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# On the homogeneity at infinity of the stationary probability for an affine random walk.

by Y. Guivarc'h, E. Le Page

## Abstract

We consider an affine random walk on  $\mathbb{R}$ . We assume the existence of a stationary probability  $\nu$  on  $\mathbb{R}$  and we describe the shape at infinity of  $\nu$ , if  $\nu$  has unbounded support. We discuss the connections of the result with geometrical or probabilistic problems.

## I - Introduction

Let  $G$  be the affine group of the line. For  $g \in G$ ,  $x \in \mathbb{R}$ , we write  $gx = a(g)x + b(g)$  with  $a(g) \in \mathbb{R}^*$ ,  $b(g) \in \mathbb{R}$ . Let  $\mu$  be a probability on  $G$ . We denote by  $P$  the Markov operator on  $\mathbb{R}$  defined by  $P\varphi(x) = \int \varphi(gx)\mu(dg)$  where  $\varphi$  is a bounded Borel function. Our hypothesis  $H_\mu$  is stated below and we observe that  $H_\mu(1)$  and  $H_\mu(2)$  imply that  $P$  has a unique stationary probability  $\nu$  (see [16]) ; if  $H_\mu(4)$  is also valid, then  $\text{supp}\nu$  is unbounded. Here we are interested in the "shape at infinity" of  $\nu$  ; we will show that for some  $\alpha > 0$ , the quantities  $|t|^\alpha \nu[t, \infty)$  and  $|t|^\alpha \nu(-\infty, t]$  have limits at infinity, we discuss their positivity and we illustrate the possible uses of this result by two corollaries in two different contexts. This "homogeneity at infinity" of  $\nu$  plays an essential role in extreme value theory (see [19]), for random variables associated with the Markov chain  $X_n^x$  with kernel  $P$  on  $\mathbb{R}$ . Also, for random walk in a random medium on  $\mathbb{Z}$  (see [21]) the slow diffusion property is closely related to this homogeneity (see [6], [17]). Furthermore the construction of  $\nu$  given here provides a natural construction of a large class of heavy tailed measures which generates "anomalous" random walks on the additive group  $\mathbb{R}$ . This class of measures appears now to be of great interest from the physical point of view (see [2]). In the geometrical context of excursions of geodesic flows on manifolds of negative curvature the "logarithm law" is well known (see [22], [18]), and we will discuss analogous properties for the Markov chain  $X_n^x$ .

We assume that  $\mu$  satisfies the following set of conditions  $H_\mu$ .

$$H_\mu(1) : \int (|\ln|a(g)|| + |\ln|b(g)||)\mu(dg) < \infty.$$

$$H_\mu(2) : \text{For some } \alpha > 0 \int |a(g)|^\alpha \mu(dg) = 1.$$

$$H_\mu(3) : \int |a(g)|^\alpha \ln|a(g)|\mu(dg) < \infty, \int |b(g)|^\alpha \mu(dg) < \infty.$$

$$H_\mu(4) : \text{The elements of } \text{supp}\mu \text{ have no common fixed point in } \mathbb{R}.$$

$$H_\mu(5) : \text{The set } \{\ln|a(g)| ; g \in \text{supp}\mu\} \text{ generates a dense subgroup of } \mathbb{R}.$$

Then we have the

## Theorem 1

Assume that  $\mu$  satisfies  $H_\mu$ . Then

1) There exists  $c_+ \geq 0$ ,  $c_- \geq 0$  such that  $\lim_{t \rightarrow \infty} |t|^\alpha \nu(t, \infty) = c_+$ ,  $\lim_{t \rightarrow -\infty} |t|^\alpha \nu(-\infty, t) = c_-$ .

Moreover  $c = c_+ + c_- > 0$ .

2) If  $\mu\{g \in G ; a(g) < 0\} > 0$ , then  $c_+ = c_- > 0$ .

- 3) If  $\mu\{g \in G ; a(g) > 0\} = 1$ , then  $c_+ > 0$  (resp  $c_- > 0$ ) if and only if the action of  $\text{supp}\mu$  on  $\mathbb{R}$  has no invariant half-line of the form  $] - \infty, k]$ , (resp  $[k, \infty[$ ) .
- 4) If  $\mu\{g \in G ; a(g) < 0\} > 0$ , then  $\text{supp}\nu = \mathbb{R}$ . Otherwise the set  $\text{supp}\nu$  is a half-line if and only if  $\text{supp}\mu$  preserves a half-line of the same form.

We denote by  $\mathbb{P}$  the product probability  $\mu^{\otimes \mathbb{N}}$  on  $\Omega = G^{\mathbb{N}}$ , where  $\mathbb{N}$  is the set of positive integers. For  $\omega \in \Omega$ , we write  $g_k = (a(g_k), b(g_k)) = (a_k, b_k)$ . Then  $X_n^x$  satisfies the stochastic recursion :

$$X_n^x = a_n X_{n-1}^x + b_n, \quad X_0^x = x.$$

The Markov chain  $X_n^x$  on  $\mathbb{R}$  will be called "affine random walk". It is well known that, for the existence and uniqueness of the stationary measure  $\nu$ , it is sufficient to assume  $H_\mu(1)$

and  $\int \ell n |a(g)| \mu(dg) < 0$  ; then  $X_n^x$  converges in law to  $R = \sum_1^\infty a_1 \dots a_{k-1} b_k$  and the law

of  $R$  is  $\nu$ . Also if  $\int (|a(g)|^\beta + |b(g)|^\beta) \mu(dg) < \infty$  for some  $\beta > 0$ , then  $\int |x|^\beta d\nu(x) < \infty$ . We observe that  $R$  can be interpreted as the sum of a "random geometric series", hence its interest for collective risk theory ([19]).

The validity of 1) and 2) was proved in [10], [16] ; in particular implicit expression for  $c_+, c_-$  were given in [10] and the relation  $c_+ + c_- > 0$  was obtained. Here we restrict our study to 3) and 4), a result which is new under hypothesis  $H_\mu$ . A different proof was sketched in [12], where a survey of the multidimensional situation was also given. We observe that the main difficulty of the proof occurs when  $\text{supp}\mu$  do not preserve a half line and  $a(g) > 0$   $\mu - a.e$  ; in this case we have  $c_+ > 0, c_- > 0$ . If  $\text{supp}\mu$  has compact support a short complex analytic proof of this fact, depending of a Lemma of E. Landau, well known in analytic number theory, is given in [11] (see also [5]).

We recall that Fréchet's law with parameter  $\gamma$  is the probability  $\Phi_\alpha^\gamma$  on  $\mathbb{R}_+$  given by  $\Phi_\alpha^\gamma(0, t) = e^{-\gamma t^{-\alpha}}$  where  $\gamma > 0, \alpha > 0$ . This family of laws is one of the three families of max-infinitely divisible laws of extreme value theory ([9], [19]). The following is shown in [14].

### **Corollary 2**

For  $x \in \mathbb{R}$  we denote

$$M_n^x = \sup\{|X_k^x| ; 1 \leq k \leq n\}, \quad {}_+M_n^x = \sup\{X_n^x ; 1 \leq k \leq n\}.$$

Then the sequence  $n^{-\alpha} M_n^x$  (resp  $n^{-\alpha} {}_+M_n^x$ ) converges in  $\mathbb{P}$ -law to  $\Phi_\alpha^{c\theta}$  with  $0 < \theta < 1$  (resp  $\Phi_\alpha^{c_+\theta_+}$  with  $0 < \theta_+ < 1$ , if  $c_+ > 0$ ).

Closely related properties have been intensively studied in the context of extreme value theory (see [19]). The positive number  $\theta$  is the so-called extremal index of the stochastic process  $X_n^x$  ; its inverse  $\theta^{-1}$  gives a measure of the clustering of the exceptionally large values of the process. If the random variables  $X_n^x$  were i.i.d. with law  $\nu$ , one would have  $\theta = 1$  (see [9]). If  $a(g) > 0, b(g) > 0$  for  $g \in \text{supp}\mu$ , the above corollary is proved in [15]. It is also known (see [13]) that, under hypothesis  $H_\mu$ , the normalized Birkhoff sum of  $X_n^x$

converges in law to a stable law of index  $\alpha$  if  $\alpha < 2$ . As mentioned in ([3], remark 4.8), this convergence is a consequence of extreme value properties of  $X_n^x$ , at least for  $\alpha < 1$ . The analysis of random walk in a random medium on  $\mathbb{Z}$  developed in [6] is closely related to such properties for the sojourn time of the particle at a site in  $\mathbb{Z}$ , instead of its hitting time as in [17], where Birkhoff sums as above played a dominant role. The following logarithm law is an easy consequence of Corollary 2.

**Corollary 3**

For any  $x \in \mathbb{R}$ , we have the following  $\mathbb{P} - a.e$  convergences :

$$\limsup_{n \rightarrow \infty} \frac{\ell n |X_n^x|}{\ell n(n)} = \frac{1}{\alpha}, \quad \limsup_{n \rightarrow \infty} \frac{\ell n^+(X_n^x)}{\ell n(n)} = \frac{1}{\alpha} \text{ if } c_+ > 0.$$

The so-called "logarithm law" for excursions of geodesic flow around the cusps on hyperbolic manifolds was proved in [22] and extended to more general situations in [18]. It was observed in [20] that in case of the modular surface, it is a simple consequence of Fréchet's law for geodesic flow which follows from already known extreme value properties of the continuous fraction expansion of a number  $x$  uniformly distributed on  $[0, 1]$  (see [8]).

**II - Calculation of invariant measures on  $\mathbb{R}$  in a special case**

The Lie algebra of  $G$  is generated by the vector fields  $X = a \frac{\partial}{\partial a}$ ,  $Y = \frac{\partial}{\partial b}$ . We consider the convolution semi-group of probability measures on  $G$  with infinitesimal generator  $D = X^2 + Y^2 - (\beta + 1)X$ . This operator is elliptic and we denote by  $p^t(t \geq 0)$  the associated semi-group of probability measures.

We have  $\int \ln a(g) p^t(dg) = -t(\beta + 1)$  in particular  $\int \ell n a(g) p(dg)$  is negative if  $\beta > -1$ , hence  $p^t$  has a stationary probability  $\nu$  on  $\mathbb{R}$  in this case. We consider more generally, for any  $\beta$ , the action of  $p^t$  and  $X, Y, D$  on positive measures of the form  $\nu = f(x)dx$  on the line. We denote by  $X^*, Y^*, D^*$  the operators adjoint to  $X, Y, D$ . Then the extremal solutions of the equation  $D^*f = 0$  ( $f \geq 0$ ) are described by the

**Proposition 1**

With the above notations, the equation  $D^*f = 0$  has the following normalized extremal solutions :

$$\begin{aligned} \beta \geq -1 : f(x) &= (1 + x^2)^{-(1+\beta/2)}, \\ \beta < -1 : f_+(x) &= (1 + x^2)^{-(1+\beta/2)} \int_{-\infty}^x (1 + t^2)^{\beta/2} dt, \\ &\text{and } f_-(x) = (1 + x^2)^{-(1+\beta/2)} \int_x^{\infty} (1 + t^2)^{\beta/2} dt. \end{aligned}$$

If  $\beta > -1$ , then  $\int f(x)dx < \infty$ . If  $\beta \leq -1$  then  $\int f(x)dx = \int f_+(x)dx = \int_- f(x)dx = \infty$ .

**Proof**

We calculate the action of  $X, Y$  on the measure  $\nu = f dx$  as follows.

Since  $dx$  is translation-invariant and the action of the one parameter group  $x \rightarrow x + b$  is by translation we get  $Y^*f = -f'$ .

Since  $X\varphi(x) = x\varphi'(x)$ , we get also  $X^*f(x) = -(xf(x))'$ . It follows  $D^*f(x) = (x(xf(x))' +$

$f'' + (\beta + 1)(xf)'$ , so that the equation  $D^*f = 0$  implies :

$$x(xf)' + f'(x) + (\beta + 1)(xf) = k,$$

for a certain constant  $k$ , i.e :

$$(1 + x^2)f' + (\beta + 2)(xf) = k.$$

With  $u(x) = (1 + x^2)^{-(1+\beta/2)}$  we have  $(1 + x^2)u'(x) + (\beta + 2)xu(x) = 0$ , hence the above differential equation has the solutions :  $f = u(d + kv)$  with  $v(x) = \int_0^x (1 + t^2)^{\beta/2} dt$  and  $d$  is a constant.

For  $\beta \geq -1$ , we have  $\lim_{x \rightarrow \infty} v(x) = \infty$ , hence the condition  $f \geq 0$  implies  $k = 0$ . In this case the equation  $D^*f = 0$  has only positive extremal solutions of the form  $f(x) = d(1 + x^2)^{-(1+\beta/2)}$ . For  $\beta = 0$ ,  $D$  is the hyperbolic Laplacian and we recover the Cauchy law on  $\mathbb{R}$  with density  $\frac{1}{\pi} \frac{1}{1+x^2}$ . For  $\beta > -1$ , we get a probability law with density proportional to  $(1 + x^2)^{-(1+\beta/2)}$ .

We verify that for  $\beta < -1$ , the equation  $D^*f = 0$  has two basic extremal solutions :

$$f_+(x) = (1 + x^2)^{-(1+\beta/2)} \int_{-\infty}^x (1 + t^2)^{\beta/2} dt,$$

$$f_-(x) = (1 + x^2)^{-(1+\beta/2)} \int_x^{\infty} (1 + t^2)^{\beta/2} dt.$$

The measure  $\nu$  corresponding to  $f_+$  has infinite mass and satisfies :

$$\lim_{t \rightarrow -\infty} |t|^{2+\beta} \nu(-\infty, t) = c- > 0$$

At  $+\infty$   $f_+(x)$  is asymptotic to  $c_+ x^{-1}$  with  $c_+ > 0$ . Analogous properties are valid for  $f_-$ .

Also, at  $\infty$ ,  $f(x)$  is asymptotic to  $c|x|^{-1}$  ( $c > 0$ )  $\square$

### **Remark**

The case  $\beta > -1$  corresponds to the situation of the theorem with  $\alpha = \beta + 1$ .

The case  $\beta = -1$  corresponds to the (critical) situation of [1], [4]. Then the unique basic extremal solution behaves at infinity like multiplicative Lebesgue measure on  $\mathbb{R}^*$ .

The situation  $\beta < -1$ , with two extremal solutions, corresponds to a so-called phase transition in *P.D.E* theory, for example in the context of non linear Schrödinger equations.

### **III - Proof of theorem 1**

The proofs of 1) and 2) in [10] are based on the first renewal equation in Lemma 1 below. A delicate point in [10] for the use of the renewal theorem (see [7]) is solved by replacing  ${}^\alpha f(t) = e^{\alpha t} f(t)$  by a related directly Riemann-integrable function. Here we give only the proofs of 3) and 4). We will now assume  $\mu\{g ; a(g) > 0\} = 1$  and we will only study the non vanishing of  $c_+$ . To do that we need some preliminary notations and results.

Let  $T$  be the stopping time on  $\Omega$  defined by :

$$T = \{n \geq 1 ; g_1 g_2 \cdots g_n \in G_+\}, \quad T = \infty \text{ if } \{n \geq 1 ; g_1 g_2 \cdots g_n \in G_+\} = \phi,$$

where  $G_+ = \{b(g) > 0\}$ .

We denote by  $\bar{\mu}$  the probability on the additive group  $\mathbb{R}$  given by  $\bar{\mu}(A) = \mu\{\ell n a(g) \in A\}$

Moreover we denote by  $\mu_T$  the positive measure on  $\mathbb{R}$  defined by :

$$\mu_T(A) = \mathbb{P}\{T < +\infty ; \ell n(a_1 a_2 \cdots a_T) \in A\},$$

where  $A$  is a Borel subset of  $\mathbb{R}$ . We have  $\mu_T(\mathbb{R}) = \mathbb{P}(T < +\infty) \leq 1$ , and we denote by  $\mu_T^n$  the  $n^{\text{th}}$  convolution power of  $\mu_T$  on the additive group  $\mathbb{R}$ . Define  $f$  by

$$f(t) = \mathbb{P}\{R > e^t\} = \nu(]e^t, +\infty[) \quad t \in \mathbb{R},$$

and write  $R_n = \sum_{k=1}^n a_1 a_2 \cdots a_{k-1} b_k$ ,  $S_n = \sum_{k=1}^n \ell n(a_k)$ .

Then we have the :

**Lemma 1**

1) For every real  $t$ , we have  $f(t) = \bar{\mu} * f(t) + f_1(t) = \mu_T * f(t) + h_1(t)$  where :

$$f_1(t) = \mathbb{P}\{R - b_1 > e^t\} - \mathbb{P}\{\mathbb{R} > e^t\}, \quad h_1(t) = \mathbb{E}\{1_{[T < +\infty]} \nu(]e^{-S_T}(e^t - R_T), e^{t-S_T}])\}$$

2) For every real  $t$ , we have  $f(t) = \sum_{n=0}^{+\infty} \mu_T^n * h_1(t) = \sum_{n=0}^{\infty} \mu^n * f_1(t)$ .

If  $p$  is a bounded measure on  $\mathbb{R}$  and  $\varphi$  is a positive Borel function, we write

$$p * \varphi(t) = \int \varphi(t-x)p(dx), \quad t \in \mathbb{R}.$$

We denote by  ${}^\alpha\mu$ , the probability measure on  $G$  defined by :  ${}^\alpha\mu(dg) = a^\alpha(g)\mu(dg)$ .

We define the probability  ${}^\alpha\mathbb{P}$  on  $G^{\mathbb{N}}$  by  ${}^\alpha\mathbb{P} = {}^\alpha\mu^{\otimes \mathbb{N}}$  and we write  ${}^\alpha\mathbb{E}$  for the corresponding expectation.

The measure  ${}^\alpha\mu_T$  on  $\mathbb{R}$  is defined by  ${}^\alpha\mu_T(A) = {}^\alpha\mathbb{E}(1_A(\ell n(a_1 \cdots a_T)))$ ,

and we write  ${}^\alpha h_1(t) = e^{\alpha t} h_1(t) \quad t \in \mathbb{R}$ .

Then from lemma 1 we get :

**Lemma 2**

For every real  $t$  we have  ${}^\alpha f(t) = \sum_{n=0}^{+\infty} {}^\alpha\mu_T^n * {}^\alpha h_1(t)$

Now we are going to study some properties of  $T$  and  $\ell n(a_1 a_2 \cdots a_T)$  under  ${}^\alpha\mathbb{P}$ . For that purpose we consider the new random variables  $g'_i (i \geq 1)$  defined by  $g'_i = (a_i^{-1}, b_i a_i^{-1})$ . Under  ${}^\alpha\mathbb{P}$ , there random variables are i.i.d with law  ${}^\alpha\mu'$ . We have :

$$g'_n g'_{n-1} \cdots g'_1 = ((a_1 a_2 \cdots a_n)^{-1}, R_n(a_1 \cdots a_n)^{-1}),$$

hence for  $T' = \text{Inf}\{n ; g'_n g'_{n-1} \cdots g'_1 \in G_+\}$  we have  $T' = T$ . It follows that  $T$  can be interpreted as the entrance time in  $\mathbb{R}_+ = ]0, \infty[$  of the affine random walk on  $\mathbb{R}$  defined by  ${}^\alpha\mu'$ , starting from 0. We denote by  ${}^\alpha Q$  the Markov kernel of this affine random walk, and for  $p \in \mathbb{R}$  we write  $p_n = g'_n g'_{n-1} \cdots g'_1 p$ .

**Lemma 3**

1) There exists a unique probability measure  ${}^\alpha\nu'$  on  $\mathbb{R}$  such that  ${}^\alpha Q({}^\alpha\nu') = {}^\alpha\nu'$ . The probability  ${}^\alpha\nu'$  has no atoms.

2) If  ${}^\alpha\nu'([0, +\infty]) > 0$  then  $0 < {}^\alpha\mathbb{E}(T') < \infty$ .

Now we complete the proof of Theorem 1 using the above Lemmas.

For assertion 3, there are two cases.

First case  ${}^\alpha\nu'([0, +\infty]) > 0$ .

Then by Lemma 3 and the observation before Lemma 3,  ${}^\alpha\mathbb{E}(T) = {}^\alpha\mathbb{E}(T') < \infty$ ,  ${}^\alpha\mu_T(\mathbb{R}) = 1$ . By Wald's lemma (see [7]), since  $T' < \infty$   $\alpha\mathbb{P} - a.e$  :

$${}^\alpha\mathbb{E}\{\ell n(a_1 a_2 \cdots a_T)\} = {}^\alpha\mathbb{E}(\ell n(a_1)) {}^\alpha\mathbb{E}(T)$$

where  ${}^\alpha\mathbb{E}(\ell n(a_1)) = \mathbb{E}(a_1^\alpha \ell n(a_1))$  is finite and positive, hence  ${}^\alpha\mathbb{E}(S_T)$  is finite and positive. Assume  $c_+ = 0$ , hence  $\lim_{t \rightarrow \infty} {}^\alpha f(t) = 0$ . Then, if we denote by  ${}^\alpha h_{1,L}$  ( $L > 0$ ) the function  $t \rightarrow {}^\alpha h_1(t) 1_{[-L,L]}(t)$ , we have using Lemma 2 and Proposition A below : for every  $L > 0$ ,

$$0 = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sum_{n=0}^{+\infty} {}^\alpha \mu_T^n * {}^\alpha h_{1,L}(s) ds = \frac{1}{{}^\alpha\mathbb{E}(\ell n(a_1)) {}^\alpha\mathbb{E}(T)} \int_{-L}^L {}^\alpha h_1(s) ds,$$

hence  $0 = \int_{\mathbb{R}} {}^\alpha h_1(s) ds$ . Since  ${}^\alpha h_1$  and  $h_1$  are non negative we get  $h_1 = 0$   $a.e$ , hence Lemma 1 implies  $f(t) = 0$ ,  $dt - a.e$ .

We conclude that for almost every real  $s$  :

$$f(s) = \mathbb{P}(R > e^s) = 0,$$

and so  $\mathbb{P}(R \leq 0) = 1$ , hence  $supp\nu \subset ]-\infty, 0]$ . It follows that  $supp\mu$  preserves an interval  $(-\infty, v_0)$  with  $v_0 \leq 0$ .

Second case  ${}^\alpha\nu'([0, +\infty]) = 0$ .

Denote by  $v_0 \leq 0$  the upper bound of the support of the probability  ${}^\alpha\nu'$ . Then by the stationarity property of  ${}^\alpha\nu'$  we can write that for every  $n \geq 1$  :

$${}^\alpha\mathbb{P}\{g'_n g'_{n-1} \cdots g'_1 v_0 \leq v_0\} = 1, \quad 1 = \mathbb{E}(a_1^\alpha \cdots a_n^\alpha 1_{\{v_0 + R_n \leq a_1 \cdots a_n v_0\}}),$$

which implies that for every integer  $n \geq 1$ ,  $\mathbb{P}(R_n \leq -v_0) = 1$  since  $\mathbb{E}(a_1^\alpha) = 1$ . Since  $R_n$  converges  $\mathbb{P} - a.e$  to  $R$  we have  $\mathbb{P}(R \leq -v_0) = 1$  hence  $c_+ = 0$ .

In conclusion we see that  $c_+ = 0$  if and only if the upper bound of  $supp\nu$  is finite i.e if  $supp\mu$  preserves an interval  $] - \infty, -v_0]$ .

In order to show assertion 4 we will distinguish the 2 cases  $c_+ > 0$ ,  $c_- = 0$ ,  $c_+ > 0$ , and  $c_- > 0$ . We observe that  $supp\nu$  is invariant under  $supp\mu$  and condition  $\int \ell n(a(g)) d\mu(g) < 0$  implies that for some  $g \in (supp\mu)^2$  we have  $0 < a(g) < 1$ . Also the complement of  $supp\nu$  is invariant under  $(supp\mu)^{-1}$ . We denote by  $T_\mu$  the closed subsemigroup of  $G$  generated by  $supp\mu$ , and by  $\Delta \subset \mathbb{R}$  the closure of the set of attractive fixed points of the elements of  $T_\mu$ . We observe that  $T_\mu \Delta \subset \Delta$ . Since for any  $x \in \Delta$  the law of  $g_n \cdots g_1 x$  is supported by  $\Delta$  and converges to  $\nu$ , we obtain that  $\Delta \supset supp\nu$ . Since the attractive fixed points of  $T_\mu$  belong to  $supp\nu$ , we conclude that  $\Delta = supp\nu$ . Then, for any open interval  $I = [a, b] \subset \mathbb{R}$ ,  $n < 0$ ,  $g^n(I)$  is an interval of length  $a^n(g)(b - a)$  which converges to  $+\infty$ ,  $-\infty$  or  $\mathbb{R}$ , depending of the relative positions of  $I$  and the fixed point  $x_0$  of  $g$ . If  $c_+ > 0$  and  $c_- = 0$ , then from above  $supp\mu$  preserves the interval  $[\tau, \infty[$  with  $\tau = Inf(supp\nu)$ . Since  $\Delta = supp\nu$  we can choose  $g \in (supp\mu)^2$  such that its fixed point  $x_0 \in supp\nu$  is arbitrary close to  $\tau$ , and in particular  $\tau \leq x_0 < a$ . If  $I \subset ]\tau, \infty[$  satisfies  $\nu(I) = 0$  then  $\nu(g^n(I)) = 0$  for  $n < 0$  ; since the length of the interval  $g^n(I)$  is  $a^n(g)(b - a)$  and  $\lim_{n \rightarrow -\infty} a^n(g) = \infty$  this contradicts  $c_+ > 0$ .

If  $c_+ > 0$ ,  $c_- > 0$  the same argument is valid for any interval  $I$  with  $\nu(I) = 0$ .  $\square$

We now give the proofs of the above lemmas.

**Proof of Lemma 1**

1) Denote  $R^n = \sum_{k=n}^{+\infty} a_{n+1} \cdots a_k b_{k+1}$ . Under  $\mathbb{P}$  the law of  $R_n$  is  $\nu$  and moreover  $R^n$  is independant of the random variables  $g_i (1 \leq i \leq n)$ .

The formula  $R = R_n + a_1 \cdots a_n R^n$  gives  $R - b_1 = a_1 R^1$ , hence :

$$\mathbb{P}\{R - b_1 > e^t\} = \mathbb{P}\{R^1 > e^t a_1^{-1}\} = \mu * f(t), \quad f(t) = \mu * f(t) + f_1(t)$$

We have also from above

$$\begin{aligned} \{R > e^t\} &= \{R_T + a_1 a_2 \cdots a_T R^T > e^t, T < \infty\} \\ &= \{R^T > e^{t - \ell n(a_1 a_2 \cdots a_T)}; T < \infty\} \cup \{e^{t - \ell n(a_1 a_2 \cdots a_T)} < R^T \leq e^{t - \ell n(a_1 a_2 \cdots a_T)}; T < \infty\} \end{aligned}$$

Using the fact that  $T$  is a stopping time we have

$$f(t) = \mathbb{P}\{R > e^t\} = \mathbb{P}\{R > e^t, T < \infty\} = \mu_T * f(t) + h_1(t)$$

where  $h_1(t) = \mathbb{E}(1_{\{T < \infty\}} \nu] e^{t - \ell n(a_1 \cdots a_T)} - R_T, e^{t - \ell n(a_1 \cdots a_T)}]$ ,

It follows :

$$f = \sum_{k=0}^n \bar{\mu}^k * f_1 + \bar{\mu}^{n+1} * f$$

where  $\bar{\mu}^{n+1} * f(t) = \mathbb{P}\{R > e^t (a_1 \cdots a_{n+1})^{-1}\}$ . The condition  $\mathbb{E}(\ell n(a_1)) < 0$  implies the  $\mathbb{P} - a.e$  convergence of  $(a_1 \cdots a_{n+1})^{-1}$  to  $\infty$ , hence  $\lim_{n \rightarrow \infty} \bar{\mu}^{n+1} * f(t) = 0$ . The first part of the formula follows.

2) From above we deduce that for every integer  $n$  and  $t \in \mathbb{R}$ .

$$f(t) = \sum_{j=0}^n \mu_T^j * h_1(t) + \mu_T^{n+1} * f(t).$$

We now prove that  $\lim_{n \rightarrow +\infty} \mu_T^{n+1} * f_1(t) = 0$ .

There are two cases

Case 1)  $\mathbb{P}(T < \infty) < 1$

We have

$$0 \leq \mu_T^{n+1} * f(t) \leq (\mathbb{P}(T < \infty))^n$$

hence  $\lim_{n \rightarrow \infty} \mu_T^{n+1} * f(t) = 0$ .

Case 2)  $\mathbb{P}(T < \infty) = 1$

Define the shift  $\theta$  on  $\Omega$  by  $\theta(\omega) = (g_{i+1}(\omega), i \geq 1)$  where  $\omega = (g_i(\omega), i \geq 1)$  and consider the sequence  $(T_n(\omega))_{n \geq 1}$  of random times defined  $\mathbb{P} - a.e$  by  $T_{n+1} = T_0 \theta^{T_n}$ ,  $T_1 = T$ . Under  $\mathbb{P}$  the sequence of random variables  $[(T_1, S_{T_1}), \cdots, (T_{n+1} - T_n, S_{T_{n+1}} - S_{T_n})]$ , is i.i.d and the law of  $S_{T_n}$  is  $\mu_T^n$ . Because  $\mathbb{E}(\ell n(a_1)) < 0$ , we have  $\mathbb{P} - a.e \lim_{n \rightarrow \infty} S_n = -\infty$  and moreover

$\lim_{n \rightarrow \infty} T_n = \infty$  hence  $\mathbb{P} - a.e, \lim_{n \rightarrow \infty} S_{T_n} = -\infty$ . We have that

$$\mu_T^{n+1} * f(t) = \mathbb{E}(f(t - S_{T_{n+1}})),$$

and  $\lim_{t \rightarrow \infty} f(t) = 0$ . So, using Lebesgue's theorem, we can conclude that  $\lim_{n \rightarrow \infty} \mu_T^{n+1} * f(t) = 0$

□

**Proof of lemma 2**

Lemma 2 us a direct consequence of the formula  ${}^\alpha \mu_T^n * {}^\alpha h(t) = e^{\alpha t} \mu_T^n * h_1(t)$ , Lemma 1 part 2, and the fact that  $h_1$  is non negative. □



### **Proof of lemma 3**

The definition of  ${}^\alpha\mu'$  and the condition  $H_\mu(3)$  imply  $\int |\ell n(a(g))| {}^\alpha\mu'(dg) < \infty$ . The strict convexity of the function  $\ell n \int a^s(g)\mu(dg)$  ( $s > 0$ ) gives  $\int \ell n(a(g)) {}^\alpha\mu'(dg) < 0$ .

It follows  $\int |\ell n|b(g)| {}^\alpha\mu'(dg) < \infty$ .

As observed above, the existence and uniqueness of  ${}^\alpha\nu'$  follows.

If  $x_0$  is a fixed point of  $\text{supp}{}^\alpha\mu'$  then for any  $(a, b) \in \text{supp}\mu$  ;

$$a^{-1}x_0 + ba^{-1} = x_0, \text{ i.e } x_0(a - 1) = b.$$

This implies that  $-x_0$  is a fixed point of  $\text{supp}\mu$ , which contradicts  $H_\mu(4)$ . Hence, as it well known (see [5]),  ${}^\alpha\nu'$  has no atom.

In order to show  ${}^\alpha\mathbb{E}(T') < \infty$  we consider the space  ${}^a\Omega^\# = \mathbb{R} \times G^{\mathbb{Z}}$  and the extended bilateral shift defined by  ${}^a\theta(p, \omega) = (p_1, \theta\omega)$  where  $p_1 = g'_1(p)$  and  $\theta$  is the bilateral shift on  $G^{\mathbb{Z}}$ . We endow  ${}^a\Omega^\#$  with the Markov measure  $\kappa^\#$  associated with the  ${}^\alpha Q$ -invariant probability  ${}^\alpha\nu'$ . Clearly  $\kappa^\#$  is  ${}^a\theta$ -invariant and ergodic. Also we consider the fibered bilateral Markov chain  $(p_n, V_n)$  on  $\mathbb{R} \times \mathbb{R}^*$  where  $V_n = p^{-1}p_n(a_1a_2 \cdots a_n) = p^{-1}(p + R_n)$ . Let  $\tau$  be the first "ladder epoch" of  $(p_n, V_n)$  (see [7]), i.e  $\tau = \text{Inf}\{n \geq 1 ; V_n > 1\}$ , hence  $p^{-1}p_\tau > 0$  and  $\tau = T$  if  $p > 0$ . We observe that the conditions in  $H_\mu(3)$  implies  $\int |p|^\varepsilon {}^\alpha\nu'(dp) < \infty$  for some  $\varepsilon > 0$ , hence  $\limsup_{|n| \rightarrow \infty} \frac{1}{|n|} \ell n|p_n| \leq 0$ . Since  ${}^\alpha\mathbb{E}(\ell n(a_1)) > 0$  the

ergodic theorem gives  $\lim_{n \rightarrow \infty} |V_n| = \infty$ ,  $\lim_{n \rightarrow -\infty} |V_n| = 0$ , in particular  $\tau$  is finite  $\kappa^\# - a.e$ .

Since  ${}^\alpha\nu'(\mathbb{R}_+) > 0$  we can consider the Markov kernel  ${}^\alpha Q_+$  induced by  ${}^\alpha Q$  on  $\mathbb{R}_+$  ; the normalized restriction  ${}^\alpha\nu'_+$  of  ${}^\alpha\nu'$  to  $\mathbb{R}_+$  is  ${}^\alpha Q_+$ -invariant and ergodic. We denote by  ${}^a\Omega^\#_+$  the subset of  ${}^a\Omega^\#$  defined by the conditions  $p_n > 0$  infinitely often for  $n = n_k > 0$  and  $n = n_{-k} < 0$ . Since  ${}^\alpha\nu'(\mathbb{R}_+) > 0$ ,  ${}^a\Omega^\#_+$  has positive  $\kappa^\#$ -measure and we denote by  $\kappa^\#_+$  the normalized restriction of  $\kappa^\#$  to  ${}^a\Omega^\#_+$  ; then  $\kappa^\#_+$  is invariant and ergodic under the corresponding induced shift  ${}^a\theta_+$ . From above we know that  $\lim_{k \rightarrow \infty} V_{n_{-k}} = 0$ , hence the time

$\tau_+(\omega) = n_{-j}$  ( $j \geq 0$ ), of the last strict maximum of  $V_{n_{-k}}$  is finite  $\kappa^\#_+ - a.e$ . We define  ${}^a\Omega^\#_0 = \{\sup_{k>0} V_{n_{-k}} < 1\} = \{\tau_+ = 0\}$ . Then we have  ${}^a\kappa^\#_+$  ( ${}^a\Omega^\#_0$ )  $> 0$  since, by  ${}^a\theta_+$ -invariance

of  $\kappa^\#_+$  :

$$\begin{aligned} 1 &= \kappa^\#_+\{\tau_+ > -\infty\} = \sum_{n \geq 0} \kappa^\#_+\{\tau_+ = -n\} \leq \sum_{n \geq 0} \kappa^\#_+\{V_{-n} > \sup_{n_{-k} < -n} V_{n_{-k}}\} = \\ &\sum_{n \geq 0} \kappa^\#_+\{0 > \sup_{k > 0} V_{n_{-k}}\} \leq \infty \kappa^\#_+({}^a\Omega^\#_0). \end{aligned}$$

On the other hand, the definition of  $\tau$  shows that for  $\omega \in {}^a\Omega^\#_0$ ,  $\tau(\omega)$  is the first return time of  ${}^a\theta^k(\omega)$  to  ${}^a\Omega^\#_0$ , so that  ${}^a\theta^\tau$  is the transformation of  ${}^a\Omega^\#_0$  induced by  ${}^a\theta$  on  ${}^a\Omega^\#_0$ . Then Kac's theorem (see [23]) implies that  ${}^a\theta^\tau$  is ergodic with respect to the normalized restriction  $\kappa^\#_0$  of  $\kappa^\#_+$  to  ${}^a\Omega^\#_0$  and  ${}^\alpha\mathbb{E}(\tau) = \int \tau(\omega)\kappa^\#_0(d\omega) < \infty$ . Also we denote by  ${}^\alpha\nu^\tau_+$  the push forward of  $\kappa^\#_0$  to  $\mathbb{R}_+$  under the map  $\omega \rightarrow p_0(\omega)$ . Since the stopped kernel  ${}^\alpha Q^\tau_+$  and the map  $\tau$  commute with  $p_0$ , the measure  $\kappa^\tau_+$  is  ${}^\alpha Q^\tau_+$ -invariant, ergodic and absolutely continuous with respect to  ${}^\alpha\nu_+$  with  ${}^\alpha\mathbb{E}_0(\tau) = {}^\alpha\mathbb{E}(T) < \infty$ .  $\square$

**Remark**

A different proof of  ${}^\alpha\mathbb{E}(T) < \infty$  uses the interpretation of  $T = T'$  as hitting time of the open set  $\mathbb{R}_+$  by the Markov chain with kernel  ${}^\alpha Q$  starting from 0.

Since  $\int a^\delta(g) {}^\alpha\mu'(dg) < \infty$ ,  $\int |b^\delta(g)| {}^\alpha\mu'(dg) < \infty$  for  $0 < \delta < \alpha$ , the operator defined by  ${}^\alpha Q$  on a space of Hölder functions on  $\mathbb{R}$  (as in [13]) has a spectral gap. This implies  ${}^\alpha\mathbb{E}(T') < \infty$ . The proof given above extends to the multidimensional case.

**IV - Appendix : a weak renewal theorem**

Proposition A Let  $(Z_n)_{n \geq 1}$  a sequence of independant, identically distributed real random variables on  $\mathbb{R}$  with law  $\eta$ . Assume that  $\int |z|\eta(dz) < +\infty$  and that  $\gamma = \int z\eta(dz) > 0$ .

Let  $\psi$  a bounded non negative Borel function which is supported on  $[-a, a]$ .

Then the potential  $U\psi = \sum_{n=0}^{+\infty} \eta^n * \psi$  is a bounded function and we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t U\psi(s) ds = \frac{1}{\gamma} \int_{\mathbb{R}} \psi(t) dt$$

**Proof**

If  $\Sigma_n = \sum_{i=1}^n Z_i$ , we have :  $U\psi(s) = \sum_{n=0}^{+\infty} E[\psi(s - \Sigma_n)]$ .

Because  $\gamma = \int z\eta(dz) > 0$  the random walk on  $\mathbb{R}$  with law  $\eta$  is transient and using the maximum principle we have that  $\sup_{s \in \mathbb{R}} U\psi(s) < +\infty$ .

For  $\varepsilon > 0$ ,  $t > 0$  denote

$$\begin{aligned} n_1(t) &= [\frac{1}{\gamma}\varepsilon t] = n_1, & n_2(t) &= [\frac{1}{\gamma}(1 - \varepsilon)t] = n_2, \\ U_n\psi &= \sum_0^{n-1} \eta^k * \psi, & U^n\psi &= \sum_{n+1}^{\infty} \eta^k * \psi, & U_n^m\psi &= \sum_n^m \eta^k * \psi. \end{aligned}$$

Then we have

$$I(t) = \frac{1}{t} \int_0^t U\psi(s) ds = \sum_1^3 I_k(t) - I_4(t)$$

where

$$\begin{aligned} I_1(t) &= \frac{1}{t} \int_{\mathbb{R}} U_{n_1}^{n_2} \psi(s) ds, & I_2(t) &= \frac{1}{t} \int_{\mathbb{R}} U_{n_1} \psi(s) ds, \\ I_3(t) &= \frac{1}{t} \int_{\mathbb{R}} U^{n_2} \psi(s) ds, & I_4(t) &= \frac{1}{t} \int_{\mathbb{R}[0,t]} U_{n_1}^{n_2} \psi(s) ds. \end{aligned}$$

We have

$$\begin{aligned} I_1(t) &= \frac{n_2 - n_1 + 1}{t} (\int_{\mathbb{R}} \psi(s) ds) \text{ hence } \lim_{t \rightarrow +\infty} I_1(t) = \frac{(12\varepsilon)}{\gamma} \int_{\mathbb{R}} \psi(s) ds, \\ 0 \leq I_4(t) &\leq (\frac{(n_2 - n_1 + 1)}{t} \sup_{s \in \mathbb{R}} |\psi(s)|) \sup_{n_1 \leq n \leq n_2} [\mathbb{P}(\Sigma_n \leq a) + \mathbb{P}(t - \Sigma_n \leq a)]. \end{aligned}$$

By the law of large numbers we know that  $\mathbb{P} - a.e$ ,  $\lim_{n \rightarrow +\infty} \frac{\Sigma_n}{n} = \gamma > 0$ , hence :

$$\lim_{t \rightarrow +\infty} \sup_{n_1 \leq n \leq n_2} (\mathbb{P}\{\Sigma_n \leq a\} + P\{t - \Sigma_n \leq a\}) = 0.$$

Hence  $\lim_{t \rightarrow +\infty} I_4(t) = 0$  and :  $0 \leq \frac{I_2(t)}{t} \leq \frac{\varepsilon}{\gamma} \times \int \psi(s) ds$ .

Consider now  $I_3(t)$  and denote for  $n \in \mathbb{N}$ ,  $s > 0$  :  $\rho_n^s = \text{Inf}\{k \geq n ; |V_n - s| \leq a\}$ .

We use the interpretation of  $U^n\psi$  as the expected number of visits to  $\psi$  after time  $n$  :  $U^n\psi(x) \leq (U\psi)\mathbb{P}\{\rho_n^s < \infty\}$  with  $n \frac{[(1+\varepsilon)t]}{\gamma} = n_2$ , hence :

$$I_3(t) \leq |U\psi|\mathbb{P}\{\Sigma_k \leq t + a \text{ for some } k \geq \frac{(1+\varepsilon)t}{\gamma}\}.$$

Since  $\frac{\Sigma_n}{n}$  converges to  $\gamma$ ,  $\mathbb{P} - a.e$ , we get  $\lim_{t \rightarrow \infty} I_3(t) = 0$ .

Since  $\varepsilon$  is arbitrary we get finally :  $\lim_{t \rightarrow \infty} I(t) = \frac{1}{\gamma} \int \psi(s)ds$ .  $\square$

## References

- [1] Babillot M., Bougerol P., Elie L. : The random difference equation  $X_n = A_n X_{n-1} + B_n$  in the critical case. Ann. Probab. 25,1 (478-493) (1997).
- [2] Bardou F., Bouchaud J.P., Aspect A., Cohen-Tannoudji C. : "Levy Statistics and Laser Cooling". Cambridge Univ. Press (2002).
- [3] Basrak B., Segers J. : Regularly varying time series. Stochastic Pocess Appl. 119, 1055-1080 (2009).
- [4] Brofferio S., Buraczewski D., Damek E. : On the invariant measure of the random difference equation  $X_n = A_n X_{n-1} + B_n$  in the critical case. Ann. I.H.P. 48, 377-395 (2012).
- [5] Buraczewski D. and Damek E.,Guivarc'h Y., Hulanicki A., Urban R. : Tail-homogeneity of stationary measures for some multidimensional stochastic recursions. Probab. Theory Relat. Fields 145 n°3-4 385-420 (2009).
- [6] Dolgopyat D., Goldsheid I. : Quenched limit theorems for nearest neighbour random walks in 1D random environment. Comm. Math. Phys. 315, 1, 247-277 (2012).
- [7] Feller W. : "An Introduction to Probability Theory and its Applications". vol. II, 2nd edn. Wiley New-York (1971).
- [8] Galambos J. : The distribution of the largest coefficient in the continued fraction expansion. Quart. J. Math. 23, 147-151 (1972).
- [9] Gnedenko B. : Sur la distribution limite d'une série aléatoire. Annals of Math. 44 n°3, 423-453 (1943).
- [10] Goldie C.M. : Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. 1(1) 126-166 (1991).

- [11] Guivarc'h Y. : Heavy tail properties of stationary solutions of multidimensional stochastic recursions. In "Dynamics and Stochastics". IMS Lecture Notes Monograph Series, vol. 48, pp 85-99. Institute of Mathematical Statistics, Beachwood (2006).
- [12] Guivarc'h Y., Le Page E. : Homogeneity at infinity of stationary solutions of Multivariate Affine Stochastic Recursions. In "Random Matrices and Iterated Random functions". Alsmeyer G. - Löwe M. Eds. Springer Proceedings in Mathematics and Statistics 53, pp 119-135 (2013).
- [13] Guivarc'h Y., Le Page E. : On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks, Ergod. Th. Dynam. Sys. 28, 423-446 (2008).
- [14] Guivarc'h Y., Le Page E. : Asymptotique des valeurs extrêmes pour les marches aléatoires affines. C.R.A.S (2013).
- [15] Haan L., Resnick S.I., Rootzen H., Vries C.G. : Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. Stochastic Process. Appl. 32, 213-224 (1989).
- [16] Kesten H. : Random difference equations and renewal theory for products of random matrices. Acta Mathematica 131, 207-248 (1973).
- [17] Kesten H., Kozlov M., Spitzer F. : A limit law for random walk in a random environment, Compositio Math. 30 145-168 (1975).
- [18] Kleinbock D.Y., Margulis G.A. : Logarithm laws for flows on homogeneous spaces, Inv. Math. 138, 451-494 (1999).
- [19] Leadbetter M.R., Lindgren G., Rootzen H. : "Extremes and related properties of random sequences and processes", Springer, Berlin (1983).
- [20] Pollicott M. : Limiting distributions for geodesic excursions on the modular surface, Contemp. Math. 469, 117-185 (2008).
- [21] Solomon F. : Random walks in a random environment. Ann. Proba. 3, 1-31 (1975).
- [22] Sullivan D. : Disjoint spheres, approximation by imaginary quadratic numbers and the logarithm law for geodesics, Acta Math. 149, 215-237 (1982).
- [23] Wolfowitz J. : Remarks on the notion of recurrence. Bull A.M.S. 55, 394-395 (1949).