\rho_Q(t)$ is divergent. In order to control this divergence, we set $\alpha = e^2$ the (bare) fine structure constant and fix $\alpha \log \Lambda$ so that $\Lambda \to +\infty$ implies $\alpha \to 0$. In the spirit of [8, 5, 6, 18], we expect this limit to be convergent. Let us now explain the difficulties of rigorously proving this convergence. Consider the relation (19). Then, for all $t \neq 0$ and for all $k \in \mathbb{R}^3$,

$$4\pi e \widehat{\rho_{Q_1}}(t,k) = -\alpha |k|^2 \widehat{V}(t,k) B^0_{\Lambda}(0) + \alpha |k|^2 \widehat{V}(t,k) C_{\Lambda}(|k|) + \alpha |k|^2 \widehat{V}(0,k) B_{\Lambda}(t,k) + \alpha \int_0^t |k|^2 (\partial_t \widehat{V})(s,k) B_{\Lambda}(t-s,k) \, \mathrm{d}s,$$

where $C_{\Lambda}(|k|) := B^{0}_{\Lambda}(0) - B^{0}_{\Lambda}(k)$. The function C has been studied in [6] and satisfies the estimate

$$\forall \Lambda \ge 1, \ \forall r \ge 0, \quad 0 \le C_{\Lambda}(r) \le C \log(2 + r^2), \tag{21}$$

for some universal constant C (in particular, independent of Λ). Combining the estimates (20) and (21) and assuming that \widehat{V} and $\partial_t \widehat{V}$ are decaying rapidly enough, one obtains the following convergence as $\alpha \to 0$ with $\alpha \log \Lambda$ fixed, for fixed time $t \neq 0$:

$$4\pi e \rho_{Q_1}(t) \simeq \left(\frac{2}{3\pi} \alpha \log \Lambda\right) \Delta V,$$

for instance in $L^2(\mathbb{R}^3)$. Assuming furthermore that $4\pi(\rho_Q(t) - \rho_{Q_1}(t)) \simeq 0$ in the same limit, we infer that V satisfies the following wave equation

$$\partial_t^2 V - \left(1 + \frac{2}{3\pi}\alpha \log\Lambda\right)\Delta_x V = 0, \tag{22}$$

in the limit $\alpha \to 0$ with $\alpha \log \Lambda$ fixed. We see from this formulation that this limit leads to a renormalization of the propagation speed of the wave in the vacuum, very much alike to the charge renormalization formula in the stationary case. Of course, justifying all the steps leading to this limit is the hard part. In particular, in the spirit of [18], one should begin with deriving a priori estimates on the solution (Q, V) to (7), for instance in the fixed point argument of the proof of Proposition 1. The typical a priori estimate one would need is the existence of T > 0 and C > 0 depending only on α and $\alpha \log \Lambda$ such that

$$\forall t \in [-T, T], \quad \|\rho_Q(t)\|_{L^2} + \|V(t)\|_{H^1} + \|\partial_t V(t)\|_{L^2} \leqslant C.$$
(23)

The time T built in the proof of Proposition 1 satisfies $T \to 0$ as $\alpha \to 0$ with $\alpha \log \Lambda$ fixed, so that it does not imply (23). However, as shown by (19), one cannot expect such a good estimate as (23) since $\rho_{Q_1}(t)$ diverges as $\log \Lambda$. One way to circumvent this issue, in the spirit of [18], would be to use a fixed point argument on $\tilde{Q} := Q - Q_1$ which satisfies the equation

$$\widetilde{Q}(t) = e^{-itD^0} \widetilde{Q}_0 e^{itD^0} - ie \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda}[V(t'), \widetilde{Q}(t')] \Pi_{\Lambda} e^{i(t-t')D^0} dt' + Q_2(t), \quad (24)$$

where

$$Q_2(t) := -ie \int_0^t e^{-i(t-t')D^0} \Pi_{\Lambda}[V(t'), Q_1(t')] \Pi_{\Lambda} e^{i(t-t')D^0} dt'.$$

To build \widetilde{Q} by a fixed point argument, one needs estimates on Q_2 and ρ_{Q_2} that depend only on α and $\alpha \log \Lambda$, and one also needs to improve estimates as (11) to control the second term in the right side of (24). We hope to come back to this problem in the future.

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