Optimal Results On TV Bounds For Scalar Conservation Laws With Discontinuous Flux

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Abstract

This paper is concerned with the total variation of the solution of scalar conservation law with discontinuous flux in one space dimension. One of the main unsettled questions concerning conservation law with discontinuous flux was the boundedness of the total variation of the solution near interface. In [1], it has been shown by a counter example at $T = 1$, that the total variation of the solution blows up near interface, but in that example the solution become bounded variation after time $T > 1$. So the natural question is that what happens to the BV ness of the solution for large time. Here we give a complete picture of the bounded variation of the solution for all time. For a uniform convex flux with only L^∞ data, we obtain a natural smoothing effect in BV for all time $t > T_0$. Also we give a counter example (even for a BV data) to show that the assumptions which has been made are optimal.

Key words: Hamilton-Jacobi equation, scalar conservation laws, discontinuous Flux, explicit Formula, characteristic lines, BV function.

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1 Introduction:

In this paper, we investigate the total variation bound of the following scalar conservation laws

$$\begin{array}{rcll} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} F(x,u) & = & 0 & \text{if } x \in I\!R, t > 0 \\ u(x,0) & = & u_0(x) & \text{if } x \in I\!R \end{array} \tag{1.1}$$

where the flux function $F(x,u)$ is a discontinuous function of x given by $F(x,u) = H(x)f(u) + (1-H(x))g(u)$, H is the Heaviside function. Here we consider f, g to be strictly convex function with superlinear growth. That is

$$\lim_{|u|\to\infty} \left(\frac{f(u), g(u)}{|u|} \right) = (\infty, \infty). \tag{1.2}$$

Equation (1.1) has been extensively studied since few decades from both the theoretical and numerical points of view. Notice that a very few results are known regarding the total variation of the solution of (1.1) near interface.

A conservation laws with a discontinuous flux of the form (1.1), is a first order hyperbolic model, which arises in many applicative problems. It has a huge application in fluid flows in heterogeneous media such as two-phase flow in a porous medium, which arises in the petroleum industry. (1.1) also arises while dealing with modeling gravity, continuous sedimentation in a clarifier thickener unit [16, 17, 15, 14, 14, 23, 24]. Some other Applications are in the model of car traffic flow on a highway (see [47]) and in ion etching in the semiconductor industry (see [48]).

It is well understood that even if F and u_0 are smooth, the solution of (1.1) may not admit classical solution in finite time, hence one must define the notion of weak solutions. In general, weak solutions of (1.1) are not unique. Due to this fact, one has to put some extra condition so called "entropy condition" in order to get the uniqueness. When $f = g$, Kruzkov[40] has introduced the most general entropy condition in order to prove the uniqueness by using doubling variable technique. For $f \neq g$, the general entropy condition has been established in [5], [8], [7], [18], [10], [45] in order to prove the uniqueness.

When $f = g$, existence of the solution has been studied in several ways, namely vanishing viscosity method (see [40]), convergence of numerical schemes, front tracking method(see [31]) and via Hamilton-Jacobi equation (see [25], [42]).

In general, when $f \neq g$, (1.1) may not admit any solutions, hence for existence, some extra assumptions are required. Under the assumption that the fluxes f and g coincide at least two points Gimse-Risebro [28, 27], Diehl [22] obtained a solution for Riemann data. (also see [39, 23]). A theory by using front tracking method has been developed in [27], [35],[39]. Under the assumptions that the fluxes f and g are strictly

convex and C^2, (1.1) also has been studied in [5]. They obtained a Lax-Oleinik type formula satisfying the following interface entropy condition

$$\text{meas}\big\{t : f'(u^+(t)) > 0, g'(u^-(t)) < 0\big\} = 0 \tag{1.3}$$

and Lax-Olenik entropy condition for $x \neq 0$. Also they prove the L^1-contraction semi-group. On the otherhand, Karlsen, Risebro, Towers [35, 36], used a modified Kruzkhov type entropy condition and they proved L^1 stability of entropy solutions. In general, these solutions admit undercompressive waves at the interface $x = 0$ which is not allowed in the classical theory of Lax-Olenik and Kruzkov (see, [40]). From the model of capillary diffusion, Kaasschieter [34] studied this problem by using a different diffusion term and he noticed that the solution satisfies interface entropy condition (1.3). Some cases like clarifier-thickner model also allows undercompressive at the interface [14, 15, 16, 13, 17, 35, 19, 21, 20, 36].

In [8], Adimurthi et. al. characterized an infinitely many stable semigroups of entropy solutions ((A, B) entropy) in terms of explicit Hopf-Lax type formulas under the assumption that either the fluxes are strictly convex or strictly concave. This general theory known as (A, B) interface entropy conditions. It was shown that (A, B)-entropy solution exists and forms an L^1-contractive semi-group and is unique. Convergence of a numerical scheme that approximates entropy solutions of type (A, B) for any connection (A, B) has been encountered in [7, 18]. Proof of convergence for the the Godunov or Enquist-Osher type scheme has been encountered as in [6, 44, 37, 53, 52, 13, 9] and the Lax-Friedrichs type as in [38]. Very recently, Andreianov et.al. [10] characterized a more general concept of "vanishing viscosity" germs and they proved the L^1 contraction property.

BV regularity of a solution for conservation laws is an important phenomenon in order to understand the convergence and the existence of traces of the solution. For $f = g$, it is well understood that the solution is TVD, that is $TV(u(\cdot, t)) \leq TV(u_0)$, for all $t > 0$. By Lax-Oleinik type formula it is easy to see that under the assumption $f'' \geq \alpha$, for some $\alpha > 0$, $u(\cdot, t) \in BV_{loc}$, even if $u_0 \notin BV$. We ask the question that can one expect the similar result when $f \neq g$? When $f = g$, it has been noticed in [3] that the assumption of uniformly convexity is optimal in order to prove the BV regularity without the the assumption that $u_0 \notin BV$. For the case when $f \neq g$, we cannot expect total variation diminishing property due to the fact that constant data gives rise to a non-constant solution and hence the total variation increases. Due to the lack of the total variation bound the convergence theory for the discontinuous flux has been studied by using different technique, namely, singular mapping technique in [43, 52, 6, 39], [7]. It has been noticed in [12] that the solution is of total variation bounded away from the interface $x = 0$. But the information at the interface $x = 0$ was completely unknown. The open problem was **what happens to the total variation of the solution near interface?**

A counter example has been given in [1] by choosing a proper initial data $u_0 \in$

BV and proved that $u(\cdot,1) \notin BV$, but in that example the solution become BV due to the fact that the characteristics intersects after time $T = 1$ and the solution become smoother. Now repeatation of this kind of initial data will not work in order to get $u(\cdot,t) \notin BV$ for $t > 1$, due to the fact that the intersection of characteristics makes the solution smoother near interface. So we need to create some "free region", such a way that we can put the right initial data further to obtain $u(\cdot,T_n) \notin BV$ with $\lim_{n\to\infty} T_n = \infty$. This idea of creating free region has been encountered in [4]. In [1], they have shown that the (A,B) entropy solutions are of bounded variation in the interface if $A \neq \theta_g$ and $B \neq \theta_f$, that is if (A,B) are away from the critical points of f and g, then the associated singular mappings are invertible and Lipschitz continuous which allows (in [1]) to prove the bounded variation of the solution near interface. When $A = \theta_g$ or $B = \theta_f$, the total variation of the solution near interface has been proved in [1], under the assumption that $f^{-1}g(u_0), g^{-1}f(u_0)$ and $u_0 \in BV$. In this paper, we relax this condition and allow only $u_0 \in L^\infty$. It has been noticed in [1] that if (A,B) are away from the critical points, then the solution is of total bounded variation for all time $t > 0$.

Aim of this paper is to understand the bounded variation of the solution for all time and for all A, B connection. In this paper, we have noticed a very surprising result that if the lower height of the fluxes f, g are same i.e. if $f(\theta_f) = g(\theta_g)$, then the solution is of bounded variation near interface for all time $t > 0$, even if **the initial condition does not belongs to BV**. When $f(\theta_f) \neq g(\theta_g)$, then $u(\cdot,t) \in BV$ for $t > T_0$, for some T_0. We assume the fluxes f, g to be convex, superlinear growth and we use the Lax-Oleinik type formulas for the discontinuous flux introduced in [5].

Under the suitable condition on f, g and u_0, Adimurthi et.al [1] proved the following.

THEOREM 1.1 *(Adimurthi et.al [1]) Let $u_0 \in L^\infty(\mathbb{R})$ and u be the solution obtained in Theorem Let $t > 0,\ \epsilon > 0, M > \epsilon,\ I(M,\epsilon) = \{x : \epsilon \leq |x| \leq M\}$. Then*

(1). Suppose there exists an $\alpha > 0$ such that $f'' \geq \alpha,\ g'' \geq \alpha$, then there exist $C = C(\epsilon, M, \alpha)$ such that

$$TV\big(u(.,t), I(M,\epsilon)\big) \leq C(\epsilon, M, t).$$

(2). Suppose $u_0 \in BV$, and $T > 0$. Then there exists $C(\epsilon, T)$ such that for all $0 < t \leq T$

$$TV\big(u(.,t), |x| > \epsilon\big) \leq C(\epsilon, t)\, TV(u_0) + 4||u_0||_\infty$$

(3). Let $u_0 \in BV$, $T > 0$ and $A \neq \theta_g$ and $B \neq \theta_f$. Then there exists $C > 0$ such that for all $0 < t \leq T$,

$$TV\big(u(.,t)\big) \leq C\ TV(u_0) + 6||u_0||_\infty.$$

(4). Let $u_0,\ f_+^{-1}(g(u_0)),\ g_-^{-1}(f(u_0)) \in BV$, $T > 0$ *and* $A = \theta_g$. *Then for all* $0 < t \leq T$,

$$TV\big(u(.,t)\big)$$

$$\leq TV(u_0) + \max\Big(TV(f_+^{-1}(g(u_0))), TV(g_-^{-1}(f(u_0)))\Big) + 6||u_0||_\infty.$$

(5). For a certain choice of fluxes f *and* g *there exists* $u_0 \in BV \cap L^\infty$ *such that* $TV(u(.,1))$
$= \infty$ *if* $A = \theta_g$ *or* $B = \theta_f$.

In this context, we ask the following questions

Problem I. When can we say $u(\cdot,t) \in BV$ near interface, even if $u_0 \notin BV$?

Problem II. Is it possible to choose a $u_0 \in BV$ such that solution of (1.1) $\notin BV$ for large time?

Here in this paper, we answer the above problems in a very general setting. This paper is organized in the following way. Section 2 has been devoted for the preliminaries to make this article self contained. There we have recollected some properties of the characteristics and the entropy condition (see [5] for details). Section 3 deals with the main theorems. Theorem 3.1, (i) (3.1) proves the fact that $u(\cdot,t) \in BV$ near interface for the case when $f(\theta_f) \neq g(\theta_g)$ and Theorem 3.1, ((i) and (ii)(3.3), (3.4) and (3.5)) deals with the case when $f(\theta_f) = g(\theta_g)$ without the assumption that $u_0 \in BV$. We give a counter example by choosing a proper initial data $u_0 \in BV$ such that $u(x,t) \notin BV$ for lagre $t > 0$, which has been explained by splitting in to several steps, has been put in section 4.

2 Preliminaries:

We assume the following

(i). f and g are strictly convex, C^2 and of superlinear growth.

(ii). $u_0 \in L^\infty$ and let v_0 be its primitive given by

$$v_0(x) = \int_0^x u_0(\theta)d\theta.$$

Let $f(\theta_f) = \inf_{\theta \in I\!R} f(\theta)$, $g(\theta_g) = \inf_{\theta \in I\!R} g(\theta)$ be the point of minima of f and g. Let f^* and g^* be their respective convex duals defined by

$$f^*(x) = \sup_{y \in I\!R}\{xy - f(y)\}.$$

If f is strictly convex and super linear growth then f and f^* satisfies the following:
(a) $f^*(0) = -\min f$, is finite
(b) f^* is strictly convex and super linear growth and satisfy

$$f(y) = \sup_{x \in I\!R}\{xy - f^*(x)\}.$$

(c) $(f^*)^* = f$.
(d) $(f')^{-1} = (f^*)'$.
(e) $f^*(f(p)) = pf(p) - f(p), f(f^*(p)) = pf^*(p) - f^*(p)$.

Let us recall some of the definitions and notations from [5].

DEFINITION 2.1 ***Interior entropy condition*** *(E_i) : u is said to satisfy the entropy condition (E_i) (Lax-Oleinik entropy conditions) if for all $t > 0$,*

$$\lim_{0<z\to 0} \leq \lim_{0<z\to 0} u(x-z,t) \quad \text{if } x > 0, \tag{2.1}$$

$$\lim_{0<z\to 0} \leq \lim_{0<z\to 0} u(x-z,t) \quad \text{if } x < 0. \tag{2.2}$$

DEFINITION 2.2 ***Interface Entropy Condition*** *(E_b) : At $x = 0$, $u(0+,t) = \lim_{x\to 0+} u(x,t)$, $u(0-,t) = \lim_{x\to 0^-} u(x,t)$ exist for a.e. $t > 0$. Furthermore, for a.e. $t > 0$ the following condition must hold:*

$$meas\big\{t : f'(u^+(t)) > 0, g'(u^-(t)) < 0\big\} = 0 \tag{2.3}$$

DEFINITION 2.3 *((A,B)* ***Connection*** *). Let $(A,B) \in I\!R^2$. Then (A,B) is called a connection if it satisfies*
(i). $f(B) = g(A)$.
(ii). $f'(B) \geq 0,\ g'(A) \leq 0$.

DEFINITION 2.4 ***(Interphase entropy functional)****. Let $u \in L^\infty_{loc}(I\!R \times I\!R_+)$ such that $u^\pm(t) = u(0\pm,t)$ exist a.e. $t > 0$. Then we define $I_{AB}(t)$, the interface entropy functional by*

$$I_{AB}(t) = (g(u^-(t)) - g(A))sign(u^-(t) - A) - (f(u^+(t)) - f(B))sign(u^+(t) - B). \tag{2.4}$$

DEFINITION 2.5 ***(Interphase entropy condition)****. Let $u \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm(t)$ exist a.e. $t > 0$. Then u is said to satisfy Interfase entropy condition relative to a connection (A, B) if for a.e. $t > 0$*

$$I_{AB}(t) \geq 0. \tag{2.5}$$

Notice that when $A = \theta_g$ or $B = \theta_f$, (2.5) and (2.3) coincide. In this paper, we deal with the case when $A = \theta_g$ or $B = \theta_f$, hence we use (2.3) to be the entropy condition through out this paper.

DEFINITION 2.6 ***Entropy Solution :*** *Let*

$$F(x, u) = H(x)f(u) + (1 - H(x))g(u).$$

Let $u_0 \in L^\infty_{loc}(\mathbb{R})$. Then $u \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}_+)$ is said to be an entropy solution if:

(i). u is a weak solution of

$$\begin{array}{lcll} u_t + F(x,u)_x & = & 0 & \text{if } x \in \mathbb{R}, t > 0, \\ u(x,0) & = & u_0(x) & \text{if } x \in \mathbb{R}, t = 0. \end{array} \tag{2.6}$$

(ii). u satisfies the Lax-Oleinik-Kruzkov entropy condition away from the interface $x = 0$, i.e. u must satisfy (2.1) and (2.2).

(iii). At the interface $x = 0$, u satisfies the interface entropy condition (2.3).

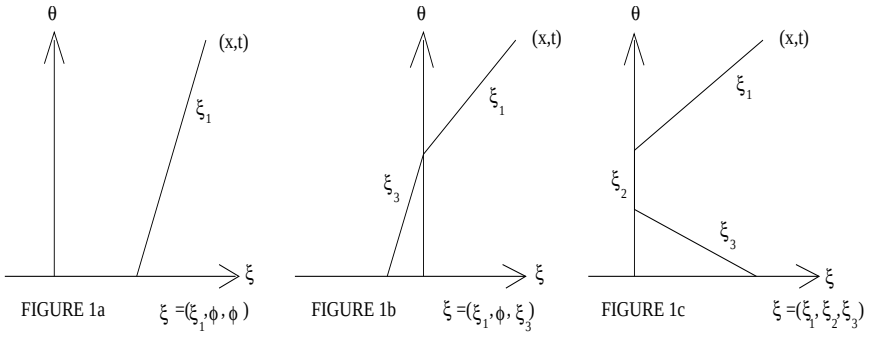

Figure 1: Characteristic curves

DEFINITION 2.7 ***Admissible curves :*** *Let $0 \leq s < t$ and $\xi \in c([s, t], \mathbb{R})$. ξ is called an admissible curve if the following holds.*

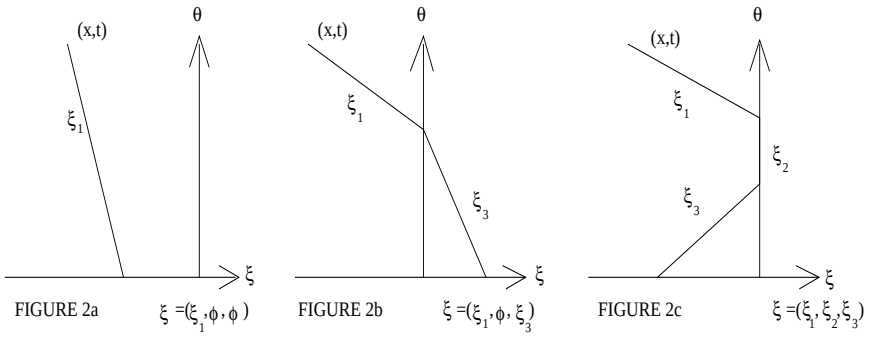

Figure 2: Characteristic curves

(i). ξ consists of atmost three linear curves (see Figure 1a, Figure 1b, Figure 1c, for $x \geq 0$, Figure 2a, Figure 2b, Figure 2c, for $x \leq 0$) and each segment lies completely in either $x \geq 0$ or $x \leq 0$.

(ii). Let $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be such that for $i = 1, 2, 3, \quad \xi_i = \xi|_{[t_i, t_{i-1}]}$ be the linear parts of ξ. If ξ consists of three linear curves then $\xi_2 = 0$ (see Figure 1c, Figure 2c) and $\xi_1, \xi_3 > 0$ or $\xi_1, \xi_3 < 0$.

Represent an admissible curve $\xi = \{\xi_1, \xi_2, \xi_3\}$. Let

$$\begin{aligned} c(x,t,s) &= \{\xi \in c([s,t], I\!R); \xi(t) = x, \xi \text{ is an admissible curve}\} \\ c(x,t) &= c(x,t,0). \end{aligned}$$

Divide $c(x,s,t)$ into three categories defined as below.

$$\begin{aligned} c_0(x,t,s) &= \{\xi \in c(x,t,s); \xi \textit{ is exactly one linear curve}\}. \\ &\quad (\text{ see Figure 1a and Figure 2a}). \\ c_b(x,t,s) &= \{\xi \in c(x,t,s); \xi_2 = \phi \ , \xi_1(\theta)\xi_3(\theta) \leq 0, \textit{ for } \theta \in (s,t)\}. \\ &\quad (\text{ see Figure 1b and Figure 2b}). \\ c_r(x,t,s) &= \{\xi \in c(x,t,s); \xi \text{ consists of three pieces and } x\xi(\theta) \geq 0 \ \forall \ \theta \in [s,t]\}. \\ &\quad (\text{ see Figure 1c and Figure 2c}). \end{aligned}$$

DEFINITION 2.8 *Let $t > 0$, define*

$$\begin{aligned} R_1(t) &= \inf\{x \ ; \ x \geq 0, \ ch(x,t) \subset c_0(x,t)\}, \\ R_2(t) &= \begin{cases} \inf\{x \ ; \ 0 \leq x \leq R_1(t), \ ch(x,t) \cap c_r(x,t) \neq \phi\}, \\ R_1(t) \quad \textit{if the above set is empty.} \end{cases} \\ L_1(t) &= \sup\{x \ ; \ x \leq 0, \ ch(x,t) \subset c_0(x,t)\}, \\ L_2(t) &= \begin{cases} \sup\{x \ ; \ L_1(t) \leq x \leq 0, \ ch(x,t) \cap c_r(x,t) \neq \phi\}, \\ L_1(t) \quad \textit{if the above set is empty.} \end{cases} \end{aligned}$$

See [2],[4] for the finer properties of this curves.

Now recall from [5], [8], [1] the existence and uniqueness of entropy solution. With out loss of generality we can assume that $g(\theta_g) \geq f(\theta_f)$.

THEOREM 2.1 *(Adimurthi, Gowda [5],[8]) Let $u_0 \in L^\infty(\mathbb{R})$, then there exists an entropy solution u of (2.6) with $u \in L^\infty(\mathbb{R})$ and is unique under mild regularity assumption (see Remark 2.12 of [1]). Furthermore, the solution can be described explicitly by Lax-Olenik type formula as follows. For each $t > 0$ there exists*

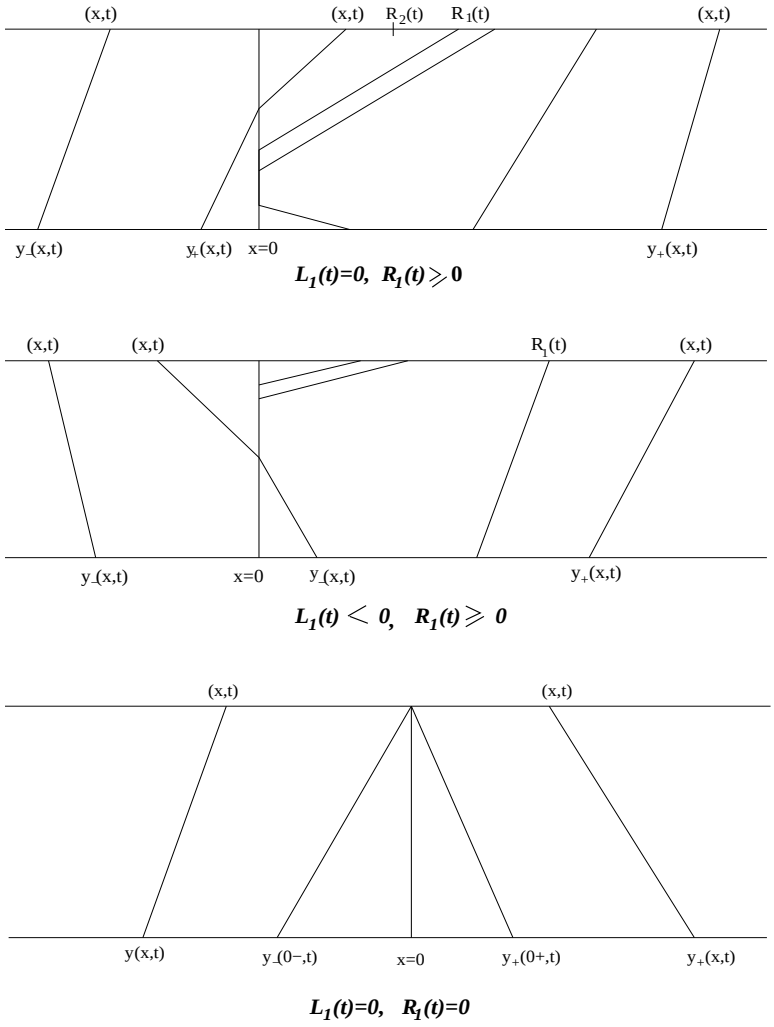

Figure 3:

$R_1(t),\ R_2(t) \geq 0,\ L_1(t) \leq 0,\ L_2(t) \leq 0$ *(for $i = 1, 2,\ R_i, L_i$ are Lipschitz continuous functions) and monotone functions $y_\pm(x,t),\ t_\pm(x,t)$ such that*

(i). For $x \in [R_1(t), \infty)$, $y_+(x,t) \geq 0$ is a non-decreasing function and for $x \in [0, R_1(t))$, $0 \leq t_+(x,t) < t$ is a non-increasing function such that for $x > 0$,(see page 16, equation (44), [8])

$$u(x,t) = \begin{cases} (f')^{-1}\left(\frac{x-y_+(x,t)}{t}\right) = u_0(y_+(x,t)) & \text{if} \quad x \geq R_1(t), \\ (f')^{-1}\left(\frac{x}{t-t_+(x,t)}\right) & \text{if} \quad 0 \leq x < R_1(t). \end{cases} \tag{2.7}$$

(ii). For $x \in (-\infty, L_1(t)]$, $y_-(x,t) \leq 0$, is a non-decreasing function and for $x \in (L_1(t), 0]$, $0 \leq t_-(x,t) < t$, $t_-(x,t)$ is non-increasing function such that for $x < 0$,

$$u(x,t) = \begin{cases} (g')^{-1}\left(\frac{x-y_-(x,t)}{t}\right) = u_0(y_-(x,t)) & \text{if} \quad x \leq L_1(t), \\ (g')^{-1}\left(\frac{x}{t-t_-(x,t)}\right) & \text{if} \quad L_1(t) < x < 0. \end{cases} \tag{2.8}$$

(iii). Furthermore, we have the following three cases
Case I. *$L_1(t) = 0$, $R_1(t) \geq 0$ (see page 53, equation (4.21), (4.22), [5])*

$$u(x,t) = \begin{cases} f_+^{-1}(g(u_0(y_+(x,t)))) & \text{if} \quad 0 < x < R_2(t) \\ f_+^{-1}(g(\theta_g)) & \text{if} \quad R_2(t) \leq x < R_1(t), \\ u_0(y_-(x,t)) & \text{if} \quad x < L_1(t) = 0. \end{cases} \tag{2.9}$$

Case II. *$L_1(t) < 0$, $R_1(t) \geq 0$ (see Lemma 4.8 and page 55, equation (4.30), [5])*

$$u(x,t) = \begin{cases} g_-^{-1}(f((u_0(y_-(x,t))))) & \text{if} \quad 0 > x > L_1(t) = L_2(t) \\ f_+^{-1}(g(\theta_g)) & \text{if} \quad 0 < x < R_1(t) = R_2(t). \end{cases} \tag{2.10}$$

Case III. *$L_1(t) = 0$, $R_1(t) = 0$ (see page 53, equation (4.20) , [5])*

$$u(x,t) = \begin{cases} u_0(y_+(x,t)) & \text{if} \quad x > R_1(t) = 0, \\ u_0(y_-(x,t)) & \text{if} \quad x < L_1(t) = 0. \end{cases} \tag{2.11}$$

See Figure (3) for clear illustrations.

3 Main Theorems

In this section, we prove our main results for the connection $A = \theta_g$ or $B = \theta_f$ (where θ_f, θ_g are the respective critical point of the fluxes). For the other connection proof can be done similarly and the sketch of the proof has been given in the Appendix.

THEOREM 3.1 *Let $u_0 \in L^\infty(\mathbb{R})$ and u be a solution obtained in Theorem 2.6. Let $t > 0,\ \epsilon > 0,\ M > \epsilon$ and*

$$\begin{aligned} I(M) &= \{x \ : \ |x| < M\}, \\ I(R_1(t)) &= \{x > 0 \ : \ x < R_1(t)\},\ I(L_1(t)) = \{x < 0 \ : \ x > L_1(t)\}, \end{aligned}$$

(i). Let $f(\theta_f) \neq g(\theta_g)$ and $f'' \geq \alpha,\ g'' \geq \alpha$, for some $\alpha > 0$, also assuming the fact that Supp $u_0 \subset [-K, K]$, for some $K > 0$, then there exists a $T_0 > 0$ such that for all $t > T_0$,

$$TV\left(u(\cdot\,, t), I(M)\right) \leq C(M, t). \tag{3.1}$$

As a consequence we have, for all $t > T_0$,

$$TV(u(\cdot\,, t), \mathbb{R}) \leq C(t), \tag{3.2}$$

where $C(t), C(M, t) > 0$ are some constants.

(ii). Let $f(\theta_f) = g(\theta_g)$ then for all $t > 0$,

$$TV\left(u(\cdot\,, t), I(R_1(t)) \cup I(L_1(t))\right) \leq C(t). \tag{3.3}$$

In addition if $f'' \geq \alpha,\ g'' \geq \alpha$, for some $\alpha > 0$, then for all $t > 0$,

$$TV\left(u(\cdot\,, t), I(M)\right) \leq C(M, t). \tag{3.4}$$

As a consequence, if Supp $u_0 \subset [-K, K]$, for some $K > 0$, then for all $t > 0$,

$$TV(u(\cdot\,, t), \mathbb{R}) \leq C(t), \tag{3.5}$$

(iii). Let $f(\theta_f) = g(\theta_g)$ and $u_0 \in BV(\mathbb{R})$ then for all $t > 0$,

$$TV\left(u(\cdot\,, t)\right) \leq C(t)(TV(u_0) + 1) + 4\|u_0\|\infty. \tag{3.6}$$

Proof of (i). With out loss of generality we can assume that $g(\theta_g) > f(\theta_f)$. If $M > R_1(t)$, then for $x > R_1(t)$, $u(x, t) = f^{*'}\left(\frac{x - y_+(x,t)}{t}\right)$. Let $x_1 = R_1(t) < x_2 < \cdots < x_{N+1} = M$ be any partition in $(R_1(t), M)$. Then by using the Lipschitz continuity of $f^{*'}$(which is bounded by $\frac{1}{\alpha}$) and monotonicity of y_+, we obtain

$$\begin{aligned} \sum_{i=1}^{N} |u(x_i, t) - u(x_{i+1}, t)| &\leq \tfrac{1}{\alpha} \sum_{i=1}^{N} \left| \tfrac{x_i - y_+(x_i,t)}{t} - \tfrac{x_{i+1} - y_+(x_{i+1},t)}{t} \right| \\ &\leq \tfrac{1}{\alpha t} \sum_{i=1}^{N} (|x_i - x_{i+1}| + |y_+(x_i, t) - y_+(x_{i+1}, t)|) \\ &\leq \tfrac{1}{\alpha t}((M - R_1(t)) + (y_+(M, t) - y_+(R_1(t), t)) \\ &\leq C(t). \end{aligned} \tag{3.7}$$

Hence

$$TV(u(\cdot\,,t),(R_1(t),M)) \le C(t) \tag{3.8}$$

and similarly

$$TV(u(\cdot\,,t),(-M,L_1(t))) \le C(t). \tag{3.9}$$

So from now onwards we can assume that $M < R_1(t)$. We consider the following three cases.

Case 1: $L_1(t) = 0,\ R_1(t) \ge 0$. From (2.9), if $x \in (R_2(t), R_1(t))$, $u(x,t) = f^{-1}g(\theta_g) =$ a constant, hence u is bounded variation in $(R_2(t), R_1(t))$. Let $\bar{\theta}, \tilde{\theta}$ be such that $g(\theta_g) = f(\bar{\theta}) = f(\tilde{\theta})$ with $f'(\bar{\theta}) < 0, f'(\tilde{\theta}) > 0$. At first, we prove total variation of u in $(0,\epsilon)$ and then in (ϵ, M), where $\epsilon > 0$ be such that $0 < \epsilon < R_2(t)$. Now

$$u(x,t) = f^{*\prime}\left(\frac{x}{t - t_+(x,t)}\right) \qquad \text{if} \quad 0 < x < \epsilon < R_2(t), \tag{3.10}$$

where $t_+(x,t)$ is a non-increasing function of x.

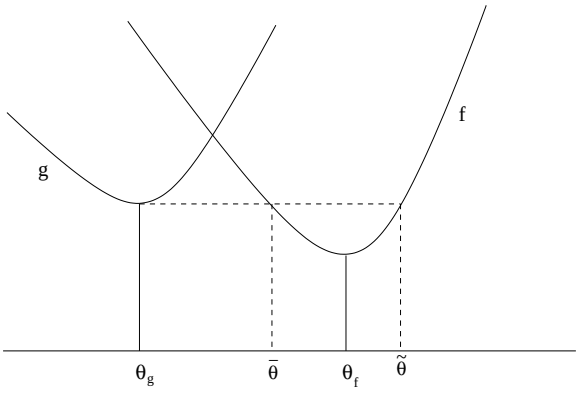

Figure 4:

Claim:

$$t_+(x,t) \to \infty \ \text{ as } \ t \to \infty, \quad \text{if} \quad x \in (0,\epsilon). \tag{3.11}$$

If possible, let

$$t_+(x,t) \le \bar{C}, \quad x \in (0,\epsilon), \ \text{for all } t > T_0, \tag{3.12}$$

for some constants $\bar{C}, T_0 > 0$. Then from (3.10),(3.12) there exists a large $T_1 > T_0$ (denoting T_1 by T_0 only) such that

$$u(x,t) \in (\theta_f, \tilde{\theta}) \quad \text{if} \quad x \in (0,\epsilon),\ t > T_0. \tag{3.13}$$

Again

$$\begin{aligned} u(x,t) &= u(0+, t_+(x,t)) \ \text{ [see page 15,16 equations 42-46 in [8]]} \\ &= f^{-1}g(u(0-, t_+(x,t))) \ \text{ [by R-H condition]} \\ &> \tilde{\theta} \ \text{[since } g(\theta_g) > f(\theta_f)\text{]}, \end{aligned} \tag{3.14}$$

which contradicts (3.13). Hence the claim.

For $x \in (0, R_2(t))$,

$$\begin{aligned} u(x,t) &= f^{-1}g(u(0-, t_+(x,t))) \\ &= f^{-1}g\left(g^{*\prime}\left(\tfrac{-y_+(0,t_+(x,t))}{t_+(x,t)}\right)\right) \ \text{[see page 53, equation (4.22) and step 1 of Lemma 4.10 in [5]]} \\ &= f^{-1}g(u_0(y_+(x,t))) \qquad \text{[see page 53, equation (4.22) in [5]]} \end{aligned} \tag{3.15}$$

where $y_+(0,z)$ is a non-decreasing function of $z > 0$ and $y_+(x,t)$ is a non-decreasing function of $x > 0$.

If $y_+(0, t_+(\bar{x}, t)) < -K$ for some $\bar{x} \in (0, R_2(t))$, then by monotoncity of t_+ and y_+ we have

$$y_+(0, t_+(x,t)) < -K, \quad \text{for all} \quad x \in (0, \bar{x}). \tag{3.16}$$

Since Supp $u_0 \subset [-K, K]$ therefore by (3.15),(3.16) $u(x,t) = f^{-1}g(0)$ =constant, hence u is of bounded variation in $(0, \bar{x})$. Since t_+ is non increasing function of x, hence $0 < \beta < \frac{x}{t - t_+(x,t)}$ for all $x \in (\bar{x}, M)$, for some β. Let $x_1 = \bar{x} < x_2 < \cdots < x_{N+1} = M$ be any partition in $(\bar{x}, M)$. We obtain

$$\begin{aligned} \sum_{i=1}^{N} |u(x_i,t) - u(x_{i+1},t)| &\leq \tfrac{1}{\alpha} \sum_{i=1}^{N} \left| \tfrac{x_i}{t - t_+(x_i,t)} - \tfrac{x_{i+1}}{t - t_+(x_{i+1},t)} \right| \\ &\leq \tfrac{1}{\alpha t} \tfrac{Mt}{t-\beta}. \end{aligned} \tag{3.17}$$

Hence

$$TV(u(\cdot\,, t), (0, M) \leq C(t) \tag{3.18}$$

Now we assume $-K \leq y_+(0, t_+(x,t)) \leq 0$, for all $x \in (0, R_2(t))$. Hence by (3.11)

$$\frac{y_+(0, t_+(x,t))}{t_+(x,t)} \to 0 \ \text{ as } \ t \to \infty. \tag{3.19}$$

Since $g^{*\prime}(0) = \theta_g$, therefore by (3.19)

$$g^{*\prime}\left(-\frac{y_+(0, t_+(x,t))}{t_+(x,t)}\right) \to \theta_g \ \text{ as } \ t \to \infty \quad \text{if } x \in (0, M). \tag{3.20}$$

Hence from (3.20), it is easy to see that there exists a small $\delta_1 > 0$ and a large $T_0 > 0$ such that

$$g^{*\prime}\left(-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}\right) \in (\theta_g, \theta_g + \delta_1) \quad \text{if} \quad x \in (0,M), t > T_0. \tag{3.21}$$

Since $f(\theta_f) < g(\theta_g)$, therefore from (3.21) we deduce that $g\left(g^{*\prime}\left(-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}\right)\right)$ avoids critical point of f hence $f^{-1}g\left(g^{*\prime}\left(-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}\right)\right)$ is Lipschitz continuous for $x \in (0,M), t > T_0$. Again $g^{*\prime} = (g')^{-1}$ is Lipschitz continuous and Lipschitz constant is bounded by $\frac{1}{\alpha}$. Let $0 < x_1 < x_2 < \cdots < x_{N+1} = M$ be a partition. Therefore by using the fact y_+, t_+ are monotone also $y_+ \in [-K, 0]$, we conclude from (3.11) and (3.15) that

$$\begin{aligned}
&\sum_{i=1}^{N} |u(x_i,t) - u(x_{i+1},t)| \\
\leq\ & C\sum_{i=1}^{N}\left|\left(-\frac{y_+(0,t_+(x_{i+1},t))}{t_+(x_{i+1},t)}\right) - \left(-\frac{y_+(0,t_+(x_i,t))}{t_+(x_i,t)}\right)\right| \\
\leq\ & C\sum_{i=1}^{N}\frac{1}{t_+(x_i,t)}\left|y_+(0,t_+(x_{i+1},t)) - y_+(0,t_+(x_i,t))\right| + \\
&\sum_{i=1}^{N} C\left|y_+(0,t_+(x_{i+1},t))\right|\left|\tfrac{1}{t_+(x_i,t)} - \tfrac{1}{t_+(x_{i+1},t)}\right| \\
\leq\ & C\sum_{i=1}^{N}\left|y_+(0,t_+(x_{i+1},t)) - y_+(0,t_+(x_i,t))\right| + \\
&\sum_{i=1}^{N} CK\left|\tfrac{1}{t_+(x_i,t)} - \tfrac{1}{t_+(x_{i+1},t)}\right| \\
\leq\ & CK + CK\left|\tfrac{1}{t_+(x_1,t)} - \tfrac{1}{t_+(x_{N+1},t)}\right| \\
\leq\ & CK(1+\tfrac{2}{t}).
\end{aligned} \tag{3.22}$$

So for $t > T_0$,

$$TV(u(\cdot,t), 0 < x < M) \leq C(M,t). \tag{3.23}$$

Since $L_1(t) = 0$, so for $-M < x < 0$,

$$u(x,t) = g^{*\prime}\left(\frac{x - y_-(x,t)}{t}\right),$$

where $y_-(x,t)$ is a non-decreasing function of x. Hence from (3.9)

$$TV(u(\cdot\,,t) :\ -M < x < 0) \leq C(M,t). \tag{3.24}$$

Hence (3.1) follows.

From the finite speed of propagation, we know (see Lemma 4.2, page no. 38 in [5]) that there exist a $S > 0$ such that

$$\left\| \frac{d\xi}{d\theta} \right\|_\infty \leq S \quad \text{for all} \quad \xi \in ch(x,t). \tag{3.25}$$

By (3.25) it is clear that characteristic originating from the point $(K,0)$ of x axis can travel with speed at most S and characteristic originating from the point $(-K,0)$ can travel with speed not less than $-S$. Since Supp $u_0 \subset [-K,K]$, therefore for each $t > 0$, there exists $l(t) > 0$ such that $u(x,t)$ is constant outside $(-l(t), l(t))$. Choosing $M = l(t)$ in (3.23), we conclude

$$TV(u(\cdot\,,t) : I\!R) \leq C + C(\epsilon, t). \tag{3.26}$$

Case 2 : $L_1(t) = L_2(t) < 0,\ R_1(t) \geq 0$.
In this case $R_2(t) = 0$, so for $x \in (0, R_1(t)),\ u(x,t) = f^{-1}g(\theta_g)$, hence $u(x,t)$ is bounded variation in $(0, R_1(t))$. From the explicit formulas, we get

$$\begin{aligned} u(x,t) &= g^{*\prime}\left(\tfrac{x}{t - t_-(x,t)}\right) \quad \text{if} \quad 0 > x > -\epsilon > L_1(t), \\ &= u(0-, t_-(x,t)) \\ &= g^{-1} f\left(f^{*\prime}\left(-\tfrac{y_-(0-,t_-(x,t))}{t_-(x,t)}\right)\right) \quad \text{[by R-H condition]} \\ &= f^{-1} g(u_0(y_-(x,t))) \end{aligned} \tag{3.27}$$

where $t_-(x,t)$ is a non-decreasing function of x and $y_-(0,z)$ is a non-decreasing function of $z < 0$.

If $y_-(0, t_-(\bar{x}, t)) > K$, for some $\bar{x} \in (L_1(t), 0)$, then u is of bounded variation in $(-\epsilon, 0)$ (choosing $\epsilon = -\bar{x}$). So we let $0 \leq y_-(0, t_-(x,t)) \leq K$, for all $x \in (L_1(t), 0)$.
Claim:

$$|t_-(x,t)| \leq \bar{C} \quad \text{if} \quad x \in (L_1(t), 0),\ t > T_0, \tag{3.28}$$

for some constants $\bar{C}, T_0 > 0$.

Suppose for some $\epsilon > 0$,

$$t_-(x,t) \to \infty \text{ as } t \to \infty \quad \text{if} \quad x \in (-\epsilon, 0). \tag{3.29}$$

Therefore we deduce

$$f^{*\prime}\left(\frac{-y_-(0, t_-(x,t))}{t_-(x,t)}\right) \to \theta_f \quad \text{as} \quad t \to \infty \text{ [since } f^{*\prime}(0) = \theta_f]. \tag{3.30}$$

By (3.29), there exists a small $\delta_2(t)$ and a large $T_0 > 0$ such that

$$f^{*\prime}\left(\frac{-y_-(0, t_-(x,t))}{t_-(x,t)}\right) \in (\theta_f - \delta_2(t), \theta_f) \quad \text{if} \quad x \in (-\epsilon, 0), t > T_0. \tag{3.31}$$

If $x \in (-\epsilon, 0)$, then from (3.27),

$$g(u(x,t)) = f\left(f^{*\prime}\left(-\frac{y_-(0,t_-(x,t))}{t_-(x,t)}\right)\right) \leq \bar{\theta} \text{ [since } g(\theta_g) > f(\theta_f)],$$

which contradicts (3.31). Hence the claim.

Let $-M = x_{N+1} < x_N < \cdots < x_1 < 0$ be a partition then by (3.27)

$$\begin{aligned} \sum_{i=1}^{N} |u(x_i,t) - u(x_{i+1},t)| &= \sum_{i=1}^{N} \left| g^{*\prime}\left(\frac{x_i}{t - t_-(x_i,t)}\right) - g^{*\prime}\left(\frac{x_{i+1}}{t - t_-(x_{i+1},t)}\right)\right| \\ &\leq C \sum_{i=1}^{N} \left| \frac{x_i}{t - t_-(x_i,t)} - \frac{x_{i+1}}{t - t_-(x_{i+1},t)}\right|. \end{aligned} \tag{3.32}$$

Now by using (3.28) and monotoncity of t_-, we conclude

$$\frac{x_{i+1}}{t - t_-(x_{i+1},t)} - \frac{x_i}{t - t_-(x_i,t)} \geq \frac{x_{i+1} - x_i}{t - t_-(x_{i+1},t)} \tag{3.33}$$

and

$$\begin{aligned} \frac{x_{i+1}}{t - t_-(x_{i+1},t)} - \frac{x_i}{t - t_-(x_i,t)} &\leq \frac{x_{i+1}(t_-(x_{i+1},t) - t_-(x_i,t))}{(t - t_-(x_{i+1},t))(t - t_-(x_i,t))} \\ &\leq \frac{x_{i+1}(t_-(x_{i+1},t) - t_-(x_i,t))}{(t - \bar{C})(t - \bar{C})}. \end{aligned} \tag{3.34}$$

By using, (3.33) and (3.34) we deduce that for $t > T_0$,

$$\textstyle\sum_{i=1}^{N} \left| \frac{x_i}{t - t_-(x_i,t)} - \frac{x_{i+1}}{t - t_-(x_{i+1},t)}\right| \leq C \text{ Max } \{\frac{1}{t}, \epsilon\}. \tag{3.35}$$

Hence for $t > T_0$,

$$TV(u(\cdot,t), -M < x < 0) \leq C(M).$$

From (3.24), we obtain

$$TV(u(\cdot,t), I(M)) \leq C(M,t). \tag{3.36}$$

Since Supp $u_0 \subset [-K, K]$, hence from (3.25) and (3.36) we conclude

$$TV(u(\cdot,t), I\!R) \leq C(t).$$

Case 3 : $L_1(t) = 0,\ R_1(t) = 0$. In this case

$$u(x,t) = \begin{cases} f^{*\prime}\left(\frac{x - y_+(x,t)}{t}\right) & \text{if} \quad x > 0, \\ g^{*\prime}\left(\frac{x - y_-(x,t)}{t}\right) & \text{if} \quad x < 0. \end{cases} \tag{3.37}$$

By using Lipschitz continuity of $f^{*\prime}, g^{*\prime}$ and monotoncity of y_+, y_- it is easy to see that for $t > 0$,

$$TV(u(\cdot,t) \ : \ -M < x < M) \leq C(M,t).$$

Again using (3.25) and Supp $u_0 \subset [-K, K]$, we derive

$$TV(u(\cdot,t), I\!R) \leq C(t).$$

This proves (i).

Proof of (ii) and (iii). It is enough to prove the result for the case when $L_1(t) = 0,\ R_1(t) \geq 0$, other cases follows similarly. Let $0 < \epsilon < R_2(t)$. To prove (3.3), atfirst we prove total variation of u in $(0,\epsilon)$ then in $(\epsilon, R_2(t))$. Consider a characteristic $\xi \in ch(\epsilon,t)$. Then by monotoncity of t_+ and y_+ we have the following

$$\begin{aligned}
t_+(\epsilon,t) &\leq t_+(x,t) && \text{if} \quad x \in (0,\epsilon), && (3.38)\\
t_+(\epsilon,t) &\geq t_+(x,t) && \text{if} \quad x \in (\epsilon, R_2(t)), && (3.39)\\
y_+(0,t_+(\epsilon,t)) &\geq y_+(0,t_+(x,t)) && \text{if} \quad x \in (0,\epsilon), && (3.40)\\
y_+(0,t_+(\epsilon,t)) &\leq y_+(0,t_+(x,t)) && \text{if} \quad x \in (\epsilon, R_2(t)). && (3.41)
\end{aligned}$$

Let $0 < x_1 < x_2 < \cdots < x_{N+1} = \epsilon < R_2(t)$ be a partition. From (3.40), there exists $\delta_3(t) > 0$ (i.e. there exists a neighbourhood of θ_g) such that for $t > 0$

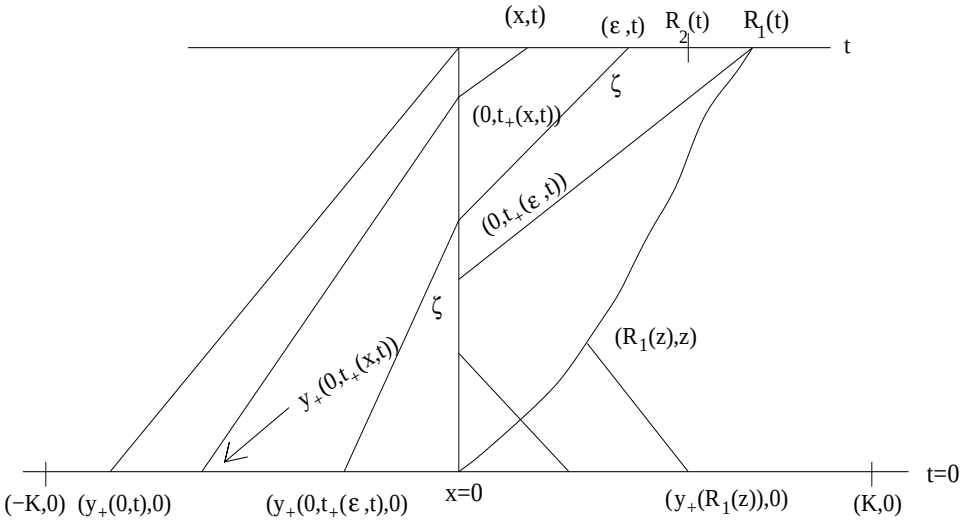

Figure 5:

$$-\frac{y_+(0,t_+(x,t))}{t_+(x,t)} \notin (\theta_g - \delta_3(t), \theta_g + \delta_3(t)) \quad \text{if} \quad x \in (0,\epsilon). \tag{3.42}$$

Therefore from (3.42), we conclude that $-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}$ avoids critical point of g, for $x \in (0,\epsilon)$, so it is easy to see that for $x \in (0,\epsilon)$, $g^{*\prime}\left(-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}\right)$ is Lipschitz continuous. Again from (3.42), there exists a $\delta_4(t) > 0$ such that

$$g\left(g^{*\prime}\left(-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}\right)\right) \notin (g(\theta_g) - \delta_4(t), g(\theta_g) + \delta_4(t)) \quad \text{if} \quad x \in (0,\epsilon). \tag{3.43}$$

Since $g(\theta_g) = f(\theta_f)$, hence from (3.43), we deduce that $f^{-1}g\left(g^{*\prime}\left(-\frac{y_+(0,t_+(x,t))}{t_+(x,t)}\right)\right)$ is

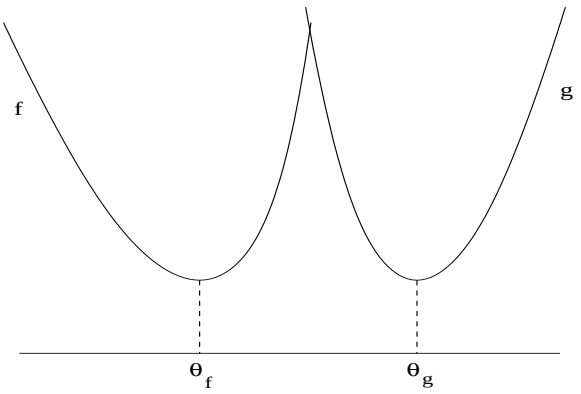

Figure 6:

Lipschitz continuous for $x \in (0,\epsilon), t > 0$. Now from (3.15) and above arguments we estimate similarly as in (3.22) to obtain

$$\begin{aligned}
&\sum_{i=1}^{N} |u(x_i,t) - u(x_{i+1},t)| \\
&= \sum_{i=1}^{N} \left| f^{-1}g(g^{*\prime}(-\frac{y_+(0,t_+(x_{i+1},t))}{t_+(x_{i+1},t)})) - f^{-1}g(g^{*\prime}(-\frac{y_+(0,t_+(x_i,t))}{t_+(x_i,t)})) \right| \qquad (3.44)\\
&\leq C(t)|y_+(0,t)| + C(t)|y_+(0,t)| \left| \tfrac{1}{t_+(x_{N+1},t)} - \tfrac{1}{t_+(x_1,t)} \right| \\
&\leq C(t).
\end{aligned}$$

Therefore

$$TV(u(\cdot,t) \ : \ 0 < x < \epsilon) \leq C(t). \tag{3.45}$$

From (3.39), there exists $\delta_5(t) > 0$ such that

$$\frac{x}{t - t_+(x,t)} \notin (\theta_f, \theta_f + \delta_5(t)) \quad \text{if} \quad x \in (0,\epsilon), \ t > 0. \tag{3.46}$$

Hence $f^{*\prime}\left(\frac{x}{t-t_+(x,t)}\right)$ is Lipschitz continuous for $x \in (\epsilon, R_2(t)), t > 0$. Let $\epsilon = x_1 < x_2 < \cdots < x_{N+1} = R_2(t)$ be a partition. Therefore

$$\sum_{i=1}^{N} |u(x_i,t) - u(x_{i+1},t)| \;=\; \sum_{i=1}^{N} \left| f^{*\prime}\left(\frac{x_i}{t - t_+(x_i,t)}\right) - f^{*\prime}\left(\frac{x_{i+1}}{t - t_+(x_{i+1},t)}\right)\right| \quad (3.47)$$

(3.33), (3.34) and (3.39) yields

$$\sum_{i=1}^{N} \left| \left(\frac{x_i}{t - t_+(x_i,t)}\right) - \left(\frac{x_{i+1}}{t - t_+(x_{i+1},t)}\right)\right| \leq \text{Max}\{\frac{1}{t - t_+(\epsilon,t)}, R_2(t)\} \quad (3.48)$$

Then from (3.47) and (3.48) we have , for $t > 0$,

$$\sum_{i=1}^{N} |u(x_i,t) - u(x_{i+1},t)| \leq C(t). \quad (3.49)$$

Hence

$$TV(u(\cdot,t) \;:\; 0 < x < R_2(t)) \leq C(t). \quad (3.50)$$

This proves (3.3).

If $M < R_1(t)$, then (3.3) follows from (3.50), so let $M > R_1(t)$. In this case (3.3) follows by using Lipschitz continuity of $f^{*\prime}$ and monotoncity of y_+. (3.4) follows similarly as in (3.26). This proves (ii).

In addition, if $u_0 \in BV(I\!R)$ (see Theorem (2.13), (ii) in [1]) then for $t > 0$,

$$TV(u(\cdot,t) \;:\; |x| > \epsilon) \leq C(\epsilon,t)TV(u_0) + 4\|u_0\|_\infty \quad (3.51)$$

Hence (3.6) follows from (3.3) and (3.51). This proves (iii).

4 Construction of the counter example

In this section we focus on the counter example of the blow up of TV bound for large time. In order to provide the example we need to use the following Lemma.

LEMMA 4.1 *Consider the following problem*

$$\begin{array}{rcl} u_t + f(u)_x &=& 0 \quad \textit{if} \quad x > 0, t > 0, \\ u_t + g(u)_x &=& 0 \quad \textit{if} \quad x < 0, t > 0, \end{array} \quad (4.52)$$

where $f(u) = (u-1)^2 - 1,\ g = u^2$ *and the initial data* u_0 *is given by*

$$u_0(x) = \begin{cases} 0 & \textit{if} \quad x < c_1 < 0, \\ \bar{u}_0(x) & \textit{if} \quad c_1 < x < 0, \\ \tilde{v} & \textit{if} \quad x \geq 0 \end{cases} \quad (4.53)$$

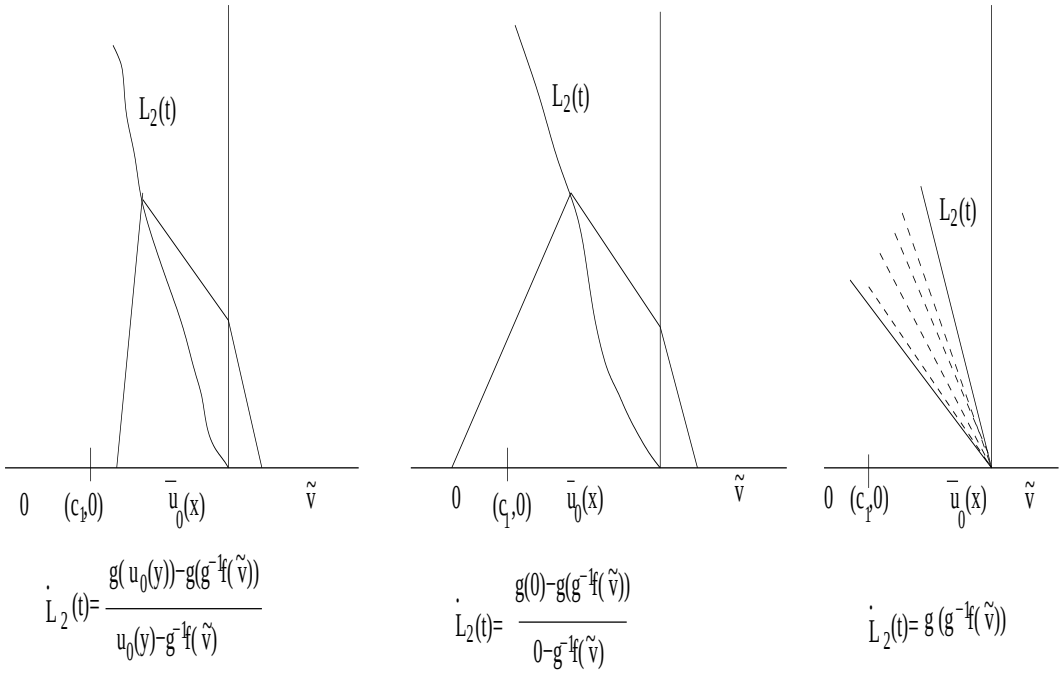

Figure 7:

where $c_1 < 0, c_2 > 0$ *are constants and* $\bar{u}_0, \tilde{b}$ *satisfies the following property*

$$g'(\bar{u}_0(x)) < 0 \quad \textit{for all } x \in (c_1, 0) \tag{4.54}$$

$$g^{-1}f(\tilde{v}) < 0, f'(\tilde{v}) < 0, g'(\tilde{v}) < 0, \tag{4.55}$$

$$|u_0(x) - g^{-1}f(\tilde{v})| > c_2 \quad \textit{for all } x \in (c_1, 0), \tag{4.56}$$

then there exists a $m < 0$*, such that the solution of (4.52) and (4.53) is given by*

$$u(x,t) = \begin{cases} g^{-1}f(\tilde{v}) & \textit{if} \quad mt < x < 0, \\ \tilde{v} & \textit{if} \quad x > 0. \end{cases} \tag{4.57}$$

PROOF : *Since* $g'(\bar{u}_0(x)) < 0, g'(\tilde{v}) < 0$ *and* $g'(0) = 0$*, so* $L_2(t) < 0$*. Hence by RH condition we have*

$$\begin{aligned} \dot{L}_2(t) &= \tfrac{g(L_2(t)-)-g(L_2(t)+)}{(L_2(t)-)-(L_2(t)+)} \textit{ or } g'(g^{-1}f(\tilde{v})) \textit{ (see Figure 7)} \\ &= \tfrac{g(\bar{u}_0(z))-g(g^{-1}f(\tilde{v}))}{\bar{u}_0(z)-g^{-1}f(\tilde{v})} \textit{ or } \tfrac{g(0)-g(g^{-1}f(\tilde{v}))}{0-g^{-1}f(\tilde{v})} \textit{ or } g'(g^{-1}f(\tilde{v})). \end{aligned} \tag{4.58}$$

for some $z \in (c_1, 0)$*. In any case , by (4.54), (4.55), (4.56) and (4.58) we have*

$$\dot{L}_2(t) < m < 0, \textit{ for some } m < 0. \tag{4.59}$$

Hence by (4.59) we have

$$L_2(t) < mt < 0. \tag{4.60}$$

then by R-H condition the solution is given by

$$u(x,t) = \begin{cases} g^{-1}f(\tilde{v}) & \text{if } \ mt < x < 0, \\ \tilde{v} & \text{if } \ x > 0. \end{cases} \tag{4.61}$$

Hence the Lemma.

Counter Example : If we do not assume Supp $u_0 \subset [-K, K]$, for some $K > 0$, then for a certain choice of fluxes f, g there exists $u_0 \in L^\infty(I\!R) \cap BV(I\!R)$ such that $TV(u(\cdot, T_n)) = \infty$, for all n, with $\lim_{n\to\infty} T_n = \infty$ (This violets Theorem 3.1 (i) without those assumptions).

We have devided this proof into several steps.

Step 1: In order to construct this counter example, first we study the following initial value problem

$$\begin{array}{rcl} u_t + f(u)_x &=& 0 \ \text{ if } \ x > 0, t > 0, \\ u_t + g(u)_x &=& 0 \ \text{ if } \ x > 0, t > 0, \end{array} \tag{4.62}$$

where $f = (u-1)^2 - 1, \ g(u) = u^2$ with the initial data u_0 as follows

$$u_0(x) = \begin{cases} 0 & \text{if } \ x < 0, \\ a_1 & \text{if } \ 0 < x < x_1, \\ a_2 & \text{if } \ x_1 < x < x_2, \\ a_3 & \text{if } \ x > x_2, \end{cases} \tag{4.63}$$

where $0 < x_1 < x_2, \ a_1 < a_2 < 0, \ a_2 > a_3$ and we are going to choose x_i, a_i in a proper way.

Since $a_1 < a_2$ and $a_2 > a_3$, so it creats a rarefaction at $x = x_1$ and a shock at $x = x_2$. Now we choose a_1, a_2, a_3 such a way that characteristics do not meet in $x > 0, t > 0$. Therefore for $x > 0$, the solution of (4.62), (4.63) is given by

$$u(x,t) = \begin{cases} a_1 & \text{if } \ 0 < x < 2(a_1-1)t + x_1, \\ \frac{1}{2}\left(\frac{x-x_1}{t} + 2\right) & \text{if } \ 2(a_1-1)t + x_1 < x << 2(a_2-1)t + x_1, \\ a_2 & \text{if } \ 2(a_1-1)t + x_2 < x < (a_2+a_3-2)t + x_2, \\ a_3 & \text{if } \ x > (a_2+a_3-2)t + x_2. \end{cases} \tag{4.64}$$

Since $g'(0) = 0$ and $f'(a_i) < 0$ for all $i = 1, 2, 3$, so (4.64) yields

$$u(0+,t) = \begin{cases} a_1 & \text{if } \ 0 < t < \frac{x_1}{2(1-a_1)} = t_1(\text{say}), \\ \frac{1}{2}\left(\frac{x_1}{t} + 2\right) & \text{if } \ \frac{x_1}{2(1-a_1)} < t < \frac{x_1}{2(1-a_2)} = t_2(\text{say}), \\ a_2 & \text{if } \ \frac{x_1}{2(1-a_2)} < t < \frac{x_2}{2-(a_2+a-3)} = t_3(\text{say}), \\ a_3 & \text{if } \ \frac{x_2}{2-(a_2+a_3)} < t. \end{cases} \tag{4.65}$$

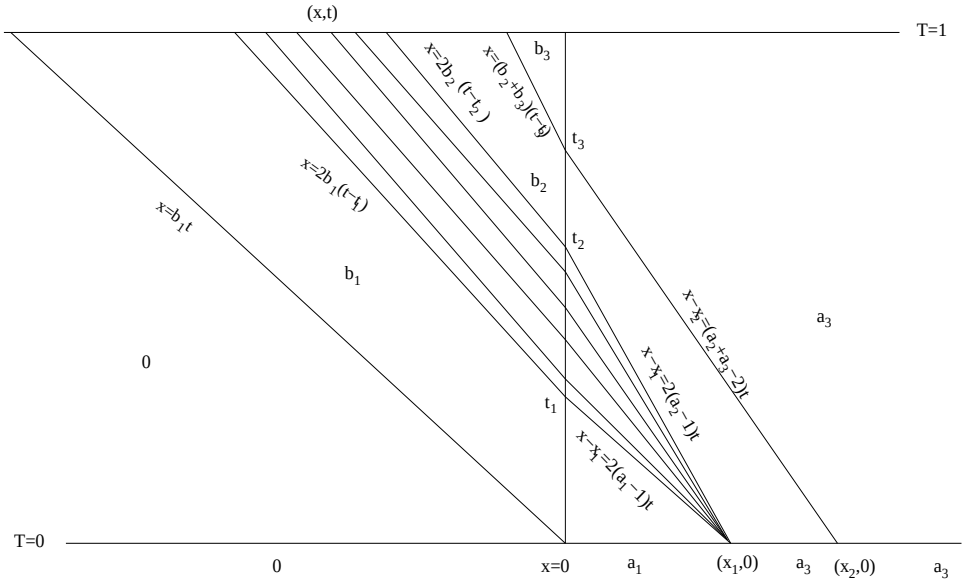

Figure 8:

By R-H condition, (4.65) yields

$$u(0-,t) = \begin{cases} g^{-1}f(a_1) = b_1 \text{ say} & \text{if } \ 0 < t < t_1, \\ g^{-1}f\left(\frac{1}{2}\left(-\frac{x_1}{t}+2\right)\right) & \text{if } \ t_1 < t < t_2, \\ g^{-1}f(a_2) = b_2 \text{ (say)} & \text{if } \ t_2 < t < t_3, \\ g^{-1}f(a_3) = b_3 \text{ (say)} & \text{if } \ t_3 < t. \end{cases} \tag{4.66}$$

$u(0+,t)$ is an increasing function of t in the interval (t_1,t_2) so as $u(0-,t)$, therefore the outgoing characteristic from (t_1,t_2) will never intersect. Now we modify a_1,a_2,a_3 (such a modification is obvious) such a way that characteristics do not meet in $0 < t \le 1, x \in I\!R$.

Then the solution of (4.62),(4.63) at $T=1$ is given by

$$u(x,1) = \begin{cases} 0 & \text{if } \ x < b_1, \\ b_1 & \text{if } \ b_1 < x < 2b_1(1-t_1), \\ u(0-,t_+(x,1)) & \text{if } \ 2b_1(1-t_1) < x < 2b_2(1-t_2), \\ b_2 & \text{if } \ 2b_2(1-t_2) < x < (b_2+b_3)(1-t_3), \\ b_3 & \text{if } \ (b_2+b_3)(1-t_3) < x < 0, \\ a_3 & \text{if } \ x > 0. \end{cases} \tag{4.67}$$

where $t_+(x,1)$ is an increasing function of x (this is possible due to the fact that $u(0-,t)$ is an increasing function of t in (t_1,t_2)).

See Figure (8) for clear illustrations.

Step 2 : Let us denote

$$\mathcal{D} = \left\{ \sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2 - 1},\ \sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2 - 1} :\ i \in \mathbb{N} \right\}.$$

Let $v_1 < 0$, such that

$$\begin{array}{lcll} v_1 & < & -\frac{1}{(i+1)^2} & \text{for all } i \geq i_0, \\ |v_1| & < & \frac{1}{2^{l_0}} & \text{for some } \ l_0 \in \mathbb{N}, \\ v_1 & \notin & \mathcal{D}, & \end{array} \tag{4.68}$$

where $i_0 \in \mathbb{N}$ is large and we are going to choose it later.

Let $T_1 > 0$ be a positive no. which is going to be choose later. Let us Denote

$$\begin{aligned} A_1 &= 2T_1,\ a_{2i-1} = -\frac{1}{(i+1)^2},\ a_{2i} = -\frac{1}{(i+1)^4},\ b_i = g^{-1}f(a_i) \text{ for } i \geq 1, \\ x_{1,2i_0-1} &= A_1 - \sum_{j=2i_0-1}^{\infty} \frac{1}{j^{3/2}},\ x_{1,i} = x_{1,2i_0-1} + \sum_{j=2i_0}^{i} \frac{1}{j^{3/2}},\ \text{for } i \geq 2i_0, \\ s_1 &= \frac{(v_1-1)^2-1}{v_1},\ B_1 = \frac{1-(v_1-1)^2}{v_1} T_1. \end{aligned}$$

Then by definition, $x_{1,2i_0-1} < x_{1,2i_0} < \cdots < A_1$, and $A_1 = \sum_{j=2i_0-1}^{\infty} x_{1,j}$.

Now consider the following initial data

$$u_0(x) = \begin{cases} 0 & \text{if } \ x \leq 0, \\ a_{2i_0-1} & \text{if } \ 0 < x < x_{1,2i_0-1}, \\ a_{2i} & \text{if } \ x_{1,2i-1} < x < x_{1,2i},\ \text{for } i \geq i_0, \\ a_{2i+1} & \text{if } \ x_{1,2i} < x < x_{1,2i+1},\ \text{for } i \geq i_0, \\ 0 & \text{if } \ A_1 < x < B_1, \\ v_1 & \text{if } \ x \geq B_1. \end{cases} \tag{4.69}$$

Since $a_{2i-1} < a_{2i},\ a_{2i} > a_{2i+1}$, so it creats rarefaction at $x = x_{1,2i-1}$ and shock at $x = x_{1,2i}$. For $i \geq i_0$, let $\tilde{t}_i$ be the time when the lines $x = 2(a_{2i}-1)t + x_{1,2i-1}$ and $x = (a_{2i} + a_{2i+1} - 2)t + x_{1,2i}$ meets, then we have

$$\tilde{t}_{1,i} = \frac{x_{1,2i} - x_{1,2i-1}}{a_{2i} - a_{2i+1}} = \frac{1(2i)^{3/2}}{-\frac{1}{(i+1)^4} + \frac{1}{(i+2)^2}}. \tag{4.70}$$

Let $\bar{t}_{1,i}$ be the time when the lines $x = (a_{2i} + a_{2i+1} - 2)t + x_{1,2i}$ and $x = 2(a_{2i+1} - 1)t + x_{1,2i+1}$ meets, then we have

$$\bar{t}_{1,i} = \frac{x_{1,2i+1} - x_{1,2i}}{a_{2i} - a_{2i+1}} = \frac{1(2i+1)^{3/2}}{-\frac{1}{(i+1)^4} + \frac{1}{(i+2)^2}}. \tag{4.71}$$

It is clear from (4.70) and (4.71) that we can choose a large i_0 such that

$$\tilde{t}_{1,i}, \bar{t}_{1,i} > T_1 \quad \text{for all } i \geq i_0. \tag{4.72}$$

Since $a_i < 0$, for all i, hence there will be no shocks at $x = A_1$ and the characteristic speed at $x = A_1$ is given by $f'(0) = -2$. This characteristic hits the line $x = 0$ at time $t = T_1$. Again $v_1 < 0$, so at $x = B_1$ it creats a shock with speed s_1 and this shock also hits the line $x = 0$ at time $t = T_1$. By (4.72) we see that for $x \geq 0$ no characteristics intersects before $t = T_1$, i.e. shocks and rarefactions do not meet before the time $t = T_1$.

For $x > 0, t < T_1$, the solution of (4.62),(4.69) is given by

$$u(x,t) = \begin{cases} a_{2i_0-1} & \text{if} \quad 0 < x < 2(a_{2i_0-1} - 1)t + x_{1,2i_0-1}, \\ \frac{1}{2}\left(\frac{x - x_{1,2i-1}}{t} + 2\right) & \text{if} \quad 2(a_{2i-1} - 1)t + x_{1,2i-1} < x \\ & < \quad 2(a_{2i} - 1)t + x_{1,2i-1}, \ i \geq i_0, \\ a_{2i} & \text{if} \quad 2(a_{2i} - 1)t + x_{1,2i-1} < x \\ & < \quad (a_{2i} + a_{2i+1} - 2)t + x_{1,2i}, \quad i \geq i_0, \\ a_{2i+1} & \text{if} \quad (a_{2i} + a_{2i+1} - 2)t + x_{1,2i} < x \\ & < \quad 2(a_{2i+1} - 1)t + x_{1,2i+1}, \quad i \geq i_0, \\ 0 & \text{if} \quad A_1 - 2t < x < s_1 + B_1, \\ v_1 & \text{if} \quad x > s_1 + B_1. \end{cases} \tag{4.73}$$

For $i \geq i_0$, let $t_{1,i}$ be the time when the characteristics originating from $x = x_{1,i}$ hits the line $x = 0$, then

$$\begin{aligned} t_{1,3i} &= \frac{x_{1,2i}}{2 - (a_{2i} + a_{2i+1})}, \\ t_{1,3i-1} &= \frac{x_{1,2i-1}}{2(1 - a_{2i})}, \\ t_{1,3i-2} &= \frac{x_{1,2i-1}}{2(1 - a_{2i-1})}. \end{aligned}$$

Since no characteristic intersects in the region $\{(x,t) \ : \ x > 0, 0 < t < T_1\}$, therefore

$$t_{1,3i-2} < t_{1,3i-1} < t_{1,3i} < T_1 \quad \text{for all } i \geq i_0. \tag{4.74}$$

Now $f'(a_i) < 0$ for all $i \geq 1$, therefore by using (4.73), we obtain

$$u(0+,t) = \begin{cases} a_{2i_0-1} & \text{if} \quad 0 < t < \frac{x_{1,2i_0-1}}{2(1-a_{2i_0-1})} = t_{1,3i_0-2}, \\ \frac{1}{2}\left(-\frac{x_{1,2i-1}}{t} + 2\right) & \text{if} \quad t_{1,3i-2} < t < t_{1,3i-1}, \quad \text{for } i \geq i_0, \\ a_{2i} & \text{if} \quad t_{1,3i-1} < t < t_{1,3i}, \quad \text{for } i \geq i_0, \\ a_{2i+1} & \text{if} \quad t_{1,3i} < t < t_{1,3i+1}, \quad \text{for } i \geq i_0, \\ v_1 & \text{if} \quad t > T_1. \end{cases} \tag{4.75}$$

R-H condition and (4.75) yields

$$u(0-,t)=\begin{cases} b_{2i_0-1} & \text{if } \ 0<t<t_{1,3i_0-2}, \\ g^{-1}f\left(\frac{1}{2}\left(-\frac{x_{1,2i-1}}{t}+2\right)\right) & \text{if } \ t_{1,3i-2}<t<t_{1,3i-1}, \quad \text{for } i\geq i_0, \\ b_{2i} & \text{if } \ t_{1,3i-1}<t<t_{1,3i}, \quad \text{for } i\geq i_0, \\ b_{2i+1} & \text{if } \ t_{1,3i}<t<t_{1,3i+1}, \quad \text{for } i\geq i_0, \\ g^{-1}f(v_1)=w_1(\text{say}) & \text{if } \ t>T_1. \end{cases} \tag{4.76}$$

Since $b_{2i}>b_{2i+1}$, it creats shock at $t=t_{1,3i}$. $g^{-1}f\left(\frac{1}{2}\left(-\frac{x_{1,2i-1}}{t}+2\right)\right)$ is an increasing function of t in $(t_{1,3i-2},t_{1,3i-1})$, so outgoing characteristic from $(t_{1,3i-2},t_{1,3i-1})$ will never meet. Let $t_{1,i}^{'}$ be the time when the lines $x=(b_{2i}+b_{2i+1})(t-t_{1,3i})$ and $x=2b_{2i}(t-t_{1,3i-1})$ meets, then we have

$$t_{1,i}^{'}=\frac{\frac{x_{1,2i}}{2+\frac{1}{(i+1)^2}+\frac{1}{(i+1)^4}}\left[\frac{\sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2-1}}{\sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2-1}}+1\right]-\frac{x_{1,2i-1}}{1+\frac{1}{(i+1)^4}}}{\left[\frac{\sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2-1}}{\sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2-1}}-1\right]} \tag{4.77}$$

Since

$$\lim_{i\to\infty}\frac{\sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2-1}}{\sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2-1}}=\infty.$$

Therefore

$$\lim_{i\to\infty}t_{1,i}^{'}=\lim_{i\to\infty}\frac{x_{1,2i}}{2+\frac{1}{(i+1)^2}+\frac{1}{(i+1)^4}}. \tag{4.78}$$

Let $t_{1,i}^{''}$ be the time when the lines $x=(b_{2i}+b_{2i+1})(t-t_{1,3i})$ and $x=2b_{2i+1}(t-t_{1,3i+1})$ meet, then we have

$$t_{1,i}^{''}=\frac{\frac{-x_{1,2i+1}}{(1+\frac{1}{(i+1)^2})}+\frac{x_{1,2i}}{2+\frac{1}{(i+1)^4}+\frac{1}{(i+1)^2}}\left[\frac{\sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2-1}}{\sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2-1}}+1\right]}{\left[\frac{\sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2-1}}{\sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2-1}}+1\right]} \tag{4.79}$$

Since

$$\lim_{i\to\infty}\frac{\sqrt{\left(\frac{1}{(i+1)^2}+1\right)^2-1}}{\sqrt{\left(\frac{1}{(i+1)^4}+1\right)^2-1}}=0,$$

therefore

$$\begin{aligned}\lim_{i\to\infty} t^{''}_{1,i} &= \lim_{i\to\infty} \frac{-x_{1,2i+1}}{(1+\frac{1}{(i+1)^2})} + \frac{x_{1,2i}}{2+\frac{1}{(i+1)^4}+\frac{1}{(i+1)^2}} \\ &= \lim_{i\to\infty} -\frac{x_{1,2i}}{2}.\end{aligned} \tag{4.80}$$

From (4.77) and (4.78), there exists a large $\tilde{i}_0$ (denoting $\tilde{i}_0$ by i_0 only) and a $T_1 > 0$ such that

$$t^{'}_{1,i} > T_1 > 0 \ \text{ for all } i \geq i_0. \tag{4.81}$$

Now from (4.79), (4.80), there exists a large $\tilde{\tilde{i}}_0$ (denoting $\tilde{\tilde{i}}_0$ by i_0 only) so that

$$t^{''}_{1,i} < 0 \ \text{ for all } i \geq i_0. \tag{4.82}$$

By using (4.72)(4.81)(4.82) it is clear that no characteristics meet in the region

$$\text{(say) } F_1 = \{(x,t) : \ x \in I\!R, 0 < t < T_1.\} \tag{4.83}$$

Hence the solution of (4.59),(4.69) for $t = T_1$ is given by

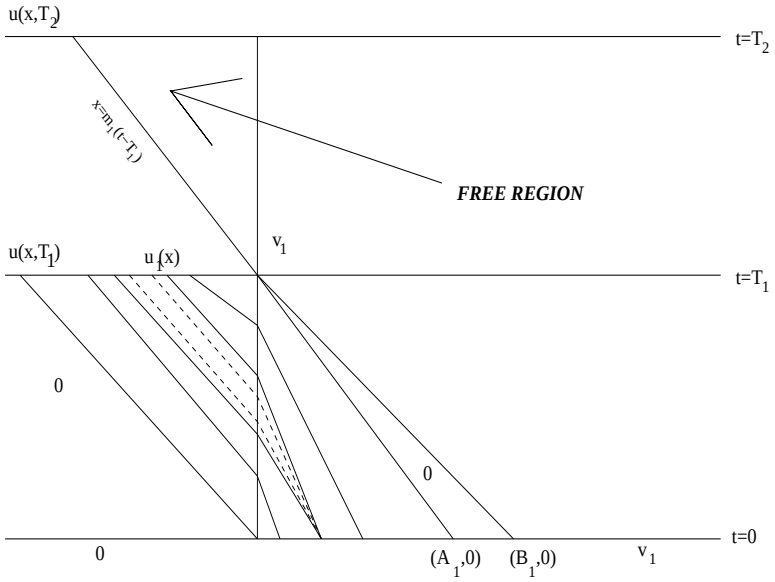

Figure 9:

$$u(x,B_1)=u_1(x)=\begin{cases} 0 & \text{if} \quad 0<b_{2i_0-1}T_1, \\ b_{2i_0-1} & \text{if} \quad b_{2i_0-1}T_1<x<2b_{2i_0-1}(t-t_{1,3i_0-2}), \\ u(0-,t_+(x,T_1)) & \text{if} \quad 2b_{2i-1}(t-t_{1,3i-2})<x \\ & < \quad 2b_{2i}(t-t_{1,3i-1}), \text{ for } i\geq i_0, \\ b_{2i} & \text{if} \quad 2b_{2i}(t-t_{1,3i-1})<x \\ & < \quad (b_{2i}+b_{2i+1})(t-t_{1,3i}), \text{ for } i\geq i_0, \\ b_{2i+1} & \text{if} \quad (b_{2i}+b_{2i+1})(t-t_{1,3i})<x \\ & < \quad b_{2i+1}(t-t_{1,3i+1}), \text{ for } i\geq i_0, \\ v_1 & \text{if} \quad x>0, \end{cases} \tag{4.84}$$

where $t_+(x,T_1)$ is an increasing function of x (this is possible due to the fact that $u(0-,t)$ is an increasing function of t in $(t_{1,3i-2},t_{1,3i-1})$).

Step 3 : From (4.84) we can choose a partition such that $P_{1,2i_0-1}<P_{1,2i_0}<\cdots<0$, with $\lim\limits_{j\to\infty}P_{1,j}=0$ such that $u(P_{1,i},T_1)=b_i$ for all $i\geq 2i_0-1$. Hence

$$\begin{aligned} TV(u(\cdot\,,T_1)) &\geq \sum_{j=2i_0-1}^{\infty}|u(P_{1,j},T_1)-u(P_{1,j+1},T_1)| \\ &= \sum_{j=2i_0-1}^{\infty}|b_j-b_{j+1}| \\ &\geq \sum_{j=2i_0-1}^{\infty}\sqrt{\left(1+\frac{1}{(j+1)^2}\right)^2-1} \\ &- \sum_{j=2i_0-1}^{\infty}\sqrt{\left(1+\frac{1}{(j+1)^4}\right)^2-1} \\ &\geq \sum_{j=2i_0-1}^{\infty}\frac{1}{j+1}-\sum_{j=2i_0-1}^{\infty}\sqrt{\left(1+\frac{1}{(j+1)^4}\right)^2-1} \\ &= \infty. \end{aligned} \tag{4.85}$$

Now our aim is to find a $T_2>T_1$ and a $u_0\in BV(I\!R)$ such that $u(x,T_1)\notin BV$ as well as $u(x,T_2)\notin BV$.

Step 4 : Consider the following problem

$$\begin{aligned} u_t+((u-1)^2-1)_x=0 \qquad & \text{if} \qquad x>0, t>T_1, \\ u_t+(u^2)_x=0 \qquad & \text{if} \qquad x<0, t>T_1, \\ u(x,T_1)=u_1(x) \quad & \text{for all} \quad x\in I\!R. \end{aligned} \tag{4.86}$$

Now $u_1(x)$ satisfies all the condition of Lemma 3.2, therefore there exists a $m_1<0$ such that the solution of (4.86) satisfies

$$\begin{aligned} u(x,t)=g^{-1}f(v_1)=w_1(\text{say}) \quad & \text{if} \quad m_1(t-T_1)<x<0, \\ =v_1 \quad & \text{if} \quad x>0. \end{aligned} \tag{4.87}$$

So we do get a free region (see Figure 9). Let $\tilde{A}_1 > B_1, T_2 > \frac{\tilde{A}_1}{2(1-v_1)}$, where $\tilde{A}_1, T_2$ are going to be choose in a proper way. We choose v_2 such that

$$\begin{array}{rll} v_1 < v_2 < -\frac{1}{(i+1)^2} & \text{for all} & i \geq i_1, \\ |v_2| < \frac{1}{2^{l_1}} \text{ and } v_2 \notin \mathcal{D}, & \text{for some} & l_1 \in N \text{ with } l_0 < l_1, \end{array} \tag{4.88}$$

where i_1 is a large natural number with $i_1 > i_0$ and we are going to choose i_1 later. Denote

$$\begin{aligned} A_2 &= 2T_2,\ B_2 = T_1\frac{1-(v_2-1)^2}{v_2},\ s_2 = \frac{(v_2-1)^2-1}{v_2}, \\ \beta_1 &= 2\sqrt{(v_1-1)^2-1},\ \bar{\beta}_1 = \frac{\tilde{A}_1}{1-v_1},\ \gamma_1 = 2\sqrt{(a_{2i_1-1}-1)^2-1}, \\ \bar{\gamma}_1 &= \frac{\tilde{A}_1}{1-a_{2i_1-1}},\ \tilde{A}_1 + x_{2,2i_1-1} = A_2 - \sum_{j=2i_1-1}^{\infty}\frac{1}{j^{3/2}}, \\ x_{2,2i} &= x_{2,2i_1-1} - \sum_{j=2i}^{i}\frac{1}{j^{3/2}} \text{ for } i \geq i_1. \end{aligned}$$

Then by definition

$$x_{2,2i_1-1} < x_{2,2i_1} < \text{.....} < x_{2,i}\text{.....} < A_2 \text{ and } A_2 = \sum_{j=2i_1-1}^{\infty} x_{2,j}.$$

Now consider the following initial value problem

$$\begin{array}{rl} u_t + ((u-1)^2-1)_x = 0, & \text{if } x > 0, t > 0 \\ u_t + (u^2)_x = 0 & \text{if } x < 0, t > 0, \end{array} \tag{4.89}$$

with initial data u_0 as follows

$$u_0(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ a_{2i_0-1} & \text{if } 0 < x < x_{1,2i_0-1}, \\ a_{2i} & \text{if } x_{1,2i-1} < x < x_{1,2i}, \text{ for } i \geq i_0, \\ a_{2i+1} & \text{if } x_{1,2i} < x < x_{1,2i+1}, \text{ for } i \geq i_0, \\ 0 & \text{if } A_1 < x < B_1, \\ v_1 & \text{if } B_1 < x < \tilde{A}_1, \\ a_{2i_1-1} & \text{if } \tilde{A}_1 < x < \tilde{A}_1 + x_{2,2i_1-1}, \\ a_{2i} & \text{if } \tilde{A}_1 + x_{2,2i-1} < x < \tilde{A}_1 + x_{2,2i}, \text{ for } i \geq i_1, \\ a_{2i+1} & \text{if } \tilde{A}_1 + x_{2,2i} < x < \tilde{A}_1 + x_{2,2i+1}, \text{ for } i \geq i_1, \\ 0 & \text{if } A_2 < x < B_2, \\ v_2 & \text{if } x \geq B_2. \end{cases} \tag{4.90}$$

The lines $x = f'(v_1)t + \tilde{A}_1$ and $x = s_1 t + T_1$ meet at time

$$t = -\frac{\tilde{A}_1 - T_1}{v_1} \tag{4.91}$$

The lines $x = g'(g^{-1}f(v_1))(t - \frac{\tilde{A}_1}{2(1-v_1)})$ and $x = m_1(t - T_1)$ meet at time

$$t = \frac{2\sqrt{(v_1-1)^2-2}\frac{\tilde{A}_1}{2(v_1-1)} + m_1T_1}{m_1 - 2\sqrt{(v_1-1)^2-1}}. \tag{4.92}$$

From (4.91) and (4.92) it is clear that if we choose $\tilde{A}_1$ large such that $\tilde{A}_1 > 1$, then the above lines do not intersects before time $T_2 = \delta + \frac{\tilde{A}_1}{2(1-v_1)}$, for some $\delta > 0$, which we are going to choose later. Denote for $i \geq i_1$, $t_{2,i}$ be the time when the characteristics originating from $x = x_{2,i}$ hits the line $x = 0$, then

$$t_{2,3i} = \frac{x_{2i} + \tilde{A}_1}{2 - (a_{2i} + a_{2i+1})},\ t_{2,3i-1} = \frac{x_{2i-1} + \tilde{A}_1}{2(1 - a_{2i})},\ t_{2,3i-2} = \frac{x_{2i-1} + \tilde{A}_1}{2(1 - a_{2i-1})}.$$

Let the lines $x = 2(a_{2i}-1)t + \tilde{A}_1 + x_{2,2i-1}$ and $x = (a_{2i}+a_{2i+1}-2)t + \tilde{A}_1 + x_{2,2i}$ meet at time $t = \tilde{t}_{2,i}$, the lines $x = (a_{2i}+a_{2i+1}-2)t + \tilde{A}_1 + x_{2,2i}$ and $x = 2(a_{2i+1}-1)t + \tilde{A}_1 + x_{2,2i+1}$ meet at time $t = \bar{t}_{2,i}$, the lines $x = (b_{2i} + b_{2i+1})(t - t_{2,3i})$ and $x = 2b_{2i}(t - t_{2,3i})$ meet at time $t = t'_{2,i}$, the lines $x = (b_{2i} + b_{2i+1})(t - t_{2,3i})$ and $x = 2b_{2i+1}(t - t_{2,3i+1})$ meet at time $t = t''_{2,i}$, then similarly as in (4.72), (4.81), (4.82) it is clear that there exists $T_2 > \frac{\tilde{A}_1}{2(1-v_1)}$ and a large $i_1 \in \mathbb{N}$ such that no characteristics insersects in the region

$$\text{(say) } F_2 = \{(x,t) :\ m_1(t - T_1) < x < 0,\ T_1 < t < T_2\}. \tag{4.93}$$

By (4.93) the solution of (4.59), (4.90) as follows (see Figure 10)

$$u(x, T_1) \ = \ u_1(x) \text{ for all } x.$$

For $m_1(t - T_1) < x,\ t = T_2$,

$$u(x, T_2) = u_2(x) = \begin{cases} g^{-1}f(v_1) & \text{if} \quad m_1(T_2 - T_1) < x < 2\beta(T_2 - \bar{\beta}_1), \\ u(0+, t_+(x, T_2)) & \text{if} \quad 2\beta(T_2 - \bar{\beta}_1) < x < 2\gamma(T_2 - \bar{\gamma}_1), \\ b_{2i_1-1} & \text{if} \quad b_{2i_1-1}B_1 < x \\ & < \quad 2b_{2i_1-1}(T_2 - t_{2,3i_1-2}), \\ u(0-, t_+(x, T_2)) & \text{if} \quad 2b_{2i-1}(T_2 - t_{2,3i-2}) < x \\ & < \quad 2b_{2i}(T_2 - t_{2,3i-1}),\ \text{for } i \geq i_1, \\ b_{2i} & \text{if} \quad 2b_{2i}(T_2 - t_{2,3i-1}) < x \\ & < \quad (b_{2i} + b_{2i+1})(T_2 - t_{2,3i}),\ \text{for } i \geq i_1, \\ b_{2i+1} & \text{if} \quad (b_{2i} + b_{2i+1})(T_2 - t_{2,3i}) < x \\ & < \quad b_{2i+1}(T_2 - t_{2,3i+1}),\ \text{for } i \geq i_1, \\ v_2 & \text{if} \quad x > 0. \end{cases} \tag{4.94}$$

Now by (4.94) we can choose a partition such that

$$\begin{aligned} &P_{2,2i_1-1} < P_{2,2i_1} < \cdots < 0,\ \text{with } \lim_{j\to\infty} P_{2,j} = 0 \\ &\text{such that } u(P_{2,i}, T_2) = b_i,\ \text{for all } i \geq 2i_1 - 1. \end{aligned} \tag{4.95}$$

Hence

$$\begin{aligned} TV(u(\cdot\,,T_2)) &\geq \sum_{j=2i_1-1}^{\infty} |u(P_{2,j},T_2) - u(P_{2,j+1},T_2)| \\ &= \sum_{j=2i_1-1}^{\infty} |b_j - b_{j+1}| \\ &= \infty. \end{aligned} \tag{4.96}$$

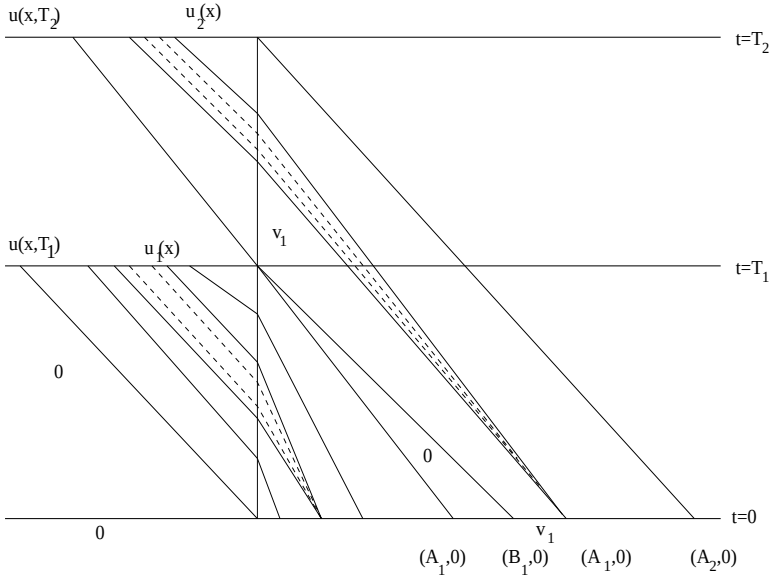

Figure 10:

Step 5 : Again we Consider the following problem

$$\begin{aligned} u_t + ((u-1)^2 - 1)_x &= 0 \qquad \text{if } \ x > 0, t > T_2, \\ u_t + (u^2)_x &= 0 \qquad \text{if } \ x < 0, t > T_2, \\ u(x,T_1) &= u_2(x) \quad \text{for all } \ x \in I\!R. \end{aligned} \tag{4.97}$$

Now $u_2(x)$ satisfies all the condition of Lemma 3.2, therefore there exists a $m_2 < 0$ such that the solution of (4.97) satisfies

$$\begin{aligned} u(x,t) &= g^{-1}f(v_2) = w_2(\text{say}) \ \text{ if } \ m_2(t-T_2) < x < 0, \\ &= v_2 \ \text{ if } \ x > 0. \end{aligned} \tag{4.98}$$

Then we can proceed as in Step 4 to get

$$T_3 > T_2 > T_1 \text{ such that } u(x,T_i) \notin BV, \text{ for } i = 1,2,3.$$

Step 6 : In general, we consider the following problem to get free region

$$\begin{array}{rlll} u_t + ((u-1)^2-1)_x = 0 & \text{if} & x>0, t>T_n, \\ u_t + (u^2)_x = 0 & \text{if} & x<0, t>T_n, \\ u(x,T_n) = u_n(x) & \text{for all} & x \in I\!R. \end{array} \tag{4.99}$$

Now $u_n(x)$ satisfies all the condition of Lemma 4.1, therefore there exists a $m_n < 0$ such that the solution of (4.99) satisfies

$$\begin{array}{rcl} u(x,t) & = & g^{-1}f(v_n) = w_n(\text{say}) \quad \text{if} \quad m_n(t-T_n) < x < 0, \\ & = & v_n \ \text{ if } \ x>0. \end{array} \tag{4.100}$$

Let $\tilde{A}_n > B_n, T_{n+1} > \frac{\tilde{A}_n}{2(1-v_n)}$ then $A_1 < B_1 < \tilde{A}_1 < A_2 < B_2 < \tilde{A}_2 < A_3 < \cdots < B_n < \tilde{A}_n$, where $n < \tilde{A}_n$ and we are going to be choose $\tilde{A}_n, T_{n+1}$ in a suitable way. We choose v_{n+1} such that

$$\begin{array}{lll} v_1 < v_2 < ..... < v_n < v_{n+1} < -\frac{1}{(i+1)^2} & \text{for all} & i \geq i_n, \\ |v_{n+1}| < \frac{1}{2^{l_n}} \text{ and } v_{n+1} \notin \mathcal{D}, & \text{for some} & l_1 \in \mathbb{N} \\ \text{with } l_0 < l_1 < .... < l_{n-1} < l_n, & & \end{array} \tag{4.101}$$

where i_n is a large natural number with $i_n > i_{n-1}$ and we are going to choose i_n later.

Denote

$$\begin{aligned} A_{n+1} &= 2T_{n+1},\ B_{n+1} = T_n\frac{1-(v_{n+1}-1)^2}{v_{n+1}},\ s_{n+1} = \frac{(v_{n+1}-1)^2-1}{v_{n+1}}, \\ \beta_n &= 2\sqrt{(v_n-1)^2-1},\ \bar{\beta}_n = \frac{\tilde{A}_n}{1-v_n},\ \gamma_n = 2\sqrt{(a_{2i_n-1}-1)^2-1}, \\ \bar{\gamma}_n &= \frac{\tilde{A}_n}{1-a_{2i_n-1}},\ \tilde{A}_n + x_{n+1,2i_n-1} = A_{n+1} - \sum_{j=2i_n-1}^{\infty}\frac{1}{j^{3/2}}, \\ x_{n+1,2i} &= x_{n+1,2i_n-1} - \sum_{j=2i}^{i}\frac{1}{j^{3/2}},\ \text{for } i \geq i_n. \end{aligned}$$

Then by definition

$$x_{n+1,2i_n-1} < x_{n+1,2i_n} < ..... < x_{n+1,i}..... < A_{n+1} \text{ and } A_{n+1} = \sum_{j=2i_n-1}^{\infty} x_{n+1,j}.$$

Denote for $i \geq i_n$, $t_{n+1,i}$ be the time when the characteristics originating from $x = x_{n+1,i}$, hits the line $x = 0$, then

$$\begin{aligned} t_{n+1,3i} &= \frac{x_{n+1,2i} + \tilde{A}_n}{2 - (a_{2i} + a_{2i+1})}, \\ t_{n+1,3i-1} &= \frac{x_{n+1,2i-1} + \tilde{A}_n}{2(1 - a_{2i})}, \\ t_{n+1,3i-2} &= \frac{x_{n+1,2i-1} + \tilde{A}_n}{2(1 - a_{2i-1})}. \end{aligned}$$

Finally we consider the following initial value problem

$$\begin{aligned} u_t + ((u-1)^2 - 1)_x = 0 \quad \text{if} \quad x > 0, t > 0 \\ u_t + (u^2)_x = 0 \quad \text{if} \quad x < 0, t > 0. \end{aligned} \tag{4.102}$$

with the general initial data u_0 as follows

$$u_0(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ a_{2i_0-1} & \text{if} \quad 0 < x < x_{1,2i_0-1}, \\ a_{2i} & \text{if} \quad x_{1,2i-1} < x < x_{1,2i}, \text{ for } i \geq i_0, \\ a_{2i+1} & \text{if} \quad x_{1,2i} < x < x_{1,2i+1}, \text{ for } i \geq i_0, \\ 0 & \text{if} \quad A_1 < x < B_1, \\ v_1 & \text{if} \quad B_1 < x < \tilde{A}_1, \\ a_{2i_1-1} & \text{if} \quad \tilde{A}_1 < x < \tilde{A}_1 + x_{2,2i_1-1}, \\ a_{2i} & \text{if} \quad \tilde{A}_1 + x_{2,2i-1} < x < \tilde{A}_1 + x_{2,2i}, \text{ for } i \geq i_1, \\ a_{2i+1} & \text{if} \quad \tilde{A}_1 + x_{2,2i} < x < \tilde{A}_1 + x_{2,2i+1}, \text{ for } i \geq i_1, \\ 0 & \text{if} \quad A_2 < x < B_2, \\ v_2 & \text{if} \quad x \geq B_2, \\ . & \\ . & \\ . & \\ a_{2i_n-1} & \text{if} \quad \tilde{A}_n < x < \tilde{A}_n + x_{n+1,2i_n-1}, \\ a_{2i} & \text{if} \quad \tilde{A}_n + x_{n+1,2i-1} < x < \tilde{A}_n + x_{n+1,2i}, \text{ for } i \geq i_n, \\ a_{2i+1} & \text{if} \quad \tilde{A}_n + x_{n+1,2i} < x < \tilde{A}_n + x_{n+1,2i+1}, \text{ for } i \geq i_n, \\ 0 & \text{if} \quad A_{n+1} < x < B_{n+1}, \\ v_{n+1} & \text{if} \quad x \geq B_{n+1}, \\ . & \\ . & \\ . & \end{cases} \tag{4.103}$$

By similar methods as in Step 4, we can choose T_{n+1} and a i_n large so that no characteristic intersects in the region

$$\text{(say) } F_{n+1} = \{(x,t) : \ m_n(t - T_n) < x < 0, \ T_n < t < T_{n+1}\}. \tag{4.104}$$

Therefore by (4.104) the solution of (4.62), (4.103) as follows

$$\begin{aligned} u(x,T_1) &= u_1(x) \quad \text{for } x \in I\!R, \\ u(x,T_2) &= u_2 \quad \text{for } m_1(T_2-T_1) < x, \\ &\vdots \\ u(x,T_n) &= u_n \quad \text{for } m_{n-1}(T_n-T_{n-1}) < x, \end{aligned}$$

and for $m_n(T_{n+1}-T_n) < x,\ t = T_{n+1}$

$$u(x,T_{n+1}) = u_{n+1}(x) = \begin{cases} g^{-1}f(v_n) & \text{if } m_n(T_{n+1}-T_n) < x \\ & < 2\beta(T_{n+1}-\bar{\beta}_n), \\ u(0+,t_+(x,T_{n+1})) & \text{if } 2\beta(T_{n+1}-\bar{\beta}_n) < x \\ & < 2\gamma(T_{n+1}-\bar{\gamma}_n), \\ b_{2i_n-1} & \text{if } b_{2i_n-1}B_n < x \\ & < 2b_{2i_n-1}(T_{n+1}-t_{n+1,3i_n-2}), \\ u(0-,t_+(x,T_{n+1})) & \text{if } 2b_{2i-1}(T_{n+1}-t_{n+1,3i-2}) < x \\ & < 2b_{2i}(T_{n+1}-t_{n+1,3i-1}), \\ & \text{for } i \geq i_n, \\ b_{2i} & \text{if } 2b_{2i}(T_{n+1}-t_{n+1,3i-1}) < x \\ & < (b_{2i}+b_{2i+1})(T_{n+1}-t_{n+1,3i}), \\ & \text{for } i \geq i_n, \\ b_{2i+1} & \text{if } (b_{2i}+b_{2i+1})(T_{n+1}-t_{n+1,3i}) \\ & < x < b_{2i+1}(T_{n+1}-t_{n+1,3i+1}), \\ & \text{for } i \geq i_n, \\ v_{n+1} & \text{if } x > 0. \end{cases} \tag{4.105}$$

Now from (4.105) (similarly as in step 3) we can choose a partition such that

$$\begin{aligned} &P_{n+1,2i_n-1} < P_{n+1,2i_n} < \cdots < 0, \text{ with } \lim_{j\to\infty} P_{n+1,j} = 0 \\ &\text{such that } u(P_{n+1,i},T_{n+1}) = b_i \text{ for all } i \geq 2i_n - 1. \end{aligned} \tag{4.106}$$

Hence

$$\begin{aligned} TV(u(\cdot\,,T_{n+1})) &\geq \sum_{j=2i_n-1}^{\infty} |u(P_{n+1,j},T_{n+1}) - u(P_{n+1,j+1},T_{n+1})| \\ &= \sum_{j=2i_n-1}^{\infty} |b_j - b_{j+1}| \\ &= \infty. \end{aligned} \tag{4.107}$$

Now

$$T_n = \frac{A_{n+1}}{2} = \tilde{A}_n,\ n < \tilde{A}_n \ \text{ for all } \ n \in \mathbb{N},$$

therefore

$$\lim_{n\to\infty} \tilde{A}_n = \infty \ \text{ and } \ \lim_{n\to\infty} T_n = \infty.$$

Hence the result.

5 Appendix

THEOREM 5.1 *Let $A \neq \theta_g$ and $B \neq \theta_f$. If $f'', g'' \geq \alpha > 0, u_0 \in L^\infty(I\!R)$ then for all $t > 0$,*

$$TV\left(u(\cdot\,,t), I(\epsilon) \cup I(M,\epsilon)\right) \leq C_1(\epsilon) + C_2(\epsilon, M, t). \tag{5.108}$$

As a consequence, if $u_0 \in BV(I\!R)$ then for all $t > 0$,

$$TV\left(u(\cdot\,,t)\right) \leq C(\epsilon,t)(TV(u_0)+1) + 4\|u_0\|\infty. \tag{5.109}$$

PROOF : Without loss of generality we can prove the result for $x > 0$, i.e. we prove $u(\cdot,t) \in BV(I(\epsilon) \cup I(M,\epsilon)) \cap I\!R_+$, similar result holds in $(I(\epsilon) \cup I(M,\epsilon)) \cap I\!R_-$. From [8] (equation no. (44)), we have for $x > 0$,

$$u(x,t) = \begin{cases} f^{*'}\left(\frac{x-y_+(x,t)}{t}\right) \text{if} y_+(x,t) > 0, \quad \text{i,e. if } \ 0 \leq R(t) \leq x \\ \lambda_+(t_+(x,t)) = f^{*'}\left(\frac{-y_+(t_+)}{t_+}\right) = u(0+,t) \\ \text{if } 0 < t_+(x,t) < t \ \ \text{i,e. if } \ 0 \leq x \leq R(t). \end{cases} \tag{5.110}$$

Then we have the following two cases.

Case I: $R(t) = 0$. In this case $u(x,t) = f^{*'}\left(\frac{x-y_+(x,t)}{t}\right)$ for all $x > 0$, hence by using the Lipschitz continuity of $f^{*'}$ and monotonicity of y_+, we obtain

$$TV\left(u((\cdot\,,t), I(\epsilon) \cup I(M,\epsilon)) \cap I\!R_+\right) \leq \frac{C}{t}. \tag{5.111}$$

Where the constant depends only on the $f^{*'}$ and M.

Case II: $R(t) > 0$. From Lemma 2.10 of [1], we have $f(\lambda_+) > f(B)$, when $B \neq \theta_f$. Again R-H condition yields $g(\lambda_-(t)) = f(\lambda_+(t))$. Since $f'(\lambda_+(t)) > f'(B)$ and $g'(\lambda_-(t)) < A$, hence the characteristics speed is never zero near interface, which allows $\lambda_+(t_+(x,t)) = f^{*'}\left(\frac{-y_+(t_+)}{t_+}\right) \in BV$ by using the Lipschitz continuity of $f^{*'}$ and the monotoncity of t_+, y_+. Hence the result.

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