



**HAL**  
open science

# Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions

Pierre-Yves Madec

► **To cite this version:**

Pierre-Yves Madec. Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions. *Stochastic Processes and their Applications*, 2015, 125 (5), pp.1821-1860. 10.1016/j.spa.2014.11.015 . hal-00868129v3

**HAL Id: hal-00868129**

**<https://hal.science/hal-00868129v3>**

Submitted on 15 Jan 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions

P.Y. Madec<sup>a,\*</sup>

<sup>a</sup>IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.

---

## Abstract

We study a class of ergodic BSDEs related to PDEs with Neumann boundary conditions. The randomness of the driver is given by a forward process under weakly dissipative assumptions with an invertible and bounded diffusion matrix. Furthermore, this forward process is reflected in a convex subset of  $\mathbb{R}^d$  not necessarily bounded. We study the link of such EBSDEs with PDEs and we apply our results to an ergodic optimal control problem.

*Keywords:* backward stochastic differential equations, weakly dissipative drift, Neumann boundary conditions, ergodic partial differential equations, optimal ergodic control problem

---

## 1. Introduction

In this paper we study the following ergodic backward stochastic differential equation (EBSDE in what follows) in finite dimension and in infinite horizon:  $\forall t, T \in \mathbb{R}_+, 0 \leq t \leq T < +\infty$ :

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds + \int_t^T [g(X_s^x) - \mu] dK_s^x - \int_t^T Z_s^x dW_s, \quad (1.1)$$

where the given data satisfy:

- $W$  is an  $\mathbb{R}^d$ -valued standard Brownian motion;
- $G = \{\phi > 0\}$  is an open convex subset of  $\mathbb{R}^d$  with smooth boundary;
- $x \in G$ ;
- $X^x$  is a  $\bar{G}$ -valued process starting from  $x$ , and  $K^x$  is a non decreasing real valued process starting from 0 such that the pair  $(X^x, K^x)$  is a solution of the following reflected stochastic differential equation (SDE in what follows):

$$X_t^x = x + \int_0^t f(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s + \int_0^t \nabla \phi(X_s^x) dK_s^x, \quad t \geq 0,$$
$$K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}} dK_s^x, \quad K^x \text{ is non-decreasing,}$$

- $\psi : \mathbb{R}^d \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  is measurable and  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable;

---

\*Tel.: +33 02 23 23 53 05

Email address: pierre-yves.madec@univ-rennes1.fr (P.Y. Madec)

- $\lambda$  and  $\mu$  belong both to  $\mathbb{R}$ . If  $\lambda$  is given then  $\mu$  is unknown and if  $\mu$  is given then  $\lambda$  is unknown.

Therefore, the unknown is either the triplet  $(Y^x, Z^x, \lambda)$  if  $\mu$  is given or the triplet  $(Y^x, Z^x, \mu)$  if  $\lambda$  is given, where:

- $Y^x$  is a real-valued progressively measurable process;
- $Z^x$  is an  $\mathbb{R}^{1 \times d}$ -valued progressively measurable process.

We recall that a function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to be strictly dissipative if there exists a constant  $\eta > 0$  such that,  $\forall x, y \in \mathbb{R}^d$ ,

$$(h(x) - h(y), x - y) \leq -\eta|x - y|^2.$$

Richou in the paper [14] studied the case when  $\overline{G}$  is bounded and with the assumptions that  $f$  and  $\sigma$  are Lipschitz and:

$$\sup_{x, y \in \overline{G}, x \neq y} \left\{ \frac{t(x - y)(f(x) - f(y))}{|x - y|^2} + \frac{\text{Tr}[(\sigma(x) - \sigma(y))^t(\sigma(x) - \sigma(y))]}{2|x - y|^2} \right\} < -K_{\psi, z}K_{\sigma}$$

where  $K_{\psi, z}$  is the Lipschitz constant of  $\psi$  in  $z$  and  $K_{\sigma}$  is the Lipschitz constant of  $\sigma$ . Note that this assumption implies that  $f$  is strictly dissipative. However this hypothesis on  $f$  is not very natural because it supposes a dependence between parameters of the problem. Thanks to this condition it is possible to establish one of the key results: the strong estimate on the exponential decay in time of two solutions of the forward equation starting from different points. Indeed, it is used to construct, by a diagonal procedure, a solution to the EBSDE. Note that, in this work,  $\overline{G}$  is assumed to be bounded.

In the paper [3], Debussche, Hu and Tessitore were concerned with the study of EBSDE in a weakly dissipative environment. This means that the driver of the forward process is assumed to be the sum of a strictly dissipative term and a perturbation term which is Lipschitz and bounded. In their infinite dimensional framework, they supposed that the dissipative term is linear. In addition,  $\sigma$  is constant, and the forward process is not reflected. Finally the coefficients of the forward process are assumed to be Gâteaux differentiable to obtain an estimate which is needed to prove the existence of a solution in this framework. In this context, the weaker assumption on  $f$  makes the strong estimate on the exponential decay in time of two solutions of the forward equation impossible. However it is possible to substitute this result by a weaker result, called "basic coupling estimate" which involves the Kolmogorov semigroups of the forward process  $X^x$  and which is enough to prove the existence of a solution to the EBSDE.

In this paper we extend the framework of [14] to the case of an unbounded domain  $\overline{G}$  for a driver weakly dissipative. Namely, we assume that  $f = d + b$  where  $d$  is locally Lipschitz and dissipative with polynomial growth and  $b$  is Lipschitz and bounded. The price to pay is that  $\sigma$  is assumed to be Lipschitz, invertible and such that  $\sigma$  and  $\sigma^{-1}$  are bounded. We do not need more regularity than continuous coefficients for this study, because we treat this problem by a regularization procedure. As the basic coupling estimate of [3] holds for a non reflected process, we start by studying the following forward process,  $\forall t \geq 0$ ,

$$V_t^x = x + \int_0^t f(V_s^x)ds + \int_0^t \sigma(V_s^x)dW_s,$$

with  $f$  and  $\sigma$  defined as before. We show that the coupling estimate still holds in our framework with constants which depend on  $d$  only through its dissipativity coefficient. Once this is

established, we apply this result to establish existence and uniqueness (of  $\lambda$ ) of solutions to the following EBSDE:

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (1.2)$$

Then we want to obtain the same result when the process  $V^x$  is replaced by a reflected process  $X^x$  in  $\overline{G}$ , namely:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (1.3)$$

For this purpose, we use a penalization method to construct a sequence of processes  $X^{x,n}$  defined on the whole  $\mathbb{R}^d$  and which converges to the reflected process  $X^x$ . More precisely, we denote by  $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$  the solution of the following BSDE with regularized coefficients  $\psi^\varepsilon, d^\varepsilon, F_n^\varepsilon$  and  $b^\varepsilon$  by convolution with a sequence approximating the identity,  $\forall t, T \in \mathbb{R}_+, 0 \leq t \leq T < +\infty$ :

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}] ds - \int_t^T Z_s^{x,\alpha,n,\varepsilon} dW_s, \quad (1.4)$$

where  $X^{x,n,\varepsilon}$  is the strong solution of the SDE:

$$X_t^{x,n,\varepsilon} = x + \int_0^t (d^\varepsilon + F_n^\varepsilon + b^\varepsilon)(X_s^{x,n,\varepsilon}) ds + \int_0^t \sigma^\varepsilon(X_s^{x,n,\varepsilon}) dW_s.$$

Note that as  $F_n$  is dissipative with a dissipative constant equal to 0,  $d + F_n$  remains dissipative with a dissipative coefficient equal to  $\eta$ . Then, making  $\varepsilon \rightarrow 0, n \rightarrow +\infty$  and  $\alpha \rightarrow 0$ , it is possible to show that, roughly speaking,  $(Y_t^{x,\alpha,n,\varepsilon} - Y_0^{x,\alpha,n,\varepsilon}, Z_t^{x,\alpha,n,\varepsilon}, Y_0^{x,\alpha,n,\varepsilon}) \rightarrow (Y_t^x, Z_t^x, \lambda)$  which is solution of EBSDE (1.3). Once a solution  $(Y, Z, \lambda)$  is found for the EBSDE (1.3) we study existence and uniqueness of solutions of the type  $(Y, Z, \lambda)$  and  $(Y, Z, \mu)$  of the EBSDE (1.1). Here we only manage to find solutions which are not Markovian and which are not bounded in expectation. Then we show that the function defined by  $v(x) := Y_0^x$ , where  $Y$  is a solution of EBSDE (1.3) is a viscosity solution of the following partial differential equation (PDE in what follows) :

$$\begin{cases} \mathcal{L}v(x) + \psi(x, \nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \quad (1.5)$$

where:

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(\sigma(x)^t \sigma(x) \nabla^2 u(x)) + {}^t f(x) \nabla u(x).$$

Note that the boundary ergodic problem

$$\begin{cases} F(D^2v, Dv, x) & = \lambda \text{ in } \overline{G} \\ L(Dv, x) & = \mu \end{cases}$$

were studied in [1] by Barles, Da Lio, Lions and Souganidis when  $\overline{G}$  is a smooth, periodic, half-space-type domain and  $F$  a periodic function. They found a constant  $\mu$  such that there exists a bounded viscosity solution  $v$  of the above problem.

At last we show that we can use the theory of EBSDE to solve an optimal ergodic control problem.  $R : U \rightarrow \mathbb{R}^d$  is assumed to be bounded and  $L$  is assumed to be Lipschitz and bounded. We define the ergodic cost:

$$I(x, \rho) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_T^\rho \left[ \int_0^T L(X_s^x, \rho_s) ds \right], \quad (1.6)$$

where  $\rho$  is an adapted process with values in a separable metric space  $U$  and  $\mathbb{E}_T^\rho$  is the expectation with respect to the probability measure under which  $W_t^\rho = W_t + \int_0^t R(\rho_s) ds$  is a Brownian motion on  $[0, T]$ . Defining

$$\psi(x, z) = \inf_{u \in U} \{L(x, u) + zR(u)\}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^{1 \times d},$$

it is possible to show that, for any admissible control  $\rho$ ,  $I(x, \rho) \geq \lambda$ . That is why  $\lambda$  is called ergodic cost. In a similar way,  $\mu$  is called boundary ergodic cost.

The paper is organized as follows. In section 2, we study the forward SDE under the hypothesis that the drift is weakly dissipative and that the diffusion matrix is invertible and bounded. In this section we prove that the estimates we establish depend on  $d$  through its dissipativity coefficient. In section 3, we use the basic coupling estimate to study existence and uniqueness of an EBSDE with zero Neumann boundary conditions with a forward process weakly dissipative but non-reflected. In section 4, we use a penalization method to show that the same result holds for a reflected process in a convex set not necessarily bounded. Then, we establish the link between the EBSDE with zero Neumann boundary condition and a PDE. Finally, we apply our results to an optimal ergodic control problem. Some technical proofs are given in the Appendix.

## 2. The forward SDE

### 2.1. General notation

The canonical scalar product on  $\mathbb{R}^d$  is denoted by  $(\cdot, \cdot)$  and the associated norm is denoted by  $|\cdot|$ . Given a matrix  $\sigma \in \mathbb{R}^{d \times d}$ , we define by  $\|\cdot\|$  its operator norm. Let  $\mathcal{O}$  be an open connected subset of  $\mathbb{R}^d$ . We denote by  $\mathcal{C}_b^k(\mathcal{O})$  the set of real functions of class  $\mathcal{C}^k$  on  $\mathcal{O}$  with bounded partial derivatives. We denote by  $\mathcal{C}_{\text{lip}}^k$  the set of real functions whose partial derivatives of order less than or equal to  $k$  are Lipschitz. We denote by  $B_b(\mathcal{O})$  the set of Borel measurable bounded functions defined on  $\mathcal{O}$ .

$(\Omega, \mathcal{F}, \mathbb{P})$  denotes a complete probability space,  $(W_t)_{t \geq 0}$  denotes an  $\mathbb{R}^d$ -valued standard Brownian motion defined on this space and  $(\mathcal{F}_t)_{t \geq 0}$  is the natural filtration of  $W$  augmented by  $\mathbb{P}$ -null sets. Then  $(\mathcal{F}_t)_{t \geq 0}$  satisfies the usual condition.

$\mathcal{S}^2$  denotes the space of real-valued adapted continuous processes  $Y$  such that for all  $T > 0$ ,  $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < +\infty$ .

$\mathcal{S}^p$  denotes the space of real-valued adapted continuous processes  $Y$  such that for all  $T > 0$ ,  $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^p] < +\infty$ .

$\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^k)$  denotes the space consisting of all progressively measurable processes  $X$ , with value in  $\mathbb{R}^k$  such that, for all  $T > 0$ ,

$$\mathbb{E} \left[ \int_0^T |X_s|^2 ds \right] < +\infty.$$

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  be two locally Lipschitz functions. We denote by  $(V_t^x)_{t \geq 0}$  the strong solution of the following SDE:

$$V_t^x = x + \int_0^t f(V_s^x) ds + \int_0^t \sigma(V_s^x) dW_s. \quad (2.1)$$

**Lemma 2.1.** *Assume that  $\exists a \in \mathbb{R}^d$ ,  $\eta_1, \eta_2 > 0$  such that,  $\forall y \in \mathbb{R}^d$ ,*

$$(f(y), y - a) \leq -\eta_1 |y - a|^2 + \eta_2,$$

*and that  $|\sigma|$  is bounded by  $\sigma_\infty$ , then there exists a strong solution  $(V_t^x)_{t \geq 0}$  to (2.1) which is pathwise unique and for which the explosion time is almost surely equal to infinity. Furthermore the following estimate holds  $\forall t \geq 0$ :*

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-2\eta_1 t}),$$

*where  $C$  is a constant which does not depend on time  $t$  and depends on  $f$  only through  $\eta_1$  and  $\eta_2$ , and on  $\sigma$  only through  $\sigma_\infty$ . Furthermore, for all  $p > 2$ , for all  $0 < \beta < p\eta_1$ ,*

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-\beta t}),$$

*where  $C$  is a constant which does not depend on time  $t$ , depends on  $f$  only through  $\eta_1$  and  $\eta_2$ , and on  $\sigma$  only through  $\sigma_\infty$ . We also have the following inequality, for all  $p \geq 1$ ,*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |V_t^x|^p \right] \leq C(1 + |x|^p),$$

*where  $C$  depends in this case, on time  $T$ .*

*Proof.* The proof is given in the appendix. □

We recall that a function is weakly dissipative if it is a sum of an  $\eta$ -dissipative function (namely  $\forall x, x' \in \mathbb{R}^d$ ,  $(d(x) - d(x'), x - x') \leq -\eta|x - x'|^2$ ), and a bounded function. Thus we write  $f = d + b$ , with  $d$   $\eta$ -dissipative and  $|b|$  bounded by  $B$ .

**Hypothesis 2.1.**

- $f = d + b$  is weakly dissipative,
- $d$  is locally Lipschitz and have polynomial growth: there exists  $\nu > 0$  such that for all  $x \in \mathbb{R}^d$ ,  $|d(x)| \leq C(1 + |x|^\nu)$ ,
- $b$  is Lipschitz,
- $\sigma$  is Lipschitz, invertible, and  $|\sigma|$  and  $|\sigma^{-1}|$  are bounded by  $\sigma_\infty$ ,
- there exists  $\Lambda \geq 0$  such that for every  $x, y \in \mathbb{R}^d$ ,

$$|(y, \sigma(x + y) - \sigma(x))| \leq \Lambda|y|$$

and there exists  $\lambda > 0$  such that

$$2(\lambda - \lambda^2 \Lambda^2) > \|\sigma^{-1}\|^2.$$

**Remark 2.2.** Note that when  $\sigma(x)$  does not depend on  $x$ , then  $\Lambda = 0$  and the last assumption of Hypothesis 2.1 is satisfied for  $\lambda$  large enough. We give an example of  $\sigma$  depending on  $x$  and satisfying the last assumption. In the one dimensional case, take  $\sigma(x) = 10\mathbb{1}_{\{x \leq 0\}} + (10 + \frac{1}{10}x)\mathbb{1}_{\{0 < x < 1\}} + (101/10)\mathbb{1}_{\{x \geq 1\}}$ . Then  $\Lambda = 1/10$  and  $\|\sigma^{-1}\| \leq 1/10$ . Clearly, the assumption is satisfied for  $\lambda = 1$ .

**Remark 2.3.** It is clear that if  $f$  satisfies Hypothesis 2.1 then  $f$  satisfies the assumption of Lemma 2.1. Indeed, let us suppose that  $f$  satisfies Hypothesis 2.1. Let  $a \in \mathbb{R}^d$ , then  $\forall y \in \mathbb{R}^d$ ,

$$\begin{aligned} (f(y) - f(a), y - a) &= (d(y) - d(a), y - a) + (b(y) - b(a), y - a) \\ \Rightarrow (f(y), y - a) &\leq -\eta|y - a|^2 + 2B|y - a| + |f(a)||y - a| \\ \Rightarrow (f(y), y - a) &\leq -\eta|y - a|^2 + \frac{(2B + |f(a)|)^2}{2\varepsilon} + \frac{\varepsilon|y - a|^2}{2}, \end{aligned}$$

which gives us the desired result, for  $\varepsilon$  small enough.

**Lemma 2.4.** Assume that Hypothesis (2.1) holds true but this time with  $b$  replaced by  $b_2$  which is only bounded measurable and not Lipschitz anymore. Then the solution of (2.1) with  $b$  replaced by  $b_2$  still exists but in the weak sense, namely there exists a new Brownian motion  $(\hat{W}_t)_{t \geq 0}$  with respect to a new probability measure  $\hat{\mathbb{P}}$  under which equation (2.1) is satisfied by  $(V_t^x)_{t \geq 0}$  with  $(W_t)_{t \geq 0}$  replaced by  $(\hat{W}_t)_{t \geq 0}$ . Such a process is unique in law and the estimates of Lemma (2.1) are still satisfied under the new probability  $\hat{\mathbb{P}}$ .

*Proof.* It is enough to write:

$$\begin{aligned} dV_t^x &= [d(V_t^x) + b(V_t^x)]dt + \sigma(V_t^x)dW_t \\ &= [d(V_t^x) + b_2(V_t^x)]dt + \sigma(V_t^x)[\sigma^{-1}(V_t^x)(b(V_t^x) - b_2(V_t^x)) + dW_t] \\ &= [d(V_t^x) + b_2(V_t^x)]dt + \sigma(V_t^x)d\hat{W}_t, \end{aligned}$$

where  $d\hat{W}_t = \sigma^{-1}(V_t^x)(b(V_t^x) - b_2(V_t^x)) + dW_t$  is the new Brownian motion thanks to the Girsanov theorem (note that  $\sigma^{-1}$ ,  $b$  and  $b_2$  are measurable and bounded by hypothesis).  $\square$

**Lemma 2.5.** Assume that Hypothesis 2.1 holds true. Then there exist  $C > 0$  and  $\mu > 0$  such that  $\forall \Phi \in B_b(\mathbb{R}^d)$ ,

$$|\mathcal{P}_t[\Phi](x) - \mathcal{P}_t[\Phi](x')| \leq C(1 + |x|^2 + |x'|^2)e^{-\mu t}|\Phi|_0 \quad (2.2)$$

where  $\mathcal{P}_t[\Phi](x) = \mathbb{E}\Phi(V_t^x)$  is the Kolmogorov semigroup associated to (2.1). We stress the fact that the constants  $C$  and  $\mu$  depend on  $f$  only through  $\eta$  and  $B$ .

*Proof.* The proof is given in the appendix.  $\square$

**Remark 2.6.** The importance of the dependency of  $C$  and  $\mu$  only through some parameters of the problem will appear in Remark 4.1.

**Corollary 2.7.** The estimate (2.2) can be extended to the case in which  $b$  is only bounded measurable and there exists a uniformly bounded sequence of Lipschitz functions  $\{b_m\}_{m \geq 1}$  (i.e.  $b_m$  is Lipschitz and  $\sup_m \sup_x |b_m(x)| < +\infty$ ) such that

$$\forall x \in \mathbb{R}^d, \quad \lim_m b_m(x) = b(x).$$

In this case, we define a semigroup relatively to the new probability measure, namely:

$$\mathcal{P}_t[\Phi](x) := \hat{\mathbb{E}}\Phi(X_t^x).$$

*Proof.* We denote by  $\mathcal{P}_t^m$  the Kolmogorov semigroup of (2.1) with  $b$  replaced by  $b_m$ , for more clarity we rewrite this equation below:  $\forall x \in \overline{G}$ ,

$$V_t^{x,m} = x + \int_0^t (d + b_m)(V_s^{x,m}) ds + \int_0^t \sigma(V_s^{x,m}) dW_s.$$

It is sufficient to prove that,  $\forall x \in \overline{G}$ ,  $\forall t \geq 0$ ,

$$\mathcal{P}_t^m[\Phi](x) \rightarrow \mathcal{P}_t[\Phi](x).$$

To do that, it is easy to adapt the proof from [3] replacing the process  $U_t^x$  by its analogue in our context. Thus we define  $U_t^x$  as the strong solution of the following SDE:

$$U_t^x = x + \int_0^t d(U_s^x) ds + \int_0^t \sigma(U_s^x) dW_s,$$

and the rest remains the same. □

### 3. The ergodic BSDE

In this section we study the following EBSDE in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (3.1)$$

At the moment, the forward process, defined as the strong solution of (2.1) is not reflected. However the existence result we are going to show in the next theorem is interesting for its own, because it gives some ideas which will be reused in the next section.

We need the following hypothesis on  $\psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  :

**Hypothesis 3.1.** There exists  $M_\psi \in \mathbb{R}$  such that:  $\forall x \in \mathbb{R}^d, \forall z, z' \in \mathbb{R}^{1 \times d}$ ,

- $\psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable,
- $\psi(\cdot, z)$  is continuous ,
- $|\psi(x, 0)| \leq M_\psi$ ,
- $|\psi(x, z) - \psi(x, z')| \leq M_\psi |z - z'|$ .

**Hypothesis 3.2.**

- $f$  is  $\mathcal{C}^1$  and all of its derivatives have polynomial growth of first order, i.e. for each  $x \in \mathbb{R}^d$  and each multi-index  $L$  with  $|L| \leq m$ ,  $m \in \{0, 1\}$ , there exist positive constants  $\gamma_m$  and  $q_m$  such that

$$|\partial_L d(x)|^2 \leq \gamma_m (1 + |x|^{q_m}).$$

Also, set  $\xi := \max_{m \in \{0, 1\}} q_m < +\infty$ .

- $b, \sigma$  and  $\psi \in \mathcal{C}_b^1$ .



Using a standard approach (see [4]), we are going to study the following BSDE in infinite horizon

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [\psi(V_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}] ds - \int_t^T Z_s^{x,\alpha} dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (3.2)$$

Such an equation was studied in [2] from which we have the following result:

**Lemma 3.1.** *Assume that hypotheses (2.1) and (3.1) hold true. Then there exists a unique solution  $(Y^{x,\alpha}, Z^{x,\alpha})$  to BSDE (3.2) such that  $Y^{x,\alpha}$  is a bounded adapted continuous process and  $Z^{x,\alpha} \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Furthermore,  $|Y_t^{x,\alpha}| \leq \frac{M_\psi}{\alpha}$ . Finally there exists a function  $v^\alpha$  such that  $Y_t^{x,\alpha} = v^\alpha(X_t^x)$   $\mathbb{P}$ -a.s. and there exists a measurable function  $\zeta^\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$  such that  $Z_t^{x,\alpha} = \zeta^\alpha(X_t^x)$   $\mathbb{P}$ -a.s.*

We will need the following lemma :

**Lemma 3.2.** *Let  $\zeta, \zeta'$  be two continuous functions:  $\mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$ . We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x, \zeta(x)) - \psi(x, \zeta'(x))}{|\zeta(x) - \zeta'(x)|^2} (\zeta(x) - \zeta'(x)), & \text{if } \zeta(x) \neq \zeta'(x), \\ 0, & \text{if } \zeta(x) = \zeta'(x). \end{cases}$$

*There exists a uniformly bounded sequence of Lipschitz functions  $(\Upsilon_n)_{n \geq 0}$  (i.e.,  $\forall n, \Upsilon_n$  is Lipschitz and  $\sup_n \sup_x |\Upsilon_n(x)| < +\infty$ ) such that  $\Upsilon_n$  converges pointwisely to  $\Upsilon$ .*

*Proof.* For all  $n \in \mathbb{N}$ , we fix infinitely differentiable functions  $\rho_n : \mathbb{R}^d \rightarrow \mathbb{R}_+$  bounded together with their derivatives of all order, such that  $\int_{\mathbb{R}^d} \rho_n(x) dx = 1$  and

$$\text{supp}(\rho_n) \subset \left\{ x \in \mathbb{R}^d, |x| \leq \frac{1}{n} \right\},$$

where  $\text{supp}$  denotes the support. Then the required functions  $\Upsilon$  can be defined as  $\Upsilon_n(x) := \int_{\mathbb{R}^d} \rho_n(x-y) \Upsilon(y) dy$ .  $\square$

The following lemma gives us the desired estimates on  $v^\alpha(x)$  which will allow us to apply a diagonal procedure.

**Lemma 3.3.** *Assume that the hypotheses (2.1), (3.1) and (3.2) hold true. Then, there exists a constant  $C > 0$  independent of  $\alpha$  and which depends on  $f$  only through  $\eta$  and  $B$ , on  $\sigma$  only through  $\sigma_\infty$  and on  $\psi$  only through  $M_\psi$  such that,  $\forall x, y \in \mathbb{R}^d$ ,*

$$\begin{aligned} |v^\alpha(x) - v^\alpha(y)| &\leq C(1 + |x|^2 + |y|^2), \\ |v^\alpha(x) - v^\alpha(y)| &\leq C(1 + |x|^2 + |y|^2)|x - y|. \end{aligned}$$

*Proof.* Let us introduce, as in the paper [15], equation (3.4), some smooth functions  $\Phi_m : \mathbb{R}^d \rightarrow \mathbb{R}$  such that defining

$$d_m(x) = \Phi_m(x) d(x), \quad \forall x \in \mathbb{R}^d,$$

$d_m$  is globally Lipschitz, continuously differentiable and for each multi-index  $L$  with  $|L| \leq 1$ ,

$$\sup_{m,x} \{ |\partial_L \phi_m(x)| + |d(x) \partial_L \phi_m(x)| \} \leq C, \quad (3.3)$$

for some  $C > 0$ . We recall that  $\phi_m = 1$  on  $A_m := \{x \in \mathbb{R}^d; |x| \leq m^\xi\}$ . Furthermore,

$$\begin{aligned}\nabla d_m(x) &= \begin{pmatrix} \frac{\partial \phi_m(x)}{\partial x_1} d_1(x) & \dots & \frac{\partial \phi_m(x)}{\partial x_d} d_1(x) \\ \vdots & & \vdots \\ \frac{\partial \phi_m(x)}{\partial x_1} d_d(x) & \dots & \frac{\partial \phi_m(x)}{\partial x_d} d_d(x) \end{pmatrix} + \phi_m(x) \nabla d(x) \\ &= d(x) \nabla \phi_m(x) + \phi_m(x) \nabla d(x).\end{aligned}$$

Now let us consider  $(V_t^{x,m})_{t \geq 0}$  the unique solution of

$$V_t^{x,m} = x + \int_0^t [d_m(V_s^{x,m}) + b(V_s^{x,m})] ds + \int_0^t \sigma(V_s^{x,m}) dW_s, \quad \forall t \geq 0. \quad (3.4)$$

We recall that, for each  $t \geq 0$  and  $p > 1$ ,  $V_t^{x,m}$  converges to  $V_t$  in  $L^p$  and almost surely. Furthermore the following estimates hold, thanks to the proof of Lemma 3.2 in [15], for every  $p \in \mathbb{N} \setminus \{0, 1\}$ , there exists  $C_p > 0$  such that

$$\sup_{m \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|V_t^{x,m}|^p] \leq C_p (1 + |x|^{p\nu}) e^{C_p T}. \quad (3.5)$$

We denote by  $\nabla V_s^{x,m} = (\nabla_1 V_s^{x,m}, \dots, \nabla_d V_s^{x,m})$  the solution of the following variational equation (see equation (2.9) in [7]):

$$\nabla_i V_s^{x,m} = e_i + \int_t^s \nabla(d_m + b)(V_r^{x,m}) \nabla_i V_r^{x,m} dr + \sum_{j=1}^d \int_t^s [\nabla \sigma^j(V_r^{x,m})] \nabla_i V_r^{x,m} dW_r^j.$$

Let us mention that the following estimate holds, for every  $p > 1$ :

$$\sup_{m \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|\nabla V_t^{x,m}|^p] \leq C_{p,T}. \quad (3.6)$$

Let us denote by  $(Y_t^{x,\alpha,m}, Z_t^{x,\alpha,m})_{t \geq 0}$  the unique solution of the following monotone BSDE in infinite horizon,  $\forall 0 \leq t \leq T < +\infty$ ,

$$Y_t^{x,\alpha,m} = Y_T^{x,\alpha,m} + \int_t^T [\psi(V_s^{x,m}, Z_s^{x,\alpha,m}) - \alpha Y_s^{x,\alpha,m}] ds - \int_t^T Z_s^{x,\alpha,m} dW_s.$$

We recall that if we denote by  $v^{\alpha,m}$  the following quantity:

$$v^{\alpha,m}(x) := Y_0^{x,\alpha,m}$$

then  $Y_s^{x,\alpha,m} = v^{\alpha,m}(V_s^x)$ .

Now remark that Theorem 4.2 in [7] asks for the terminal condition of the BSDE to be Lipschitz, whereas in our case, the terminal condition  $v^{\alpha,m}$  is only continuous and bounded. However, using the same method as that in Theorem 4.2 in [5], we can readily extend Theorem 4.2 in [7] for a Markovian terminal condition which is only continuous with polynomial growth, which is our case here. So, by Theorem 4.2 in [7],  $v^{\alpha,m}$  is continuously differentiable,

$$Z_s^{x,\alpha,m} = \nabla v^{\alpha,m}(V_s^{x,m}) \sigma(V_s^{x,m}), \quad \forall s \in [0, T], \quad \mathbb{P}\text{-a.s.},$$

and

$$\nabla v^{\alpha,m}(x) = \mathbb{E} \left[ v^\alpha(V_T^{x,m}) N_T^{0,m} + \int_0^T [\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m}] N_r^{0,m} dr \right],$$

where

$$N_r^{0,m} = \frac{1}{r} t \left( \int_0^r \sigma^{-1}(V_s^{x,m}) \nabla V_s^{x,m} dW_s \right).$$

We immediately deduce the following estimate

$$\begin{aligned} \mathbb{E} |N_r^{0,m}|^2 &= \frac{1}{|r|^2} \mathbb{E} \int_0^r |\sigma^{-1}(V_s^{x,m}) \nabla V_s^{x,m}|^2 ds \\ &\leq \frac{1}{|r|^2} \sigma_\infty^2 \int_0^r \mathbb{E} |\nabla V_s^{x,m}|^2 ds \\ &\leq \frac{C_T}{r}. \end{aligned} \tag{3.7}$$

As a consequence we can immediately get a uniform bound in  $m$  for  $\nabla v^{\alpha,m}(x)$ . Indeed,

$$\begin{aligned} |\nabla v^{\alpha,m}(x)| &\leq \frac{M_\psi}{\alpha} \sqrt{\mathbb{E} \left( |N_T^{0,m}|^2 \right)} + \int_0^T \sqrt{\mathbb{E} [(2M_\psi + M_\psi |Z_s^{x,\alpha,m}|)^2]} \sqrt{\mathbb{E} \left( |N_r^{0,m}|^2 \right)} dr \\ &\leq \frac{M_\psi}{\alpha} \frac{C_T}{\sqrt{T}} + \int_0^T C \left( 1 + \sqrt{\mathbb{E} |\nabla v^{\alpha,m}(V_r^{x,m})|^2} \right) \frac{1}{\sqrt{r}} dr. \end{aligned}$$

We define  $|||\nabla v^{\alpha,m}(x)||| := \sup_{x \in \mathbb{R}^d} |\nabla v^{\alpha,m}(x)|$ , then,

$$|\nabla v^{\alpha,m}(x)| \leq \frac{C_{T,\alpha}}{\sqrt{T}} + C\sqrt{T}(1 + |||\nabla v^{\alpha,m}|||),$$

which implies that, taking the supremum over  $x$  and for  $T$  small enough, for all  $x \in \mathbb{R}^d$ ,

$$|\nabla v^{\alpha,m}(x)| \leq C_{T,\alpha}. \tag{3.8}$$

Now we claim that for every  $T \geq 0$ ,

$$\mathbb{E} [|Y_T^{x,\alpha,m} - Y_T^{x,\alpha}|^2] + \mathbb{E} \int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 ds \xrightarrow{m \rightarrow +\infty} 0.$$

For that purpose, let us denote by  $(Y_t^{x,\alpha,n}, Z_t^{x,\alpha,n})$  the solution of the following finite horizon BSDE, for all  $0 \leq t \leq n$ ,

$$Y_t^{x,\alpha,n} = 0 + \int_t^n [\psi(V_s^x, Z_s^{x,\alpha,n}) - \alpha Y_s^{x,\alpha,n}] ds - \int_t^n Z_s^{x,\alpha,n} dW_s.$$

By inequality (12) in [2],  $\mathbb{P}$ -a.s., for all  $0 \leq t \leq n$ ,

$$\mathbb{E} |Y_t^{x,\alpha,n} - Y_t^{x,\alpha}|^2 + \mathbb{E} \int_0^t |Z_t^{x,\alpha,n} - Z_t^{x,\alpha}|^2 ds \leq C e^{-2\alpha n},$$

where  $C$  depends only on  $M_\psi$  and  $\alpha$ . Similarly, let us denote by  $(Y_t^{x,\alpha,m,n}, Z_t^{x,\alpha,m,n})$  the solution of the following finite horizon BSDE, for all  $0 \leq t \leq n$ ,

$$Y_t^{x,\alpha,m,n} = 0 + \int_t^n [\psi(V_s^{x,m}, Z_s^{x,\alpha,m,n}) - \alpha Y_s^{x,\alpha,m,n}] ds - \int_t^n Z_s^{x,\alpha,m,n} dW_s.$$

Again, by inequality (12) in [2],  $\mathbb{P}$ -a.s., for all  $0 \leq t \leq n$ ,

$$\mathbb{E}|Y_t^{x,\alpha,m,n} - Y_t^{x,\alpha,m}|^2 + \mathbb{E} \int_0^t |Z_t^{x,\alpha,m,n} - Z_t^{x,\alpha,m}|^2 ds \leq Ce^{-2\alpha n},$$

where  $C$  depends only on  $M_\psi$  and  $\alpha$ .

Furthermore, thanks to the continuity of  $\psi$  in  $x$ , the following stability result for BSDEs in finite horizon holds (see for example Lemma 2.3 in [2]), for all  $0 \leq t \leq n$ :

$$\mathbb{E}|Y_t^{x,\alpha,m,n} - Y_t^{x,\alpha,n}|^2 + \mathbb{E} \int_0^t |Z_t^{x,\alpha,m,n} - Z_t^{x,\alpha,n}|^2 ds \xrightarrow{m \rightarrow \infty} 0.$$

Then,

$$\begin{aligned} & \mathbb{E} [|Y_T^{x,\alpha,m} - Y_T^{x,\alpha}|^2] + \mathbb{E} \int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 ds \\ & \leq 3\mathbb{E} [|Y_T^{x,\alpha,m} - Y_T^{x,\alpha,m,n}|^2] + 3\mathbb{E} [|Y_T^{x,\alpha,m,n} - Y_T^{x,\alpha,n}|^2] \\ & \quad + 3\mathbb{E} [|Y_T^{x,\alpha,n} - Y_T^{x,\alpha}|^2] + 3\mathbb{E} \int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha,m,n}|^2 ds \\ & \quad + 3\mathbb{E} \int_0^T |Z_s^{x,\alpha,m,n} - Z_s^{x,\alpha,n}|^2 ds + 3\mathbb{E} \int_0^T |Z_s^{x,\alpha,n} - Z_s^{x,\alpha}|^2 ds \\ & \leq Ce^{-2\alpha n} + 3\mathbb{E} [|Y_T^{x,\alpha,m,n} - Y_T^{x,\alpha,n}|^2] + 3\mathbb{E} \int_0^T |Z_s^{x,\alpha,m,n} - Z_s^{x,\alpha,n}|^2 ds. \end{aligned}$$

Now, for every  $\varepsilon > 0$ , we pick  $n$  large enough such that  $2\frac{M_\psi}{\alpha}e^{-\alpha n} < \varepsilon/2$ . Then, we choose  $m$  large enough such that  $3\mathbb{E} [|Y_T^{x,\alpha,m,n} - Y_T^{x,\alpha,n}|^2] + 3\mathbb{E} \int_0^T |Z_s^{x,\alpha,m,n} - Z_s^{x,\alpha,n}|^2 ds < \varepsilon/2$ . This shows that, for all  $x \in \mathbb{R}^d$  and  $T \geq 0$ ,

$$\mathbb{E} [|Y_T^{x,\alpha,m} - Y_T^{x,\alpha}|^2] + \mathbb{E} \int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 ds \xrightarrow{m \rightarrow +\infty} 0. \quad (3.9)$$

In particular taking  $T = 0$  we deduce that for every  $x \in \mathbb{R}^d$ ,

$$\lim_{m \rightarrow +\infty} v^{\alpha,m}(x) = v^\alpha(x).$$

Therefore, if we show that  $\lim_{m \rightarrow +\infty} \nabla v^{\alpha,m}(x) = h^\alpha(x)$  for some function  $h^\alpha(x)$  then this will imply that  $v^\alpha$  is continuously differentiable and  $\nabla v^\alpha = h^\alpha$ . Furthermore, as  $\mathbb{E} \int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 ds \xrightarrow{m \rightarrow +\infty} 0$  and  $Z_s^{x,\alpha,m} = \nabla v^{\alpha,m}(V_s^x) \sigma(V_s^x)$ , then it will imply that for a.a.  $s \geq 0$ ,  $\mathbb{P}$ -a.s.,

$$Z_s^{x,\alpha} = \nabla v^\alpha(V_s^x) \sigma(V_s^x). \quad (3.10)$$

Now we claim that  $h^\alpha$  can be written as

$$h^\alpha(x) = \mathbb{E} \left[ v^\alpha(V_T^x) N_T^0 + \int_0^T [\psi(V_r^x, Z_r^{x,\alpha}) - \alpha Y_r^{x,\alpha}] N_r^0 dr \right],$$

where

$$N_r^0 = \frac{1}{r} t \left( \int_0^r t [\sigma^{-1}(V_s^x) \nabla V_s^x] dW_s \right).$$

Indeed, for every  $x \in \mathbb{R}^d$ ,

$$\begin{aligned}
& |\nabla v^{\alpha,m}(x) - h^\alpha(x)| \\
& \leq \mathbb{E}|v^{\alpha,m}(V_T^{x,m})N_T^{0,m} - v^\alpha(V_T^x)N_T^0| \\
& \quad + \mathbb{E} \int_0^T |(\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^{0,m} - (\psi(V_r^x, Z_r^{x,\alpha}) - \alpha Y_r^{x,\alpha})N_r^0| dr \\
& \leq \mathbb{E}|v^{\alpha,m}(V_T^{x,m})N_T^{0,m} - v^{\alpha,m}(V_T^x)N_T^0| + \mathbb{E}|v^{\alpha,m}(V_T^{x,m})N_T^0 - v^\alpha(V_T^x)N_T^0| \\
& \quad + \mathbb{E} \int_0^T |(\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^{0,m} - (\psi(V_r^x, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^0| dr \\
& \quad + \mathbb{E} \int_0^T |(\psi(V_r^x, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^0 - (\psi(V_r^x, Z_r^{x,\alpha}) - \alpha Y_r^{x,\alpha})N_r^0| dr \\
& \leq \frac{M_\psi}{\alpha} \sqrt{\mathbb{E}(|N_T^{0,m} - N_T^0|^2)} + \sqrt{\mathbb{E}(|N_T^0|^2)} \sqrt{\mathbb{E}(|v^{\alpha,m}(V_T^{x,m}) - v^\alpha(V_T^x)|^2)} \\
& \quad + \int_0^T \sqrt{C(1 + \mathbb{E}(|\nabla v^{\alpha,m}(V_s^{x,m})|^2))} \sqrt{\mathbb{E}(|N_r^{0,m} - N_r^0|^2)} dr \\
& \quad + \int_0^T \sqrt{\mathbb{E}(|N_r^0|^2)} \sqrt{\mathbb{E}(|\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m} - \psi(V_r^x, Z_r^{x,\alpha}) + \alpha Y_r^{x,\alpha}|^2)} dr \\
& \leq \frac{M_\psi}{\alpha} \sqrt{\mathbb{E}(|N_T^{0,m} - N_T^0|^2)} + \sqrt{\mathbb{E}(|N_T^0|^2)} \sqrt{\mathbb{E}(|v^{\alpha,m}(V_T^{x,m}) - v^\alpha(V_T^x)|^2)} \\
& \quad + C_{T,\alpha}(\sqrt{T} + \frac{1}{\sqrt{T}}) \int_0^T \sqrt{\mathbb{E}(|N_r^{0,m} - N_r^0|^2)} dr \\
& \quad + \int_0^T \frac{C_T}{\sqrt{r}} C_{T,\alpha} \sqrt{\mathbb{E}(|Z_r^{x,\alpha,m} - Z_r^{x,\alpha}|^2)} dr \\
& \quad + \int_0^T \sqrt{\mathbb{E}(|\psi(V_r^{x,m}, Z_r^{x,\alpha}) - \psi(V_r^x, Z_r^{x,\alpha})|^2)} dr \\
& \quad + \int_0^T \frac{C_T}{\sqrt{r}} \sqrt{\mathbb{E}(|Y_r^{x,\alpha,m} - Y_r^{x,\alpha}|^2)} dr
\end{aligned}$$

where we have used the estimate (3.8) for the last inequality.

We have

$$\mathbb{E}(|N_T^{0,m} - N_T^0|^2) = \frac{1}{T^2} \int_0^T \mathbb{E}(|\sigma^{-1} \nabla V_s^{x,m} - \sigma^{-1} \nabla V_s^x|^2) ds \xrightarrow{m \rightarrow +\infty} 0,$$

since  $\sigma^{-1}(V_s^{x,m}) \nabla V_s^{x,m} - \sigma^{-1}(V_s^x) \nabla V_s^x \xrightarrow{m \rightarrow +\infty} 0$   $\mathbb{P}$ -a.s. and since

$$\sup_m \mathbb{E}(|\sigma^{-1}(V_s^{x,m}) \nabla V_s^{x,m} - \sigma^{-1}(V_s^x) \nabla V_s^x|^4) < +\infty$$

by estimate (3.6).

The second and the third term in the sum converge toward 0 by the dominated convergence theorem. The fourth one converges toward 0 by Jensen's inequality and the dominated convergence theorem and the last two ones converge toward 0 by the dominated convergence theorem. Now the first estimate of the lemma can be established exactly as in Lemma 3.6 of [3] thanks to the representation formula (3.10).

Let us establish the second inequality of the lemma. We have, using the following notation

$$\bar{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0),$$

$$|\nabla \bar{v}^\alpha(x)| = \left| \mathbb{E} \left[ \bar{v}^\alpha(V_T^x) N_T^0 + \int_0^T [\psi(V_s^x, Z_s^x) - \alpha \bar{v}^\alpha(V_r^x) - \alpha v^\alpha(0)] N_r^0 dr \right] \right|. \quad (3.11)$$

We have, using the first inequality of the lemma and inequality (3.7):

$$\mathbb{E} |\bar{v}^\alpha(V_T^x) N_T^0| \leq C \frac{(1 + |x|^2)}{\sqrt{T}}.$$

Furthermore, since we can assume that  $\alpha \leq 1$ :

$$\begin{aligned} & \mathbb{E} \int_0^T \left| [\psi(V_r^x, Z_r^{x,\alpha}) - \alpha \bar{Y}_r^{x,\alpha} - \alpha v^\alpha(0)] N_r^0 \right| dr \\ & \leq C \mathbb{E} \int_0^T (M_\psi + M_\psi |Z_r^{x,\alpha}| + C(1 + |V_r^x|^2) + M_\psi) |N_r^0| dr \\ & \leq C \mathbb{E} \int_0^T |N_r^0| dr + C \mathbb{E} \int_0^T |Z_r^{x,\alpha}| |N_r^0| dr + C \mathbb{E} \int_0^T (1 + |V_r^x|^2) |N_r^0| dr \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

We easily get  $I_1 \leq C$ . Furthermore, thanks to the representation formula (3.10) and the fact that  $|\sigma|$  is bounded:

$$I_2 \leq C \int_0^T \sqrt{\mathbb{E} |\nabla v^\alpha(V_s^x)|^2} \frac{1}{\sqrt{r}} dr.$$

Regarding  $I_3$ , we easily get  $I_3 \leq C(1 + |x|^2)$ .

Now we define  $|||\nabla v^\alpha||| := \sup_{x \in \mathbb{R}^d} \frac{|\nabla v^\alpha(x)|}{1 + |x|^2}$ , then coming back to equation (3.11), we have

$$\begin{aligned} |\nabla v^\alpha(x)| & \leq C \left( 1 + |x|^2 + \frac{1 + |x|^2}{\sqrt{T}} \right) + C \int_0^T \sqrt{\mathbb{E}(1 + |V_s^x|^2)^2} |||\nabla v^\alpha||| \frac{1}{\sqrt{r}} dr \\ & \leq C \left( 1 + |x|^2 + \frac{1 + |x|^2}{\sqrt{T}} \right) + C \int_0^T (1 + |x|^2) |||\nabla v^\alpha||| \frac{1}{\sqrt{r}} dr. \end{aligned}$$

This implies

$$|||\nabla v^\alpha||| \leq C \left( 1 + \frac{1}{\sqrt{T}} \right) + C\sqrt{T} |||\nabla v^\alpha|||.$$

Thus, for  $T$  small enough:

$$|||\nabla v^\alpha||| \leq C \left( 1 + \frac{1 + \sqrt{T}}{\sqrt{T}(1 - C\sqrt{T})} \right),$$

which implies that, for all  $x \in \mathbb{R}^d$ ,

$$|\nabla v^\alpha(x)| \leq C(1 + |x|^2).$$

This last estimate gives us, for all  $x, y \in \mathbb{R}^d$ ,

$$|v^\alpha(x) - v^\alpha(y)| \leq C(1 + |x|^2 + |y|^2)|x - y|.$$

□

Thanks to this estimate, it is possible to get an existence result for EBSDE (3.1). Here Hypothesis 3.2 can be removed thanks to a convolution argument which will appear in the proof.

**Theorem 3.4.** *Assume that the hypotheses (2.1) and (3.1) hold true. Then there exists a solution  $(\bar{Y}^x, \bar{Z}^x, \bar{\lambda})$  to EBSDE (3.1) such that  $\bar{Y}^x = \bar{v}(V^x)$  with  $\bar{v}$  locally Lipschitz, and there exists a measurable function  $\bar{\xi} : \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$  such that  $\bar{Z}^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  and  $\bar{Z}^x = \bar{\xi}(V^x)$ .*

*Proof.* We start by regularizing  $f$  and  $\psi$  thanks to classical convolution arguments. For all  $k \in \mathbb{N}^*$  let us denote by  $\rho_\varepsilon^k : \mathbb{R}^k \rightarrow \mathbb{R}_+$  the classical mollifier for which the support is the ball of center 0 and radius  $\varepsilon$ . Let us denote for a sequence  $(\varepsilon_n)_{n \in \mathbb{N}} \in \mathbb{R}_+$  such that  $\varepsilon_n \xrightarrow{n \rightarrow +\infty} 0$ ,  $d^{\varepsilon_n} := d * \rho_{\varepsilon_n}^d$ ,  $b^{\varepsilon_n} := b * \rho_{\varepsilon_n}^d$ ,  $\psi^{\varepsilon_n} := \psi * \rho_{\varepsilon_n}^{d,d}$  and  $\sigma^{\varepsilon_n} := \sigma * \rho_{\varepsilon_n}^{d \times d}$ . Those functions are  $\mathcal{C}^1$  and satisfies:

- $d^{\varepsilon_n}$  is  $\eta$ -dissipative;
- $|d^{\varepsilon_n}(x)| \leq C(1 + |x|)^p$ , for a  $p \geq 0$ ;
- $|\nabla d^{\varepsilon_n}(x)| \leq C_\varepsilon(1 + |x|^q)$ , for a  $q \geq 0$ ;
- $b^{\varepsilon_n}$  is bounded by  $B$ ;
- $\psi^{\varepsilon_n}$  satisfies Hypothesis 3.1;
- $\sigma^{\varepsilon_n}$  is invertible;
- $d^{\varepsilon_n} \rightarrow d$ ,  $b^{\varepsilon_n} \rightarrow b$ ,  $\psi^{\varepsilon_n} \rightarrow \psi$ ,  $\sigma^{\varepsilon_n} \rightarrow \sigma$  pointwisely as  $\varepsilon_n \rightarrow 0$ .

Note now that Hypothesis 3.2 is satisfied by the regularized functions defined above, therefore Lemma 3.3 can be applied. We just precise that the pointwise convergence of the regularized functions is a consequence of the continuity of the functions  $d$ ,  $b$ ,  $\psi$  and  $\sigma$ . We denote by  $Y_t^{x,\varepsilon_n}$  the solution of (2.1) with  $f$  replaced by  $f^{\varepsilon_n}$  and  $\sigma$  replaced by  $\sigma^{\varepsilon_n}$ . The same notation is used for the regularized BSDE, we denote by  $(Y_t^{x,\alpha,\varepsilon_n}, Z_t^{x,\alpha,\varepsilon_n})$  the solution in  $\mathcal{S}^2 \times \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  of BSDE (3.2) with  $\psi$  replaced by  $\psi^{\varepsilon_n}$  (existence and uniqueness of such a solution is guaranteed by Lemma 3.1), namely  $\forall 0 \leq t \leq T < +\infty$ :

$$Y_t^{x,\alpha,\varepsilon_n} = Y_T^{x,\alpha,\varepsilon_n} + \int_t^T (\psi(V_s^{x,\varepsilon_n}, Z_s^{x,\alpha,\varepsilon_n}) - \alpha Y_s^{x,\alpha,\varepsilon_n}) ds - \int_t^T Z_s^{x,\alpha,\varepsilon_n} dW_s. \quad (3.12)$$

Then we define  $v^{\alpha,\varepsilon_n}(x) := Y_0^{x,\alpha,\varepsilon_n}$  and  $\bar{Y}_t^{x,\alpha,\varepsilon_n} = Y_t^{x,\alpha,\varepsilon_n} - \alpha v^{\alpha,\varepsilon_n}(0)$ . We can rewrite the BSDE and we get:

$$\begin{aligned} \bar{Y}_t^{x,\alpha,\varepsilon_n} &= \bar{Y}_T^{x,\alpha,\varepsilon_n} + \int_t^T (\psi(V_s^{x,\varepsilon_n}, Z_s^{x,\alpha,\varepsilon_n}) - \alpha \bar{Y}_s^{x,\alpha,\varepsilon_n} - \alpha v^{\alpha,\varepsilon_n}(0)) ds \\ &\quad - \int_t^T Z_s^{x,\alpha,\varepsilon_n} dW_s, \quad 0 \leq t \leq T < +\infty. \end{aligned}$$

Uniqueness of solutions implies that  $v^{\alpha,\varepsilon_n}(V_s^{x,\varepsilon_n}) = Y_s^{x,\alpha,\varepsilon_n}$ . Now, in a very classical way, we set  $\bar{v}^{\alpha,\varepsilon_n}(x) = v^{\alpha,\varepsilon_n}(x) - v^{\alpha,\varepsilon_n}(0)$ . Thanks to the fact that  $\alpha |v^{\alpha,\varepsilon_n}(0)| \leq M_\psi$  and by Lemma 3.3 we can extract a subsequence  $\beta(\varepsilon_n) \xrightarrow{n \rightarrow +\infty} 0$  such that  $\forall \alpha > 0$ ,  $\forall x \in D$  a countable subset of  $\mathbb{R}^d$ :

$$\bar{v}^{\alpha,\beta(\varepsilon_n)}(x) \xrightarrow{n \rightarrow +\infty} \bar{v}^\alpha(x) \quad \text{and} \quad \alpha v^{\alpha,\beta(\varepsilon_n)}(0) \xrightarrow{n \rightarrow +\infty} \bar{\lambda}^\alpha,$$

for a suitable function  $\bar{v}$  and a suitable real  $\bar{\lambda}^\alpha$ . Now thanks to the estimates from Lemma 3.3 we have  $\forall \alpha > 0$ ,  $|\bar{v}^{\alpha,\beta(\varepsilon_n)}(x) - \bar{v}^{\alpha,\beta(\varepsilon_n)}(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|$  for all  $x, x' \in \mathbb{R}^d$ . Therefore

extending  $\bar{v}^\alpha$  to the whole  $\mathbb{R}^d$  by setting  $\bar{v}^\alpha(x) = \lim_{x_p \rightarrow x} \bar{v}^\alpha(x_p)$  we still have the following estimates: for all  $x, x' \in \mathbb{R}^d$ ,

$$|\bar{v}^\alpha(x) - \bar{v}^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.$$

In addition, we also have

$$|\bar{\lambda}^\alpha| \leq M_\psi.$$

Now let us define  $\forall t \geq 0, \bar{Y}_t^{x,\alpha} = \bar{v}^\alpha(V_t^x)$ . Let us show that

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_s^{x,\alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_T^{x,\alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

First we write:

$$\begin{aligned} |\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)}) - \bar{v}^\alpha(V_s^x)| &\leq |\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)}) - \bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^x)| \\ &\quad + |\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^x) - \bar{v}^\alpha(V_s^x)| \\ &\leq C(1 + |V_s^{x,\beta(\varepsilon_n)}|^2 + |V_s^x|^2) |V_s^{x,\beta(\varepsilon_n)} - V_s^x| \\ &\quad + |\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^x) - \bar{v}^\alpha(V_s^x)|, \end{aligned}$$

which shows the convergence of  $\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})$  toward  $\bar{v}^\alpha(V_s^x)$  almost surely, up to a subsequence (it is well known that  $\forall T > 0, E \sup_{0 \leq t \leq T} |V_t^{x,\beta(\varepsilon_n)} - V_t^x|^2 \xrightarrow{n \rightarrow +\infty} 0$ ). Then, due to the fact that

$|\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})| \leq M_\psi/\alpha$   $\mathbb{P}$ -a.s., we can apply the dominated convergence theorem to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_s^{x,\alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_T^{x,\alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Now we show that  $(Z^{x,\alpha,\beta(\varepsilon_n)})_n$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . We denote

$$\tilde{V}_t = V_t^{x,\beta(\varepsilon_n)} - V_t^{x,\beta(\varepsilon_n)'};$$

$$\tilde{Y}_t = \bar{Y}_t^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_t^{x,\alpha,\beta(\varepsilon_n)'};$$

$$\tilde{Z}_t = \bar{Z}_t^{x,\alpha,\beta(\varepsilon_n)} - \bar{Z}_t^{x,\alpha,\beta(\varepsilon_n)'};$$

and

$$\tilde{\lambda} = \alpha v^{\alpha,\beta(\varepsilon_n)}(0) - \alpha v^{\alpha,\beta(\varepsilon_n)'}(0).$$



Itô's formula applied to  $|\tilde{Y}_t|^2$  gives us, for all  $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ :

$$\begin{aligned} |\tilde{Y}_t|^2 + \int_t^T |\tilde{Z}_s|^2 ds &= |\tilde{Y}_T|^2 + 2 \int_t^T \tilde{Y}_t [\psi(V_s^{x,\beta(\varepsilon_n)}, Z_s^{x,\alpha,\beta(\varepsilon_n)}) - \psi(V_s^{x,\beta(\varepsilon_n)'}, Z_s^{x,\alpha,\beta(\varepsilon_n)'}) \\ &\quad - (\alpha \bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \alpha \bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)'}) - \tilde{\lambda}] ds \\ &\quad - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s \\ &\leq |\tilde{Y}_T|^2 + (\varepsilon_1 M_\psi + \varepsilon_2 M_\psi + \varepsilon_3) \int_t^T |\tilde{Y}_s|^2 ds + \frac{M_\psi}{\varepsilon_1} \int_t^T |\tilde{V}_s|^2 ds \\ &\quad + \frac{M_\psi}{\varepsilon_2} \int_t^T |\tilde{Z}_s|^2 ds + \frac{1}{\varepsilon_3} \int_t^T |\tilde{\lambda}|^2 ds + c \int_t^T |\tilde{Y}_s| ds - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s, \end{aligned}$$

because  $\alpha|v^{\alpha,\varepsilon}(0)| \leq M_\psi$ . Thus, taking the expectation and for  $\varepsilon_2$  large enough we get

$$\mathbb{E} \int_0^T |\tilde{Z}_s|^2 ds \leq \mathbb{E} |\tilde{Y}_T|^2 + c \left( \mathbb{E} \left[ \int_0^T |\tilde{Y}_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T |\tilde{V}_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T |\tilde{Y}_s| ds \right] + T |\tilde{\lambda}|^2 \right),$$

which proves that  $(Z_s^{x,\alpha,\beta(\varepsilon_n)})_{\beta(\varepsilon_n)}$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Now we pass to the limit in equation (3.12) to obtain:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(V_s^x, Z_s^{x,\alpha}) - \bar{\lambda}^\alpha) ds - \int_t^T Z_s^{x,\alpha} dW_s, \quad 0 \leq t \leq T < +\infty.$$

Now we reiterate the above method. Thanks to the following estimates:  $\forall x, x' \in \mathbb{R}^d$ ,

$$|\bar{v}^\alpha(x) - \bar{v}^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|,$$

and

$$|\bar{\lambda}^\alpha| \leq M_\psi,$$

it is possible, by a diagonal procedure, to construct a sequence  $(\alpha_n)_n$  such that

$$\begin{aligned} \bar{v}^{\alpha_n}(x) &\xrightarrow{n \rightarrow +\infty} \bar{v}(x) \\ \bar{\lambda}^{\alpha_n} &\xrightarrow{n \rightarrow +\infty} \bar{\lambda}. \end{aligned}$$

We define  $\bar{Y}_t^x := \bar{v}(V_t^x)$ . Let us just precise why

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha} - \bar{Y}_s^x|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha} - \bar{Y}_T^x|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

First the convergence of  $\bar{v}^{\alpha_n}(V_s^x)$  toward  $\bar{v}(V_s^x)$  is clear. Secondly, we have

$$|v^{\alpha_n}(V_s^x)| \leq C(1 + |V_s^x|^2).$$

Therefore the dominated convergence theorem can be applied to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_s^{x,\alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_T^{x,\alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Then, just as before, it is possible to show that  $(Z^{x,\alpha_n})_{\alpha_n}$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . We denote its limit by  $\bar{Z}_s^x$ .

The end of the proof is very classical, it suffices to apply BDG's inequality to show that  $\mathbb{E} \sup_{0 \leq t \leq T} |\bar{Y}_t^x|^2 < +\infty$ ,  $\forall T > 0$ . To show that  $\bar{Z}^x$  is Markovian, just apply the same method as in the proof of Theorem 4.4 in [4].  $\square$

**Remark 3.5.** It is clear that we do not have uniqueness of the solutions of EBSDE (3.1) because if  $(Y, Z, \lambda)$  is a solution then  $(Y + \theta, Z, \lambda)$  is another solution, for all  $\theta \in \mathbb{R}$ . However we have a uniqueness property for  $\lambda$  under the following polynomial growth property:

$$|Y_t^x| \leq C(1 + |V_t^x|^2).$$

One can notice that the solution  $\bar{Y}_t^x = \bar{v}(V_t^x)$  constructed in the proof of Theorem (3.4) satisfies such a growth property.

**Theorem 3.6.** (*Uniqueness of  $\lambda$* ). Assume that the hypotheses (2.1) and (3.1) hold true. Let us suppose that we have two solutions of EBSDE (4.3) denoted by  $(Y, Z, \lambda)$  and  $(Y', Z', \lambda')$  where  $Y$  and  $Y'$  are progressively measurable continuous processes,  $Z$  and  $Z' \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  and  $\lambda, \lambda' \in \mathbb{R}$ . Finally assume that the following growth properties hold:

$$\begin{aligned} |Y_t| &\leq C(1 + |V_t^x|^2) \\ |Y'_t| &\leq C'(1 + |V_t^x|^2). \end{aligned}$$

Then  $\lambda = \lambda'$ .

*Proof.* It suffices to adapt the proof of Theorem 4.6 of [4]. With the same notations one can write:

$$\begin{aligned} \tilde{\lambda} &= T^{-1} \mathbb{E}^{\mathbb{P}^h} [\tilde{Y}_T - \tilde{Y}_0] \\ &\leq T^{-1} \mathbb{E}^{\mathbb{P}^h} ((C + C')(1 + |V_T^x|^2)) + T^{-1} \mathbb{E}^{\mathbb{P}^h} ((C + C')(1 + |x|^2)). \end{aligned}$$

To conclude, just use the estimates from Lemma 2.1, and let  $T \rightarrow +\infty$ .  $\square$

#### 4. The ergodic BSDE with zero and non-zero Neumann boundary conditions in a weakly dissipative environment

In this section we replace the process  $(V_t^x)_{t \geq 0}$  by the process  $(X_t^x)_{t \geq 0}$ , which is solution of a stochastic differential equation reflected in the closure of an open convex subset  $G$  of  $\mathbb{R}^d$  with regular boundary, namely, we consider the following stochastic equation for a pair of unknown processes  $(X_t^x, K_t^x)_{t \geq 0}$  such that, for every  $x \in G$ ,  $t \geq 0$  :

$$\begin{cases} X_t^x = x + \int_0^t f(X_s^x) ds + \int_0^t \sigma(X_s^x) ds + \int_0^t \nabla \phi(X_s^x) dK_s^x, \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}} dK_s^x, \end{cases} \quad (4.1)$$

where  $f$  is weakly dissipative.

As far as we know, there is no result regarding such diffusions. That is why it is necessary to adapt a result of Menaldi in [11] where an existence and uniqueness result is stated by a penalization method for a diffusion reflected in a convex and bounded set under Lipschitz assumptions

for the drift. Therefore, it is necessary to adapt this result in our framework, namely when the set is not bounded anymore but with weakly dissipative assumptions for the drift.

We denote by  $\Pi(x)$  the projection of  $x \in \mathbb{R}^d$  on  $\overline{G}$ . Let us denote by  $(X_t^{x,n})_{t \geq 0}$  the unique strong solution of the following penalized problem associated to the reflected problem (4.1) :

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n}) ds + \int_0^t \sigma(X_s^{x,n}) dW_s, \quad (4.2)$$

where  $\forall x \in \mathbb{R}^d$ ,  $F_n(x) = -2n(x - \Pi(x))$ .

**Remark 4.1.** The functions  $d + F_n + b$  and  $\sigma$  satisfy Hypothesis 2.1. Indeed, from [6],  $F_n$  is 0-dissipative therefore  $d + F_n$  remains  $\eta$ -dissipative thus the estimate of Lemma 2.5 holds with constants which do not depend on  $n$ . Furthermore one can remark that for all  $\xi \in \mathbb{R}^d$ ,  $\xi \nabla F_n(x) \xi \leq 0$ , for all  $x \in \mathbb{R}^d$  (see for example [6]). Finally, taking  $a \in \overline{G}$  (thus  $F_n(a) = 0$ ) in Remark 2.3 shows us that the estimate of Lemma 2.1 holds with constants that do not depend on  $n$ .

We need the following assumptions on  $G$ :

**Hypothesis 4.1.**  $G$  is an open convex set of  $\mathbb{R}^d$ .

**Hypothesis 4.2.** There exists a function  $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$  such that  $G = \{\phi > 0\}$ ,  $\partial G = \{\phi = 0\}$  and  $|\nabla \phi(x)| = 1$ ,  $\forall x \in \partial G$ .

The following Lemma states that the penalized process is Cauchy in the space of predictable continuous process for the norm  $\mathbb{E} \sup_{0 \leq t \leq T} |\cdot|^p$ , for every  $p > 2$  and that it converges to the reflected process solution of (4.1) for a process  $K^x$  with bounded variations.

**Lemma 4.2.** Assume that the hypotheses (2.1), (4.1) and (4.2) hold true. Then for every  $x \in \overline{G}$ , there exists a unique pair of processes  $\{(X_t^x, K_t^x)_{t \geq 0}\}$  with values in  $(\overline{G} \times \mathbb{R}_+)$  and which belong to the space  $\mathcal{S}^p \times \mathcal{S}^p$ ,  $\forall 1 \leq p < +\infty$ , satisfying (4.1) and such that

$$\eta_t^x := \int_0^t \nabla \phi(X_s^x) dK_s^x \quad \text{has bounded variation on } [0, T], 0 < T < \infty, \eta_0^x = 0$$

and for all process  $z$  continuous and progressively measurable taking values in the closure  $\overline{G}$  we have

$$\int_0^T (X_s^x - z_s) d\eta_s^x \leq 0, \quad \forall T > 0.$$

Finally the following estimate hold for the convergence of the penalized process, for any  $1 < q < p/2$ , for any  $T \geq 0$  there exists  $C > 0$  such that

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{x,n} - X_t^x|^p \leq C \left( \frac{1}{n^q} \right),$$

*Proof.* The proof is given in Appendix. □

#### 4.1. The ergodic BSDE with zero Neumann boundary conditions in a weakly dissipative environment

In a first time we are concerned with the following EBSDE with zero Neumann condition in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty, \quad (4.3)$$

where the unknown is the triplet  $(Y^x, Z^x, \lambda)$ .  $(X_t^x)_{t \geq 0}$  is the solution of (4.1).

To study the problem of existence of a solution to such an equation, we are going to study the following BSDE, with monotonic drift in  $y$ , regularized coefficients and penalized generator, namely:  $\forall 0 \leq t \leq T < +\infty$ ,

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}] ds - \int_t^T Z_s^{x,\alpha,n,\varepsilon} dW_s, \quad (4.4)$$

where the process  $(X_t^{x,n,\varepsilon})$  is the solution of the following SDE:

$$X_t^{x,n,\varepsilon} = x + \int_0^t (f^\varepsilon(X_s^{x,n,\varepsilon}) + F_n^\varepsilon(X_s^{x,n,\varepsilon})) ds + \int_0^t \sigma(X_s^{x,n,\varepsilon}) dW_s.$$

**Remark 4.3.**  $F_n$  is regularized like other regularized functions. Thanks to convolutions arguments it is possible to construct a sequence of functions  $F_n^\varepsilon$  which converges pointwisely toward  $F_n$  and such that for all  $\varepsilon$ ,  $F_n^\varepsilon$  is 0-dissipative and  $4n$ -Lipschitz.

Now we can state the existence theorem for EBSDE (4.3).

**Theorem 4.4.** *Assume that the hypotheses 2.1, 3.1, 4.1 and 4.2 hold. Then there exists a solution  $(\bar{Y}_t^x, \bar{Z}_t^x, \bar{\lambda})$  to EBSDE (4.3) such that  $\bar{Y}_t^x = \bar{v}(V_t^x)$  with  $\bar{v}$  locally Lipschitz, and there exists a measurable function  $\bar{\xi} : \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$  such that  $\bar{Z}^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  and  $\bar{Z}_t^x = \bar{\xi}(X_t^x)$ .*

*Proof.* We give the main ideas, because the proof is very similar to the proof of Theorem 3.4. The beginning of the proof is the same as the proof of Theorem 3.4. Lemma 3.1 gives us the existence and uniqueness of the solution  $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$  of BSDE (4.4) in  $\mathcal{S}^2 \times \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Then, as the function  $d + F_n$  is still  $\eta$ -dissipative and as the work in the previous section involves  $d$  only through its dissipativity constant  $\eta$ , we can apply previous results. As always we define  $v^{\alpha,n,\varepsilon}(x) := Y_0^{\alpha,n,\varepsilon}$ . By Lemma 3.3 we have the following estimate:  $\forall x, x' \in \mathbb{R}^d$ :

$$|v^{\alpha,n,\varepsilon}(x) - v^{\alpha,n,\varepsilon}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.$$

In addition we also have

$$|\alpha v^{\alpha,n,\varepsilon}(0)| \leq M_\psi.$$

As those inequalities are uniform in  $\varepsilon$  it is possible to construct by a diagonal procedure a subsequence  $\varepsilon_p \rightarrow +0$  such that  $\forall n \in \mathbb{N}, \alpha > 0$ :

$$\bar{v}^{\alpha,n,\varepsilon_p}(x) \xrightarrow{p \rightarrow +\infty} \bar{v}^{\alpha,n}(x) \quad \text{and} \quad \alpha v^{\alpha,n,\varepsilon_p}(0) \xrightarrow{p \rightarrow +\infty} \bar{\lambda}^{\alpha,n},$$

We recall the fact that the function  $\bar{v}^{\alpha,n}$  is locally Lipschitz on  $\mathbb{R}^d$  and that we keep the following estimates:

$$|\bar{v}^{\alpha,n}(x) - \bar{v}^{\alpha,n}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|;$$

$$|\bar{\lambda}^{\alpha,n}| \leq M_\psi.$$

Now let us define  $\forall t \geq 0, \bar{Y}_t^{x,\alpha,n} := \bar{v}^{\alpha,n}(V_t^{x,n})$ . Let us show that

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,n,\varepsilon_p} - \bar{Y}_s^{x,\alpha,n}|^2 ds \xrightarrow{p \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,n,\varepsilon_p} - \bar{Y}_T^{x,\alpha,n}|^2 \xrightarrow{p \rightarrow +\infty} 0.$$

First we write:

$$\begin{aligned} |\bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n,\varepsilon_p}) - \bar{v}^{\alpha,n}(X_s^{x,n})| &\leq |\bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n,\varepsilon_p}) - \bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n})| \\ &\quad + |\bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n}) - \bar{v}^{\alpha,n}(X_s^{x,n})| \\ &\leq C(1 + |X_s^{x,n,\varepsilon_p}|^2 + |X_s^{x,n}|^2)|X_s^{x,n,\varepsilon_p} - X_s^{x,n}|, \end{aligned}$$

which shows the pointwise convergence of  $\bar{v}^{\alpha,n,\varepsilon_p}(V_s^{x,n,\varepsilon_p})$  toward  $\bar{v}^{\alpha,n}(V_s^{x,n})$  almost surely when  $p \rightarrow +\infty$ . Then, due to the fact that  $|\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})| \leq M_\psi/\alpha$   $\mathbb{P}$ -a.s., we can apply the dominated convergence theorem to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,n,\varepsilon_p} - \bar{Y}_s^{x,\alpha,n}|^2 ds \xrightarrow{p \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,n,\varepsilon_p} - \bar{Y}_T^{x,\alpha,n}|^2 \xrightarrow{p \rightarrow +\infty} 0.$$

In addition it is possible to show as in Theorem 3.4 that  $(Z^{x,\alpha,n,\varepsilon_p})_p$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Note that we keep the estimates  $\forall x, x' \in \mathbb{R}^d$ :

$$|\bar{v}^{\alpha,n}(x) - \bar{v}^{\alpha,n}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|,$$

and

$$|\bar{\lambda}^{\alpha,n}| \leq M_\psi.$$

Therefore, again, by a diagonal procedure, it is possible to extract a subsequence  $(\beta(n))_n$  such that

$$v^{\alpha,\beta(n)}(x) \rightarrow \bar{v}^\alpha(x).$$

And thanks to Lemma 4.2, one can apply the dominated convergence theorem to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,\beta(n)} - \bar{Y}_s^{x,\alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,\beta(n)} - \bar{Y}_T^{x,\alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Finally a last diagonal procedure in  $\alpha$  allow us to conclude (see the end of the proof of Theorem 3.4). □

Once again, we notice that the solution we have constructed satisfies the following growth property:

$$|\bar{Y}_t^x| \leq C(1 + |X_t^x|^2),$$

so it is natural to establish the following theorem under the same growth properties.

**Theorem 4.5.** (*Uniqueness of  $\lambda$* ). *Assume that the hypotheses 2.1 and 3.1 hold true. Let  $(Y, Z, \lambda)$  be a solution of EBSDE (4.3). Then  $\lambda$  is unique among solutions  $(Y, Z, \lambda)$  such that  $Y$  is a bounded continuous process and  $Z \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Finally assume that we have the following growth property*

$$\begin{aligned} |Y_t| &\leq C(1 + |X_t^x|^2), \\ |Y_t'| &\leq C'(1 + |X_t^x|^2). \end{aligned}$$

Then  $\lambda = \lambda'$ .

*Proof.* Simply, adapt the proof of Theorem 4.6 of [4]. With the same notations we can write:

$$\begin{aligned}\tilde{\lambda} &= T^{-1}\mathbb{E}^{\mathbb{P}^h}[\tilde{Y}_T - \tilde{Y}_0] \\ &\leq (C + C')T^{-1}(2 + |x|^2 + \mathbb{E}^{\mathbb{P}^h}|X_T^x|^2) \\ &\leq (C + C')T^{-1}(2 + |x|^2 + \mathbb{E}^{\mathbb{P}^h}|X_T^{x,n}|^2 + \mathbb{E}^{\mathbb{P}^h}|X_T^x - X_T^{x,n}|^2)\end{aligned}$$

To conclude, just use the first estimate from Lemma 2.1, the estimate from Lemma 4.2 and let  $T \rightarrow +\infty$ .  $\square$

*4.2. The ergodic BSDE with non-zero Neumann boundary conditions in a weakly dissipative environment*

We are now concerned by the following EBSDE in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]ds + \int_t^T [g(X_s^x) - \mu]dK_s^x - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty, \quad (4.5)$$

where  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable and such that the term  $\int_t^T [g(X_s^x) - \mu]dK_s^x$  is well defined for all  $0 \leq t \leq T < +\infty$ .

**Proposition 4.6.** (*Existence of a Solution  $(Y, Z, \lambda)$* ). *Assume that the hypothesis 2.1, 3.1 and 4.2 hold true. Then for any  $\mu \in \mathbb{R}$  there exists  $\lambda \in \mathbb{R}$ ,  $Y^x$  continuous adapted process and  $Z^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  such that the triple  $(Y, Z, \lambda)$  is a solution of EBSDE (4.5).*

*Proof.* The Theorem 4.4 gives us the existence of a solution  $(Y^x, Z^x, \lambda)$  of the following EBSDE

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.6)$$

Now, defining  $\hat{Y}_t^x = Y_t^x - \int_0^t [g(X_s^x) - \mu]dK_s^x$ , it is easy to see that  $(\hat{Y}^x, Z^x, \lambda)$  is a solution of the EBSDE (4.5) with  $\mu$  fixed.  $\square$

**Remark 4.7.** The constructed solution  $\hat{Y}^x$  is not Markovian anymore. Furthermore, it satisfies the following growth property:  $\forall t \geq 0, |\hat{Y}_t^x| \leq C(1 + |X_t^x|^2 + K_t^x)$ . This dependence on  $K_t^x$  prevents us to get the uniqueness of  $\lambda$  among the space of solutions satisfying such a growth property.

Similarly, for every  $\lambda \in \mathbb{R}$ , an existence result can be stated for a solution  $(Y, Z, \mu)$ .

**Proposition 4.8.** (*Existence of a Solution  $(Y, Z, \mu)$* ). *Assume that the hypotheses 2.1, 3.1 and 4.2 hold true. Then for any  $\lambda \in \mathbb{R}$  there exists a continuous adapted process  $Y$  and  $Z^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  such that for all  $\mu \in \mathbb{R}$  the triple  $(Y, Z, \mu)$  is a solution of EBSDE (4.5).*

*Proof.* From Theorem 4.4, we have constructed a solution  $(Y^{x,0}, Z^{x,0}, \lambda^0)$  of the following EBSDE

$$Y_t^{x,0} = Y_T^{x,0} + \int_t^T [\psi(X_s^x, Z_s^{x,0}) - \lambda^0]ds - \int_t^T Z_s^{x,0} dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.7)$$

Then setting  $\hat{Y}_t^x := Y_t^{x,0} + (\lambda - \lambda^0)t - \int_0^t [g(X_s^x) - \mu]dK_s^x$ , the triple  $(\hat{Y}^x, Z^{x,0}, \mu)$  is solution of EBSDE (4.5).  $\square$

**Remark 4.9.** The constructed solution satisfies the following growth property:

$$|\widehat{Y}_t| \leq C(1 + |X_t^x|^2 + K_t^x + t), \mathbb{P}\text{-a.s.}$$

Again, this solution does not allow us to establish a result of uniqueness for  $\mu$  among the space of solutions satisfying such a growth property.

**Remark 4.10.** If the convex  $\overline{G}$  is assumed to be bounded, it is possible, following [14] to show that there exists a Markovian solution  $(Y, Z, \lambda)$  when  $\mu$  is fixed or  $(Y, Z, \mu)$  when  $\lambda$  is fixed exists, for a driver weakly dissipative. The proofs are the same as in [14].

#### 4.3. Probabilistic interpretation of the solution of an elliptic PDE with zero Neumann boundary condition

We are concerned with the following semi-linear elliptic PDE:

$$\begin{cases} \mathcal{L}v(x) + \psi(x, \nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \quad (4.8)$$

where:

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(\sigma(x)^t \sigma(x) \nabla^2 u(x)) + \nabla u(x) f(x).$$

The unknowns of this equation is the couple  $(v, \lambda)$ . Now we show that the pair  $(v, \lambda)$  defined in Theorem 4.6 is a viscosity solution of the PDE (4.8).

**Theorem 4.11.** *Assume that hypotheses of Theorem 4.4 hold. Then  $(v, \lambda)$  is a viscosity solution of the elliptic PDE (4.8) where  $(v, \lambda)$  is defined in Theorem 4.4.*

*Proof.* Just adapt the proof of Theorem 4.3 from [13]. □

#### 4.4. Optimal ergodic control

We make the standard assumption for optimal ergodic control, namely we consider  $U$  a separable metric space, which is the state space of the control process  $\rho$ .  $\rho$  is assumed to be  $(\mathcal{F}_t)$ -progressively measurable. We introduce  $R : U \rightarrow \mathbb{R}^d$  and  $L : \mathbb{R}^d \times U \rightarrow \mathbb{R}$  two continuous functions such that, for some constants  $M_R > 0$  and  $M_L > 0$ ,  $\forall u \in U, \forall x, x' \in \mathbb{R}^d$ ,

- $|R(u)| \leq M_R$ ,
- $|L(x, u)| \leq M_L$ ,
- $|L(x, u) - L(x', u)| \leq M_L |x - x'|$ .

For an arbitrary control  $\rho$ , the cost will be evaluated relatively to the following Girsanov density:

$$\Gamma_T^\rho = \exp \left( \int_0^T R(\rho_s) dW_s - \frac{1}{2} \int_0^T |R(\rho_s)|^2 ds \right).$$

We denote by  $\mathbb{P}_T^\rho$  the associated probability measure, namely:  $d\mathbb{P}_T^\rho = \Gamma_T^\rho d\mathbb{P}$  on  $\mathcal{F}_T$ . Now we define the ergodic costs, relatively to a given control  $\rho$  and a starting point  $x \in \mathbb{R}^d$ , by:

$$I(x, \rho) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_T^\rho \left[ \int_0^T L(X_s^x, \rho_s) ds \right], \quad (4.9)$$

where  $\mathbb{E}_T^\rho$  denotes expectation with respect to  $\mathbb{P}_T^\rho$ . We notice that the process  $W_t^\rho := W_t - \int_0^t R(\rho_s) ds$  is a Wiener process on  $[0, T]$  under  $\mathbb{P}_T^\rho$ . We define the Hamiltonian in the usual way:

$$\psi(x, z) = \inf_{u \in U} \{L(x, u) + zR(u)\}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^{1 \times d}, \quad (4.10)$$

and we remark that if, for all  $x, z$ , the infimum is attained in (4.10) then, according to Theorem 4 of [10], there exists a measurable function  $\gamma : \mathbb{R}^d \times \mathbb{R}^{1 \times d} \rightarrow U$  such that:

$$\psi(x, z) = L(x, \gamma(x, z)) + zR(\gamma(x, z)). \quad (4.11)$$

One can verify that  $\gamma$  is a Lipschitz function. Now we can prove the following theorem, exactly like in [14].

**Theorem 4.12.** *Assume that the hypotheses of Theorem 4.4 hold true. Let  $(Y, Z, \lambda)$  be a solution of EBSDE (4.5) with  $\mu$  fixed. Then:*

1. *For arbitrary control  $\rho$  we have  $I(x, \rho) \geq \lambda$ .*
2. *If  $L(X_t^x, \rho_s) + Z_t^x R(\rho_t) = \psi(X_t^x, Z_t^x)$ ,  $\mathbb{P}$ -a.s. for almost every  $t$  then  $I(x, \rho) = \lambda$ .*
3. *If the infimum is attained in (4.11) then the control  $\bar{\rho}_t = \gamma(X_t^x, Z_t^x)$  verifies  $I(x, \bar{\rho}) = \lambda$ .*

**Remark 4.13.** When the Neumann conditions are different from 0, we need regularity on the solution  $Y_t^x$  in order to state the same result. Again the degeneracy of the solution constructed in Proposition 4.6 or 4.8 does not allow us to conclude.

## 5. Appendix

### Appendix A. Proof of Lemma 2.1

Let us define  $\varphi(x) = |x - a|^p$  for  $p \geq 1$ . We recall the following formulas for derivatives of  $\varphi$ , for  $p \geq 2$ .

$$\begin{aligned} \nabla \varphi(x) &= p(x - a)|x - a|^{p-2}. \\ \partial^2 \varphi(x) / \partial x_i \partial x_j &= \begin{cases} p|x - a|^{p-2} + p(p-2)(x_i - a_i)^2|x - a|^{p-4} & \text{if } i = j, \\ p(p-2)(x_i - a_i)(x_j - a_j)|x - a|^{p-4} & \text{if } i \neq j. \end{cases} \end{aligned}$$

Therefore we have the following estimate

$$|\nabla^2 \varphi(x)| \leq K|x - a|^{p-2}, \quad (A.1)$$

for a constant  $K$  which depends only on  $p$  and  $d$ . Under the hypothesis of this Lemma, it is well known that a unique strong solution for which the explosion time is almost surely equal to infinity exists (see [8] for example). By Itô's formula we get, for  $p = 2$ , for all  $t \geq 0$ ,

$$\begin{aligned} |V_t^x - a|^2 e^{2\eta_1 t} &= |x - a|^2 + 2 \int_0^t e^{2\eta_1 s} (V_s^x - a, f(V_s^x)) ds + \sigma(V_s^x) dW_s \\ &\quad + 2\eta_1 \int_0^t |V_s^x - a|^2 e^{2\eta_1 s} ds + \int_0^t \sum_i (\sigma(V_s^x)^t \sigma(V_s^x))_{i,i} e^{2\eta_1 s} ds \\ &\leq |x - a|^2 + 2 \int_0^t {}^t(V_s^x - a) \sigma(V_s^x) dW_s + \frac{2\eta_2 + d|\sigma|_\infty}{2\eta_1} (e^{2\eta_1 t} - 1). \end{aligned} \quad (A.2)$$



Taking the expectation, we get:

$$\mathbb{E}|V_t^x - a|^2 \leq |x - a|^2 e^{-2\eta_1 t} + \frac{2\eta_2 + d|\sigma|_\infty^2}{2\eta_1} (1 - e^{-2\eta_1 t}). \quad (\text{A.3})$$

Therefore:

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-2\eta_1 t}), \quad (\text{A.4})$$

where  $C$  is a constant that depends only on  $a$ ,  $\eta_1$ ,  $\eta_2$  and  $\sigma$  but not on the time  $t$ .

Let  $0 < \delta < p\eta_1$ . For  $p > 2$ , Itô's formula gives us, for a generic constant  $C$  which depends only on  $p$ ,  $d$ ,  $|\sigma|_\infty$ ,  $\eta_2$ ,  $\varepsilon$  (defined later):

$$\begin{aligned} |V_t^x - a|^p e^{(p\eta_1 - \delta)t} &\leq |x - a|^p + p \int_0^t e^{(p\eta_1 - \delta)s} |V_s^x - a|^{p-2} (V_s^x - a, f(V_s^x)) ds + \sigma(V_s^x) dW_s \\ &\quad + (p\eta_1 - \delta) \int_0^t |V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds \\ &\quad + \frac{1}{2} \int_0^t \text{Tr}(\sigma(V_s^x)^t \sigma(V_s^x) \nabla^2 \varphi(V_s^x)) e^{(p\eta_1 - \delta)s} ds. \end{aligned}$$

Then, taking the expectation, using the assumption on  $f$  and using estimate (A.1) we have

$$\begin{aligned} \mathbb{E}|V_t^x - a|^p e^{(p\eta_1 - \delta)t} &\leq |x - a|^p + C \int_0^t \mathbb{E}|V_s^x - a|^{p-2} e^{(p\eta_1 - \delta)s} ds \\ &\quad - \delta \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds. \end{aligned}$$

The Young inequality  $ab \leq a^p/p + b^q/q$  for  $1/p + 1/q = 1$  with  $p$  replaced by  $p/(p-2)$  and  $q$  replaced by  $p/2$  applied to the last term of the above inequality allows us to write:

$$|V_s^x - a|^{p-2} \leq (p-2)\varepsilon |V_s^x - a|^p/p + 2/(p\varepsilon^{(p-2)/2}),$$

hence,

$$\begin{aligned} \mathbb{E}|V_t^x - a|^p e^{(p\eta_1 - \delta)t} &\leq |x - a|^p + \varepsilon C \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds + C/\varepsilon^{(p-2)/2} \\ &\quad - \delta \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds. \end{aligned}$$

We choose  $\varepsilon = \delta/C$ , then:

$$\mathbb{E}|V_t^x - a|^p e^{(p\eta_1 - \delta)t} \leq |x - a|^p C.$$

Therefore:

$$\mathbb{E}|V_t^x - a|^p \leq C(1 + |x|^p e^{-(p\eta_1 - \delta)t}).$$

This can be rewritten:

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-(p\eta_1 - \delta)t}),$$

where  $C$  is a constant which depends on  $p$ ,  $d$ ,  $\sigma_\infty$ ,  $\eta_1$ ,  $\eta_2$ ,  $\varepsilon$  and  $a$ . Finally, note that this result holds for any  $0 < \delta < p\eta_1$ .

Now, let us come back to (A.2), we have, for all  $r > 1$ ,  $2r = p$

$$\begin{aligned}\mathbb{E}\left[\sup_{0 \leq t \leq T} |V_t^x|^p\right] &\leq C \left(1 + |x|^p + \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t (V_s^x - a)\sigma(V_s^x) dW_s \right|^p\right]\right) \\ &\leq C \left[1 + |x|^p + \mathbb{E} \left( \left( \int_0^T |(V_s^x - a)\sigma(V_s^x)|^2 ds \right)^{r/2} \right)\right]\end{aligned}$$

by BDG's inequality. Now distinguish the cases  $r/2 < 1$  or  $r/2 \geq 1$ , we readily get, for each  $p > 2$ ,

$$\begin{aligned}\mathbb{E}\left[\sup_{0 \leq t \leq T} |V_t^x|^p\right] &\leq C(T)(1 + |x|^p + |x|^r) \\ &\leq C(T)(1 + |x|^p).\end{aligned}$$

Once this is established, one can readily extend this estimate to the case  $p \geq 1$ . Indeed, for  $0 < \alpha < 1$ , we have, by Jensen's inequality

$$\begin{aligned}\mathbb{E}\left[\sup_{0 \leq t \leq T} |V_t^x|^{p\alpha}\right] &\leq \left(\mathbb{E}\left[\sup_{0 \leq t \leq T} |V_t^x|^p\right]\right)^\alpha \\ &\leq C(T)(1 + |x|^{p\alpha}).\end{aligned}$$

□

## Appendix B. Proof of Lemma 2.5

We adapt the proof of Theorem 2.4 from [3]. We give the full proof for reader convenience. In this proof,  $\kappa_i$ ,  $i = 0, 1, \dots$  denotes a constant which depends only on  $\eta$ ,  $B$ ,  $\sigma_\infty$  and the dimension  $d$ . There are three steps in this proof. In the first step, we show that the process  $V^x$  enters a fixed ball quickly enough. In the second step, we construct a coupling of solutions starting from two different points in this ball and we show that the probability of the constructed solutions to be equal after a time  $T$  (given in the proof) is positive. Iterating this argument in step 3, we obtain the result.

**Step 1 :** By Remark 2.3, one can take  $a = 0$  in equation (A.3) and then:

$$\mathbb{E}|V_t^x|^2 \leq |x|^2 e^{-\eta_1 t} + \kappa_1.$$

By the Markov property :  $\forall k \in \mathbb{N}$ ,

$$\mathbb{E}[|V_{(k+1)T}^x|^2 | \mathcal{F}_{kT}] \leq |V_{kT}^x|^2 e^{-\eta_1 T} + \kappa_1. \quad (\text{B.1})$$

Let us define for  $R \geq 0$ ,

$$C_k = \{|V_{kT}^x|^2 \geq R\}, \quad B_k = \bigcap_{j=0}^k C_j.$$

By Markov's inequality

$$\begin{aligned}\mathbb{P}(C_k + 1 | \mathcal{F}_{kT}) &\leq \frac{\mathbb{E}(|V_{(k+1)T}^x|^2 | \mathcal{F}_{kT})}{R} \\ &\leq \frac{|V_{kT}^x|^2 e^{-\eta_1 T} + \kappa_1}{R} + \frac{\kappa_1}{R}.\end{aligned} \quad (\text{B.2})$$

Let us multiply (B.1) and (B.2) by  $\mathbb{1}_{B_k}$  and let us take the expectation to obtain, since  $\mathbb{1}_{B_{k+1}} \leq \mathbb{1}_{B_k}$

$$\left( \frac{\mathbb{E}(|V_{(k+1)T}^x|^2 \mathbb{1}_{B_{k+1}})}{\mathbb{P}(B_{k+1})} \right) \leq A \left( \frac{\mathbb{E}(|V_{kT}^x|^2 \mathbb{1}_{B_k})}{\mathbb{P}(B_k)} \right)$$

where

$$A = \begin{pmatrix} e^{-\eta_1 T} & \kappa_1 \\ \frac{e^{-\eta_1 T}}{R} & \frac{\kappa_1}{R} \end{pmatrix}.$$

The eigenvalues of  $A$  are 0 and  $e^{-\eta_1 T} + \frac{\kappa_1}{R}$ . So for every  $T > 0$  we can pick  $R$  large enough such that  $\lambda_{T,R} := e^{-\eta_1 T} + \frac{\kappa_1}{R} < 1$ .

We deduce that

$$\mathbb{P}(B_k) \leq \kappa_2 (\lambda_{T,R})^k (1 + |x|^2).$$

Defining,

$$\tau = \inf\{kT; |V_{kT}^x|^2 \leq R\},$$

it follows that

$$\mathbb{P}(\tau \geq kT) \leq \mathbb{P}(B_k) \leq \kappa_2 (\lambda_{T,R})^k (1 + |x|^2).$$

Thus, if we pick  $\alpha > 0$  small enough such that  $e^{\alpha T} \lambda_{T,R} < 1$ , we obtain:

$$\mathbb{E}(e^{\alpha \tau}) \leq \kappa_2 (1 + |x|^2) \sum_{k=0}^{+\infty} (e^{\alpha T} \lambda_{T,R})^k \leq \kappa_3 (1 + |x|^2). \quad (\text{B.3})$$

**Step 2.** In this step, we construct a coupling of processes starting respectively from  $x, y \in B_R$ , the ball of center 0 and radius  $R$ , such that the probability that they take the same value at time  $T$  is strictly positive. We denote by  $\mu^x$  the law of  $V^x$  under  $\mathbb{P}$  and by  $\mu^y$  the law of  $V^y$  under  $\mathbb{P}$  on  $[0, T]$ . Let us define

$$\tilde{V}_t = V_t^y + Y_t^{x,y}$$

where  $Y_t^{x,y}$  is the solution of the following SDE, for all  $t < T$ ,

$$\begin{cases} dY_t^{x,y} = \left[ d(V_t^y + Y_t^{x,y}) - d(V_t^y) - \sigma(V_t^y + Y_t^{x,y}) \sigma^{-1}(V_t^y) \frac{LY_t^{x,y}}{T-t} \right] dt \\ \quad + [\sigma(V_t^y + Y_t^{x,y}) - \sigma(V_t^y)] dW_t, \\ Y_0^{x,y} = x - y, \end{cases} \quad (\text{B.4})$$

and where  $L > 0$ . We denote by  $\tilde{\mu}$  the law of  $\tilde{V}$  on  $[0, T]$ . The process  $\tilde{V}_t$  satisfies for all  $t < T$ :

$$\begin{cases} d\tilde{V}_t = f(\tilde{V}_t) dt + \sigma(\tilde{V}_t) \left[ dW_t + \left( \sigma^{-1}(\tilde{V}_t) (b(V_t^y) - b(\tilde{V}_t)) - \sigma^{-1}(V_t^y) \frac{LY_t^{x,y}}{T-t} \right) dt \right] \\ \quad + [\sigma(V_t^y + Y_t^{x,y}) - \sigma(V_t^y)] dW_t, \\ \tilde{V}_0 = x, \end{cases} \quad (\text{B.5})$$

Since all the coefficients are locally Lipschitz,  $Y_t^{x,y}$  (and thus  $\tilde{V}_t$ ) are well-defined continuous process for  $t \leq T \wedge \zeta$  where  $\zeta$  is the explosion time of  $Y_t^{x,y}$ ; namely,  $\zeta := \lim_{n \rightarrow +\infty} \zeta_n$  for

$$\zeta_n := \inf\{t \in [0, T) : |Y_t^{x,y}| \geq n\}$$

where we set  $\inf\{\emptyset\} = T$ . We define

$$h(t) = \sigma^{-1}(\tilde{V}_t) \left( b(V_t^y) - b(\tilde{V}_t) \right) - \sigma^{-1}(V_t^y) \frac{LY_t^{x,y}}{T-t}, \quad t \leq T \wedge \zeta,$$

$$d\tilde{W}_t = dW_t + h(t)dt, \quad t \leq T \wedge \zeta,$$

and

$$I_t = \int_0^t h(s) dW_s, \quad t \leq T \wedge \zeta.$$

If  $\zeta = T$  and

$$R_t := \exp\left(-I_t - \frac{1}{2}\langle I, I \rangle_t\right)$$

is a uniformly integrable martingale for  $t \in [0, T)$ , then by the martingale convergence theorem,  $R_T := \lim_{t \nearrow T} R_t$  exists and  $(R_t)_{t \in [0, T]}$  is a martingale. In this case, by the Girsanov theorem  $(\tilde{W}_t)_{t \in [0, T]}$  is a standard Brownian motion under the probability  $R_T \mathbb{P}$ . Rewrite (B.4) as

$$\begin{cases} dY_t^{x,y} = \left[ d(\tilde{V}_t) - d(V_t^y) - \left( \sigma(\tilde{V}_t) - \sigma(V_t^y) \right) (b(\tilde{V}_t) - b(V_t^y)) \right] dt - \frac{LY_t^{x,y}}{T-t} dt \\ \quad + \left[ \sigma(\tilde{V}_t) - \sigma(V_t^y) \right] d\tilde{W}_t, \\ Y_0^{x,y} = x - y, \end{cases} \quad (\text{B.6})$$

Now we would like to apply the Girsanov theorem. The following lemma is a direct adaptation of a result in [16]. However we give the proof for completeness.

**Lemma AppendixB.1.** *Assume Hypothesis 2.1 hold true and let  $x, y \in \mathbb{R}^d$  and  $T > 0$  be fixed. Then*

1. *There holds*

$$\sup_{t \in [0, T), n \geq 1} \mathbb{E} R_{t \wedge \zeta_n} \log R_{t \wedge \zeta_n} < +\infty.$$

*Consequently,*

$$R_{t \wedge \zeta} := \lim_{n \nearrow \infty} R_{t \wedge \zeta_n \wedge (T-1/n)}, \quad t \in [0, T], \quad R_{T \wedge \zeta} := \lim_{s \nearrow T} R_{s \wedge \zeta}$$

*exist such that  $(R_{s \wedge \zeta})_{s \in [0, T]}$  is a uniformly martingale.*

2. *Let  $\mathbb{Q} = R_{T \wedge \zeta} \mathbb{P}$ . Then  $\mathbb{Q}(\zeta = T) = 1$  so that  $\mathbb{Q} = R_T \mathbb{P}$ .*
3.  *$Y_T^{x,y} = 0$ ,  $\mathbb{Q}$ -a.s.*

*Proof.* (1) let  $t \in [0, T)$  and be fixed. By an Itô's formula,

$$\begin{aligned} \frac{|Y_{t \wedge \zeta_n}^{x,y}|^2}{T-s} &= \frac{|x-y|^2}{T} + 2 \int_0^{t \wedge \zeta_n} \left( \frac{Y_s^{x,y}}{T-s}, d(\tilde{V}_t) - d(V_s^y) - \frac{LY_s^{x,y}}{T-s} \right) ds \\ &\quad - 2 \int_0^{t \wedge \zeta_n} \left( \frac{Y_s^{x,y}}{T-s}, (\sigma(\tilde{V}_t) - \sigma(V_s^y)) (b(\tilde{V}_t) - b(V_t^y)) \right) ds \\ &\quad + \int_0^{t \wedge \zeta_n} \frac{|Y_s^{x,y}|^2}{|T-s|^2} ds + 2 \int_0^{t \wedge \zeta_n} \left( \frac{Y_s^{x,y}}{T-s}, [\sigma(\tilde{V}_s) - \sigma(V_s^y)] d\tilde{W}_s \right) \\ &\quad + \int_0^{t \wedge \zeta_n} \frac{1}{T-s} \text{Tr} \left[ (\sigma(\tilde{V}_s) - \sigma(V_s^y))^t (\sigma(\tilde{V}_s) - \sigma(V_s^y)) \right] ds \end{aligned}$$

By standard computations, since  $d$  is dissipative, since  $b$  is bounded and since  $\sigma$  is Lipschitz and bounded, for every  $\varepsilon > 0$ , we get

$$\begin{aligned} \frac{|Y_{t \wedge \zeta_n}^{x,y}|^2}{T-s} + (2L-1-\varepsilon) \int_0^{t \wedge \zeta_n} \frac{|Y_s^{x,y}|^2}{|T-s|^2} ds &\leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon} (t \wedge \zeta_n) \\ &\quad + 2 \int_0^{t \wedge \zeta_n} \left( \frac{Y_s^{x,y}}{T-s}, [\sigma(\tilde{V}_s) - \sigma(V_s^y)] d\tilde{W}_s \right). \end{aligned} \tag{B.7}$$

By the Girsanov theorem,  $(\tilde{W}_s)_{s \leq t \wedge \zeta_n}$  is a standard Brownian motion under the probability measure  $R_{t \wedge \zeta_n} \mathbb{P}$ . So, taking expectation  $\mathbb{E}^{t,n}$  with respect to  $R_{t \wedge \zeta_n} \mathbb{P}$ , we arrive at

$$\mathbb{E}^{t,n} \int_0^{t \wedge \zeta_n} \frac{|Y_s^{x,y}|^2}{|T-s|^2} ds \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon} T, \quad s \in [0, T], n \geq 1. \tag{B.8}$$

By the definitions of  $R_t$  and  $\tilde{W}_t$ , we have for every  $s \leq t \wedge \zeta_n$ ,

$$\begin{aligned} \log R_s &= - \int_0^s h(r) d\tilde{W}_r + \frac{1}{2} \int_0^s |h(r)|^2 dr \\ &\leq - \int_0^s h(r) d\tilde{W}_r + C \left( 1 + \int_0^s \frac{L^2 |Y_r^{x,y}|^2}{|T-r|^2} dr \right) \end{aligned}$$

And then, taking the expectation, we obtain with (B.8):

$$\mathbb{E} R_{t \wedge \zeta_n} \log R_{t \wedge \zeta_n} = \mathbb{E}^{t,n} \log R_{t \wedge \zeta_n} \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon} T, \quad t \in [0, T], n \geq 1.$$

By the martingale convergence theorem and the Fatou lemma,  $(R_{s \wedge \zeta})_{s \in [0, T]}$  is a well-defined martingale with

$$\mathbb{E} R_{t \wedge \zeta} \log R_{t \wedge \zeta} \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon} T, \quad t \in [0, T].$$

To see that  $(R_{s \wedge \zeta})_{s \in [0, T]}$  is a martingale, let  $0 \leq s < t \leq T$ . By the dominated convergence theorem and the martingale property of  $(R_{t \wedge \zeta_n \wedge (T-1/n)})$ , we have

$$\begin{aligned} \mathbb{E}(R_{t \wedge \zeta} | \mathcal{F}_s) &= \mathbb{E} \left( \lim_{n \rightarrow +\infty} R_{t \wedge \zeta_n \wedge (T-1/n)} | \mathcal{F}_s \right) = \lim_{n \rightarrow +\infty} \mathbb{E}(R_{t \wedge \zeta_n \wedge (T-1/n)} | \mathcal{F}_s) \\ &= \lim_{n \rightarrow +\infty} R_{s \wedge \zeta_n} = R_{s \wedge \zeta}. \end{aligned}$$

(2) Since  $\widetilde{W}_t$  is a standard Brownian motion up to  $T \wedge \zeta$ , it follows from (B.7) that

$$\frac{n^2}{T} \mathbb{Q}(\zeta_n \leq t) \leq \mathbb{E}^{\mathbb{Q}} \frac{|Y_{t \wedge \zeta_n}^{x,y}|^2}{T - (t \wedge \zeta_n)} \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon} T$$

holds for  $n \geq 1$  and  $t \in [0, T)$ . By letting  $n \nearrow +\infty$ , we obtain  $\mathbb{Q}(\zeta \leq t) = 0$  for all  $t \in [0, T)$ . This is equivalent to  $\mathbb{Q}(\zeta = T) = 1$  according to the definition of  $\zeta$ .

(3) Let

$$\bar{\zeta} = \inf\{t \in [0, T] : Y_t^{x,y} = 0\}$$

and set  $\inf \emptyset = +\infty$  by convention. We claim that  $\bar{\zeta} \leq T$  and thus,  $Y_T^{x,y} = 0$ ,  $\mathbb{Q}$ -a.s. Indeed, if for some  $w \in \Omega$  such that  $\bar{\zeta} > T$ , by the continuity of the process we have

$$\inf_{t \in [0, T]} |Y_t^{x,y}|^2(w) > 0.$$

So,

$$\int_0^T \frac{|Y_s^{x,y}|^2}{|T-s|^2} ds = \infty$$

holds on the set  $\{\bar{\zeta} > T\}$ . Now, since inequality (B.7) still hold with  $\zeta_n$  replaced by  $\zeta$  and since  $\zeta = T$   $\mathbb{Q}$ -a.s., if we take the expectation with respect to  $\mathbb{Q}$ , then

$$\mathbb{E}^{\mathbb{Q}} \int_0^T \frac{|Y_s^{x,y}|^2}{|T-s|^2} ds \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon} T < +\infty.$$

Therefore  $\mathbb{Q}(\bar{\zeta} > T) = 0$ . Therefore,  $Y_T^{x,y} = 0$ ,  $\mathbb{Q}$ -a.s. □

Hence we can apply the Girsanov theorem. Therefore, there exist a new probability measure  $\mathbb{Q}$  under which  $\widetilde{W}$  is a Brownian motion. Thus, under  $\mathbb{Q}$ ,  $\widetilde{V}$  has the law  $\mu^x$  whereas under  $\mathbb{P}$  it has the law  $\tilde{\mu}$ . Of course  $\mu^x$  and  $\tilde{\mu}$  are equivalent. We deduce that for every  $\delta > 0$ ,

$$\begin{aligned} \int_{\mathcal{C}([0, T], \mathbb{R}^d)} \left( \frac{d\mu^x}{d\tilde{\mu}} \right)^{2+\delta} d\tilde{\mu} &= \int_{\Omega} \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{2+\delta} d\mathbb{P} \\ &= \int_{\Omega} \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{1+\delta} d\mathbb{Q} \\ &= \mathbb{E}^{\mathbb{Q}} \left( \exp \left( (1+\delta)I_T - \frac{1+\delta}{2} \langle I, I \rangle_T \right) \right) \\ &\leq \sqrt{\mathbb{E}^{\mathbb{Q}} (\exp(2(1+\delta)I_T - 2(1+\delta)^2 \langle I, I \rangle_T))} \\ &\quad \times \sqrt{\mathbb{E}^{\mathbb{Q}} (\exp((2(1+\delta)^2 - (1+\delta)) \langle I, I \rangle_T))} \\ &= \sqrt{\mathbb{E}^{\mathbb{Q}} (\exp((1+\delta)(1+2\delta) \langle I, I \rangle_T))}. \end{aligned} \tag{B.9}$$

We are going to show that we can pick  $L > 0$  and  $\delta > 0$  such that:

$$\mathbb{E}^{\mathbb{Q}} (\exp((1+\delta)(1+2\delta) \langle I, I \rangle_T)) < +\infty.$$

Indeed, we have, for every  $\iota > 0$

$$\mathbb{E}^{\mathbb{Q}}(\exp((1+\delta)(1+2\delta)\langle I, I \rangle_T)) \leq C\mathbb{E}^{\mathbb{Q}}\left(\exp\left(\left((1+\delta)(1+2\delta)\|\sigma^{-1}\|^2\left((1+\iota)\int_0^T \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2} ds\right)\right)\right)\right)$$

Now, by Itô's formula (using B.6):

$$\begin{aligned} \frac{L^2|Y_t^{x,y}|^2}{T-t} &= \frac{L^2|x-y|^2}{T} + 2\int_0^t \left(\frac{L^2Y_s^{x,y}}{T-s}, \left(d(V_s^y + Y_s^{x,y}) - d(V_s^y) - \frac{LY_s^{x,y}}{T-s}\right) ds\right) \\ &\quad - 2\int_0^t \left(\frac{L^2Y_s^{x,y}}{T-s}, (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))(b(V_s^y + Y_s^{x,y}) - b(V_s^y)) ds\right) \\ &\quad + 2\int_0^t \left(\frac{L^2Y_s^{x,y}}{T-s}, (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))d\widetilde{W}_s\right) + \int_0^t \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2} ds \\ &\quad + \int_0^t \frac{L^2}{T-s} \text{Tr} [\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)]^t [\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)] ds. \end{aligned}$$

Therefore, since  $d$  is dissipative, since  $b$  is bounded and since  $\sigma$  is Lipschitz and bounded,

$$\frac{L^2|Y_t^{x,y}|^2}{T-t} + 2\left(L - \frac{1}{2}\right)\int_0^t \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2} ds \leq \frac{L^2|x-y|^2}{T} + J_t + CL^2\int_0^t \frac{|Y_s^{x,y}|}{T-s} ds$$

where  $J_t = 2\int_0^t \left(\frac{L^2Y_s^{x,y}}{T-s}, (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))d\widetilde{W}_s\right)$ . Note that

$$\begin{aligned} d\langle J, J \rangle_t &= 4L^4 \frac{|(Y_s^{x,y})(\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))|^2}{|T-s|^2} dt \\ &\leq 4L^4\Lambda^2 \frac{|Y_s^{x,y}|^2}{|T-s|^2} dt. \end{aligned}$$

Therefore, taking  $t = T$ , for every  $\varepsilon > 0$ , for every  $\gamma > 0$ , we have

$$2\left(L - \frac{1}{2} - \frac{\varepsilon}{4} - \gamma L^2\Lambda^2\right)\int_0^T \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2} ds \leq \frac{L^2|x-y|^2}{T} + \frac{C^2}{2\varepsilon}T + J_T - \frac{\gamma}{2}\langle J, J \rangle_T.$$

Therefore, for  $\varepsilon$  small enough, we obtain by multiplying by  $\gamma$

$$2\gamma\left(L - \frac{1}{2} - \frac{\varepsilon}{4} - \gamma L^2\Lambda^2\right)\int_0^T \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2} ds \leq \frac{\gamma L^2|x-y|^2}{T} + \frac{\gamma C^2}{2\varepsilon}T + \gamma J_T - \frac{\gamma^2}{2}\langle J, J \rangle_T.$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left(2\gamma\left(L - \frac{1}{2} - \frac{\varepsilon}{4} - \gamma L^2\Lambda^2\right)\int_0^T \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2} ds\right)\right) \leq \exp\left(\frac{\gamma L^2|x-y|^2}{T} + \frac{\gamma C^2}{2\varepsilon}T\right)$$

Since by Hypothesis 2.1, there exists  $\lambda > 0$  such that

$$2(\lambda - \lambda^2\Lambda^2) > \|\sigma^{-1}\|^2,$$

it is enough to take  $\iota > 0$ ,  $\varepsilon > 0$ ,  $\delta$  small enough,  $\gamma L = \lambda$  and  $\gamma$  small enough such that:

$$2\left(\gamma L - \frac{\gamma}{2} - \frac{\gamma\varepsilon}{4} - \gamma^2 L^2\Lambda^2\right) = 2\left(\lambda - \frac{\gamma}{2} - \frac{\gamma\varepsilon}{4} - \lambda^2\Lambda^2\right) > (1+\delta)(1+2\delta)(1+\iota)\|\sigma^{-1}\|^2,$$

which shows that:

$$\mathbb{E}^{\mathbb{Q}}(\exp((1 + \delta)(1 + 2\delta)\langle I, I \rangle_T)) < +\infty. \quad (\text{B.10})$$

Therefore, with (B.9):

$$\int_{\mathcal{C}([0, T], \mathbb{R}^d)} \left( \frac{d\tilde{\mu}}{d\mu^x} \right)^{2+\delta} d\mu^x \leq \kappa_5. \quad (\text{B.11})$$

We recall the following result (see for instance [9]).

**Proposition AppendixB.2.** *Let  $(\mu_1, \mu_2)$  be two probability measures on a same space  $(E, \mathcal{E})$  then*

$$\|\mu_1 - \mu_2\|_{TV} = \min \mathbb{P}(Z_1 \neq Z_2)$$

where the minimum is taken on all coupling  $(Z_1, Z_2)$  of  $(\mu_1, \mu_2)$ . Moreover, there exists a coupling which realizes the infimum. We say that it is a maximal coupling. It satisfies

$$\mathbb{P}(Z_1 = Z_2, Z_1 \in \Gamma) = \mu_1 \wedge \mu_2(\Gamma), \quad \Gamma \in \mathcal{B}(E).$$

Moreover, if  $\mu_1$  and  $\mu_2$  are equivalent and if

$$\int_E \left( \frac{d\mu_2}{d\mu_1} \right)^{p+1} d\mu_1 \leq C$$

for some  $p > 1$  and  $C > 1$  then

$$\mathbb{P}(Z_1 = Z_2) = \mu_1 \wedge \mu_2(E) \geq \left(1 - \frac{1}{p}\right) \left(\frac{1}{pC}\right)^{1/(p-1)}.$$

Let us mention the following proposition which can be found in [12] under the name of Corollary 1.5.

**Proposition AppendixB.3.** *Let  $E$  be a Polish space,  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $(U_1, U_2, \tilde{U})$  be three random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking value in  $E$ .*

*Then there exists a triple  $(u_1, u_2, \tilde{u})$  such that  $(u_2, \tilde{u})$  is a maximal coupling of  $(\mathcal{D}(U_2), \mathcal{D}(\tilde{U}))$  and such that the law of  $(u_1, \tilde{u})$  is  $\mathcal{D}(U_1, \tilde{U})$ .*

Let us apply Proposition AppendixB.3 to the three random variables  $(Y^{x,y}, V^x, \tilde{V})$  : there exists  $(\hat{Y}^{x,y}, V^{1,x,y}, \tilde{V}^{2,x,y})$  such that  $(V^{1,x,y}, \tilde{V}^{2,x,y})$  is a maximal coupling of  $(\mu^x, \tilde{\mu})$  on  $[0, T]$  and  $\mathcal{D}(\hat{Y}^{x,y}, \tilde{V}^{2,x,y}) = \mathcal{D}(Y^{x,y}, \tilde{V})$ . Therefore,

$$\mathcal{D}(\tilde{V}^{2,x,y} - \hat{Y}^{x,y}) = \mathcal{D}(\tilde{V} - Y^{x,y}) = \mathcal{D}(V^y) := \mu^y,$$

and

$$\mathcal{D}(\hat{Y}^{x,y}) = \mathcal{D}(Y^{x,y}).$$

Note that the last inequality implies that  $\hat{Y}_T^{x,y} = 0$ ,  $\mathbb{P}$ -a.s.

Now remark that,  $(V^{1,y,x}, V^{2,x,y} := \tilde{V}^{2,x,y} - \hat{Y}^{x,y})$  is a coupling of  $(\mu^x, \mu^y)$  and

$$\mathbb{P}(V_T^{1,y,x} = V_T^{2,x,y}) = \mathbb{P}(V_T^{1,y,x} = \tilde{V}_T^{2,x,y}) \geq \mathbb{P}(V^{1,y,x} = \tilde{V}^{2,x,y}).$$



Now, remarking that  $(\widehat{V}^x, \widehat{V})$  is a maximal coupling of  $(\mu^x, \tilde{\mu})$  and applying Proposition AppendixB.2 thanks to equation (B.11) we get that

$$\mathbb{P}(V^{1,y,x} = \widetilde{V}^{2,x,y}) \geq \frac{1}{4\kappa_5},$$

which leads to

$$\mathbb{P}(V_T^{1,y,x} = V_T^{2,x,y}) \geq \frac{1}{4\kappa_5}. \quad (\text{B.12})$$

**Step 3.** We construct a coupling for any initial value and any date. For  $x = y$ , we set

$$(U_t^1, U_t^2) = (V_t^x, V_t^x), \quad t \in [0, T].$$

If  $x$  or  $y$  is not in  $B_R$ , then

$$(U_t^1, U_t^2) = (V_t^x, \overline{V}_t^y), \quad t \in [0, T]$$

where  $\overline{V}_t^y$  is the solution of equation (2.1) driven by a Wiener process  $\overline{W}$  independent of  $W$ . The coupling of the laws of  $V^x, V^y$  is defined as follows. Assuming that we have built  $(U_t^1, U_t^2)$  on  $[0, nT]$ , we take  $(V^{1,x,y}, V^{2,x,y})$  as above independent on  $(U_t^1, U_t^2)$  on  $[0, nT]$  and set

$$(U_{nT+t}^1, U_{nT+t}^2) = \left( V_t^{1, U_{nT}^1, U_{nT}^2}, V_t^{2, U_{nT}^1, U_{nT}^2} \right), \quad \forall t \in [0, T].$$

The Markov property of  $(U^1, U^2)$  implies that  $(U^1, U^2)$  is a coupling of  $(\mathcal{D}(V^x), \mathcal{D}(V^y))$  on  $[0, (n+1)T]$ .

We then define the following sequence of stopping times

$$L_m = \inf\{l > L_{m-1}, U_{lT}^1, U_{lT}^2 \in B_R\}$$

with  $L_0 = 0$ . Evidently, (B.3) can be generalized to two solutions and we have:

$$\mathbb{E}(e^{\alpha L_1 T}) \leq \kappa_3(1 + |x|^2 + |y|^2)$$

and

$$\mathbb{E}\left(e^{\alpha(L_{m+1}-L_m)T} \mid \mathcal{F}_{L_m T}\right) \leq \kappa_3(1 + |U_{L_m T}^1|^2 + |U_{L_m T}^2|^2).$$

It follows that

$$\begin{aligned} \mathbb{E}(e^{\alpha L_{m+1} T}) &\leq \kappa_3 \mathbb{E}(e^{\alpha L_m T} (1 + |U_{L_m T}^1|^2 + |U_{L_m T}^2|^2)) \\ &\leq \kappa_3(1 + 2R^2) \mathbb{E}(e^{\alpha L_m T}) \end{aligned}$$

and

$$\mathbb{E}(e^{\alpha L_m T}) \leq \kappa_3^m (1 + 2R^2)^{m-1} (1 + |x|^2 + |y|^2).$$

Now set

$$\ell_0 = \inf\{\ell, U_{(\ell+1)T}^1 = U_{(\ell+1)T}^2\}.$$

Since  $U_{L_\ell T}^1, U_{L_\ell T}^2 \in B_R$ , we have by (B.12),

$$\mathbb{P}(\ell_0 > \ell + 1 | \ell_0 > \ell) \leq \left(1 - \frac{1}{\kappa_5}\right).$$

Since  $\mathbb{P}(\ell_0 > \ell + 1) = \mathbb{P}(\ell_0 > \ell + 1 | \ell_0 > \ell) \mathbb{P}(\ell_0 > \ell)$ , we obtain

$$\mathbb{P}(\ell_0 > \ell) \leq \left(1 - \frac{1}{4\kappa_5}\right)^\ell.$$

Then for  $\gamma \geq 0$

$$\begin{aligned} \mathbb{E}(e^{\gamma L_{\ell_0 T}}) &\leq \sum_{\ell \geq 0} \mathbb{E}(e^{\gamma L_{\ell T}} \mathbf{1}_{\ell = \ell_0}) \\ &\leq \sum_{\ell \geq 0} \mathbb{P}(\ell = \ell_0)^{1-\gamma/\alpha} (\mathbb{E}(e^{\alpha L_{\ell T}}))^{\gamma/\alpha} \\ &\leq \sum_{\ell \geq \ell_0} \left(1 - \frac{1}{\kappa_5}\right)^{(\ell-1)(1-\gamma/\alpha)} [\kappa_3^\ell (1 + 2R^2)^{\ell-1} (1 + |x|^2 + |y|^2)]^{\gamma/\alpha}. \end{aligned}$$

We choose  $\gamma \leq \alpha$  such that

$$\left(1 - \frac{1}{4\kappa_5}\right)^{1-\gamma/\alpha} [\kappa_3 (1 + 2R^2)]^{\gamma/\alpha} < 1$$

and deduce that

$$\mathbb{E}(e^{\gamma L_{\ell_0 T}}) \leq \kappa_6 (1 + |x|^2 + |y|^2).$$

Since

$$n_0 = \inf \{k, U_{kT}^1 = U_{kT}^2\} \leq L_{\ell_0} + 1$$

it follows that

$$\mathbb{E}(e^{\gamma n_0 T}) \leq \kappa_6 (1 + |x|^2 + |y|^2)$$

and

$$\begin{aligned} \mathbb{P}(U_{kT}^1 \neq U_{kT}^2) &= \mathbb{P}(k < n_0) \\ &\leq \kappa_6 (1 + |x|^2 + |y|^2) e^{-\gamma k T}. \end{aligned}$$

Moreover, for all  $t \in [0, T)$

$$\begin{aligned} \mathbb{P}(U_{kT+t}^1 \neq U_{kT+t}^2) &\leq \mathbb{P}(U_{kT}^1 \neq U_{kT}^2) \\ &\leq \kappa_6 (1 + |x|^2 + |y|^2) e^{-\gamma k T} \\ &\leq \kappa_7 (1 + |x|^2 + |y|^2) e^{-\gamma(kT+t)}. \end{aligned}$$

We deduce for  $\phi \in B_b(\mathbb{R}^d)$

$$|\mathcal{P}_t[\Phi](x) - \mathcal{P}_t[\Phi](y)| \leq C(1 + |x|^2 + |y|^2) e^{-\mu t} |\Phi|_0.$$

□

### Appendix C. Proof of Lemma 4.2

We follow the proof of the part 3 of [11]. We need to adapt this proof because in our case, the set in which the process is reflected is not bounded. Therefore convergences are not uniform in  $x$  anymore. In our case, the dissipativity of the process is enough to avoid the boundedness of  $\overline{G}$ . We will use the following notation  $\beta(x) = (x - \Pi(x))$ . Note that  $F_n(x) = -2n\beta(x)$ . We recall the following properties of the penalization term:

$$(x' - x, \beta(x)) \leq 0, \quad \forall x \in \mathbb{R}^d, \forall x \in \overline{G}, \quad (\text{C.1})$$

$$(x' - x, \beta(x)) \leq (\beta(x'), \beta(x)), \quad \forall x, x' \in \mathbb{R}^n, \quad (\text{C.2})$$

$$\exists c \in \overline{G}, \quad \gamma > 0, \quad \forall x \in \mathbb{R}^n, \quad (x - c, \beta(x)) \geq \gamma|\beta(x)|. \quad (\text{C.3})$$

In what follow,  $C$  is a constant which may vary from line to line and which may depends on  $x$  and  $T$  but which is independent on  $n$ .

In a first time, we show that for any  $1 \leq p < \infty$ , there exists  $C > 0$

$$\mathbb{E} \left[ \left( n \int_0^T |\beta(X_t^{x,n})| ds \right)^p \right] \leq C, \forall n \in \mathbb{N}. \quad (\text{C.4})$$

For  $p = 2$ , Itô's formula gives us:

$$\begin{aligned} |X_t^{x,n} - c|^2 &= |x - c|^2 + 2 \int_0^t (X_s^{x,n} - c, f(X_s^{x,n}) ds + F_n(X_s^{x,n}) ds + \sigma(X_s^{x,n}) dW_s) \\ &\quad + \int_0^t \sum_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} ds. \end{aligned}$$

Using inequality (C.3), the fact that  $\sigma$  is bounded and Remark (2.3) we deduce:

$$4n\gamma \int_0^t |\beta(X_s^{x,n})| ds \leq |x - c|^2 + 2 \int_0^t C ds + 2 \int_0^t (X_s^{x,n} - c, \sigma(X_s^{x,n}) dW_s). \quad (\text{C.5})$$

By a BDG inequality, using the fact that  $|\sigma|$  is bounded:

$$\mathbb{E} \left[ \left| \int_0^T (X_s^{x,n} - c, \sigma(X_s^{x,n}) dW_s) \right|^p \right] \leq C \mathbb{E} \left[ \left( \int_0^T |X_s^{x,n} - c|^2 ds \right)^{p/2} \right].$$

Now, by Lemma 2.1, it follows that the process  $X^{x,n}$  has bounded moments of all orders independent of  $n$ , thus

$$\mathbb{E} \left[ \left| \int_0^T (X_s^{x,n} - c, \sigma(X_s^{x,n}) dW_s) \right|^p \right] \leq C,$$

which leads to

$$\mathbb{E} \left[ \left( n \int_0^T |\beta(X_t^{x,n})| ds \right)^p \right] \leq C, \forall n \in \mathbb{N},$$

for a constant  $C$  which does not depend on  $n$ .

Now we prove that  $\forall p > 2$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |\beta(X_t^{x,n})|^p \right) \leq \frac{C}{n^{p/2-1}}.$$

We apply Itô's formula to the function  $\varphi(x) = |x - \Pi(x)|^p$  where  $\beta(x) = x - \Pi(x)$ . Note that  $F_n(x) = -2n\beta(x)$ . It is well known that for a regular boundary, for all  $p > 2$ ,  $\varphi$  is  $\mathcal{C}^2$  on  $\mathbb{R}^d$  and  $\nabla\varphi(x) = 2(x - \Pi(x))$ . We recall the following formulas for the derivatives of  $\varphi$ ,

$$\nabla\varphi(x) = p|\beta(x)|^{p-2}\beta(x),$$

$$\nabla^2\varphi(x) = p|\beta(x)|^{p-2}\nabla\beta(x) + p(p-2)|\beta(x)|^{p-4}(\beta(x)^t\beta(x)).$$

As  $\nabla\beta(x)$  is a numerical matrix one can deduce the following inequality:

$$|\nabla^2\varphi(x)| \leq C|\beta(x)|^{p-2},$$

for a constant  $C$  which depends only on  $p$  and  $d$ .

We use Itô's formula, for all  $p > 2$ ,

$$\begin{aligned} \varphi(X_t^{x,n}) &= \int_0^t (\nabla\varphi(X_s^{x,n}), d(X_s^{x,n}) + b(X_s^{x,n}) + F_n(X_s^{x,n})) ds \\ &\quad + \int_0^t (\nabla\varphi(X_s^{x,n}), \sigma(X_s^{x,n})) dW_s \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j} (\nabla^2\varphi(X_s^{x,n}))_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi(X_t^{x,n}) + 2pn \int_0^t \varphi(X_s^{x,n}) ds &\leq \int_0^t (\nabla\varphi(X_s^{x,n}), d(X_s^{x,n}) + b(X_s^{x,n})) ds \\ &\quad + \int_0^t (\nabla\varphi(X_s^{x,n}), \sigma(X_s^{x,n})) dW_s + C \int_0^t |\beta(X_s^{x,n})|^{p-2} ds. \end{aligned} \tag{C.6}$$

From Young's inequality:  $ab \leq a^q/q + b^{q'}/q'$  for some real numbers  $q$  and  $q'$  such that  $1/q + 1/q' = 1$ , we choose  $q = p/(p-2)$  and  $q' = p/2$  so that, for  $\alpha > 0$ :

$$\begin{aligned} |\beta(X_s^{x,n})|^{p-2} &= \alpha n^{(p-2)/p} |\beta(X_s^{x,n})|^{p-2} \times \frac{1}{\alpha n^{(p-2)/p}} \\ &\leq \alpha^{p/(p-2)} \frac{p-2}{p} n |\beta(X_s^{x,n})|^p + \frac{2}{p} \left( \frac{1}{\alpha n^{(p-2)/p}} \right)^{p/2} \\ &\leq \alpha^{p/(p-2)} \frac{p-2}{p} n \varphi(X_s^{x,n}) + \frac{2}{p} \frac{1}{\alpha^{p/2} n^{(p-2)/2}}, \end{aligned}$$

and another Young's inequality applied with this time  $q = p/(p-1)$  and  $q' = p$  gives us:

$$\begin{aligned} |(\nabla\varphi(X_s^{x,n}), d(X_s^{x,n}) + b(X_s^{x,n}))| &\leq p |\beta(X_s^{x,n})|^{p-1} \times |(d(X_s^{x,n}) + b(X_s^{x,n}))| \\ &\leq \alpha^{p/(p-1)} n (p-1) p^{1/p} |\beta(X_s^{x,n})|^p \\ &\quad + |d(X_s^{x,n}) + b(X_s^{x,n})|^p / (pn^{p-1} \alpha^p) \\ &\leq \alpha^{p/(p-1)} n (p-1) p^{1/p} \varphi(X_s^{x,n}) \\ &\quad + (1 + |X_s^{x,n}|^{p\nu}) / (pn^{p-1} \alpha^p). \end{aligned}$$

Therefore using the second inequality of Lemma 2.1 and the two above inequality we deduce, for  $\alpha$  small enough:

$$\mathbb{E} \left( n \int_0^t \varphi(X_s^{x,n}) ds \right) \leq C \left( \frac{1}{n^{p-1}} + \frac{1}{n^{(p-2)/2}} \right) t,$$

therefore,

$$\mathbb{E} \left( \int_0^T \varphi(X_s^{x,n}) ds \right) \leq \frac{C}{n^{p/2}}. \quad (\text{C.7})$$

Now we come back to equation (C.6). Taking the supremum over time and the expectation and using a BDG inequality we get:

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \varphi(X_t^{x,n}) &\leq \mathbb{E} \int_0^T |\nabla \varphi(X_s^{x,n})| \times |d(X_s^{x,n}) + b(X_s^{x,n})| ds \\ &\quad + C \mathbb{E} \left[ \left( \int_0^T |\nabla \varphi(X_s^{x,n}) \sigma(X_s^{x,n})|^2 ds \right)^{1/2} \right] + C \mathbb{E} \int_0^T |\beta(X_s^{x,n})|^{p-2} ds. \end{aligned} \quad (\text{C.8})$$

We call respectively  $I_1$ ,  $I_2$  and  $I_3$  the three terms of the right hand side of (C.8). We have

$$\begin{aligned} I_1 &\leq C \sqrt{\mathbb{E} \int_0^T |\nabla \varphi(X_s^{x,n})|^2 ds} \times \sqrt{\mathbb{E} \int_0^T |1 + |X_s^{x,n}|^\nu|^2 ds} \\ &\leq C \sqrt{\mathbb{E} \int_0^T |\beta(X_s^{x,n})|^{2p-2} ds} \\ &\leq \frac{C}{n^{(p-1)/2}}, \end{aligned}$$

by using inequality (C.7) and Lemma 2.1. We also have

$$\begin{aligned} I_2 &\leq C \mathbb{E} \left[ \left( \int_0^T |\beta(X_s^{x,n})|^{2p-2} ds \right)^{1/2} \right] \\ &\leq \frac{1}{\sqrt{2}} \mathbb{E} \sup_{0 \leq t \leq T} |\beta(X_s^{x,n})|^p + C' \mathbb{E} \left[ \int_0^T |\beta(X_s^{x,n})|^{p-2} ds \right], \end{aligned}$$

thanks to Young's inequality. Applying inequality (C.7) to the second member gives us:

$$I_2 \leq \frac{1}{\sqrt{2}} \mathbb{E} \sup_{0 \leq t \leq T} |\beta(X_s^{x,n})|^p + \frac{C}{n^{(p-2)/2}}.$$

Finally applying inequality (C.7) once again gives us:

$$I_3 \leq \frac{C}{n^{(p-2)/2}}.$$

The above estimates of  $I_1$ ,  $I_2$  and  $I_3$  give us the following inequality, for all  $p > 2$

$$\mathbb{E} \sup_{0 \leq t \leq T} \varphi(X_t^{x,n}) \leq \frac{C}{n^{(p-2)/2}}. \quad (\text{C.9})$$

Now, we claim that for all  $1 \leq p < +\infty$ ,  $0 < 2q < p$ ,  $n, m \in \mathbb{N}$ ,

$$\mathbb{E} \left[ \left( m \int_0^T |\beta(X_s^{x,n}) \beta(X_s^{x,m})| ds \right)^p \right] \leq \frac{C}{n^q}. \quad (\text{C.10})$$

Indeed, we have

$$\begin{aligned} m \int_0^T |\beta(X_s^{x,n}) \beta(X_s^{x,m})| ds &\leq AB \\ &:= \left( \sup_{0 \leq s \leq T} |\beta(X_s^{x,n})| \right) \left( m \int_0^T |\beta(X_s^{x,m})| ds \right). \end{aligned}$$

Since, for  $r > 2$

$$E[(AB)^p] \leq (E(A^{rp}))^{1/r} (E(B^{r'p}))^{1/r'}, \quad r' = \frac{r}{r-1},$$

from (C.4) and (C.9), we get

$$\mathbb{E}((AB)^p) \leq \frac{C}{n^{(rp-2)/(2r)}} = \frac{C}{n^{p/2-1/r}},$$

which implies (C.10) for  $r$  large enough.

Now we will prove that if  $2 < 2q < p < \infty$ , there exists a constant  $C$  independent on  $n$  and  $m$  such that

$$\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_s^{x,n} - X_s^{x,m}|^p \right] \leq C \left( \frac{1}{n} + \frac{1}{m} \right)^q, \quad \forall n, m \in \mathbb{N}^*. \quad (\text{C.11})$$

Indeed, applying Itô's formula, for all  $0 \leq t \leq T < +\infty$ :

$$\begin{aligned} |X_t^{x,n} - X_t^{x,m}|^2 &= 2 \int_0^t (X_s^{x,n} - X_s^{x,m}) ((d+b)(X_s^{x,n}) - (d+b)(X_s^{x,m})) ds \\ &\quad - 4n \int_0^t (X_s^{x,n} - X_s^{x,m}) \beta(X_s^{x,n}) ds \\ &\quad + 4m \int_0^t (X_s^{x,n} - X_s^{x,m}) \beta(X_s^{x,m}) ds \\ &\quad + 2 \int_0^t (X_s^{x,n} - X_s^{x,m}) (\sigma(X_s^{x,n}) - \sigma(X_s^{x,m})) dW_s \\ &\quad + \int_0^t \sum_i [(\sigma(X_s^{x,n}) - \sigma(X_s^{x,m}))^t (\sigma(X_s^{x,n}) - \sigma(X_s^{x,m}))]_{i,i} ds. \end{aligned}$$

By hypothesis on  $d$ ,  $b$  and  $\sigma$  and thanks to equation (C.2) we obtain

$$\begin{aligned} |X_t^{x,n} - X_t^{x,m}|^2 &\leq C \int_0^t |X_s^{x,n} - X_s^{x,m}|^2 ds \\ &\quad + 4\mathbb{E}n \int_0^t \beta(X_s^{x,n}) \beta(X_s^{x,m}) ds + 4\mathbb{E}m \int_0^t \beta(X_s^{x,n}) \beta(X_s^{x,m}) ds \\ &\quad + 2 \int_0^t (X_s^{x,n} - X_s^{x,m}) (\sigma(X_s^{x,n}) - \sigma(X_s^{x,m})) dW_s. \end{aligned} \quad (\text{C.12})$$

Now, if  $I_t$  denotes the stochastic integral in (C.12), we have for  $r > 1$

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |I_s|^r \right] \leq CE \left[ \left( \int_0^t |X_s^{x,n} - X_s^{x,m}|^2 ds \right)^{r/2} \right].$$

Therefore, by virtue of property (C.10) and for a new constant  $C$ ,  $2r = p$ ,  $q < p/2$  and  $0 \leq t \leq T$ , we deduce from (C.12)

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} [|X_s^{x,n} - X_s^{x,m}|^p] &\leq C \int_0^t E [|X_s^{x,n} - X_s^{x,m}|^p] ds + C \left( \frac{1}{n^q} + \frac{1}{m^q} \right) \\ &\leq C \int_0^t E \sup_{0 \leq u \leq s} [|X_u^{x,n} - X_u^{x,m}|^p] ds + C \left( \frac{1}{n^q} + \frac{1}{m^q} \right), \end{aligned}$$

which implies (C.11) after using Gronwall's Lemma. inequality (C.11) shows us that the process  $(X^{x,n})_n$  is Cauchy in  $\mathcal{S}^p$ ,  $p > 2$ . Of course, taking the power  $1/r$  for  $r > 1$  in equation (C.11) shows us that  $(X^{x,n})_n$  is Cauchy in  $\mathcal{S}^p$ , for all  $p \geq 1$ . Therefore we can define in  $\mathcal{S}^p$ , for all  $p \geq 1$ :

$$X^x := \lim_{n \rightarrow +\infty} X^{x,n}.$$

Furthermore, defining

$$\eta_t^{x,n} = -2n \int_0^t \beta(X_s^{x,n}) ds,$$

and remarking that the process  $\eta^{x,n}$  satisfies the following equation

$$\eta_t^{x,n} = X_t^{x,n} - \int_0^t (d+b)(X_s^{x,n}) ds - \int_0^t \sigma(X_s^{x,n}) dW_s,$$

we can define in  $\mathcal{S}^p$ , for all  $p \geq 1$ :

$$\eta^x := \lim_{n \rightarrow +\infty} \eta^{x,n}.$$

Clearly, estimate (C.4) shows that  $\eta^x$  has locally bounded variation almost surely, and condition (C.9) implies that  $X^x$  belongs to  $\overline{G}$  almost surely. Now remark that property (C.1) implies that, for all  $T > 0$ , for all progressively measurable process  $z$  taking values in the closed set  $\overline{G}$

$$\int_0^T {}^t(X_s^{x,n} - z_s) d\eta_s^{x,n} = -2n \int_0^T {}^t(X_s^{x,n} - z_s) \beta(X_s^{x,n}) ds \leq 0. \quad (\text{C.13})$$

Now we would like to pass to the limit into the previous inequality. For that purpose let us recall the following deterministic Lemma.

**Lemma AppendixC.1.** *Let  $y_n$  be a sequence of functions in  $\mathcal{C}([0, T], \mathbb{R}^k)$  which converges uniformly to  $y$ . Let  $\eta_n$  be a sequence of functions in  $\mathcal{C}([0, T], \mathbb{R}^k)$  which converges uniformly to  $\eta$  and such that there exists  $C > 0$  such that  $\|\eta_n\|_{TV} \leq C$ . Then*

$$\|\eta\|_{TV} \leq C \quad \text{and} \quad \int_0^T y_n d\eta_n \xrightarrow{n \rightarrow +\infty} \int_0^T y d\eta.$$

*Proof.* See Lemma 5.7 in [6]. □

Now we claim that we can extract a subsequence of  $\left\{ \int_0^T {}^t(X_s^{x,n} - z_s) d\eta_s^{x,n} \right\}_n$  which converges  $\mathbb{P}$ -a.s. to  $\int_0^T {}^t(X_s^x - z_s) d\eta_s^x$ . First, by equation (C.5), we have

$$2\gamma \|\eta^{x,n}\|_{TV} ds \leq |x - c|^2 + 2 \int_0^t C ds + 2 \int_0^t (X_s^{x,n} - c, \sigma(X_s^{x,n}) dW_s).$$

Now remark that the right member of the inequality converges in probability to  $|x - c|^2 + 2 \int_0^t C ds + 2 \int_0^t (X_s^x - c, \sigma(X_s^x) dW_s)$ . Therefore, there exists a subsequence  $(\phi(n))_n$  such that the right member of the inequality converges  $\mathbb{P}$ -a.s. and then  $\|\eta^{x,\phi(n)}\|_{TV}$  is bounded  $\mathbb{P}$ -a.s. Furthermore, as  $\mathbb{E} \sup_{0 \leq t \leq T} |X_t^{x,n} - X_t^x|^2 \xrightarrow{n \rightarrow +\infty} 0$ , and  $\mathbb{E} \sup_{0 \leq t \leq T} |\eta_t^{x,n} - \eta_t^x|^2 \xrightarrow{n \rightarrow +\infty} 0$  we can extract a subsequence  $(\psi(n))_n$  of  $(\phi(n))_n$  such that  $X^{x,\psi(n)}$  converge uniformly to  $X^x$   $\mathbb{P}$ -a.s. and  $\eta^{x,\psi(n)}$  converge uniformly to  $\eta^x$   $\mathbb{P}$ -a.s. Now, it is enough to apply the previous deterministic Lemma  $\mathbb{P}$ -a.s. to obtain the result, namely, for all  $T > 0$ , for all progressively measurable process  $z$  taking values in the closed set  $\overline{G}$

$$\int_0^T {}^t(X_s^x - z_s) d\eta_s^x \leq 0.$$

By Lemma 2.1 in [6], this implies that

$$\int_0^T \mathbb{1}_{\{X_s^x \in G\}} d\eta_s^x = 0, \quad \mathbb{P}\text{-a.s.}$$

and

$$\eta_t^x = \int_0^t \nabla \phi(X_s^x) d\|\eta^x\|_{TV,s}.$$

We define  $dK_s := d\|\eta^x\|_{TV,s}$ . This shows that  $(X^x, K^x)$  satisfies the SDE (4.1).

Now we prove that the solution of the SDE (4.1) is unique. Let us assume that  $(X^1, K^1)$  and  $(X^2, K^2)$  are two solutions of (4.1) such that  $K^1$  and  $K^2$  have bounded variation on  $[0, T]$ , for all  $T > 0$  and such that for all  $i = 1, 2$ , for all continuous and progressively measurable process  $z$  taking values in the closure  $\overline{G}$  we have,

$$\int_0^T {}^t(X_s^i - z_s) d\eta_s^i \leq 0,$$

where  $\eta_t^i = \int_0^t \nabla \phi(X_s^i) dK_s^i$ . Applying Itô's formula, one has

$$\begin{aligned} |X_t^1 - X_t^2|^2 &= 2 \int_0^t (X_s^1 - X_s^2)(d(X_s^1) - d(X_s^2)) ds + 2 \int_0^t (X_s^1 - X_s^2)(b(X_s^1) - b(X_s^2)) ds \\ &\quad + 2 \int_0^t (X_s^1 - X_s^2)(d\eta_s^1 - d\eta_s^2) ds + 2 \int_0^t (X_s^1 - X_s^2)(\sigma(X_s^1) - \sigma(X_s^2)) dW_s \\ &\quad + \int_0^t \text{Tr}[(\sigma(X_s^1) - \sigma(X_s^2))^t(\sigma(X_s^1) - \sigma(X_s^2))] ds. \end{aligned}$$



Since,  $2 \int_0^t (X_s^1 - X_s^2)(d\eta_s^1 - d\eta_s^2)ds \leq 0$ , we easily get

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^1 - X_t^2|^2 \right] &\leq C \mathbb{E} \left[ \int_0^T |X_s^1 - X_s^2|^2 ds \right] \\ &\leq C \left[ \int_0^T \mathbb{E} \sup_{0 \leq u \leq s} |X_u^1 - X_u^2|^2 ds \right]. \end{aligned}$$

Applying Gronwall's lemma allows us to conclude. □

### Acknowledgements

The author would like to thank his Ph.D. adviser Ying Hu for his helpful remarks and comments during the preparation of this paper. The author also thanks the Lebesgue Center of Mathematics for its support (program "Investissements d'avenir" — ANR-11-LABX-0020-01) and the anonymous referee for the helpful comments.

- [1] G. Barles, F. Da Lio, P.-L. Lions, and P. E. Souganidis. Ergodic problems and periodic homogenization for fully nonlinear equations in half-space type domains with Neumann boundary conditions. *Indiana Univ. Math. J.*, 57(5):2355–2375, 2008.
- [2] Philippe Briand and Ying Hu. Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. *J. Funct. Anal.*, 155(2):455–494, 1998.
- [3] Arnaud Debussche, Ying Hu, and Gianmario Tessitore. Ergodic BSDEs under weak dissipative assumptions. *Stochastic Process. Appl.*, 121(3):407–426, 2011.
- [4] Marco Fuhrman, Ying Hu, and Gianmario Tessitore. Ergodic BSDEs and optimal ergodic control in Banach spaces. *SIAM J. Control Optim.*, 48(3):1542–1566, 2009.
- [5] Marco Fuhrman and Gianmario Tessitore. The Bismut-Elworthy formula for backward SDEs and applications to nonlinear Kolmogorov equations and control in infinite dimensional spaces. *Stoch. Stoch. Rep.*, 74(1-2):429–464, 2002.
- [6] A. Gégout-Petit and E. Pardoux. Équations différentielles stochastiques rétrogrades réfléchies dans un convexe. *Stochastics Stochastics Rep.*, 57(1-2):111–128, 1996.
- [7] Jin Ma and Jianfeng Zhang. Representation theorems for backward stochastic differential equations. *Ann. Appl. Probab.*, 12(4):1390–1418, 2002.
- [8] Xuerong Mao and Lukasz Szpruch. Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients. *J. Comput. Appl. Math.*, 238:14–28, 2013.
- [9] Jonathan C. Mattingly. Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics. *Comm. Math. Phys.*, 230(3):421–462, 2002.
- [10] E. J. McShane and R. B. Warfield, Jr. Addenda and corrigenda to “On Filippov’s implicit functions lemma”. *Proc. Amer. Math. Soc.*, 21:496–498, 1969.
- [11] José-Luis Menaldi. Stochastic variational inequality for reflected diffusion. *Indiana Univ. Math. J.*, 32(5):733–744, 1983.
- [12] Cyril Odasso. Exponential mixing for stochastic PDEs: the non-additive case. *Probab. Theory Related Fields*, 140(1-2):41–82, 2008.
- [13] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
- [14] Adrien Richou. Ergodic BSDEs and related PDEs with Neumann boundary conditions. *Stochastic Process. Appl.*, 119(9):2945–2969, 2009.
- [15] M. Tahmasebi and S. Zamani. Weak differentiability of solutions to sdes with semi-monotone drifts. arXiv:1309.0619, 2013.
- [16] Feng-Yu Wang. Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds. *Ann. Probab.*, 39(4):1449–1467, 2011.