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# Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions

Pierre-Yves MADEC \*

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## Abstract

We study a class of ergodic BSDEs related to PDEs with Neumann boundary conditions. The randomness of the driver is given by a forward process under weakly dissipative assumptions with an invertible and bounded diffusion matrix. Furthermore, this forward process is reflected in a convex subset of  $\mathbb{R}^d$  not necessarily bounded. We study the link of such EBSDEs with PDEs and we apply our results to an ergodic optimal control problem.

## 1 Introduction

In this paper we study the following ergodic backward stochastic differential equation (EBSDE in what follows) in finite dimension and in infinite horizon:  $\forall t, T \in \mathbb{R}_+, 0 \leq t \leq T < +\infty$ :

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds + \int_t^T [g(X_s^x) - \mu] dK_s^x - \int_t^T Z_s^x dW_s, \quad (1.1)$$

where the given data satisfy:

- $W$  is an  $\mathbb{R}^d$ -valued standard Brownian motion;
- $G = \{\phi > 0\}$  is an open convex subset of  $\mathbb{R}^d$  with smooth boundary;
- $x \in G$ ;
- $X^x$  is a  $\overline{G}$ -valued process starting from  $x$ , and  $K^x$  is a non decreasing real valued process starting from 0 such that the pair  $(X^x, K^x)$  is solution of the following reflected stochastic differential equation (SDE in what follows):

$$\begin{aligned} X_t^x &= x + \int_0^t f(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s + \int_0^t \nabla \phi(X_s^x) dK_s^x, \quad t \geq 0, \\ K_t^x &= \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}} dK_s^x, \quad K^x \text{ is non decreasing,} \end{aligned}$$

- $\psi : \mathbb{R}^d \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  is  $K_\psi$ -Lipschitz in  $x$  and  $z$  and  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable;
- $\lambda$  and  $\mu$  belong both to  $\mathbb{R}$ . If  $\lambda$  is given then  $\mu$  is unknown and if  $\mu$  is given then  $\lambda$  is unknown.

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Therefore, the unknown is either the triplet  $(Y^x, Z^x, \lambda)$  if  $\mu$  is given or the triplet  $(Y^x, Z^x, \mu)$  if  $\lambda$  is given, where:

- $Y^x$  is a real-valued progressively measurable process;
- $Z^x$  is an  $\mathbb{R}^{1 \times d}$ -valued progressively measurable process.

We recall that a function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to be strictly dissipative if there exists a constant  $\eta > 0$  such that,  $\forall x, y \in \mathbb{R}^d$ ,

$$(h(x) - h(y), x - y) \leq -\eta|x - y|^2.$$

Richou in the paper [11] studied the case when  $\overline{G}$  is bounded and with the assumptions that  $f$  and  $\sigma$  are Lipschitz and:

$$\sup_{x, y \in \overline{G}, x \neq y} \left\{ \frac{t(x - y)(b(x) - b(y))}{|x - y|^2} + \frac{\text{Tr}[(\sigma(x) - \sigma(y))^t(\sigma(x) - \sigma(y))]}{2|x - y|^2} \right\} < -K_{\psi, z}K_{\sigma}$$

where  $K_{\psi, z}$  is the Lipschitz constant of  $\psi$  in  $z$  and  $K_{\sigma}$  is the Lipschitz constant of  $\sigma$ . Note that this assumption implies that  $f$  is strictly dissipative. However this hypothesis on  $f$  is not very natural because it supposes a dependence between parameters of the problem. Thanks to this condition it is possible to establish one of the key results: the strong estimate on the exponential decay in time of two solutions of the forward equation starting from different points. Indeed, it is used to construct, by a diagonal procedure, a solution to the EBSDE. Note that, in this work,  $\overline{G}$  is assumed to be bounded.

In the paper [3], Debussche, Hu and Tessitore were concerned with the study of EBSDE in a weakly dissipative environment. This means that the driver of the forward process is assumed to be the sum of a strictly dissipative term and a perturbation term which is Lipschitz and bounded. In their infinite dimensional framework, they supposed that the dissipative term is linear. In addition,  $\sigma$  is constant, and the forward process is not reflected. Finally the coefficients of the forward process are assumed to be Gâteaux differentiable to obtain an estimate which is needed to prove the existence of a solution in this framework. In this context, the weaker assumption on  $f$  makes the strong estimate on the exponential decay in time of two solutions of the forward equation impossible. However it is possible to substitute this result by a weaker result, called "basic coupling estimate" which involves the Kolmogorov semigroups of the forward process  $X^x$  and which is enough to prove the existence of a solution to the EBSDE.

In this paper we extend the framework of [11] to the case of an unbounded domain  $\overline{G}$  for a driver weakly dissipative. Namely, we assume that  $f = d + b$  where  $d$  is locally Lipschitz and dissipative with polynomial growth and  $b$  is Lipschitz and bounded. The price to pay is that  $\sigma$  is assumed to be Lipschitz, invertible and such that  $\sigma$  and  $\sigma^{-1}$  are bounded. We do not need more regularity than continuous coefficients for this study, because we treat this problem by a regularization procedure. As the basic coupling estimate of [3] holds for a non reflected process, we start by studying the following forward process,  $\forall t \geq 0$ ,

$$V_t^x = x + \int_0^t f(V_s^x)ds + \int_0^t \sigma(V_s^x)dW_s,$$

with  $f$  and  $\sigma$  defined as before. We show that the coupling estimate still holds in our framework with constants which depend on  $d$  only through its dissipativity coefficient. Once this is established, we apply this result to establish existence and uniqueness (of  $\lambda$ ) of solutions to the following EBSDE:

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda]ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (1.2)$$

Then we want to obtain the same result when the process  $V^x$  is replaced by a reflected process  $X^x$  in  $\overline{G}$ , namely:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (1.3)$$

For this purpose, we use a penalization method to construct a sequence of processes  $X^{x,n}$  defined on the whole  $\mathbb{R}^d$  and which converges to the reflected process  $X^x$ . More precisely, we denote by  $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$  the solution of the following BSDE with regularized coefficients  $\psi^\varepsilon$ ,  $d^\varepsilon$ ,  $F_n^\varepsilon$  and  $b^\varepsilon$  by convolution with a sequence approximating the identity,  $\forall t, T \in \mathbb{R}_+$ ,  $0 \leq t \leq T < +\infty$ :

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}] ds - \int_t^T Z_s^{x,\alpha,n,\varepsilon} dW_s, \quad (1.4)$$

where  $X^{x,n,\varepsilon}$  is the strong solution of the SDE:

$$X_t^{x,n,\varepsilon} = x + \int_0^t (d^\varepsilon + F_n^\varepsilon + b^\varepsilon)(X_s^{x,n,\varepsilon}) ds + \int_0^t \sigma^\varepsilon(X_s^{x,n,\varepsilon}) dW_s.$$

Note that as  $F_n$  is dissipative with a dissipative constant equal to 0,  $d + F_n$  remains dissipative with a dissipative coefficient equal to  $\eta$ . Then, making  $\varepsilon \rightarrow 0$ ,  $n \rightarrow +\infty$  and  $\alpha \rightarrow 0$ , it is possible to show that, roughly speaking,  $(Y_t^{x,\alpha,n,\varepsilon} - Y_0^{x,\alpha,n,\varepsilon}, Z_t^{x,\alpha,n,\varepsilon}, Y_0^{x,\alpha,n,\varepsilon}) \rightarrow (Y_t^x, Z_t^x, \lambda)$  which is solution of EBSDE (1.3). Once a solution  $(Y, Z, \lambda)$  is found for the EBSDE (1.3) we study existence and uniqueness of solutions of the type  $(Y, Z, \lambda)$  and  $(Y, Z, \mu)$  of the EBSDE (1.1). Here we only manage to find solutions which are not Markovian and which are not bounded in expectation. Then we show that the function defined by  $v(x) := Y_0^x$ , where  $Y$  is a solution of EBSDE (1.3) is a viscosity solution of the following partial differential equation (PDE in what follows) :

$$\begin{cases} \mathcal{L}v(x) + \psi(x, {}^t\nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \quad (1.5)$$

where:

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(\sigma(x) {}^t\sigma(x) \nabla^2 u(x)) + {}^t f(x) \nabla u(x).$$

Note that the boundary ergodic problem

$$\begin{cases} F(D^2v, Dv, x) = \lambda & \text{in } \overline{G} \\ L(Dv, x) = \mu \end{cases}$$

were studied in [1] by Barles, Da Lio, Lions and Souganidis when  $\overline{G}$  is a smooth, periodic, half-space-type domain and  $F$  a periodic function. They found a constant  $\mu$  such that there exists a bounded viscosity solution  $v$  of the above problem.

At last we show that we can use the theory of EBSDE to solve an optimal ergodic control problem.  $R : U \rightarrow \mathbb{R}^d$  is assumed to be bounded and  $L$  is assumed to be Lipschitz and bounded. We define the ergodic cost:

$$I(x, \rho) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_T^\rho \left[ \int_0^T L(X_s^x, \rho_s) ds \right], \quad (1.6)$$

where  $\rho$  is an adapted process with values in a separable metric space  $U$  and  $\mathbb{E}_T^\rho$  is the expectation with respect to the probability measure under which  $W_t^\rho = W_t + \int_0^t R(\rho_s)ds$  is a Brownian motion on  $[0, T]$ . Defining

$$\psi(x, z) = \inf_{u \in U} \{L(x, u) + zR(u)\}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^{1 \times d},$$

it is possible to show that, for any admissible control  $\rho$ ,  $I(x, \rho) \geq \lambda$ . That is why  $\lambda$  is called ergodic cost and  $\mu$  is called boundary ergodic cost.

The paper is organized as follows. In section 2, we study the forward SDE under the hypothesis that the drift is weakly dissipative and that the diffusion matrix is invertible and bounded. In this section we prove that the estimates we establish depend on  $d$  through its dissipativity coefficient. In section 3, we use the basic coupling estimate to study existence and uniqueness of an EBSDE with zero Neumann boundary conditions with a forward process weakly dissipative but non-reflected. In section 4, we use a penalization method to show that the same result holds for a reflected process in a convex not necessarily bounded. In section 5, we establish the link between the EBSDE with zero Neumann boundary condition and a PDE. In section 6, we apply our results to an optimal ergodic control problem. Some technic proofs are given in the Appendix.

## 2 The forward SDE

### 2.1 General notation

The canonical scalar product on  $\mathbb{R}^d$  is denoted by  $(\cdot, \cdot)$  and the associated norm is denoted by  $|\cdot|$ . Let  $\mathcal{O}$  be an open connected subset of  $\mathbb{R}^d$ . We denote by  $\mathcal{C}_b^k(\mathcal{O})$  the set of real functions of class  $\mathcal{C}^k$  on  $\mathcal{O}$  with bounded partial derivatives. We denote by  $\mathcal{C}_{\text{lip}}^k$  the set of real functions whose partial derivatives of order less than or equal to  $k$  are Lipschitz. We denote by  $B_b(\mathcal{O})$  the set of Borel measurable bounded functions defined on  $\mathcal{O}$ .

$(\Omega, \mathcal{F}, \mathbb{P})$  denotes a complete probability space,  $(W_t)_{t \geq 0}$  denotes an  $\mathbb{R}^d$ -valued standard Brownian motion defined on this space and  $(\mathcal{F}_t)_{t \geq 0}$  is the natural filtration of  $W$  augmented by  $\mathbb{P}$ -null sets. Then  $(\mathcal{F}_t)_{t \geq 0}$  satisfies the usual condition.

$\mathcal{S}^2$  denotes the space of real-valued adapted continuous processes  $Y$  such that for all  $T > 0$ ,  $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < +\infty$ .

$\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^k)$  denotes the space consisting of all progressively measurable processes  $X$ , with value in  $\mathbb{R}^k$  such that, for all  $T > 0$ ,

$$\mathbb{E} \left[ \int_0^T |X_s|^2 ds \right] < +\infty.$$

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  be two locally Lipschitz functions. We denote by  $(V_t^x)_{t \geq 0}$  the strong solution of the following SDE:

$$V_t^x = x + \int_0^t f(V_s^x) ds + \int_0^t \sigma(V_s^x) dW_s. \quad (2.1)$$

**Lemma 2.1.** *Assume that  $\exists a \in \mathbb{R}^d$ ,  $\eta_1, \eta_2 > 0$  such that,  $\forall y \in \mathbb{R}^d$ ,*

$$(f(y), y - a) \leq -\eta_1 |y - a|^2 + \eta_2,$$

and that  $|\sigma|$  is bounded by  $\sigma_\infty$ , then there exists a solution  $(X_t^x)_t$  to (2.1) for which the explosion time is almost surely equal to infinity. Furthermore the following estimate holds  $\forall t \geq 0$ :

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-2\eta_1 t}),$$

where  $C$  is a constant which depends only on  $a, \eta_1, \eta_2$  and  $\sigma_\infty$ . Furthermore, for all  $p > 2$ , for all  $0 < \beta < p\eta_1$ , there exists  $C > 0$  which depends only on  $p, d, \eta_1, \eta_2, \sigma_\infty$  such that  $\forall t \geq 0$ :

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-\beta t}).$$

We also have the following inequality:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |V_t^x|^p \right] \leq C(1 + |x|^p),$$

where  $C$  depends on  $p, d, \eta_1, \eta_2, \sigma_\infty$  and  $T$ .

*Proof.* The proof is given in the appendix.  $\square$

We recall that a function is weakly dissipative if it is the sum of an  $\eta$ -dissipative function (namely  $\forall x, x' \in \mathbb{R}^d, (d(x) - d(x'), x - x') \leq -\eta|x - x'|^2$ ), and a bounded function. Thus we write  $f = d + b$ , with  $d$   $\eta$ -dissipative and  $|b|$  bounded by  $B$ .

### Hypothesis 2.1.

- $f = d + b$  is weakly dissipative,
- $d$  is locally Lipschitz with polynomial growth,
- $b$  is Lipschitz,
- $\sigma$  is Lipschitz, invertible, and  $|\sigma|$  and  $|\sigma^{-1}|$  are bounded by  $\sigma_\infty$ .

**Remark 2.2.** It is clear that if  $f$  satisfies Hypothesis 2.1 then  $f$  satisfies the assumption of Lemma 2.1. Indeed, let us suppose that  $f$  satisfies Hypothesis 2.1. Let  $a \in \mathbb{R}^d$ , then  $\forall y \in \mathbb{R}^d$ ,

$$\begin{aligned} (f(y) - f(a), y - a) &= (d(y) - d(a), y - a) + (b(y) - b(a), y - a) \\ \Rightarrow f(y)(y - a) &\leq -\eta|y - a|^2 + 2B|y - a| + |f(a)||y - a| \\ \Rightarrow f(y)(y - a) &\leq -\eta|y - a|^2 + \frac{(2B + |f(a)|)^2}{2\varepsilon} + \frac{\varepsilon|y - a|^2}{2}, \end{aligned}$$

which gives us the desired result, for  $\varepsilon$  small enough.

**Lemma 2.3.** Assume that Hypothesis (2.1) holds true but this time with  $b$  replaced by  $b_2$  which is only bounded measurable and not locally Lipschitz anymore. Then the solution of (2.1) still exists but in the weak sense, namely there exist a new Brownian motion  $(\hat{W}_t)_{t \geq 0}$  with respect to a new probability measure  $\hat{\mathbb{P}}$  under which equation (2.1) is satisfied by  $(X_t)_{t \geq 0}$  with  $(W_t)_{t \geq 0}$  replaced by  $(\hat{W}_t)_{t \geq 0}$ . Such a process is unique in law and the estimates of Lemma (2.1) are still satisfied under the new probability  $\hat{\mathbb{P}}$ .

*Proof.* Using the decomposition  $b_2 = b + b_2 - b$ , it is enough to apply a Girsanov theorem to get rid of the bounded term  $b_2 - b$ .  $\square$

**Lemma 2.4.** *Assume that Hypothesis 2.1 holds true. Then there exist  $C > 0$  and  $\mu > 0$  such that  $\forall \Phi \in B_b(\mathbb{R}^d)$ ,*

$$|\mathcal{P}_t[\Phi](x) - \mathcal{P}_t[\Phi](x')| \leq C(1 + |x|^2 + |x'|^2)e^{-\mu t}|\Phi|_0 \quad (2.2)$$

where  $\mathcal{P}_t[\Phi](x) = \mathbb{E}\Phi(V_t^x)$  is the Kolmogorov semigroup associated to (2.1). We stress the fact that the constants  $C$  and  $\mu$  depend on  $f$  only through  $\eta$  and  $B$ .

*Proof.* The proof is given in the appendix. □

**Corollary 2.5.** *The estimate (2.2) can be extended to the case in which  $b$  is only bounded measurable and there exists a uniformly bounded sequence of Lipschitz functions  $\{b_m\}_{m \geq 1}$  (i.e.  $b_m$  is Lipschitz and  $\sup_m \sup_x |b_m(x)| < +\infty$ ) such that*

$$\forall x \in \mathbb{R}^d, \quad \lim_m b_m(x) = b(x).$$

*In this case, we define the semigroup relatively to the new probability measure, namely:*

$$\mathcal{P}_t[\Phi](x) := \hat{\mathbb{E}}\Phi(X_t^x).$$

*Proof.* We denote by  $\mathcal{P}_t^m$  the Kolmogorov semigroup of (4.2) with  $b$  replaced by  $b_m$ , for more clarity we rewrite this equation below:  $\forall x \in \bar{G}$ ,

$$V_t^{x,m} = x + \int_0^t (d + b_m)(V_s^{x,m})ds + \int_0^t \sigma(V_s^{x,m})dW_s.$$

It is sufficient to prove that,  $\forall x \in \bar{G}$ ,  $\forall t \geq 0$ ,

$$\mathcal{P}_t^m[\Phi](x) \rightarrow \mathcal{P}_t[\Phi](x).$$

To do that, it is easy to adapt the proof from [3] replacing the process  $U_t^x$  by its analogue in our context. Thus we define  $U_t^x$  as the strong solution of the following SDE:

$$U_t^x = x + \int_0^t d(U_s^x)ds + \int_0^t \sigma(U_s^x)dW_s,$$

and the rest remains the same. □

### 3 The ergodic BSDE

In this section we study the following EBSDE in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda]ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (3.1)$$

At the moment, the forward process, defined as the strong solution of (2.1) is not reflected. However this result is interesting for its own, because it gives some ideas which will be reused in the next section.

We need the following hypothesis on  $\psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  :

**Hypothesis 3.1.**  $\psi$  is Lipschitz and there exists  $M_\psi \in \mathbb{R}$  such that:  $\forall x, x' \in \mathbb{R}^d, \forall z, z' \in \mathbb{R}^{1 \times d}$ ,

$$\begin{aligned} |\psi(x, 0)| &\leq M_\psi, \\ |\psi(x, z) - \psi(x', z)| &\leq M_\psi |x - x'|, \\ |\psi(x, z) - \psi(x, z')| &\leq M_\psi |z - z'|. \end{aligned}$$

**Hypothesis 3.2.**  $f$  is  $\mathcal{C}^1$  and  $b, \sigma$  and  $\psi \in \mathcal{C}_b^1$ .

Using the standard approach (see [4]), we are going to study the following BSDE in infinite horizon

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [\psi(V_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}] ds - \int_t^T Z_s^{x,\alpha} dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (3.2)$$

Such an equation were studied in [2] from which we have the following result:

**Lemma 3.1.** *Assume that hypotheses (2.1) and (3.1) hold true. Then there exists a unique solution  $(Y^{x,\alpha}, Z^{x,\alpha})$  to BSDE (3.2) such that  $Y^{x,\alpha}$  is a bounded adapted continuous process and  $Z^{x,\alpha} \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Furthermore,  $|Y_t^{x,\alpha}| \leq \frac{M_\psi}{\alpha}$ . Finally there exists a function  $v^\alpha$  such that  $Y_t^{x,\alpha} = v^\alpha(X_t^x)$   $\mathbb{P}$ -a.s. and there exists a measurable function  $\zeta^\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$  such that  $Z_t^{x,\alpha} = \zeta^\alpha(X_t^x)$   $\mathbb{P}$ -a.s.*

We will need the following lemma :

**Lemma 3.2.** *Let  $\zeta, \zeta'$  be two continuous functions:  $\mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$ . We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x, \zeta(x)) - \psi(x, \zeta'(x))}{|\zeta(x) - \zeta'(x)|^2} t(\zeta(x) - \zeta'(x)), & \text{if } \zeta(x) \neq \zeta'(x), \\ 0, & \text{if } \zeta(x) = \zeta'(x). \end{cases}$$

*There exists a uniformly bounded sequence of Lipschitz functions  $(\Upsilon_n)_{n \geq 0}$  (i.e.,  $\forall n, \Upsilon_n$  is Lipschitz and  $\sup_n \sup_x |\Upsilon_n(x)| < +\infty$ ) such that  $\Upsilon_n$  converges pointwisely to  $\Upsilon$ .*

*Proof.* Let us consider the following sequence of functions,  $\forall i \in \mathbb{N}$ :

$$\Upsilon^i(x) := \frac{\psi(x, \zeta(x)) - \psi(x, \zeta'(x))}{|\zeta(x) - \zeta'(x)|^2 + i^{-1}} t(\zeta(x) - \zeta'(x)).$$

Then  $\Upsilon^i$  is continuous bounded and converges pointwisely to  $\Upsilon$ . Then, by convolution arguments, it is classical to construct a sequence of Lipschitz functions  $\Upsilon^{i,\varepsilon}$  such that  $\Upsilon^{i,\varepsilon}$  is Lipschitz and uniformly bounded and  $\forall x \in \mathbb{R}^d$ ,

$$\lim_{i \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \Upsilon^{i,\varepsilon}(x) = \Upsilon(x).$$

Then the proof ends with a diagonal procedure. □

The following lemma gives us the desired estimates on  $v^\alpha(x)$  which will allow us to apply a diagonal procedure.

**Lemma 3.3.** *Assume that the hypotheses (2.1), (3.1) and (3.2) hold true. Then, there exists a constant  $C > 0$  independent of  $\alpha$  and which depends on  $f$  only through  $\eta$  and  $B$ , on  $\sigma$  only through  $\sigma_\infty$  and on  $\psi$  only through  $M_\psi$  such that,  $\forall x, x' \in \mathbb{R}^d$ ,*

$$|v^\alpha(x) - v^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2).$$



Furthermore,  $v^\alpha$  is differentiable and there exists a constant  $C > 0$  independent of  $\alpha$  and which depends on  $f$  only through  $\eta$  and  $B$ , on  $\sigma$  only through  $\sigma_\infty$  and on  $\psi$  only through  $M_\psi$  such that  $\forall x \in \mathbb{R}^d$ ,

$$|\nabla v^\alpha(x)| \leq C(1 + |x|^2).$$

*Proof.* We mix arguments from [3], [7] and [12]. First we set  $\bar{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0)$ . Then  $\bar{Y}_t^{x,\alpha} = Y_t^{x,\alpha} - Y_0^{0,\alpha} = \bar{v}^\alpha(V_t^x)$  is the unique solution of the finite horizon BSDE:

$$\begin{cases} -d\bar{Y}_1^{x,\alpha} &= \psi(V_t^x, Z_t^{x,\alpha})dt - \alpha\bar{Y}_t^{x,\alpha}dt - \alpha v^\alpha(0)dt - Z_t^{x,\alpha}dW_t, \\ \bar{Y}_1^{x,\alpha} &= \bar{v}^\alpha(V_1^x). \end{cases}$$

The representation Theorem 4.2 from [7] still holds under the hypotheses of this Lemma. Indeed the paper [12] guarantees the weak differentiability of the process  $V_t^x$ . However Theorem 4.2 in [7] asks for  $\bar{v}^\alpha$  to be Lipschitz. But applying the same regularization method as in the proof of Theorem 4.2 in [5] one can approximate  $v^\alpha$  by a sequence of functions  $v^{\alpha,n}$  which is Lipschitz and converges pointwisely to  $v^\alpha$ . This shows that  $v^\alpha$  is continuously differentiable and that Theorem 4.2 in [7] still holds under our assumptions.

As a consequence, we have the following identification  $Z_0^{x,\alpha} = \zeta(x) = {}^t v^\alpha(x)\sigma(x)$  which shows that  $\zeta$  is a continuous function. Hence we can show, as in [3] that  $\forall x, x' \in \mathbb{R}^d$ ,

$$|v^\alpha(x) - v^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2).$$

The second assertion needs to be clarified, because the dissipative term  $d$  is not linear anymore. From Theorem 4.2 of [7], it follows that:

$$\nabla v^\alpha(x) = \mathbb{E} \left\{ \bar{v}^\alpha(V_1^x)N_1^0 + \int_0^1 \left[ \psi(V_r^x, Z_r^{x,\alpha}) - \alpha\bar{Y}_r^{x,\alpha} - \alpha v^\alpha(0) \right] N_r^0 dr \right\}$$

where  $N_r^0 = \frac{1}{r} {}^t [\int_0^r {}^t [\sigma^{-1}(V_s^x)\nabla V_s^x] dW_s] [\nabla V_0^x]^{-1}$ . We recall that  $[\nabla V_0^x]^{-1} = I$ , the identity matrix in  $\mathbb{R}^{d \times d}$ . We have

$$\mathbb{E}|\bar{v}^\alpha(V_1^x)N_1^0| \leq \sqrt{\mathbb{E}(|\bar{v}^\alpha(V_1^x)|^2)} \times \sqrt{\mathbb{E}(|N_1^0|^2)}.$$

We have, from Lemma (2.1):

$$\sqrt{\mathbb{E}(|\bar{v}^\alpha(V_1^x)|^2)} \leq C(1 + |x|^2).$$

Furthermore, using BDG's inequality it is clear that

$$\sqrt{\mathbb{E}(|N_1^0|^2)} \leq C\mathbb{E} \sup_{0 \leq s \leq 1} |\nabla V_s^x|^2.$$

Now we need an estimate on  $|\nabla V_t^x|^2$ . As  $\nabla V_t^x$  is the solution of the following variational equation (see equation (2.9) in [7]):

$$\nabla_i V_t^x = e_i + \int_0^t \nabla(d+b)(V_s^x)\nabla_i V_s^x ds + \sum_{j=1}^d \int_0^t [\partial_x \sigma^j(V_s^x)]\nabla_i V_s^x dW_s^j$$

Itô's formula gives us:

$$\begin{aligned}
|\nabla_i V_t^x|^2 &= 1 + 2 \int_0^t \nabla_i V_s^x \cdot d\nabla_i V_t^x + \sum_{j=1}^d \int_0^t |[\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x|^2 ds \\
&= 1 + 2 \int_0^t \nabla_i V_s^x \cdot [\nabla(d+b)(V_s^x) \nabla_i V_s^x] ds + \sum_{j=1}^d \int_0^t |[\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x|^2 ds \\
&\quad + 2 \sum_{j=1}^d \int_0^t \nabla_i V_s^x \cdot [[\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x] dW_s^j
\end{aligned}$$

Due to the fact that  $d$  is dissipative and differentiable, we have that  $\forall \xi \in \mathbb{R}^d$ ,  ${}^t\xi \nabla d(x) \xi \leq -\eta |\xi|^2$ . Furthermore using the fact that  $b$  and  $\sigma$  belong to  $\mathcal{C}_b^1$ , one can verify by classic calculus that, for all  $t \leq 1$ :

$$\mathbb{E} |\nabla_i V_t^x|^2 \leq C,$$

for a constant  $C > 0$  which depends on  $d$  only through  $\eta$ . Then, by BDG's inequalities we can state that

$$\mathbb{E} \sup_{0 \leq s \leq 1} |\nabla_i V_s^x|^2 \leq C.$$

Finally,

$$\mathbb{E} \sup_{0 \leq s \leq 1} |\nabla V_s^x|^2 \leq C.$$

We have proved that:

$$\mathbb{E} |\bar{v}^\alpha(V_1^x) N_1^0| \leq C(1 + |x|^2).$$

Now we deal with the second term in the representation formula of  $\nabla v^\alpha$ . One can notice that the generator is  $M_\psi$ -Lipschitz in  $z$  and  $\alpha$ -Lipschitz with  $\alpha < 1$  in  $y$  and that  $\alpha |v^\alpha(0)| \leq M_\psi$ , we can apply the estimate of Lemma (2.2) from [7] to get the desired estimates on  $|\bar{Y}_t^{x,\alpha}|^2$  and  $|Z_t^{x,\alpha}|^2$ . Furthermore:

$$\begin{aligned}
\frac{1}{r} \mathbb{E} \left( \int_0^r [{}^t(\sigma^{-1}(V_s^x) \nabla V_s^x)] dW_s \right)^2 &\leq C \frac{1}{r} \mathbb{E} \sup_{0 \leq r \leq 1} \int_0^r |\sigma^{-1}(V_s^x) \nabla V_s^x|^2 ds \\
&\leq C \frac{1}{r} r \mathbb{E} \sup_{0 \leq r \leq 1} \sup_{0 \leq s \leq r} |\sigma^{-1}(V_s^x) \nabla V_s^x|^2 \\
&\leq C \mathbb{E} \sup_{0 \leq s \leq 1} |\nabla V_s^x|^2 \\
&\leq C.
\end{aligned}$$

Finally we have proved that

$$|\nabla v^\alpha(x)| \leq C(1 + |x|^2).$$

□

Thanks to this estimate, it is possible to get an existence result for EBSDE (3.1). Here Hypothesis 3.2 can be removed thanks to convolution arguments which will appear in the proof.

**Theorem 3.4.** *Assume that the hypotheses (2.1) and (3.1) hold true. Then there exists a solution  $(\bar{Y}^x, \bar{Z}^x, \bar{\lambda})$  to EBSDE (3.1) such that  $\bar{Y}^x = \bar{v}(V^x)$  with  $\bar{v}$  locally Lipschitz, and there exists a measurable function  $\bar{\xi} : \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$  such that  $\bar{Z}^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  and  $\bar{Z}^x = \bar{\xi}(V^x)$ .*

*Proof.* We start by regularizing  $f$  and  $\psi$  thanks to classical convolution arguments. For all  $k \in \mathbb{N}^*$  let us denote by  $\rho_\varepsilon^k : \mathbb{R}^k \rightarrow \mathbb{R}_+$  the classical mollifier for which the support is the ball of center 0 and radius  $\varepsilon$ . Let us denote  $\forall n \in \mathbb{N}$ ,  $d^{\varepsilon_n} := d * \rho_{\varepsilon_n}^d$ ,  $b^{\varepsilon_n} := b * \rho_{\varepsilon_n}^d$ ,  $\psi^{\varepsilon_n} := \psi * \rho_{\varepsilon_n}^{d,d}$  and  $\sigma^{\varepsilon_n} := \sigma * \rho_{\varepsilon_n}^{d \times d}$ . Those functions are  $\mathcal{C}^1$  and satisfies:

- $d^{\varepsilon_n}$  is  $\eta$ -dissipative;
- $|d^{\varepsilon_n}(x)| \leq C(1 + |x|)^p$ , for a  $p \geq 0$ ;
- $b^{\varepsilon_n}$  is bounded by  $B$ ;
- $\psi^{\varepsilon_n}$  satisfies Hypothesis 3.1;
- $\sigma^{\varepsilon_n}$  is invertible;
- $d^{\varepsilon_n} \rightarrow d$ ,  $b^{\varepsilon_n} \rightarrow b$ ,  $\psi^{\varepsilon_n} \rightarrow \psi$ ,  $\sigma^{\varepsilon_n} \rightarrow \sigma$  pointwisely as  $\varepsilon_n \rightarrow 0$ .

We just precise that the pointwise convergence of the regularized functions is a consequence of the continuity of the functions  $d$ ,  $b$ ,  $\psi$  and  $\sigma$ . Let us justify why we need polynomial growth on  $d$ ,  $\forall \varepsilon > 0$ ,

$$\begin{aligned} |d^\varepsilon(x)| &\leq \int_{\mathbb{R}^d} |d(x-y)| \rho_\varepsilon^d(y) dy \\ &\leq \int_{|y| \leq \varepsilon} C(1 + |x-y|^p) \rho_\varepsilon^d(y) dy \\ &\leq C \sup_{|y| \leq \varepsilon} (1 + |x|^p + |y|^p) \end{aligned}$$

and we can assume that  $\varepsilon \leq 1$ , so that we obtain:

$$|d^{\varepsilon_n}(x)| \leq C(1 + |x|)^p. \quad (3.3)$$

We denote by  $V_t^{x,\varepsilon_n}$  the solution of (2.1) with  $f$  replaced by  $f^{\varepsilon_n}$  and  $\sigma$  replaced by  $\sigma^{\varepsilon_n}$ . The same notation is used for the regularized BSDE, we denote by  $(Y_t^{x,\alpha,\varepsilon_n}, Z_t^{x,\alpha,\varepsilon_n})$  the solution in  $\mathcal{S}^2 \times \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  of BSDE (3.2) with  $\psi$  replaced by  $\psi^{\varepsilon_n}$  (existence and uniqueness of such a solution is guaranteed by Lemma 3.1), namely  $\forall 0 \leq t \leq T < +\infty$ :

$$Y_t^{x,\alpha,\varepsilon_n} = Y_T^{x,\alpha,\varepsilon_n} + \int_t^T (\psi(V_s^{x,\varepsilon_n}, Z_s^{x,\alpha,\varepsilon_n}) - \alpha Y_s^{x,\alpha,\varepsilon_n}) ds - \int_t^T Z_s^{x,\alpha,\varepsilon_n} dW_s. \quad (3.4)$$

Then we define  $v^{\alpha,\varepsilon_n}(x) := Y_0^{x,\alpha,\varepsilon_n}$  and  $\bar{Y}_t^{x,\alpha,\varepsilon_n} = Y_t^{x,\alpha,\varepsilon_n} - \alpha v^{\alpha,\varepsilon_n}(0)$ . We can rewrite the BSDE and we get:

$$\begin{aligned} \bar{Y}_t^{x,\alpha,\varepsilon_n} &= \bar{Y}_T^{x,\alpha,\varepsilon_n} + \int_t^T (\psi(V_s^{x,\varepsilon_n}, Z_s^{x,\alpha,\varepsilon_n}) - \alpha \bar{Y}_s^{x,\alpha,\varepsilon_n} - \alpha v^{\alpha,\varepsilon_n}(0)) ds \\ &\quad - \int_t^T Z_s^{x,\alpha,\varepsilon_n} dW_s, \quad 0 \leq t \leq T < +\infty. \end{aligned}$$

Uniqueness of solutions implies that  $v^{\alpha, \varepsilon_n}(V_s^{x, \varepsilon_n}) = Y_s^{x, \alpha, \varepsilon_n}$ . Now, in a very classical way, we set  $\bar{v}^{\alpha, \varepsilon_n}(x) = v^{\alpha, \varepsilon_n}(x) - v^{\alpha, \varepsilon_n}(0)$ . Thanks to the fact that  $\alpha|v^{\alpha, \varepsilon_n}(0)| \leq M_\psi$  and by Lemma 3.3 we can extract a subsequence  $\beta(\varepsilon_n) \xrightarrow{n \rightarrow +\infty} 0$  such that  $\forall \alpha > 0, \forall x \in D$  a countable subset of  $\mathbb{R}^d$ :

$$\bar{v}^{\alpha, \beta(\varepsilon_n)}(x) \xrightarrow{n \rightarrow +\infty} \bar{v}^\alpha(x) \quad \text{and} \quad \alpha v^{\alpha, \beta(\varepsilon_n)}(0) \xrightarrow{n \rightarrow +\infty} \bar{\lambda}^\alpha,$$

for a suitable function  $\bar{v}$  and a suitable real  $\bar{\lambda}^\alpha$ . Now thanks to the gradient estimates from Lemma 3.3 we have  $\forall \alpha > 0, |\bar{v}^{\alpha, \beta(\varepsilon_n)}(x) - \bar{v}^{\alpha, \beta(\varepsilon_n)}(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|$  for all  $x, x' \in \mathbb{R}^d$ . Therefore extending  $\bar{v}^\alpha$  to the whole  $\mathbb{R}^d$  by setting  $\bar{v}^\alpha(x) = \lim_{x_p \rightarrow x} \bar{v}^\alpha(x_p)$  we still have the following estimates: for all  $x, x' \in \mathbb{R}^d$ ,

$$|\bar{v}^\alpha(x) - \bar{v}^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.$$

In addition, we also have:

$$|\bar{\lambda}^\alpha| \leq M_\psi.$$

Now let us define  $\forall t \geq 0, \bar{Y}_t^{x, \alpha} = \bar{v}^\alpha(V_t^x)$ . Let us show that

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x, \alpha, \beta(\varepsilon_n)} - \bar{Y}_s^{x, \alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x, \alpha, \beta(\varepsilon_n)} - \bar{Y}_T^{x, \alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

First we write:

$$\begin{aligned} |\bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^{x, \beta(\varepsilon_n)}) - \bar{v}^\alpha(V_s^x)| &\leq |\bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^{x, \beta(\varepsilon_n)}) - \bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^x)| \\ &\quad + |\bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^x) - \bar{v}^\alpha(V_s^x)| \\ &\leq C(1 + |V_s^{x, \beta(\varepsilon_n)}|^2 + |V_s^x|^2) |V_s^{x, \beta(\varepsilon_n)} - V_s^x| \\ &\quad + |\bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^x) - \bar{v}^\alpha(V_s^x)| \end{aligned}$$

which shows the convergence of  $\bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^{x, \beta(\varepsilon_n)})$  toward  $\bar{v}^\alpha(V_s^x)$  almost surely (it is well known that  $\forall T > 0, \mathbb{E} \sup_{0 \leq t \leq T} |V_t^{\beta(\varepsilon_n)} - V_t^x|^2 \xrightarrow{n \rightarrow +\infty} 0$ , thanks to, in particular, estimate (3.3)).

Then, due to the fact that  $|\bar{v}^{\alpha, \beta(\varepsilon_n)}(V_s^{x, \beta(\varepsilon_n)})| \leq M_\psi/\alpha$   $\mathbb{P} - a.s.$ , we can apply the dominated convergence theorem to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x, \alpha, \beta(\varepsilon_n)} - \bar{Y}_s^{x, \alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x, \alpha, \beta(\varepsilon_n)} - \bar{Y}_T^{x, \alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Now we show that  $(Z^{x, \alpha, \beta(\varepsilon_n)})_n$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . We denote

$$\tilde{V}_t = V_t^{x, \beta(\varepsilon_n)} - V_t^{x, \beta(\varepsilon_n)'};$$

$$\tilde{Y}_t = \bar{Y}_t^{x, \alpha, \beta(\varepsilon_n)} - \bar{Y}_t^{x, \alpha, \beta(\varepsilon_n)'};$$

$$\tilde{Z}_t = \bar{Z}_t^{x, \alpha, \beta(\varepsilon_n)} - \bar{Z}_t^{x, \alpha, \beta(\varepsilon_n)'};$$

and

$$\tilde{\lambda} = \alpha v^{\alpha, \beta(\varepsilon_n)}(0) - \alpha v^{\alpha, \beta(\varepsilon_n)'}(0).$$

Itô's formula applied to  $|\tilde{Y}_t|^2$  gives us, for all  $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ :

$$\begin{aligned} |\tilde{Y}_t|^2 + \int_t^T |\tilde{Z}_s|^2 ds &= |\tilde{Y}_T|^2 + 2 \int_t^T \tilde{Y}_t [\psi(V_s^{x,\beta(\varepsilon_n)}, Z_s^{x,\alpha,\beta(\varepsilon_n)}) - \psi(V_s^{x,\beta(\varepsilon_n)'}, Z_s^{x,\alpha,\beta(\varepsilon_n)'}) \\ &\quad - (\alpha \bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \alpha \bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)'}) - \tilde{\lambda}] ds \\ &\quad - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s \\ &\leq |\tilde{Y}_T|^2 + (\varepsilon_1 M_\psi + \varepsilon_2 M_\psi + \varepsilon_3) \int_t^T |\tilde{Y}_s|^2 ds + \frac{M_\psi}{\varepsilon_1} \int_t^T |\tilde{V}_s|^2 ds \\ &\quad + \frac{M_\psi}{\varepsilon_2} \int_t^T |\tilde{Z}_s|^2 ds + \frac{1}{\varepsilon_3} \int_t^T |\tilde{\lambda}|^2 ds + c \int_t^T |\tilde{Y}_s| ds - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s, \end{aligned}$$

because  $\alpha|v^{\alpha,\varepsilon}(0)| \leq M_\psi$ . Thus, taking the expectation and for  $\varepsilon_2$  large enough we get

$$\mathbb{E} \int_0^T |\tilde{Z}_s|^2 ds \leq \mathbb{E} |\tilde{Y}_T|^2 + c \left( \mathbb{E} \left[ \int_0^T |\tilde{Y}_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T |\tilde{V}_s|^2 ds \right] + \mathbb{E} \left[ \int_0^T |\tilde{Y}_s| ds \right] + T |\tilde{\lambda}|^2 \right),$$

which proves that  $(Z_s^{x,\alpha,\beta(\varepsilon_n)})_{\beta(\varepsilon_n)}$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Now we pass to the limit in equation (3.4) to obtain:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(V_s^x, Z_s^{x,\alpha}) - \bar{\lambda}^\alpha) ds - \int_t^T Z_s^{x,\alpha} dW_s, \quad 0 \leq t \leq T < +\infty.$$

Now we reiterate the above method. Thanks to the following estimates:  $\forall x, x' \in \mathbb{R}^d$ ,

$$|\bar{v}^\alpha(x) - \bar{v}^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|,$$

and

$$|\bar{\lambda}^\alpha| \leq M_\psi,$$

it is possible, by a diagonal procedure, to construct a sequence  $(\alpha_n)_n$  such that

$$\begin{aligned} \bar{v}^{\alpha_n}(x) &\xrightarrow{n \rightarrow +\infty} \bar{v}(x) \\ \bar{\lambda}^{\alpha_n} &\xrightarrow{n \rightarrow +\infty} \bar{\lambda}. \end{aligned}$$

We define  $\bar{Y}_t^x := \bar{v}(V_t^x)$ . Let us just precise why

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha} - \bar{Y}_s^x|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha} - \bar{Y}_T^x|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

First the convergence of  $\bar{v}^{\alpha_n}(V_s^x)$  toward  $\bar{v}(V_s^x)$  is clear. Secondly, we have

$$|v^{\alpha_n}(V_s^x)| \leq C(1 + |V_s^x|^2).$$

Therefore the dominated convergence theorem can be applied to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_s^{x,\alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \bar{Y}_T^{x,\alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Then, just as before, it is possible to show that  $(Z^{x,\alpha})_{\alpha_n}$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . We denote its limit by  $\bar{Z}_s^x$ .

The end of the proof is very classical, it suffices to apply BDG's inequality to show that  $\mathbb{E} \sup_{0 \leq t \leq T} |\bar{Y}^x|^2 < +\infty$ ,  $\forall T > 0$ . To show that  $\bar{Z}^x$  is Markovian, just apply the same method as in the proof of Theorem 4.4 in [4]. □

**Remark 3.5.** It is clear that we do not have uniqueness of the solutions of EBSDE (3.1) because if  $(Y, Z, \lambda)$  is a solution then  $(Y + \theta, Z, \lambda)$  is another solution, for all  $\theta \in \mathbb{R}$ . However we have the following uniqueness property for  $\lambda$  under the following polynomial growth property:

$$|Y_t^x| \leq C(1 + |V_t^x|^2).$$

One can notice that the solution  $\bar{Y}_t^x = \bar{v}(X_t^x)$  constructed in the proof of Theorem (3.4) satisfies such a growth property.

**Theorem 3.6.** (*Uniqueness of  $\lambda$* ). Assume that the hypotheses (2.1) and (3.1) hold true. Let us suppose that we have two solutions of EBSDE (4.3) denoted by  $(Y, Z, \lambda)$  and  $(Y', Z', \lambda')$  where  $Y$  and  $Y'$  are progressively measurable continuous processes,  $Z$  and  $Z' \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  and  $\lambda, \lambda' \in \mathbb{R}$ . Finally assume that the following growth properties hold:

$$\begin{aligned} |Y_t| &\leq C(1 + |V_t^x|^2) \\ |Y'_t| &\leq C'(1 + |V_t^x|^2). \end{aligned}$$

Then  $\lambda = \lambda'$ .

*Proof.* It suffices to adapt the proof of Theorem 4.6 of [4]. With the same notations one can write:

$$\begin{aligned} \tilde{\lambda} &= T^{-1} \mathbb{E}^{\mathbb{P}^h} [\tilde{Y}_T - \tilde{Y}_0] \\ &\leq T^{-1} \mathbb{E}^{\mathbb{P}^h} ((C + C')(1 + |V_t^x|^2)) + T^{-1} \mathbb{E}^{\mathbb{P}^h} ((C + C')(1 + |x|^2)). \end{aligned}$$

To conclude, just use the estimates from Lemma 2.1, and let  $T \rightarrow +\infty$ . □

## 4 The ergodic BSDE with zero and non-zero Neumann boundary conditions in a weakly dissipative environment

In this section we replace the process  $(V_t^x)_{t \geq 0}$  by the process  $(X_t^x)_{t \geq 0}$ , which is solution of a stochastic differential equation reflected in an open convex subset  $G$  of  $\mathbb{R}^d$  with regular boundary. The randomness of EBSDE's generator that we are going to consider will be fully given by this process. We denote by  $\Pi(x)$  the projection of  $x \in \mathbb{R}^d$  on  $\bar{G}$ . We recall the following property:

$$(x' - x) \cdot (x - \Pi(x)) \leq (x' - \Pi(x')) \cdot (x - \Pi(x)).$$

We need the following assumptions on  $G$ :

**Hypothesis 4.1.**  $G$  is an open convex set of  $\mathbb{R}^d$ .

**Hypothesis 4.2.** There exists a function  $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$  such that  $G = \{\phi > 0\}$ ,  $\partial G = \{\phi = 0\}$  and  $|\nabla \phi(x)| = 1$ ,  $\forall x \in \partial G$ .

Then it is possible to establish the following result, by a penalization method.

**Lemma 4.1.** *Assume that the hypotheses (2.1), (4.1) and (4.2) hold true. Then for every  $x \in \overline{G}$ , there exists a unique adapted continuous couple of processes  $\{(X_t^x, K_t^x)_{t \geq 0}\}$  with values in  $(\overline{G} \times \mathbb{R}_+)$  such that  $\forall t \geq 0$  :*

$$\begin{aligned} X_t^x &= x + \int_0^t f(X_s^x) ds + \int_0^t \sigma(X_s^x) ds + \int_0^t \nabla \phi(X_s^x) dK_s^x, \\ K_t^x &= \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}} dK_s^x. \end{aligned} \quad (4.1)$$

*Proof.* The proof is a consequence of Lemma 4.2.  $\square$

We are going to penalize this reflected SDE in order to avoid dealing with the reflection term. By this way, we will be able to apply some of the results from the previous section. Then we consider the process  $(X_t^{x,n})_{t \geq 0}$  solution of the following penalized equation,  $\forall x \in \overline{G}$ ,

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n}) ds + \int_0^t \sigma(X_s^{x,n}) dW_s, \quad (4.2)$$

where  $\forall x \in \mathbb{R}^d$ ,  $F_n(x) = -2n(x - \Pi(x))$ .

The functions  $d + F_n + b$  and  $\sigma$  satisfy hypothesis 2.1. Indeed, from [6],  $F_n$  is 0-dissipative therefore  $b + F_n$  remains  $\eta$ -dissipative thus the estimate of Lemma 2.4 holds with constants which do not depend on  $n$ . Furthermore one can remark that for all  $\xi \in \mathbb{R}^d$ ,  ${}^t \xi \nabla F_n(x) \xi \leq 0$ , for all  $x \in \mathbb{R}^d$  (see for example [6]). Finally, taking  $a \in \overline{G}$  (thus  $F_n(a) = 0$ ) in Remark 2.2 shows us that the estimate of Lemma 2.1 holds with constants that do not depend on  $n$ .

We will need the following Lemma, proved in Appendix. It is not necessary to ask for  $f$  to be Lipschitz but only locally Lipschitz, due to the fact that we work with a weakly dissipative drift.

**Lemma 4.2.** *Assume that the hypotheses (2.1) and (4.1) hold true. Let  $(X_t^{x,n})_{t \geq 0}$  be the strong solution of equation (4.2). Then, for any  $1 < q < p/2$ , for any  $T \geq 0$ :*

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_s^{x,n} - X_s^{x,m}|^p \leq C(T)(1 + |x|^p) \left( \frac{1}{n^q} + \frac{1}{m^q} \right),$$

where  $C(T)$  is a constant which depends on  $T$ ,  $\eta$ ,  $B$  and  $\sigma_\infty$ .

*Proof.* The proof is given in the appendix.  $\square$

#### 4.1 The ergodic BSDE with zero Neumann boundary conditions in a weakly dissipative environment

In a first time we are concerned with the following EBSDE with zero Neumann condition in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty, \quad (4.3)$$

where the unknown is the triplet  $(Y^x, Z^x, \lambda)$ .  $(X_t^x)_{t \geq 0}$  is the solution of (4.1).

To do this, we are going to study the following BSDE, with monotonic drift in  $y$ , regularized coefficients and penalized generator, namely:  $\forall 0 \leq t \leq T < +\infty$ ,

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}] ds - \int_t^T Z_s^{x,\alpha,n,\varepsilon} dW_s, \quad (4.4)$$

where the process  $(X_t^{x,n,\varepsilon})$  is the solution of the following SDE:

$$X_t^{x,n,\varepsilon} = x + \int_0^t (f^\varepsilon(X_s^{x,n,\varepsilon}) + F_n^\varepsilon(X_s^{x,n,\varepsilon})) ds + \int_0^t \sigma(X_s^{x,n,\varepsilon}) dW_s.$$

**Remark 4.3.**  $F_n$  is regularized like other regularized functions. Thanks to convolutions arguments it is possible to construct a sequence of functions  $F_n^\varepsilon$  which converge pointwisely toward  $F_n$  and such that for all  $\varepsilon$ ,  $F_n^\varepsilon$  is 0-dissipative and  $4n$ -Lipschitz.

Now we can state the existence theorem for EBSDE (4.3).

**Theorem 4.4.** *Assume that the hypotheses 2.1, 3.1, 4.1 and 4.2 hold. Then there exists a solution  $(\bar{Y}_t^x, \bar{Z}_t^x, \bar{\lambda})$  to EBSDE (4.3) such that  $\bar{Y}^x = \bar{v}(V^x)$  with  $\bar{v}$  locally Lipschitz, and there exists a measurable function  $\bar{\xi} : \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}$  such that  $\bar{Z}^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  and  $\bar{Z}^x = \bar{\xi}(X^x)$ .*

*Proof.* We give the main ideas, because the proof is very similar to the proof of Theorem 3.4. The beginning of the proof is the same as the proof of Theorem 3.4. Lemma 3.1 gives us the existence and uniqueness of the solution  $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$  of BSDE (4.4) in  $\mathcal{S}^2 \times \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Then, as the function  $d + F_n$  is still  $\eta$ -dissipative and as the work in the previous section involves  $d$  only through its dissipativity constant  $\eta$ , we can apply previous results. As always we define  $v^{\alpha,n,\varepsilon}(x) := Y_0^{\alpha,n,\varepsilon}$ . By Lemma (3.3) we have the following estimate:  $\forall x, x' \in \mathbb{R}^d$ :

$$|v^{\alpha,n,\varepsilon}(x) - v^{\alpha,n,\varepsilon}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.$$

In addition we also have:

$$|\alpha v^{\alpha,n,\varepsilon}(0)| \leq M_\psi.$$

As those inequalities are uniform in  $\varepsilon$  it is possible to construct by a diagonal procedure a subsequence  $\varepsilon_p \rightarrow +0$  such that  $\forall n \in \mathbb{N}, \alpha > 0$ :

$$\bar{v}^{\alpha,n,\varepsilon_p}(x) \xrightarrow{p \rightarrow +\infty} \bar{v}^{\alpha,n}(x) \quad \text{and} \quad \alpha v^{\alpha,n,\varepsilon_p}(0) \xrightarrow{p \rightarrow +\infty} \bar{\lambda}^{\alpha,n},$$

We recall the fact that the function  $\bar{v}^{\alpha,n}$  is locally Lipschitz on  $\mathbb{R}^d$  and that we keep the following estimates:

$$|\bar{v}^{\alpha,n}(x) - \bar{v}^{\alpha,n}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|;$$

$$|\bar{\lambda}^{\alpha,n}| \leq M_\psi.$$

Now let us define  $\forall t \geq 0, \bar{Y}_t^{x,\alpha,n} := \bar{v}^{\alpha,n}(V_t^{x,n})$ . Let us show that

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,n,\varepsilon_p} - \bar{Y}_s^{x,\alpha,n}|^2 ds \xrightarrow{p \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,n,\varepsilon_p} - \bar{Y}_T^{x,\alpha,n}|^2 \xrightarrow{p \rightarrow +\infty} 0.$$

First we write:

$$\begin{aligned} |\bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n,\varepsilon_p}) - \bar{v}^{\alpha,n}(X_s^{x,n})| &\leq |\bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n,\varepsilon_p}) - \bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n})| \\ &\quad + |\bar{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n}) - \bar{v}^{\alpha,n}(X_s^{x,n})| \\ &\leq C(1 + |X_s^{x,n,\varepsilon_p}|^2 + |X_s^{x,n}|^2) |X_s^{x,n,\varepsilon_p} - X_s^{x,n}|, \end{aligned}$$



which shows the pointwise convergence of  $\bar{v}^{\alpha,n,\varepsilon_p}(V_s^{x,n,\varepsilon_p})$  toward  $\bar{v}^{\alpha,n}(V_s^{x,n})$  almost surely when  $p \rightarrow +\infty$ . Then, due to the fact that  $|\bar{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})| \leq M_\psi/\alpha$   $\mathbb{P}$ -a.s., we can apply the dominated convergence theorem to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,n,\varepsilon_p} - \bar{Y}_s^{x,\alpha,n}|^2 ds \xrightarrow{p \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,n,\varepsilon_p} - \bar{Y}_T^{x,\alpha,n}|^2 \xrightarrow{p \rightarrow +\infty} 0.$$

In addition it is possible to show as in Theorem 3.4 that  $(Z^{x,\alpha,n,\varepsilon_p})_p$  is Cauchy in  $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Note that we keep the estimates  $\forall x, x' \in \mathbb{R}^d$ :

$$|\bar{v}^{\alpha,n}(x) - \bar{v}^{\alpha,n}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.$$

In addition we also have:

$$|\bar{\lambda}^{\alpha,n}| \leq M_\psi.$$

Therefore, again, by a diagonal procedure, it is possible to extract a subsequence  $(\beta(n))_n$  such that

$$v^{\alpha,\beta(n)}(x) \rightarrow \bar{v}^\alpha(x).$$

And thanks to Lemma 4.2, one can apply the dominated convergence theorem to show that:

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha,\beta(n)} - \bar{Y}_s^{x,\alpha}|^2 ds \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x,\alpha,\beta(n)} - \bar{Y}_T^{x,\alpha}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Finally a last diagonal procedure in  $\alpha$  allow us to conclude (see the end of the proof of Theorem (3.4)). □

Once again, we notice that the solution we have constructed satisfies the following growth property:

$$|\bar{Y}_t^x| \leq C(1 + |X_t^x|^2),$$

so it is natural to establish the following theorem under the same growth properties.

**Theorem 4.5.** (*Uniqueness of  $\lambda$* ). *Assume that the hypotheses 2.1, 3.1 holds true. Let  $(Y, Z, \lambda)$  be a solution of EBSDE (4.3). Then  $\lambda$  is unique among solutions  $(Y, Z, \lambda)$  such that  $Y$  is a bounded continuous process and  $Z \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$ . Finally assume that we have the following growth property*

$$\begin{aligned} |Y_t| &\leq C(1 + |X_t^x|^2), \\ |Y_t'| &\leq C'(1 + |X_t^x|^2). \end{aligned}$$

Then  $\lambda = \lambda'$ .

*Proof.* Simply, adapt the proof of Theorem 4.6 of [4]. With the same notations once can write:

$$\begin{aligned} \tilde{\lambda} &= T^{-1} \mathbb{E}^{\mathbb{P}^h} [\tilde{Y}_T - \tilde{Y}_0] \\ &\leq (C + C')T^{-1}(2 + |x|^2 + \mathbb{E}^{\mathbb{P}^h} |X_t^x|^2) \\ &\leq (C + C')T^{-1}(2 + |x|^2 + \mathbb{E}^{\mathbb{P}^h} |X_t^{x,n}|^2 + \mathbb{E}^{\mathbb{P}^h} |X_t^x - X_t^{x,n}|^2) \end{aligned}$$

To conclude, just use the first estimate from Lemma 2.1 and the estimate from Lemma 4.2. First let  $n \rightarrow +\infty$  and then let  $T \rightarrow +\infty$ . □

## 4.2 The ergodic BSDE with non-zero Neumann boundary conditions in a weakly dissipative environment

We are now concerned by the following EBSDE in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds + \int_t^T [g(X_s^x) - \mu] dK_s^x - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty, \quad (4.5)$$

where  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable.

**Proposition 4.6.** (*Existence of a Solution  $(Y, Z, \lambda)$* ). *Assume that the hypothesis 2.1, 3.1 and 4.2 hold true. Then for any  $\mu \in \mathbb{R}$  there exists  $\lambda \in \mathbb{R}$ ,  $Y^x$  continuous adapted process and  $Z^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  such that the triple  $(Y, Z, \lambda)$  is a solution of EBSDE (4.5) with  $\mu$  fixed.*

*Proof.* The Theorem 4.4 gives us the existence of a solution  $(Y^x, Z^x, \lambda)$  of the following EBSDE

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.6)$$

Now, defining  $\widehat{Y}_t^x = Y_t^x - \int_0^t [g(X_s^x) - \mu] dK_s^x$ , it is easy to see that  $(\widehat{Y}^x, Z^x, \lambda)$  is a solution of the EBSDE (4.5) with  $\mu$  fixed.  $\square$

**Remark 4.7.** The constructed solution  $\widehat{Y}^x$  is not Markovian anymore. Furthermore, it satisfies the following growth property:  $\forall t \geq 0$ ,  $|\widehat{Y}_t^x| \leq C(1 + |X_t^x| + K_t^x)$ . This dependence on  $K_t^x$  prevents us to get the uniqueness of  $\lambda$  among the space of solutions satisfying such a growth property.

Similarly, an existence result can be stated for a solution  $(Y, Z, \mu)$ ,  $\lambda$  being fixed.

**Proposition 4.8.** (*Existence of a Solution  $(Y, Z, \mu)$* ). *Assume that the hypotheses 2.1, 3.1 and 4.2 hold true. Then for any  $\lambda \in \mathbb{R}$  there exists a continuous adapted process and  $Z^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^{1 \times d})$  such that for all  $\mu \in \mathbb{R}$  the triple  $(Y, Z, \mu)$  is a solution of EBSDE (4.5) with  $\lambda$  fixed.*

*Proof.* From Theorem 4.4, we have constructed a solution  $(Y^{x,0}, Z^{x,0}, \lambda^0)$  of the following EBSDE

$$Y_t^{x,0} = Y_T^{x,0} + \int_t^T [\psi(X_s^x, Z_s^{x,0}) - \lambda^0] ds - \int_t^T Z_s^{x,0} dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.7)$$

Then setting  $\widehat{Y}_t^x := Y_t^{x,0} + (\lambda - \lambda^0)t - \int_0^t [g(X_s^x) - \mu] dK_s^x$ , the triple  $(\widehat{Y}^x, Z^{x,0}, \mu)$  is solution of EBSDE (4.5) with  $\lambda$  fixed.  $\square$

**Remark 4.9.** The constructed solution satisfies the following growth property:

$$|\widehat{Y}_t^x| \leq C(1 + |X_t^x| + K_t^x + t), \quad \mathbb{P} - a.s.$$

Again, this solution does not allow us to establish a result of uniqueness for  $\mu$  among the space of solutions satisfying such a growth property.

**Remark 4.10.** If the convex  $\overline{G}$  is assumed to be bounded, it is possible, following [11] to show that there exists a Markovian solution  $(Y, Z, \lambda)$  when  $\mu$  is fixed or  $(Y, Z, \mu)$  when  $\lambda$  is fixed exists, for a driver weakly dissipative. The proofs are the same as in [11].

## 5 Probabilistic interpretation of the solution of an elliptic PDE with zero Neumann boundary condition

We are concerned with the following semi-linear elliptic PDE:

$$\begin{cases} \mathcal{L}v(x) + \psi(x, {}^t\nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \quad (5.1)$$

where:

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(\sigma(x) {}^t\sigma(x) \nabla^2 u(x)) + {}^t f(x) \nabla u(x).$$

The unknowns of this equation is the couple  $(v, \lambda)$ . Now we show that the function  $v$  defined in Theorem 4.6 is a viscosity solution of the PDE (5.1).

**Theorem 5.1.** *Assume that hypotheses of Theorem 4.4 hold . Then  $(v, \lambda)$  is a viscosity solution of the elliptic PDE (5.1) where  $v$  is defined in Theorem 4.4.*

*Proof.* Just adapt the proof of Theorem 4.3 from [10]. □

## 6 Optimal ergodic control

We make the standard assumption for optimal ergodic control, namely we consider  $U$  a separable metric space, which is the state space of the control process  $\rho$ .  $\rho$  is assumed to be  $(\mathcal{F}_t)$ -progressively measurable. We introduce  $R : U \rightarrow \mathbb{R}^d$  and  $L : \mathbb{R}^d \times U \rightarrow \mathbb{R}$  two continuous functions such that , for some constants  $M_R > 0$  and  $M_L > 0$ ,  $\forall u \in U, \forall x, x' \in \mathbb{R}^d$ ,

- $|R(u)| \leq M_R$ ,
- $|L(x, u)| \leq M_L$ ,
- $|L(x, u) - L(x', u)| \leq M_L |x - x'|$ .

For an arbitrary control  $\rho$ , the cost will be evaluated relatively to the following Girsanov density:

$$\Gamma_T^\rho = \exp \left( \int_0^T R(\rho_s) dW_s - \frac{1}{2} \int_0^T |R(\rho_s)|^2 ds \right).$$

We denote by  $\mathbb{P}_T^\rho$  the associated probability measure, namely:  $d\mathbb{P}_T^\rho = \Gamma_T^\rho d\mathbb{P}$  on  $\mathcal{F}_T$ . Now we define the ergodic costs, relatively to a given control  $\rho$  and a starting point  $x \in \mathbb{R}^d$ , by:

$$I(x, \rho) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_T^\rho \left[ \int_0^T L(X_s^x, \rho_s) ds \right], \quad (6.1)$$

where  $\mathbb{E}_T^\rho$  denotes expectation with respect to  $\mathbb{P}_T^\rho$ . We notice that the process  $W_t^\rho := W_t - \int_0^t R(\rho_s) ds$  is a Wiener process on  $[0, T]$  under  $\mathbb{P}_T^\rho$ . We define the Hamiltonian in the usual way:

$$\psi(x, z) = \inf_{u \in U} \{L(x, u) + zR(u)\}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^{1 \times d},$$

and we remark that if, for all  $x, z$ , the infimum is attained in (6.4) then, according to Theorem 4 of [8], there exists a measurable function  $\gamma : \mathbb{R}^d \times \mathbb{R}^{1 \times d} \rightarrow U$  such that:

$$\psi(x, z) = L(x, \gamma(x, z)) + zR(\gamma(x, z)). \quad (6.2)$$

One can verify that  $\gamma$  is a Lipschitz function. Now we can prove the following theorem, exactly like in [11].

**Theorem 6.1.** *Assume that the hypotheses of Theorem 4.4 hold true. Let  $(Y, Z, \lambda)$  be a solution of EBSDE (4.5) with  $\mu$  fixed. Then:*

1. *For arbitrary control  $\rho$  we have  $I(x, \rho) \geq \lambda$ .*
2. *If  $L(X_t^x, \rho_s) + Z_t^x R(\rho_t) = \psi(X_t^x, Z_t^x)$ ,  $\mathbb{P}$ -a.s. for almost every  $t$  then  $I(x, \rho) = \lambda$ .*
3. *If the infimum is attained in (6.2) then the control  $\bar{\rho}_t = \gamma(X_t^x, Z_t^x)$  verifies  $I(x, \bar{\rho}) = \lambda$ .*

**Remark 6.2.** When the Neumann conditions are different from 0, we need regularity on the solution  $Y_t^x$  in order to state the same result. Again the degeneracy of the solution constructed in Proposition 4.6 or 4.8 does not allow us to conclude.

## 7 Appendix

### A Proof of Lemma 2.1

Let us define  $\varphi(x) = |x - a|^p$  for  $p \geq 1$ . We recall the following formulas for derivatives of  $\varphi$ , for  $p \geq 2$ .

$$\begin{aligned} \nabla \varphi(x) &= p(x - a)|x - a|^{p-2}. \\ \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} &= \begin{cases} p|x - a|^{p-2} + p(p-2)(x_i - a_i)^2|x - a|^{p-4} & \text{if } i = j, \\ p(p-2)(x_i - a_i)(x_j - a_j)|x - a|^{p-4} & \text{if } i \neq j. \end{cases} \end{aligned}$$

Therefore we have the following estimate

$$|\nabla^2 \varphi(x)| \leq K|x - a|^{p-2}, \quad (\text{A.1})$$

for a constant  $K$  which depends only on  $p$  and  $d$ . Under the hypothesis of this Lemma, it is well known that a solution for which the explosion time is almost surely equal to infinity exists. By Itô's formula we get, for  $p = 2$ , for all  $t \geq 0$ ,

$$\begin{aligned} |V_t^x - a|^2 e^{2\eta_1 t} &= |x - a|^2 + 2 \int_0^t e^{2\eta_1 s} (V_s^x - a, f(V_s^x)) ds + \sigma(V_s^x) dW_s \\ &\quad + 2\eta_1 \int_0^t |V_s^x - a|^2 e^{2\eta_1 s} ds + \int_0^t \sum_i (\sigma(V_s^x)^t \sigma(V_s^x))_{i,i} e^{2\eta_1 s} ds \\ &\leq |x - a|^2 + 2 \int_0^t (V_s^x - a) \sigma dW_s + \frac{2\eta_2 + d|\sigma|_\infty^2}{2\eta_1} (e^{2\eta_1 t} - 1). \end{aligned}$$

Taking the expectation, we get:

$$\mathbb{E}|V_t^x - a|^2 \leq |x - a|^2 e^{-2\eta_1 t} + \frac{2\eta_2 + d|\sigma|_\infty^2}{2\eta_1} (1 - e^{-2\eta_1 t}).$$

Therefore:

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-2\eta_1 t}), \quad (\text{A.2})$$

where  $C$  is a constant that depends only on  $a$ ,  $\eta_1$ ,  $\eta_2$  and  $\sigma$  but not on the time  $t$ .

Let  $0 < \delta < p\eta_1$ . For  $p > 2$ , Itô's formula gives us, for a generic constant  $C$  which depends only on  $p$ ,  $d$ ,  $|\sigma|_\infty$ ,  $\eta_2$ ,  $\varepsilon$  (defined later):

$$\begin{aligned} |V_t^x - a|^p e^{(p\eta_1 - \delta)t} &\leq |x - a|^p + p \int_0^t e^{(p\eta_1 - \delta)s} |V_s^x - a|^{p-2} (V_s^x - a) \cdot (f(V_s^x) ds + \sigma(V_s^x) dW_s) \\ &\quad + (p\eta_1 - \delta) \int_0^t |V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds \\ &\quad + \frac{1}{2} \int_0^t \text{Tr}(\sigma(V_s^x)^t \sigma(V_s^x) \nabla^2 \varphi(V_s^x)) e^{(p\eta_1 - \delta)s} ds. \end{aligned}$$

Then, taking the expectation, using the assumption on  $f$  and using estimate A.1 we have:

$$\mathbb{E}|V_t^x - a|^p e^{(p\eta_1 - \delta)t} \leq |x - a|^p + C \int_0^t \mathbb{E}|V_s^x - a|^{p-2} e^{(p\eta_1 - \delta)s} ds - \delta \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds$$

The Young inequality  $ab \leq a^p/p + b^q/q$  for  $1/p + 1/q = 1$  with  $p$  replaced by  $p/(p-2)$  and  $q$  replaced by  $p/2$  applied to the last term of the above inequality allows us to write:

$$|V_s^x - a|^{p-2} \leq (p-2)\varepsilon |V_s^x - a|^p/p + 2/(p\varepsilon^{(p-2)/2}),$$

hence,

$$\begin{aligned} \mathbb{E}|V_t^x - a|^p e^{(p\eta_1 - \delta)t} &\leq |x - a|^p + \varepsilon C \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds + C/\varepsilon^{(p-2)/2} \\ &\quad - \delta \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1 - \delta)s} ds. \end{aligned}$$

We choose  $\varepsilon$  such that  $\varepsilon C = \delta$ , then:

$$\mathbb{E}|V_t^x - a|^p e^{(p\eta_1 - \delta)t} \leq |x - a|^p C.$$

Therefore:

$$\mathbb{E}|V_t^x - a|^p \leq C(1 + |x|^p e^{-(p\eta_1 - \delta)t}).$$

This can be rewritten:

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-(p\eta_1 - \delta)t}),$$

where  $C$  is a constant which depends on  $p$ ,  $d$ ,  $\sigma_\infty$ ,  $\eta_1$ ,  $\eta_2$ ,  $\varepsilon$  and  $a$ . Note that  $C$  does not depend on the Lipschitz constant of  $f$  but only on  $\eta_1$  and  $\eta_2$  which will be useful later. Finally, note that this result holds for any  $0 < \delta < p\eta_1$ . Now that we have this result it is classical to establish the following result, using BDG's inequality and Hölder's inequality, for all  $p \geq 1$ ,

$$\mathbb{E} \sup_{0 \leq t \leq T} |V_t^x|^p \leq C(T)(1 + |x|^p),$$

where the constant  $C$  depends this time on the time  $T$  and other constants given before.  $\square$

## B Proof of Lemma 2.4

We adapt the proof of Theorem 2.4 from [3]. From Remark 2.2 we can deduce, applying the Lemma 2.1 that

$$\mathbb{E}|V_t^x|^2 \leq \kappa_1(|x|^2 e^{-\eta t} + 1),$$

where  $\kappa_1$  is a constant which depends only on  $\eta$ ,  $B$ ,  $|\sigma|_\infty$  and  $\overline{G}$ . Then, choosing  $R = 16\kappa_1$  and  $T$  large enough we can conclude the step 1 of the proof.

The next step of the proof has to be modified too, due to the lack of linearity on  $d$ . More precisely we have to find a new coupling. Let  $x, y \in \overline{G} \subset B_R$  where  $B_R$  denotes the ball of center 0 and radius  $R$ . Here we recall the main ideas of this step. The goal is to find two processes satisfying SDE (2.1) in law, one starting from  $x$ , the other one from  $y$ , and such that an estimate of the ratio of their densities is known. We denote by  $\mu_1$  the law of  $(V_t^x)_{0 \leq t \leq T}$  and by  $\mu_2$  the law of  $(V_t^y)_{0 \leq t \leq T}$ .

Let us introduce the process  $Y_t^{x,y}$  as the solution of the following stochastic equation,  $\forall 0 \leq t < T$ :

$$\begin{cases} dY_t^{x,y} = \left[ d(V_t^y + Y_t^{x,y}) - d(V_t^y) - \frac{Y_t^{x,y}}{T-t} \right] dt + [\sigma(V_t^y + Y_t^{x,y}) - \sigma(V_t^y)] dW_t, \\ Y_0^{x,y} = x - y, \end{cases} \quad (\text{B.1})$$

where  $(V_t^x)$  is the strong solution of (2.1) starting from  $x$  at  $t = 0$  and  $(V_t^y)$  the solution of (2.1) starting from  $y$  at  $t = 0$ . It is well known that the system above admit a strong solution which is strongly unique and square integrable for all  $t < T$ . Furthermore, for all  $0 \leq t < T$ , Itô's formula gives us:

$$\begin{aligned} \frac{Y_t^{x,y}}{T-t} &= \frac{x-y}{T} + \int_0^t \frac{1}{T-s} [d(V_s^y + Y_s^{x,y}) - d(V_s^y)] ds \\ &\quad + \int_0^t \frac{1}{T-s} [\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)] dW_s. \end{aligned} \quad (\text{B.2})$$

Let us denote, for all  $i \in [1, d]$ , for all  $0 \leq t < T$ :

$$M_t^i := \left( \int_0^t \frac{1}{T-s} [\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)] dW_s \right)_i.$$

For all  $i$ ,  $M^i$  is a continuous local martingale. Let us denote by  $\tau^i$  its quadratic variation process:

$$\tau_t^i := \int_0^t \sum_{j=1}^d \frac{[\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)]_{i,j}^2}{(T-s)^2} ds.$$

Thus, by Dambis-Dubins-Schwarz's theorem, for all  $i$ , there exists a standard one-dimensional Brownian motion  $B^i$  such that, for all  $0 \leq t < T$ :

$$M_t^i = B^i(\tau_t^i).$$

Now let us suppose that  $\tau_t^i \xrightarrow{t \rightarrow T} a < +\infty$ , then it is clear that  $(T-t)B^i(\tau_t^i) \xrightarrow{t \rightarrow T} 0$ . Let us suppose that  $\tau_t^i \xrightarrow{t \rightarrow T} +\infty$ . Then, as, by definition of  $\tau_t^i$ , there exists  $C > 0$  such that, for all  $i$ :

$$\tau_t^i \leq \frac{C}{T-t},$$

and by the iterated logarithm law for  $(B^i(\tau_t^i))_{t \in [0, T]}$ , we have:

$$(T-t)B^i(\tau_t^i) \xrightarrow[t \rightarrow T]{} 0, \quad \mathbb{P} - a.s. \quad (\text{B.3})$$

Furthermore:

$$\begin{aligned} (T-t) \int_0^t \frac{1}{T-s} [d(V_s^y + Y_s^{x,y}) - d(V_s^y)] ds \\ \leq \sqrt{T-t} \int_0^t \frac{1}{\sqrt{T-s}} [d(V_s^y + Y_s^{x,y}) - d(V_s^y)] ds \\ \leq \sqrt{T-t} \int_0^T \frac{1}{\sqrt{T-s}} (1 + |V_s^y|^p + |Y_s^{x,y}|^p) ds. \end{aligned} \quad (\text{B.4})$$

Now let us show that  $\mathbb{E} \left( \int_0^T \frac{1}{\sqrt{T-s}} (1 + |V_s^y|^p + |Y_s^{x,y}|^p) ds \right) < +\infty$  which is enough to show that  $(T-t) \int_0^t \frac{1}{T-s} [d(V_s^y + Y_s^{x,y}) - d(V_s^y)] ds \xrightarrow[t \rightarrow T]{} 0$ ,  $\mathbb{P} - a.s.$  It is clear that  $\mathbb{E}[|V_s^y|^p] \leq C$  (see estimate 2.1). Let  $p \geq 2$  and let us define  $\forall y \in \mathbb{R}^d$ ,  $\varphi(y) := |y|^p$ . It's formula gives us, for all  $0 \leq t < T$ :

$$\begin{aligned} & |Y_t^{x,y}|^p \left| \frac{T}{T-t} \right|^p \\ &= |x-y|^p \\ &+ p \int_0^t |Y_s^{x,y}|^{p-2} (Y_s^{x,y}, d(V_t^y + Y_t^{x,y}) - d(V_t^y) - \frac{Y_s^{x,y}}{T-s}) \left| \frac{T}{T-s} \right|^p ds \\ &+ p \int_0^t |Y_s^{x,y}|^{p-2} (Y_s^{x,y}, \sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)) \left| \frac{T}{T-s} \right|^p dW_s \\ &+ p \int_0^t |Y_s^{x,y}|^p \frac{T^p}{|T-s|^{p+1}} ds \\ &+ \frac{1}{2} \int_0^t \text{Tr} [(\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))^t (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)) \nabla^2 \varphi(Y_s^{x,y})] \left| \frac{T}{T-s} \right|^p ds. \end{aligned}$$

Then, using the dissipativity of  $d$  and taking the expectation, it is easy to see that:

$$\mathbb{E} \left( |Y_t^{x,y}|^p \left| \frac{T}{T-t} \right|^p \right) \leq |x-y|^p + C \int_0^t \mathbb{E} |Y_s^{x,y}|^p \left| \frac{T}{T-s} \right|^p ds.$$

Hence, Gronwall's lemma gives us:

$$\mathbb{E} (|Y_t^{x,y}|^p) \leq |x-y|^p e^{Ct} \left| \frac{T-t}{T} \right|^p.$$

This shows that:

$$\mathbb{E} \left( \int_0^t \frac{1}{\sqrt{T-s}} (1 + |V_s^y|^p + |Y_s^{x,y}|^p) ds \right) \leq C(T) < +\infty.$$

Then, with (B.2), it is easy to see that:

$$Y_t^{x,y} \xrightarrow[t \rightarrow T]{} 0, \quad \mathbb{P} - a.s.$$

thanks to equation (B.2).

Now we consider the process  $\tilde{V}_t = V_t^y + Y_t^{x,y}$ . This process replaces the one denoted by  $\tilde{X}$  in the step 2 of [3]. This process satisfies the following stochastic differential equation:

$$\begin{cases} d\tilde{V}_t = f(V_t^y)dt + \sigma(V_t^y)dW_t + dY_t^{x,y}, \\ \tilde{V}_0 = x. \end{cases} \quad (\text{B.5})$$

Thus:

$$d\tilde{V}_t = f(\tilde{V}_t)dt + \sigma(\tilde{V}_t) \left[ dW_t + \sigma^{-1}(\tilde{V}_t) \left( b(V_t^y) - b(\tilde{V}_t) - \frac{Y_t^{x,y}}{T-t} \right) \right].$$

We define:  $h(t) = \sigma^{-1}(\tilde{V}_t) \left( b(V_t^y) - b(\tilde{V}_t) - \frac{cY_t^{x,y}}{T-t} \right)$ . Now, let us show that  $h$  satisfies the Novikov condition, namely  $\mathbb{E} \exp \left( \int_0^T \frac{1}{2} |h(t)|^2 dt \right) < +\infty$ . In the following of this proof, which depends on  $p, T$ , on  $f$  only through  $\eta$  and  $B$ , on  $\sigma$  only through  $\sigma_\infty$  and its Lipschitz constant. We recall that  $|\sigma^{-1}(x)|$  admit an upper bound uniform on  $x$  by hypothesis. Then, we have:

$$\begin{aligned} \mathbb{E} \exp \left( \int_0^T \frac{1}{2} |h(t)|^2 dt \right) &\leq \mathbb{E} \exp \left( \int_0^T C \left( 1 + \frac{|Y_t^{x,y}|^2}{(T-t)^2} \right) dt \right) \\ &\leq C \mathbb{E} \exp \int_0^T \frac{|Y_t^{x,y}|^2}{(T-t)^2} dt \\ &\leq C \mathbb{E} \sum_{k=0}^{+\infty} \frac{\int_0^T \frac{|Y_t^{x,y}|^{2k}}{(T-t)^{2k}} dt}{k!} \\ &\leq C \sum_{k=0}^{+\infty} \frac{\mathbb{E} \int_0^T \frac{|Y_t^{x,y}|^{2k}}{(T-t)^{2k}} dt}{k!}. \end{aligned}$$

We have, by Itô's formula, for all  $p \in \mathbb{N}$ ,  $p \geq 2$  and using the following notation:  $\phi(y) = |y|^p$ .

$$\begin{aligned} \frac{|Y_t^{x,y}|^p}{|T-t|^{p-1}} &= \frac{|x-y|^p}{|T|^{p-1}} + p \int_0^t \frac{|Y_s^{x,y}|^{p-2}}{|T-s|^{p-1}} Y_s^{x,y} dY_s^{x,y} + (p-1) \int_0^t \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} \left[ [\sigma(\tilde{V}_s) - \sigma(V_s^y)]^t [\sigma(\tilde{V}_s) - \sigma(V_s^y)] \nabla^2 \phi(Y_s^{x,y}) \right] \frac{1}{|T-s|^{p-1}} ds. \end{aligned}$$

This can be rewritten:

$$\begin{aligned} \frac{|Y_t^{x,y}|^p}{|T-t|^{p-1}} + \int_0^t \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds &= \frac{|x-y|^p}{T^{p-1}} + I_t + p \int_0^t \frac{|Y_s^{x,y}|^{p-2}}{|T-s|^{p-1}} (Y_s^{x,y}, d(\tilde{V}_s) - d(V_s^y)) ds \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} \left[ [\sigma(\tilde{V}_s) - \sigma(V_s^y)]^t [\sigma(\tilde{V}_s) - \sigma(V_s^y)] \nabla^2 \phi(Y_s^{x,y}) \right] \frac{1}{|T-s|^{p-1}} ds. \end{aligned}$$

where  $I_t$  is a stochastic integral. It is clear that:

$$p \int_0^t \frac{|Y_s^{x,y}|^{p-2}}{|T-s|^{p-1}} (Y_s^{x,y}, d(\tilde{V}_s) - d(V_s^y)) ds \leq 0.$$



Moreover,  $|\nabla^2\varphi(x)| \leq C|x|^{p-2}$ , thus, as  $\sigma$  is Lipschitz and bounded, we get:

$$\begin{aligned} & \frac{1}{2} \int_0^t \text{Tr} \left[ [\sigma(\tilde{V}_s) - \sigma(V_s^y)]^t [\sigma(\tilde{V}_s) - \sigma(V_s^y)] \nabla^2\varphi(Y_s^{x,y}) \right] \frac{1}{|T-s|^{p-1}} ds \\ & \leq C \int_0^t \frac{|Y_s^{x,y}|^{p-1}}{|T-s|^{p-1}} \\ & \leq \varepsilon C \mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds + \frac{C}{\varepsilon^{p-1}}. \end{aligned}$$

where the last inequality is established thanks to the following Young's inequality:  $ab \leq \varepsilon \frac{a^m}{m} + \frac{b^n}{n\varepsilon^{n/m}}$ , we get that, for  $m = p/(p-1)$  and for  $b = 1$ . Thus we get:

$$(1 - \varepsilon C) \mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds \leq \frac{|x-y|^p}{|T|^{p-1}} + \frac{C}{\varepsilon^{p-1}}.$$

It shows that it is possible to find  $\varepsilon_p$  small enough so that  $(1 - \varepsilon_p C) > 0$ . So,

$$\mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds \leq \frac{|x-y|^p}{(1 - C\varepsilon_p)|T|^{p-1}} + \frac{C}{(1 - C\varepsilon_p)\varepsilon_p^{p-1}}.$$

Therefore, it is easy to see that:

$$\begin{aligned} \mathbb{E} \exp \left( \int_0^T \frac{1}{2} |h(t)|^2 dt \right) & \leq C \sum_{k=0}^{+\infty} \frac{1}{k!} \left( \frac{|x-y|^{2k}}{(1 - \varepsilon_{2k} C)|T|^{2k-1}} + \frac{C}{(1 - \varepsilon_{2k} C)\varepsilon_{2k}^{2k-1}} \right) \\ & := \kappa_4 < +\infty. \end{aligned}$$

Hence the Novikov condition is satisfied. We stress the fact that  $\kappa_4$  depends on  $d$  only through its dissipativity constant. This concludes the step 2 of the proof. The step 3 does not have to be modified.  $\square$

## C Proof of Lemma 4.2

We follow the proof of the part 3 of [9]. We need to adapt this proof because in our case, the set in which the process is reflected is not bounded. Therefore convergences are not uniform in  $x$  anymore. In our case, the dissipativity of the process is enough to avoid the boundedness of  $\overline{G}$ . We will use the following notation  $\beta(x) = (x - \Pi(x))$ . Note that  $F_n(x) = -2n\beta(x)$ . We recall the following properties of the penalization term:

$$(x' - x) \cdot \beta(x) \leq \beta(x') \cdot \beta(x), \quad \forall x, x' \in \mathbb{R}^n, \quad (\text{C.1})$$

$$\exists c \in \overline{G}, \quad \gamma > 0, \quad \forall x \in \mathbb{R}^n, \quad (x - c) \cdot \beta(x) \geq \gamma |\beta(x)|. \quad (\text{C.2})$$

In a first time, we show that for any  $1 \leq p < \infty$ ,

$$\mathbb{E} \left[ \left( n \int_0^T |\beta(X_t^{x,n})| ds \right)^p \right] \leq CT(1 + |x|^p), \quad \forall n \in \mathbb{N},$$

for a constant  $C$  which does not depend on  $n$ . For  $p = 2$ , Itô's formula gives us:

$$\begin{aligned} |X_t^{x,n} - c|^2 &= |x - c|^2 + 2 \int_0^t (X_s^{x,n} - c) \cdot (f(X_s^{x,n}) ds + F_n(X_s^{x,n}) ds + \sigma(X_s^{x,n}) dW_s) \\ &\quad + \int_0^t \sum_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} ds. \end{aligned}$$

Using inequality (C.2), the fact that  $\sigma$  is bounded and Remark (2.2) we deduce:

$$4n\gamma \int_0^t |\beta(X_s^{x,n})| ds \leq |x - c|^2 + 2 \int_0^t C ds + 2 \int_0^t (X_s^{x,n} - c) \cdot \sigma(X_s^{x,n}) dW_s.$$

Using the BDG inequality:

$$\mathbb{E} \left[ \left| \int_0^T b(s) dW_s \right|^p \right] \leq C(p, T) \mathbb{E} \left[ \left( \int_0^T |b(s)|^2 \right)^{p/2} \right],$$

and our estimate of Lemma 2.1, it follows that:

$$\mathbb{E} \left[ \left( n \int_0^T |\beta(X_t^{x,n})| ds \right)^p \right] \leq CT(1 + |x|^p), \forall n \in \mathbb{N},$$

for a constant  $C$  which does not depend on  $n$ .

Now we prove that  $\forall p > 2$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |\beta(X_t^{x,n})|^p \right) \leq \frac{CT(1 + |x|^p)}{n^{p/2-1}}.$$

We apply Itô's formula to the function  $\varphi(x) = |x - \Pi(x)|^p$ . We will use the following notation  $\beta(x) = x - \Pi(x)$ . Note that  $F_n(x) = -2n\beta(x)$ . It is well known that for all  $p \geq 2$ ,  $\varphi$  is  $\mathcal{C}^2$  on  $\mathbb{R}^d$  and that  $\nabla\varphi(x) = 2(x - \Pi(x))$ . We recall the following formulas for the derivatives of  $\varphi$ ,

$$\nabla\varphi(x) = p|\beta(x)|^{p-2}\beta(x),$$

$$\nabla^2\varphi(x) = p|\beta(x)|^{p-2}\nabla\beta(x) + p(p-2)|\beta(x)|^{p-4}(\beta(x)^t\beta(x)).$$

As  $\nabla\beta(x)$  is a numerical matrix one can deduce the following inequality:

$$|\nabla^2\varphi(x)| \leq C|\beta(x)|^{p-2},$$

for a constant  $C$  which depends only on  $p$  and  $d$ .

We use Itô's formula, for all  $p > 2$ ,

$$\begin{aligned} \varphi(X_t^{x,n}) &= \int_0^t \nabla\varphi(X_s^{x,n}) \cdot (d(X_s^{x,n}) + b(X_s^{x,n}) + F_n(X_s^{x,n})) ds \\ &\quad + \int_0^t \nabla\varphi(X_s^{x,n}) \cdot \sigma(X_s^{x,n}) dW_s \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j} (\nabla^2\varphi(X_s^{x,n}))_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi(X_t^{x,n}) + 2pn \int_0^t \varphi(X_s^{x,n}) ds &\leq \int_0^t \nabla \varphi(X_s^{x,n})(d(X_s^{x,n}) + b(X_s^{x,n})) ds \\ &\quad + \int_0^t \nabla \varphi(X_s^{x,n}) \cdot \sigma(X_s^{x,n}) dW_s + C \int_0^t |\beta(X_s^{x,n})|^{p-2} ds \end{aligned} \quad (\text{C.3})$$

From Young's inequality:  $ab \leq a^q/q + b^{q'}/q'$  for reals  $q$  and  $q'$  such that  $1/q + 1/q' = 1$ , we choose  $q = p/(p-2)$  and  $q' = p/2$  so that, for  $\alpha > 0$ :

$$\begin{aligned} |\beta(X_s^{x,n})|^{p-2} &= \alpha n^{(p-2)/p} |\beta(X_s^{x,n})|^{p-2} \times \frac{1}{\alpha n^{(p-2)/p}} \\ &\leq \alpha^{p/(p-2)} \frac{p-2}{p} n |\beta(X_s^{x,n})|^p + \frac{2}{p} \left( \frac{1}{\alpha n^{(p-2)/p}} \right)^{p/2} \\ &\leq \alpha^{p/(p-2)} \frac{p-2}{p} n \varphi(X_s^{x,n}) + \frac{2}{p} \frac{1}{\alpha^{p/2} n^{(p-2)/2}}, \end{aligned}$$

and another Young's inequality applied with this time  $q = p/(p-1)$  and  $q' = p$  gives us:

$$\begin{aligned} |\nabla \varphi(X_s^{x,n}) \cdot (d(X_s^{x,n}) + b(X_s^{x,n}))| &\leq |p(|\beta(X_s^{x,n})|^{p-1})| \times |(d(X_s^{x,n}) + b(X_s^{x,n}))| \\ &\leq \alpha^{p/(p-1)} n (p-1) p^{1/p} |\beta(X_s^{x,n})|^p \\ &\quad + |d(X_s^{x,n}) + b(X_s^{x,n})|^p / (pn^{p-1} \alpha^p). \end{aligned}$$

Therefore using the second inequality of Lemma 2.1 and the two above inequality we deduce, for  $\alpha$  small enough:

$$\mathbb{E} \left( n \int_0^t \varphi(X_s^{x,n}) ds \right) \leq C \left( \frac{(1 + |x|^p)}{n^{p-1}} + \frac{1}{n^{(p-2)/2}} \right) t,$$

therefore,

$$\mathbb{E} \left( \int_0^t \varphi(X_s^{x,n}) ds \right) \leq C \left( \frac{(1 + |x|^p)}{n^{p/2}} \right) t, \quad (\text{C.4})$$

where  $C$  does not depend on  $n$ ,  $t$  and  $x$ .

Now we come back to equation (C.3). Taking the supremum over time and the expectation and using a BDG inequality we get:

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \varphi(X_t^{x,n}) &\leq \mathbb{E} \int_0^T |\nabla \varphi(X_s^{x,n})| \times |d(X_s^{x,n}) + b(X_s^{x,n})| ds \\ &\quad + C \mathbb{E} \left[ \left( \int_0^T |\nabla \varphi(X_s^{x,n}) \sigma|^2 ds \right)^{1/2} \right] + C \mathbb{E} \int_0^T |\beta(X_s^{x,n})|^{p-2} ds. \end{aligned} \quad (\text{C.5})$$

We call respectively  $I_1$ ,  $I_2$  and  $I_3$  the three terms of the right hand side of (C.5). We have

$$I_1 \leq \frac{CT(1 + |x|^p)}{n^{(p-1)/2}},$$

using Hölder's inequality, inequality (C.4) and Lemma 2.1. We have

$$\begin{aligned} I_2 &\leq C\mathbb{E} \left[ \left( \int_0^T |\beta(X_s^{x,n})|^{2p-2} ds \right)^{1/2} \right] \\ &\leq \frac{1}{\sqrt{2}} \mathbb{E} \sup_{0 \leq t \leq T} |\beta(X_s^{x,n})|^p + C' \mathbb{E} \left[ \int_0^T |\beta(X_s^{x,n})|^{p-2} ds \right], \end{aligned}$$

thanks to Young's inequality. Applying Hölder's inequality to the second member gives us:

$$I_2 \leq \frac{1}{\sqrt{2}} \mathbb{E} \sup_{0 \leq t \leq T} |\beta(X_s^{x,n})|^p + \frac{CT(1 + |x|^{p-2})}{n^{(p-2)/2}}.$$

Finally another Hölder's inequality gives us:

$$I_3 \leq \frac{CT(1 + |x|^{p-2})}{n^{(p-2)/2}}.$$

The above estimates of  $I_1$ ,  $I_2$  and  $I_3$  give us the following inequality:

$$\mathbb{E} \sup_{0 \leq t \leq T} \varphi(X_t^{x,n}) \leq \frac{CT(1 + |x|^p)}{n^{(p-2)/2}}.$$

Now, like in [9], we can write that for all  $1 \leq p < +\infty$ ,  $0 < 2q < p$ ,  $n, m \in \mathbb{N}$ ,

$$\mathbb{E} \left[ \left( m \int_0^T |\beta(X_s^{x,n}) \cdot \beta(X_s^{x,m})| ds \right)^p \right] \leq CT \frac{1 + |x|^{2p}}{n^q}.$$

Itô's formula gives us, for all  $0 \leq t \leq T < +\infty$ :

$$\begin{aligned} |X_t^{x,n} - X_t^{x,m}|^2 &= 2 \int_0^t {}^t(X_s^{x,n} - X_s^{x,m})((d+b)(X_s^{x,n}) - (d+b)(X_s^{x,m})) ds \\ &\quad - 4n \int_0^t {}^t(X_s^{x,n} - X_s^{x,m})\beta(X_s^{x,n}) ds \\ &\quad + 4m \int_0^t {}^t(X_s^{x,n} - X_s^{x,m})\beta(X_s^{x,m}) ds \\ &\quad + 2 \int_0^t {}^t(X_s^{x,n} - X_s^{x,m})(\sigma(X_s^{x,n}) - \sigma(X_s^{x,m})) dW_s \\ &\quad + \int_0^t \sum_i [(\sigma(X_s^{x,n}) - \sigma(X_s^{x,m}))^i (\sigma(X_s^{x,n}) - \sigma(X_s^{x,m}))^i]_{i,i} ds. \end{aligned}$$

We use the hypothesis on  $d$ ,  $b$  and  $\sigma$  and then we take the expectation:

$$\begin{aligned} \mathbb{E}|X_t^{x,n} - X_t^{x,m}|^2 &\leq C \int_0^t \mathbb{E}|X_s^{x,n} - X_s^{x,m}|^2 ds \\ &\quad + 4\mathbb{E}n \int_0^t {}^t\beta(X_s^{x,n})\beta(X_s^{x,m}) ds + 4\mathbb{E}m \int_0^t {}^t\beta(X_s^{x,n})\beta(X_s^{x,m}) ds. \end{aligned}$$

Now we have, for  $0 < q < 1/2$ :

$$\mathbb{E}|X_t^{x,n} - X_t^{x,m}|^2 \leq C \int_0^t \mathbb{E}|X_s^{x,n} - X_s^{x,m}|^2 ds + Ct(1 + |x|^2) \left( \frac{1}{n^q} + \frac{1}{m^q} \right).$$

Gronwall's Lemma gives us:

$$\mathbb{E}|X_t^{x,n} - X_t^{x,m}|^2 \leq (e^{Ct} - 1)(1 + |x|^2) \left( \frac{1}{n^q} + \frac{1}{m^q} \right). \quad (\text{C.6})$$

Previous estimates allow us to write that (following [9])

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_s^{x,n} - X_s^{x,m}|^p \leq C(T)(1 + |x|^p) \left( \frac{1}{n^q} + \frac{1}{m^q} \right),$$

for any  $1 < q < p/2$ . □

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