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Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions

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Abstract

We study a class of ergodic BSDEs related to PDEs with Neumann boundary conditions. The randomness of the drift is given by a forward process under weakly dissipative assumptions with an invertible and bounded diffusion matrix. Furthermore, this forward process is reflected in a convex subset of \mathbb{R}^d not necessary bounded. We study the link of such EBSDEs with PDEs and we apply our results to an ergodic optimal control problem.

1 Introduction

In this paper we study the following ergodic backward stochastic differential equation (EBSDE in what follows) in finite dimension and in infinite horizon : $\forall t, T \in \mathbb{R}_+, 0 \leq t \leq T < +\infty$:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds + \int_t^T [g(X_s^x) - \mu] dK_s^x - \int_t^T Z_s^x dW_s, \quad (1.1)$$

where the given datas are :

- W , a \mathbb{R}^d -valued standard Brownian motion;
- $G = \{\phi > 0\}$ is an open convex subset of \mathbb{R}^d with smooth boundary;
- $x \in G$;
- X^x a \bar{G} -valued process starting from x , and K^x a non decreasing \mathbb{R} -valued process starting from 0 such that the pair (X^x, K^x) is solution of the following stochastic differential equation (SDE in what follows) :

$$X_t^x = x + \int_0^t f(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s + \int_0^t \nabla \phi(X_s^x) dK_s^x, \quad t \geq 0;$$
$$K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}} dK_s^x, \quad K^x \text{ is non decreasing};$$

- $\psi : \mathbb{R}^d \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ is K_ψ -Lipschitz in x and z and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable;
- λ or μ which belong both to \mathbb{R} . If λ is given then μ is unknown and if μ is given then λ is unknown.

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Therefore, the unknown is either the triplet (Y^x, Z^x, λ) if μ is given or the triplet (Y^x, Z^x, μ) if λ is given, where :

- Y^x is a real-valued progressively measurable process;
- Z^x is a $\mathbb{R}^{1 \times d}$ -valued progressively measurable process.

We recall that a function $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to be strictly dissipative if there exists a constant $\eta > 0$ such that, $\forall x, y \in \mathbb{R}^d$,

$$(h(x) - h(y)) \cdot (x - y) \leq -\eta |x - y|^2.$$

Richou in the paper [10] has studied the case when \bar{G} is bounded and with the assumptions that f and σ are Lipschitz and :

$$\eta := \sup_{x, y \in \bar{G}, x \neq y} \left\{ \frac{t(x - y)(b(x) - b(y))}{|x - y|^2} + \frac{\text{Tr}[(\sigma(x) - \sigma(y))^t(\sigma(x) - \sigma(y))]}{2|x - y|^2} \right\} < -K_{\psi, z} K_{\sigma}$$

where $K_{\psi, z}$ is the Lipschitz constant of ψ in z and K_{σ} is the Lipschitz constant of σ . Please note that this assumption implies that f is strictly dissipative. However this condition is not very natural because it induces a dependency between parameters of the problem. In this paper we extend the framework to the case of an unbounded domain \bar{G} . We assume that f is locally Lipschitz and weakly dissipative, namely we assume that d is locally Lipschitz and dissipative and that b is locally Lipschitz and bounded. In addition, σ is Lipschitz, invertible and such that σ and σ^{-1} are bounded. In the paper [2], a similar work is made for an infinite dimensional forward process without reflection, but consequently the dissipative term d is linear, and σ is constant. Another difference is that the coefficients of the forward process are assumed to be Gâteaux differentiable, which is not the case in our framework, where we only need continuity of the coefficients. Indeed, we can get round of this problem with convolution arguments to regularize coefficients with a parameter ε and then study the limit when $\varepsilon \rightarrow 0$. In [10], the strong hypothesis on f is needed to establish one of the key results : the strong estimate on the exponential decay in time of two solutions of the forward equation starting from different points. Indeed, it allows us to construct, by a diagonal procedure, a solution to the EBSDE. In our context, the weaker assumption on f make this strong estimate impossible. However it is possible to substitute this result by a weaker result, called "Basic coupling estimate" which involves the Kolmogorov semigroups of the forward process X^x and that we can find in [2] in the case of a linear dissipativity term and when σ is constant. However this estimate holds for a non-reflected forward process. So, we start by studying the forward process, $\forall t \geq 0$,

$$V_t^x = x + \int_0^t f(V_s^x) ds + \int_0^t \sigma(V_s^x) dW_s.$$

with $f = d + b$ and σ defined as before. Namely, f is assumed to be locally Lipschitz and weakly dissipative. We show that the coupling estimate still holds in our framework with constants which depend on d only through its dissipativity constant. Once this is established, we apply this result to establish existence and uniqueness (of λ) of solutions to the following EBSDE :

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (1.2)$$

Then we want to obtain the same result when the process V^x is replaced by a reflected process X^x in \bar{G} , namely :

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (1.3)$$

For this purpose, we use a penalization method to construct a process $X^{x,n}$ defined on the whole \mathbb{R}^d and which converges to the reflected process X^x . More precisely, we denote by $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$ the solution of the following BSDE with regularized coefficients ψ^ε , d^ε , F_n^ε and b^ε by convolution with a sequence approximating the identity, $\forall t, T \in \mathbb{R}_+$, $0 \leq t \leq T < +\infty$:

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}] ds - \int_t^T Z_s^{x,\alpha,n,\varepsilon} dW_s, \quad (1.4)$$

where $X^{x,n,\varepsilon}$ is the strong solution of the SDE :

$$X_t^{x,n,\varepsilon} = x + \int_0^t (d^\varepsilon + F_n^\varepsilon + b^\varepsilon)(X_s^{x,n,\varepsilon}) ds + \int_0^t \sigma^\varepsilon(X_s^{x,n,\varepsilon}) dW_s.$$

Please note that as F_n is dissipative with a dissipative constant equal to 0. Therefore $d + F_n$ remains dissipative with a dissipative constant equal to η . Then, making $\alpha \rightarrow 0$, $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$, it is possible to show that, roughly speaking, $(Y_t^{x,\alpha,n,\varepsilon} - Y_0^{x,\alpha,n,\varepsilon}, Z_t^{x,\alpha,n,\varepsilon}, Y_0^{x,\alpha,n,\varepsilon}) \rightarrow (Y_t^x, Z_t^x, \lambda)$ which is solution of EBSDE (1.3). Once a solution (Y, Z, λ) is found for the EBSDE (1.3) we study existence and uniqueness of solutions of the type (Y, Z, λ) and (Y, Z, μ) of the EBSDE (1.1). Here we only manage to find degenerate solution, i.e. not markovian and which are not bounded in expectation. Then we show that the function defined by $v(x) := Y_0^x$, where Y is a solution of EBSDE (1.3) is a viscosity solution of the following partial differential equation (PDE in what follows) :

$$\begin{cases} \mathcal{L}v(x) + \psi(x, {}^t\nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \quad (1.5)$$

where :

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(\sigma(x) {}^t\sigma(x) \nabla^2 u(x)) + {}^t f(x) \nabla u(x).$$

At last we show that we can use the theory of EBSDE to solve an optimal ergodic control problem. We define the ergodic cost :

$$I(x, \rho) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_T^\rho \left[\int_0^T L(X_s^x, \rho_s) ds \right], \quad (1.6)$$

where ρ is an adapted process with values in a separable metric space U and \mathbb{E}_T^ρ is expectation under which $W_t^\rho = W_t + \int_0^t R(\rho_s) ds$ is a Brownian motion on $[0, T]$. $R : U \rightarrow \mathbb{R}^d$ is assumed to be bounded and L is assumed to be Lipschitz and bounded. Defining

$$\psi(x, z) = \inf_{u \in U} \{L(x, u) + zR(u)\}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^d,$$

it is possible to show that, for any admissible control ρ , $I(x, \rho) \geq \lambda$. That is why λ is called ergodic cost and μ is called boundary ergodic cost.

The paper is organized as follows. In section 2, we study the forward SDE under the hypothesis that the drift is weakly dissipative and that the diffusion matrix is invertible and bounded. In this section we take care that the estimates we establish depend on d through its dissipativity constant. In section 3, we use the basic coupling estimate to study existence and uniqueness of an EBSDE with zero Neumann boundary conditions with a forward process weakly dissipative but non-reflected. In section 4, we use a penalization method to show that the same result holds for a reflected process in an convex not necessary bounded. In section 5, we establish the link between the EBSDE with zero Neumann boundary condition and a PDE. In section 6, we apply our results to an optimal ergodic control problem.

2 The forward SDE

2.1 General notation

The canonical scalar product on \mathbb{R}^d is denoted by the dot \cdot and the associated norm is denoted by $|\cdot|$. Let \mathcal{O} be an open connected subset of \mathbb{R}^d . We denote by $\mathcal{C}_b^k(\mathcal{O})$ the set of real functions of class \mathcal{C}^k on \mathcal{O} with partial derivatives bounded. We denote by $\mathcal{C}_{\text{lip}}^k$ the set of real functions whose partial derivatives of order less than or equal to k are Lipschitz. We denote by $B_b(\mathcal{O})$ the set of borelian bounded functions defined on \mathcal{O} .

$(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space, $(\mathcal{F}_t)_{t \geq 0}$ denotes a filtration defined on this space satisfying the usual conditions and $(W_t)_{t \geq 0}$ denotes a standard Brownian motion on \mathbb{R}^d .

\mathcal{S}^2 denotes the space of real-valued adapted continuous processes Y such that for all $T > 0$, $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < +\infty$.

$\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^k)$ denotes the space consisting of all progressively measurable processes X , with value in \mathbb{R}^k such that, for all $T > 0$,

$$\mathbb{E} \left[\int_0^T |X_s|^2 ds \right] < +\infty.$$

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be two locally Lipschitz functions. We denote by $(V_t^x)_{t \geq 0}$ the strong maximal solution of the following SDE :

$$V_t^x = x + \int_0^t f(V_s^x) ds + \int_0^t \sigma(V_s^x) dW_s. \quad (2.1)$$

Lemma 2.1. *Assume that $\exists a \in \mathbb{R}^d$, $\eta_1, \eta_2 > 0$ such that, $\forall y \in \mathbb{R}^d$,*

$$f(y) \cdot (y - a) \leq -\eta_1 |y - a|^2 + \eta_2,$$

and that σ is bounded by σ_∞ , then there exists a solution $(X_t^x)_t$ to 2.1 for which the explosion time is almost surely equal to infinity. Furthermore the following estimate holds $\forall t \geq 0$:

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-\eta_1 t}),$$

where C is a constant which depends only on a , η_1 , η_2 and σ_∞ . Furthermore, for all $p > 2$, for all $0 < \beta < p\eta_1$, there exists C wich depends only on p , d , η_1 , η_2 , σ_∞ such that $\forall t \geq 0$:

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-\beta t}).$$

We also have the following inequality :

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |V_t^x|^p \right] \leq C(1 + |x|^p),$$

where C depends on p , d , η_1 , η_2 , σ_∞ and T .

Proof. The proof is given in the appendix. □

We recall that a function is weakly dissipative if it is the sum of an η -dissipative function (namely $\forall x, x' \in \mathbb{R}^d, (d(x) - d(x')) \cdot (x - x') \leq -\eta |x - x'|$), and a bounded function. Thus we write $f = d + b$, with d η -dissipative and b bounded by B .

Hypothesis 2.1.

- $f = d + b$ is weakly dissipative,
- d is locally Lipschitz with polynomial growth,
- b is Lipschitz,
- σ is Lipschitz, invertible, and σ and σ^{-1} are bounded by σ_∞ .

Remark 2.2. It is clear that if f satisfies hypothesis 2.1 then f satisfies the assumption of Lemma 2.1. Indeed, let us suppose that f satisfies the assumption of Lemma 2.4, let $a \in \mathbb{R}^d$, then $\forall y \in \mathbb{R}^d$,

$$\begin{aligned} (f(y) - f(a)) \cdot (y - a) &= (d(y) - d(a)) \cdot (y - a) + (b(y) - b(a)) \cdot (y - a) \\ \Rightarrow f(y)(y - a) &\leq -\eta|y - a|^2 + 2B|y - a| + |f(a)||y - a| \\ \Rightarrow f(y)(y - a) &\leq -\eta|y - a|^2 + \frac{(2B + |f(a)|)^2}{2\varepsilon} + \frac{\varepsilon|y - a|^2}{2}, \end{aligned}$$

which gives us the desired result, for ε small enough.

Lemma 2.3. *Assume that hypothesis (2.1) hold true but this time with b replaced by b_2 which is only bounded measurable and not locally Lipschitz anymore. Then the solution of (2.1) still exists but in the weak sense, namely there exist a new Brownian motion $(\hat{W}_t)_{t \geq 0}$ with respect to a new probability measure $\hat{\mathbb{P}}$ and a new adapted process $(\hat{X}_t)_{t \geq 0}$ such that equation (2.1) is satisfied with $(X_t)_{t \geq 0}$ replaced by $(\hat{X}_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ replaced by $(\hat{W}_t)_{t \geq 0}$. Such a process is unique in law and the estimates of Lemma (2.1) are still satisfied under the new probability $\hat{\mathbb{P}}$.*

Proof. Using the decomposition $b_2 = b + b_2 - b$, it is enough to apply a Girsanov theorem to get rid of the bounded term $b_2 - b$. \square

Lemma 2.4. *Assume that hypothesis 2.1 hold true. Then there exists $C > 0$ and $\mu > 0$ such that $\forall \Phi \in B_b(\mathbb{R}^d)$,*

$$|\mathcal{P}_t[\Phi](x) - \mathcal{P}_t[\Phi](x')| \leq C(1 + |x|^2 + |x'|^2)e^{-\mu t}|\Phi|_0 \quad (2.2)$$

where $\mathcal{P}_t[\Phi](x) = \mathbb{E}\Phi(V_t^x)$ is the Kolmogorov semigroup asociated to (2.1). We stress the fact that the constants C and μ depends on f only trough η and B .

Proof. The proof is given in the appendix. \square

Corollary 2.5. *The estimate (2.2) can be extended to the case in wich b is only bounded measurable and there exists a uniformly bounded sequence of Lipschitz functions $\{b_m\}_{m \geq 1}$ (i.e. b_m is Lipchitz and $\sup_m \sup_x |b_m(x)| < +\infty$) such that*

$$\forall x \in \mathbb{R}^d, \quad \lim_m b_m(x) = b(x).$$

In this case, we define the semigroup relatively to the new probability measure, namely, $\mathcal{P}_t[\Phi](x) := \hat{\mathbb{E}}\Phi(\hat{X}_t^x)$.

Proof. We denote by \mathcal{P}_t^m the Kolmogorov semigroup of (4.1) with b replaced by b_m , for more clarity we rewrite this equation below : $\forall x \in \overline{G}$,

$$V_t^{x,m} = x + \int_0^t (d + b_m)(V_s^{x,m}) ds + \int_0^t \sigma(V_s^{x,m}) dW_s.$$

It is sufficient to prove that, $\forall x \in \overline{G}$, $\forall t \geq 0$,

$$\mathcal{P}_t^m[\Phi](x) \rightarrow \mathcal{P}_t[\Phi](x).$$

To do that, it is easy to adapt the proof from [2] replacing the process U_t^x by its analogue in our context. Thus we define U_t^x as the strong solution of the following SDE :

$$U_t^x = x + \int_0^t d(U_s^x) ds + \int_0^t \sigma(U_s^x) dW_s,$$

and the rest remains the same. □

3 The ergodic BSDE

In this section we study the following EBSDE in infinite horizon :

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (3.1)$$

For now, the forward process, defined as the strong solution of (2.1) is not reflected. However this result is interesting for its own, because it makes appears some ideas which will be reused in the next section.

We will need the following hypothesis on $\psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$:

Hypothesis 3.1. ψ is Lipschitz and there exists $M_\psi \in \mathbb{R}$ such that : $\forall x, x' \in \mathbb{R}^d, \forall z, z' \in \mathbb{R}^{1,d}$,

$$|\psi(x, 0)| \leq M_\psi,$$

$$|\psi(x, z) - \psi(x', z)| \leq M_\psi |x - x'|,$$

$$|\psi(x, z) - \psi(x, z')| \leq M_\psi |z - z'|.$$

Hypothesis 3.2. f is \mathcal{C}^1 and b, σ and $\psi \in \mathcal{C}_b^1$.

Using the standard approach (see [3]), we are going to study the following BSDE in infinite horizon

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [\psi(V_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}] ds - \int_t^T Z_s^{x,\alpha} dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (3.2)$$

Such an equation has been studied in [1] from which we have the following result :

Lemma 3.1. *Assume that hypothesis (2.1) and (3.1) hold true. Then there exists a unique solution $(Y^{x,\alpha}, Z^{x,\alpha})$ to BSDE (3.2) such that $Y^{x,\alpha}$ is bounded adapted continuous process and $Z^{x,\alpha} \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$. Furthermore, $|Y_t^{x,\alpha}| \leq \frac{M_\psi}{\alpha}$. Finally there exists a function v^α such that $Y_t^{x,\alpha} = v^\alpha(X_t^x)$ \mathbb{P} -a.s. and there exists a measurable function $\zeta^\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $Z_t^{x,\alpha} = \zeta^\alpha(X_t^x)$ \mathbb{P} -a.s.*

We will need the following lemma :

Lemma 3.2. *Let ζ, ζ' be two measurable functions : $\mathbb{R}^d \rightarrow \mathbb{R}^d$. We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x, \zeta(x)) - \psi(x, \zeta'(x))}{|\zeta(x) - \zeta'(x)|^2} t(\zeta(x) - \zeta'(x)), & \text{if } \zeta(x) \neq \zeta'(x) \\ 0, & \text{if } \zeta(x) = \zeta'(x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_n)_{n \geq 0}$ (i.e., $\forall n, \Upsilon_n$ is Lipschitz and $\sup_n \sup_x |\Upsilon_n(x)| < +\infty$) such that Υ_n converges pointwise to Υ .

Proof. The function Υ is bounded so it is enough to regularize it by convolution with an approximation of the identity. \square

The following lemma gives us the desirate estimates on $v^\alpha(x)$ which will allow us to apply a diagonal procedure.

Lemma 3.3. *Assume that hypothesis (2.1), (3.1) and (3.2) hold true. Then, there exists a constant C independent of α and which depends on f only through η and B , on σ only through σ_∞ and on ψ only through M_ψ such that*

$$|v^\alpha(x) - v^\alpha(x')| \leq C(1 + |x|^2 + |x'|^2).$$

Suppose in addition that hypothesis (3.2) hold true. Then there exist a constant C independent of α and which depends on f only through η and B , on σ only through σ_∞ and on ψ only through M_ψ such that for all $x \in \mathbb{R}^d$,

$$|\nabla v^\alpha(x)| \leq C(1 + |x|^2).$$

Proof. The first part of the proof is identical to the proof of Lemma (3.6) in [2] with $\nabla v^\alpha(X_t^x)G$ replaced by $Z_t^{x,\alpha}$. The second assertion need to be precised, because the dissipative term d is not linear anymore. Thus we mix arguments from [2], [6] and [11]. First we set $\bar{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0)$. Then $\bar{Y}_t^{x,\alpha} = Y_t^{x,\alpha} - Y_0^{0,\alpha} = \bar{v}^\alpha(V_t^x)$ is the unique solution of the finite horizon BSDE :

$$\begin{cases} -d\bar{Y}_1^{x,\alpha} &= \psi(V_1^x, Z_1^{x,\alpha})dt - \alpha\bar{Y}_1^{x,\alpha}dt - \alpha v^\alpha(0)dt - Z_1^{x,\alpha}dW_1 \\ \bar{Y}_1^{x,\alpha} &= \bar{v}^\alpha(V_1^x). \end{cases}$$

The representation Theorem 4.2 from [6] still hold under the hypothesis of this Lemma. Indeed the paper [11] guarantees the weak differentiability of the process V_t^x . However the theorem 4.2 in [6] asks for \bar{v}^α to be Lipschitz. But applying the same regularization method as in the proof of theorem 4.2 in [4] one can approach v^α by a sequence of functions $v^{\alpha,n}$ which is Lipschitz and converge pointwise to v^α . This shows that v^α is Gâteaux and that the theorem 4.2 in [6] still hold under our assumptions. It follows that :

$$\nabla v^\alpha(x) = \mathbb{E} \left\{ \bar{v}^\alpha(V_1^x)N_1^0 + \int_0^1 \left[\psi(V_r^x, Z_r^{x,\alpha}) - \alpha\bar{Y}_r^{x,\alpha} - \alpha v^\alpha(0) \right] N_r^0 dr \right\}$$

where $N_r^0 = \frac{1}{r}t \left[\int_0^r t [\sigma^{-1}(V_s^x)\nabla V_s^x] dW_s \right] [\nabla V_0^x]^{-1}$. We recall that $[\nabla V_0^x]^{-1} = I$, the identity matrix in $\mathbb{R}^{d \times d}$, so

$$\mathbb{E}|\bar{v}^\alpha(V_1^x)N_1^0| \leq \sqrt{\mathbb{E}(|\bar{v}^\alpha(V_1^x)|^2)} \times \sqrt{\mathbb{E}(|N_1^0|^2)}.$$

We have, from Lemma (2.1) :

$$\sqrt{\mathbb{E}(|\bar{v}^\alpha(V_1^x)|^2)} \leq C(1 + |x|^2).$$

Furthermore, using a BDG inequality it is clear that

$$\sqrt{\mathbb{E}(|N_1^0|^2)} \leq C \mathbb{E} \sup_{0 \leq s \leq t} |\nabla V_s^x|^2.$$

Now we need an estimate on $|\nabla V_t^x|^2$. As ∇V_t^x is the solution of the following variational equation (see equation (2.9) in [6]) :

$$\nabla_i V_t^x = e_i + \int_0^t \nabla(d+b)(V_s^x) \nabla_i V_s^x ds + \sum_{j=1}^d \int_0^t [\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x dW_s^j$$

An Itô formula gives us :

$$\begin{aligned} |\nabla_i V_t^x|^2 &= 1 + 2 \int_0^t \nabla_i V_s^x \cdot d\nabla_i V_s^x + \sum_{j=1}^d \int_0^t |[\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x|^2 ds \\ &= 1 + 2 \int_0^t \nabla_i V_s^x \cdot [\nabla(d+b)(V_s^x) \nabla_i V_s^x] ds + \sum_{j=1}^d \int_0^t |[\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x|^2 ds \\ &\quad + 2 \sum_{j=1}^d \int_0^t \nabla_i V_s^x \cdot [[\partial_x \sigma^j(V_s^x)] \nabla_i V_s^x] dW_s^j \end{aligned}$$

Due to the fact that d is dissipative and differentiable, we have that $\forall \xi \in \mathbb{R}^d$, $\xi \nabla d(x) \xi \leq -\eta |\xi|^2$. Furthermore using the fact that b and σ belong to \mathcal{C}_b^2 , one can verify by classic calculus that, for all $t \leq 1$:

$$\mathbb{E} |\nabla_i V_t^x|^2 \leq C,$$

for a constant C which depends on d only through η . Then, by a BDG formula we can state that

$$\mathbb{E} \sup_{0 \leq s \leq 1} |\nabla_i V_s^x|^2 \leq C.$$

Finally,

$$\mathbb{E} \sup_{0 \leq s \leq 1} |\nabla V_s^x|^2 \leq C.$$

We have proved that :

$$\mathbb{E} |\bar{v}^\alpha(V_1^x) N_1^0| \leq C(1 + |x|^2).$$

Now we deal with the second term in the representation formula of ∇v^α . One can notice that the generator is M_ψ -Lipschitz in z and $\alpha < 1$ -Lipschitz in y and that $\alpha v^\alpha(0) \leq M_\psi$, we can apply the estimate of Lemma (2.2) from [6] to get the desirable estimates on $|\bar{Y}_t^{x,\alpha}|^2$ and $|Z_t^{x,\alpha}|^2$.

Furthermore :

$$\begin{aligned}
\frac{1}{r}\mathbb{E}\left(\int_0^r [\sigma^{-1}(V_s^x)\nabla V_s^x] dW_s\right)^2 &\leq C\frac{1}{r}\mathbb{E}\sup_{0\leq r\leq 1}\int_0^r |\sigma^{-1}(V_s^x)\nabla V_s^x|^2 ds \\
&\leq C\frac{1}{r}\mathbb{E}\sup_{0\leq r\leq 1}\sup_{0\leq s\leq r} |\sigma^{-1}(V_s^x)\nabla V_s^x|^2 \\
&\leq C\mathbb{E}\sup_{0\leq s\leq 1} |\nabla V_s^x|^2 \\
&\leq C.
\end{aligned}$$

Finally we have proved that

$$|\nabla v^\alpha(x)| \leq C(1 + |x|^2).$$

□

Thanks to this estimate, it is possible to get an existence result for the EBSDE (3.1). Here the hypothesis 3.2 can be removed thanks to convolutions arguments which will appear in the proof.

Theorem 3.4. *Assume that hypothesis (2.1) and (3.1) hold true. Then there exists a solution $(\bar{Y}^x, \bar{Z}^x, \bar{\lambda})$ to the EBSDE (3.1) such that $\bar{Y}^x = \bar{v}(V^x)$ with \bar{v} locally Lipschitz, and there exists a measurable function $\bar{\xi} : \mathbb{R}^d \rightarrow \mathbb{R}^{1,d}$ such that $\bar{Z}_t^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ and $\bar{Z}^x = \bar{\xi}(V^x)$.*

Proof. We start by regularizing f and ψ thanks to classical convolution arguments. For all $k \in \mathbb{N}^*$ let us denote by $\rho_\varepsilon^k : \mathbb{R}^k \rightarrow \mathbb{R}$ the classical mollifier for which the support is the ball of center 0 and radius ε . Let us denote $\forall n \in \mathbb{N}$, $f^{\varepsilon_n} := f * \rho_{\varepsilon_n}^d$, $\psi^{\varepsilon_n} := \psi * \rho_{\varepsilon_n}^{d,1}$ and $\sigma^{\varepsilon_n} := \sigma * \rho_{\varepsilon_n}^{d \times d}$. those functions are \mathcal{C}^1 and satisfies :

- d^{ε_n} is η -dissipative;
- b^{ε_n} is bounded by B ;
- ψ^{ε_n} satisfies the hypothesis 3.1;
- σ^{ε_n} is invertible
- $d^{\varepsilon_n} \rightarrow d$, $b^{\varepsilon_n} \rightarrow b$, $\psi^{\varepsilon_n} \rightarrow \psi$, $\sigma^{\varepsilon_n} \rightarrow \sigma$ pointwise as $\varepsilon_n \rightarrow 0$.

We denote by V_t^{x,ε_n} the solution of (2.1) with f replaced by f^{ε_n} and σ replaced by σ^{ε_n} . The same notation is used for the regularized BSDE, we denote by $(Y_t^{x,\alpha,\varepsilon_n}, Z_t^{x,\alpha,\varepsilon_n})$ the solution in $\mathcal{S}^2 \times \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ of BSDE (3.2) with ψ replaced by ψ^{ε_n} (existence and uniqueness of such a solution is guaranteed by Lemma 3.1), namely :

$$\begin{aligned}
Y_t^{x,\alpha,\varepsilon_n} &= Y_T^{x,\alpha,\varepsilon_n} + \int_t^T (\psi(V_s^{x,\varepsilon_n}, Z_s^{x,\alpha,\varepsilon_n}) - \alpha Y_s^{x,\alpha,\varepsilon_n}) ds \\
&\quad - \int_t^T Z_s^{x,\alpha,\varepsilon_n} dW_s, \quad 0 \leq t \leq T < +\infty.
\end{aligned}$$

Then we define $v^{\alpha,\varepsilon_n}(x) := Y_0^{x,\alpha,\varepsilon_n}$ and $\bar{Y}_t^{x,\alpha,\varepsilon_n} = Y_t^{x,\alpha,\varepsilon_n} - \alpha v^{\alpha,\varepsilon_n}(0)$. We can rewriting the BSDE, hereabove and we get :

$$\begin{aligned}
\bar{Y}_t^{x,\alpha,\varepsilon_n} &= \bar{Y}_T^{x,\alpha,\varepsilon_n} + \int_t^T (\psi(V_s^{x,\varepsilon_n}, Z_s^{x,\alpha,\varepsilon_n}) - \alpha \bar{Y}_s^{x,\alpha,\varepsilon_n} - \alpha v^{\alpha,\varepsilon_n}(0)) ds \\
&\quad - \int_t^T Z_s^{x,\alpha,\varepsilon_n} dW_s, \quad 0 \leq t \leq T < +\infty.
\end{aligned}$$

Uniqueness of solutions implies that $v^{\alpha, \varepsilon_n}(V_s^{x, \varepsilon_n}) = Y_s^{x, \alpha, \varepsilon_n}$. Now, in very classical way, we set $\bar{v}^{\alpha, \varepsilon_n}(x) = v^{\alpha, \varepsilon_n}(x) - v^{\alpha, \varepsilon_n}(0)$. Thanks to the fact that $\alpha|v^{\alpha, \varepsilon_n}(0)| \leq M_\psi$ and by Lemma 3.3 we can extract two subsequences $(\alpha_n)_{n \rightarrow +\infty} \rightarrow 0$ and $(\beta(\varepsilon_n))_{n \rightarrow +\infty} \rightarrow 0$ such that $\forall \varepsilon_n, \forall x \in D$ a countable subset of \mathbb{R}^d :

$$\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(x) \xrightarrow{n \rightarrow +\infty} \bar{v}(x) \quad \text{and} \quad \alpha_n v^{\alpha_n, \beta(\varepsilon_n)}(0) \xrightarrow{n \rightarrow +\infty} \bar{\lambda}.$$

for a suitable function \bar{v} and a suitable real $\bar{\lambda}$. Now thanks to the gradient estimates from Lemma 3.3 we have $|\bar{v}^{\alpha, \beta(\varepsilon_n)}(x) - \bar{v}^{\alpha, \beta(\varepsilon_n)}(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|$ for all $x, x' \in \mathbb{R}^d$ therefore extending \bar{v} to the whole \mathbb{R}^d setting $\bar{v}(x) = \lim_{x_p \rightarrow x} \bar{v}(x_p)$ we keep the following estimates : for all $x \in \mathbb{R}^d$,

$$|\bar{v}(x) - \bar{v}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|$$

As $|\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(x)| \leq C(1 + |x|^2)$ and with the second inequality of Lemma (2.1) we get that : $\mathbb{E} \sup_{0 \leq t \leq T} [\sup_{\alpha, \beta > 0} |\bar{Y}_t^{x, \alpha, \beta(\varepsilon_n)}|^2] < +\infty, \forall T > 0$. Furthermore, defining, $\bar{Y}^x = \bar{v}(V^x)$ the inequality

$$\begin{aligned} |\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(V_s^{x, \beta(\varepsilon_n)}) - \bar{v}(V_s^x)| &\leq |\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(V_s^{x, \beta(\varepsilon_n)}) - \bar{v}^{\alpha_n, \beta(\varepsilon_n)}(V_s^x)| \\ &\quad + |\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(V_s^x) - \bar{v}(V_s^x)| \\ &\leq C(1 + |V_s^{x, \beta(\varepsilon_n)}|^2 + |V_s^x|^2)|V_s^{x, \beta(\varepsilon_n)} - V_s^x| \\ &\quad + |\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(V_s^x) - \bar{v}(V_s^x)| \end{aligned}$$

for all $s \geq 0$ shows us the pointwise convergence of $\bar{v}^{\alpha_n, \beta(\varepsilon_n)}(V_T^{x, \beta(\varepsilon_n)})$ toward $\bar{v}(V_T^x)$ thanks to the following well known convergence : $\mathbb{E} \sup_{0 \leq t \leq T} |V_t^{x, \beta(\varepsilon_n)} - V_t^x|^2$ toward 0. Then we can apply the dominated convergence theorem to show that :

$$\mathbb{E} \int_0^T |\bar{Y}_s^{x, \alpha_n, \beta(\varepsilon_n)} - \bar{Y}_s^x|^2 ds \rightarrow 0 \quad \text{and} \quad \mathbb{E} |\bar{Y}_T^{x, \alpha_n, \beta(\varepsilon_n)} - \bar{Y}_T^x|^2 \rightarrow 0.$$

Now we show that $(Z^{x, \alpha_n, \beta(\varepsilon_n)})_n$ is Cauchy in $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$. We denote

$$\tilde{V}_t = V_t^{x, \beta(\varepsilon_n)} - V_t^{x, \beta(\varepsilon_n)'};$$

$$\tilde{Y}_t = \bar{Y}_t^{x, \alpha_n, \beta(\varepsilon_n)} - \bar{Y}_t^{x, \alpha_n', \beta(\varepsilon_n)'},$$

$$\tilde{Z}_t = \bar{Z}_t^{x, \alpha_n, \beta(\varepsilon_n)} - \bar{Z}_t^{x, \alpha_n', \beta(\varepsilon_n)'},$$

and

$$\tilde{\lambda} = \alpha_n v^{\alpha_n, \beta(\varepsilon_n)}(0) - \alpha_n' v^{\alpha_n', \beta(\varepsilon_n)'}(0).$$

An Itô formula applied to $|\tilde{Y}_t|$ gives us, for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$:

$$\begin{aligned}
|\tilde{Y}_t|^2 + \int_t^T |\tilde{Z}_s|^2 ds &= |\tilde{Y}_T|^2 + 2 \int_t^T \tilde{Y}_s [\psi(V_s^{x, \beta(\varepsilon_n)}, Z_s^{x, \alpha_n, \beta(\varepsilon_n)}) - \psi(V_s^{x, \beta(\varepsilon_n)'}, Z_s^{x, \alpha_n', \beta(\varepsilon_n)'}) \\
&\quad - (\alpha_n \bar{Y}_s^{x, \alpha_n, \beta(\varepsilon_n)} - \alpha_n' \bar{Y}_s^{x, \alpha_n', \beta(\varepsilon_n)'}) - \tilde{\lambda}] ds \\
&\quad - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s \\
&\leq |\tilde{Y}_T|^2 + (\varepsilon_1 M_\psi + \varepsilon_2 M_\psi + \varepsilon_3) \int_t^T |\tilde{Y}_s|^2 ds + \frac{M_\psi}{\varepsilon_1} \int_t^T |\tilde{V}_s|^2 ds \\
&\quad + \frac{M_\psi}{\varepsilon_2} \int_t^T |\tilde{Z}_s|^2 ds + \frac{1}{\varepsilon_3} \int_t^T |\tilde{\lambda}|^2 ds + c \int_t^T |\tilde{Y}_s| ds - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s,
\end{aligned}$$

because $\alpha v^{\alpha, \varepsilon}(0) \leq M_\psi$. Thus, taking the expectation and for ε_2 large enough we get

$$\mathbb{E} \int_0^T |\tilde{Z}_s|^2 ds \leq \mathbb{E} |\tilde{Y}_T|^2 + c \left(\mathbb{E} \left[\int_0^T |\tilde{Y}_s|^2 ds \right] + \mathbb{E} \left[\int_0^T |\tilde{V}_s|^2 ds \right] + \mathbb{E} \left[\int_0^T |\tilde{Y}_s| ds \right] + T |\tilde{\lambda}|^2 \right).$$

which proves that $(Z^{x, \alpha_n, \beta(\varepsilon_n)})_{\alpha_n, \beta(\varepsilon_n)}$ is Cauchy in $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$. The end of the proof is very classical, using a BDG inequality to show that \bar{Y}^x is Cauchy for the norm $\mathbb{E} \sup_{0 \leq t \leq T} |\cdot|^2$, $\forall T > 0$. To show that \bar{Z}^x is Markovian, just apply the same method as in the proof of Theorem 4.4 in [3]. □

Remark 3.5. Due to the fact that the estimates of Lemma (2.4) is not of the kind $|V_t^x - V_t^{x'}| \leq |x - x'|$, we can not apply the same argument as in [3] to get that \bar{Z}^x is markovian.

Remark 3.6. It is clear that we dont have uniqueness of the solutions of EBSDE (3.1) because if (Y, Z, λ) is a solution then $(Y + \theta, Z, \lambda)$ is another solution, for all $\theta \in \mathbb{R}$. However we have the following uniqueness property for λ under the following polynomial growth property :

$$|Y_t^x| \leq C(1 + |V_t^x|^2).$$

One can notice that the solution $\bar{Y}_t^x = \bar{v}(X_t^x)$ constructed in the proof Theorem (3.4) satisfies such a growth property.

Theorem 3.7. (*Uniqueness of λ*). Assume that hypothesis 2.1 and 3.1 hold true. Let us suppose that we have two solutions of the EBSDE (4.2) denoted by (Y, Z, λ) and (Y', Z', λ') where Y and Y' are progressively measurable continuous processes, Z and $Z' \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ and $\lambda, \lambda' \in \mathbb{R}$. Finally assume that the following growth properties hold :

$$\begin{aligned}
|Y_t| &\leq C(1 + |V_t^x|^2) \\
|Y'_t| &\leq C'(1 + |V_t^x|^2).
\end{aligned}$$

Then $\lambda = \lambda'$.

Proof. Simply, adapt the proof of Theorem 4.6 of [3]. With the same notations one can write :

$$\begin{aligned}
\tilde{\lambda} &= T^{-1} \mathbb{E}^{\mathbb{P}^h} [\tilde{Y}_T - \tilde{Y}_0] \\
&\leq T^{-1} \mathbb{E}^{\mathbb{P}^h} ((C + C')(1 + |V_t^x|^2)) + T^{-1} \mathbb{E}^{\mathbb{P}^h} ((C + C')(1 + |x|^2))
\end{aligned}$$

To conclude, just use the estimates from Lemma (2.1), and let $T \rightarrow +\infty$. □

4 The ergodic BSDE with zero and non-zero Neumann boundary conditions in a weakly dissipative environment

In this section we replace the process $(V_t^x)_{t>0}$ by the process $(X_t^x)_{t>0}$, which is solution of a stochastic differential equation reflected in an open convex subset G of \mathbb{R}^d with regular boundary. The randomness of EBSDE's generator that we are going to consider will be fully given by this process. We denote by $\Pi(x)$ the projection of $x \in \mathbb{R}^d$ on \overline{G} . We recall the following property :

$$(x' - x) \cdot (x - \Pi(x)) \leq (x' - \Pi(x')) \cdot (x - \Pi(x))$$

We will need the following assumptions on G :

Hypothesis 4.1. G is an open convex set of \mathbb{R}^d .

Hypothesis 4.2. There exists a function $\phi \in \mathcal{C}_b^2(\mathbb{R}^d)$ such that $G = \{\phi > 0\}$, $\partial G = \{\phi = 0\}$ and $|\nabla\phi(x)| = 1$, $\forall x \in \partial G$.

Then it is possible to establish the following result, by a penalization method.

Lemma 4.1. *Assume that hypothesis (2.1), (4.2) and hold true. Then for every $x \in \overline{G}$, there exists a unique adapted continuous couple of processes $\{(X_t^x, K_t^x)_{t \geq 0}\}$ with values in $(\overline{G} \times \mathbb{R}_+)$ such that $\forall t \geq 0$:*

$$\begin{aligned} X_t^x &= x + \int_0^t f(X_s^x) ds + \int_0^t \sigma(X_s^x) ds + \int_0^t \nabla\phi(X_s^x) dK_s^x, \\ K_t^x &= \int_0^t \mathbf{1}_{\{X_s^x \in \partial G\}} dK_s^x. \end{aligned}$$

Proof. The proof is a consequence of the following Lemma. □

We are going to penalize this reflected SDE in order to avoid dealing with the reflection term. This way, we will be able to apply some of the results from the previous section. Then we consider the process $(X_t^{x,n})_{t \geq 0}$ solution of the following penalized equation, $\forall x \in \overline{G}$,

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n}) ds + \int_0^t \sigma(X_s^{x,n}) dW_s, \quad (4.1)$$

where $\forall x \in \mathbb{R}^d$, $F_n(x) = -2n(x - \Pi(x))$.

The functions $d + F_n + b$ and σ satisfy the Hypothesis 2.1. Indeed, from [5], F_n is 0-dissipative therefore $b + F_n$ remains η -dissipative thus the estimate of Lemma 2.4 holds with constants which does not depend on n . Furthermore one can remark that for all $\xi \in \mathbb{R}^d$, $\xi \nabla F_n(x) \xi \leq 0$, for all $x \in \mathbb{R}^d$ (see for example [5]). Finally, taking $a \in \overline{G}$ (thus $F_n(a) = 0$) in Remark 2.2 shows us that the estimate of Lemma 2.1 holds with constants that does not depend on n .

We will need the following Lemma, proved in Appendix. it is not necessary to ask for f to be Lipschitz but only locally Lipschitz, due to the fact that we work with a weakly dissipative drift.

Lemma 4.2. *Assume that hypothesis (2.1) and (4.1) hold true. Let $(X_t^{x,n})_{t>0}$ be the strong solution of equation (4.1). Then for any $1 < q < p/2$,*

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_s^{x,n} - X_s^{x,m}|^p \leq C(T)(1 + |x|^p) \left(\frac{1}{n^q} + \frac{1}{m^q} \right),$$

where $C(T)$ is a constant which depends on T , η , B and σ_∞ .

Proof. The proof is given in the appendix. □

4.1 The ergodic BDSE with zero Neumann boundary conditions in a weakly dissipative environment

In a first time we are concerned with the following EBSDE with zero Neumann condition in infinite horizon :

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty, \quad (4.2)$$

where the unknown is the triplet (Y^x, Z^x, λ) . $(X_t^x)_{t>0}$ is the solution of (??).

To do that, we are going to study the following BSDE, with an artificial monotonic drift in y , regularized coefficients and penalized generator, namely : $\forall 0 \leq t \leq T < +\infty$,

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}] ds - \int_t^T Z_s^{x,\alpha,n,\varepsilon} dW_s. \quad (4.3)$$

where the process $(X_t^{x,n,\varepsilon})$ is the solution of the following SDE :

$$X_t^{x,n,\varepsilon} = x + \int_0^t f^\varepsilon(X_s^{x,n,\varepsilon}) + F_n^\varepsilon(X_s^{x,n,\varepsilon}) ds + \int_0^t \sigma(X_s^{x,n,\varepsilon}) dW_s.$$

Remark 4.3. F_n is regularized like other regularized functions. Thanks to convolutions arguments it is possible to construct a sequence of functions F_n^ε which converge pointwise toward F_n and such that for all ε , F_n^ε is 0-dissipative and $4n$ -Lipschitz.

Now we can state the existence theorem for the EBSDE (4.2).

Theorem 4.4. *Assume that Hypothesis (2.1), (3.1) and (4.1) hold. Then there exists a solution $(\bar{Y}_t^x, \bar{Z}_t^x, \bar{\lambda})$ to the EBSDE (4.2) such that $\bar{Y}^x = \bar{v}(X^x)$ with \bar{v} locally Lipschitz, $\bar{Z}^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ and $\bar{\lambda}$ real.*

Proof. We give the main ideas, because the proof is very similar to the proof of Theorem 3.4. However two extractions are required to obtain the convergence of the solution. The beginning of the proof is the same as the proof of the Theorem 3.4. The Lemma 3.1 gives us the existence and uniqueness of the solution $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$ of BSDE 4.3 in $\mathcal{S}^2 \times \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$. Then, as the function $d + F_n$ is still η -dissipative and as the work in the previous section involves d only through its dissipativity constant η , we can apply previous results. As the estimates are uniform in n , α and ε it is possible to construct by a diagonal procedure two subsequences $\alpha_n \rightarrow 0$ and $\gamma(n) \rightarrow +\infty$ such that

$$\bar{v}^{\alpha_n, \gamma(n), \varepsilon}(x) \rightarrow \bar{v}^\varepsilon(x) \quad \text{and} \quad \alpha_n v^{\alpha_n, \gamma(n), \varepsilon}(0) \rightarrow \bar{\lambda}^\varepsilon,$$

where $v^{\alpha,n,\varepsilon}(x) := Y_0^{\alpha,n,\varepsilon}$. We recall the fact that the function \bar{v}^ε is locally Lipschitz on \mathbb{R}^d and that we keep the following estimates :

$$|\bar{\lambda}^\varepsilon| \leq M_\psi,$$

$$|\bar{v}^\varepsilon(x) - \bar{v}^\varepsilon(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|.$$

Now we show that the triplet $(Y^{x,\alpha_n,\gamma(n),\varepsilon}, Z^{x,\alpha_n,\gamma(n),\varepsilon}, \alpha_n v^{\alpha_n,\gamma(n),\varepsilon}(0))$ converge towards a solution $(Y^{x,\varepsilon}, Z^{x,\varepsilon}, \lambda^\varepsilon)$ in a good sense. We define $\bar{Y}_s^{x,\varepsilon} := \bar{v}^\varepsilon(X_s^{x,\varepsilon})$. Like in the Theorem 3.4, it is easy to show that $\mathbb{E} \int_0^T |\bar{Y}_s^{x,\alpha_n,\gamma(n),\varepsilon} - \bar{Y}_s^{x,\varepsilon}|^2 ds$ and $\mathbb{E} |\bar{Y}_T^{x,\alpha_n,\gamma(n),\varepsilon} - \bar{Y}_T^{x,\varepsilon}|^2$ converge toward 0. We use the following notations $\tilde{X}_t = X_t^{x,\gamma(n),\varepsilon} - X_t^{x,\gamma(n),\varepsilon}$, $\tilde{Y}_t = \bar{Y}_t^{x,\alpha_n,\gamma(n),\varepsilon} - \bar{Y}_t^{x,\alpha_n,\gamma(n),\varepsilon}$,

$\tilde{Z}_t = \bar{Z}_t^{x, \alpha_n, \gamma(n), \varepsilon} - \bar{Z}_t^{x, \alpha'_n, \gamma(n)', \varepsilon}$ and $\tilde{\lambda} = \alpha_n v^{\alpha_n, \gamma(n), \varepsilon}(0) - \alpha'_n v^{\alpha'_n, \gamma(n)', \varepsilon}(0)$. An Itô formula applied to $|\tilde{Y}_t|^2$ gives us, for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$:

$$\begin{aligned} |\tilde{Y}_t|^2 + \int_t^T |\tilde{Z}_s|^2 ds &= |\tilde{Y}_T|^2 + 2 \int_t^T \tilde{Y}_s [\psi(V_s^{x, \gamma(n), \varepsilon}, Z_s^{x, \alpha_n, \gamma(n), \varepsilon}) - \psi(V_s^{x, \gamma(n)', \varepsilon}, Z_s^{x, \alpha'_n, \gamma(n)', \varepsilon}) \\ &\quad - (\alpha_n \bar{Y}_s^{x, \alpha_n, \gamma(n), \varepsilon_n} - \alpha'_n \bar{Y}_s^{x, \alpha'_n, \gamma(n)', \varepsilon}) - \tilde{\lambda}] ds \\ &\quad - 2 \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s \end{aligned}$$

Using the same technic as in the Theorem 3.4 we get, for all $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3 > 0$,

$$\mathbb{E} \int_0^T |\tilde{Z}_s|^2 ds \leq \mathbb{E} |\tilde{Y}_T|^2 + c \left(\mathbb{E} \left[\int_0^T |\tilde{Y}_s|^2 ds \right] + \mathbb{E} \left[\int_0^T |\tilde{X}_s|^2 ds \right] + \mathbb{E} \left[\int_0^T |\tilde{Y}_s| ds \right] + T |\tilde{\lambda}|^2 \right).$$

This last inequality shows that the process $(Z^{x, \alpha_n, \gamma(n), \varepsilon})_n$ is Cauchy in $\mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$. Then by usual techniques (BDG) it is possible to show that for all $T > 0$,

$$\mathbb{E} \sup_{0 \leq t \leq T} |\bar{Y}_s^{x, \alpha_n, \gamma(n), \varepsilon} - \bar{Y}_s^{x, \varepsilon}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Now, because the estimates

$$|\bar{\lambda}^\varepsilon| \leq M_\psi,$$

$$|\bar{v}^\varepsilon(x) - \bar{v}^\varepsilon(x')| \leq c(1 + |x|^2 + |x'|^2)|x - x'|$$

are uniform in ε , it is possible, by a diagonale procedure to extract a sequence $\beta(\varepsilon_n)$ such that

$$\bar{v}^{\beta(\varepsilon)}(x) \rightarrow \bar{v}(x) \quad \text{and} \quad \bar{\lambda}^{\beta(\varepsilon)} \rightarrow \bar{\lambda}.$$

The end of the proof is very similar to the previous developed arguments so we do not give the details here. \square

Once again, we notice that the solution we have constructed satisfies the following growth property :

$$|\bar{Y}_t^x| \leq C(1 + |X_t^x|^2),$$

so it is natural to establish the following theorem under the same growth properties.

Theorem 4.5. (*Uniqueness of λ*). *Assume that f is Lipschitz and that Hypothesis 3.1 hold true. Let (Y, Z, λ) be a solution of EBSDE (4.2). Then λ is unique among solutions (Y, Z, λ) such that Y is a bounded continuous process and $Z \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$. Finally assume that we have the following growth property*

$$\begin{aligned} |Y_t| &\leq C(1 + |X_t^x|^2) \\ |Y'_t| &\leq C'(1 + |X_t^x|^2). \end{aligned}$$

Then $\lambda = \lambda'$.

Proof. Simply, adapt the proof of Theorem 4.6 of [3]. With the same notations once can write :

$$\begin{aligned}\tilde{\lambda} &= T^{-1} \mathbb{E}^{\mathbb{P}^h} [\tilde{Y}_T - \tilde{Y}_0] \\ &\leq (C + C') T^{-1} (2 + |x|^2 + \mathbb{E}^{\mathbb{P}^h} |X_t^x|^2) \\ &\leq (C + C') T^{-1} (2 + |x|^2 + \mathbb{E}^{\mathbb{P}^h} |X_t^{x,n}|^2 + \mathbb{E}^{\mathbb{P}^h} |X_t^x - X_t^{x,n}|^2)\end{aligned}$$

To conclude, just use the first estimate from Lemma (2.1) and the estimate (C.6) which allow us to get rid of the dependency on n and t , then let $T \rightarrow +\infty$. \square

4.2 The ergodic BDSE with non-zero Neumann boundary conditions in a weakly dissipative environment

We are now concerned by the following EBSDE in infinite horizon :

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds + \int_t^T [g(X_s^x) - \mu] dK_s^x - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.4)$$

where $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable.

Proposition 4.6. (*Existence of a Solution (Y, Z, λ)*). *Assume that hypothesis (2.1), (3.1) and (4.2) hold true. Then for any $\mu \in \mathbb{R}$ there exists $\lambda \in \mathbb{R}$, Y^x continuous adapted process and $Z^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ such that the triple (Y, Z, λ) is \mathbb{P} -a.s. a solution of EBSDE (4.4) with μ fixed.*

Proof. The Theorem 4.4 gives us the existence of a solution (Y^x, Z^x, λ) of the following EBSDE

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda] ds - \int_t^T Z_s^x dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.5)$$

Now, defining $\hat{Y}_t^x = Y_t^x - \int_0^t [g(X_s^x) - \mu] dK_s^x$, it is easy to see that $(\hat{Y}^x, Z^x, \lambda)$ is a solution of the EBSDE (4.4) with μ fixed. \square

Remark 4.7. The solution constructed \hat{Y}^x is not markovian anymore. Furthermore, it satisfies the following growth property : $\forall t \geq 0, |\hat{Y}_t^x| \leq C(1 + |X_t^x| + K_t^x)$. This dependency on K_t^x prevents us to get the unicity of λ among the space of solutions satisfying such a growth property.

Similarly, an existence result can be stated for a solution (Y, Z, μ) , λ being fixed.

Proposition 4.8. (*Existence of a Solution (Y, Z, μ)*). *Assume that hypothesis (2.1), (3.1) and (4.2) hold true. Then for any $\lambda \in \mathbb{R}$ there exists a continuous adapted process and $Z^x \in \mathcal{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ such that for all $\mu \in \mathbb{R}$ the triple (Y, Z, μ) is \mathbb{P} -a.s. a solution of EBSDE (4.4) with λ fixed.*

Proof. From theorem 4.4, we have constructed a solution $(Y^{x,0}, Z^{x,0}, \lambda^0)$ of the following EBSDE

$$Y_t^{x,0} = Y_T^{x,0} + \int_t^T [\psi(X_s^x, Z_s^{x,0}) - \lambda^0] ds - \int_t^T Z_s^{x,0} dW_s, \quad \forall 0 \leq t \leq T < +\infty. \quad (4.6)$$

Then setting $\hat{Y}_t^x := Y_t^{x,0} + (\lambda - \lambda^0)t - \int_0^t [g(X_s^x) - \mu] dK_s^x$, the triple $(\hat{Y}^x, Z^{x,0}, \mu)$ is solution of the EBSDE (4.4) with λ fixed. \square

Remark 4.9. The solution constructed satisfies the following growth property :

$$|\widehat{Y}_t| \leq C(1 + |X_t^x| + K_t^x + t), \mathbb{P} - a.s.$$

Again, this solution is degenerate and does not allow us to establish a result of uniqueness for μ among the space of solutions satisfying such a growth property.

Remark 4.10. If the convex \overline{G} is assumed to be bounded, it is possible, following [10] to show that there exists a markovian solution (Y, Z, λ) when μ is fixed or (Y, Z, μ) when λ is fixed exists. The proofs are the same as in [10].

5 Probabilistic interpretation of the solution of an elliptic PDE with linear Neumann boundary condition

We are concerned with the following semi-linear elliptic PDE :

$$\begin{cases} \mathcal{L}v(x) + \psi(x, {}^t\nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \quad (5.1)$$

where :

$$\mathcal{L}u(x) = \frac{1}{2} \text{Tr}(\sigma(x) {}^t\sigma(x) \nabla^2 u(x)) + {}^t u(x) \nabla u(x).$$

The unknowns of this equation is the couple (v, λ) . Now we show that the function v defined in Theorem 4.6 is a viscosity solution of the PDE (5.1).

Theorem 5.1. *Assume that hypothesis of Theorem (4.4) hold . Then (v, λ) is a viscosity solution of the elliptic PDE (5.1) where v is defined in the Theorem (4.6).*

Proof. Just adapt the proof of Theorem 4.3 from [9]. □

6 Optimal ergodic control

We make the standard assumption for optimal ergodic control, namely we consider U a separable metric space, which is the state space of the control process ρ . ρ is assumed to be (\mathcal{F}_t) -progressively measurable. We introduce $R : U \rightarrow \mathbb{R}^d$ and $L : \mathbb{R}^d \times U \rightarrow \mathbb{R}$ two continuous functions such that , for some constants $M_R > 0$ and $M_L > 0$, $\forall u \in U, \forall x, x' \in \mathbb{R}^d$,

- $|R(u)| \leq M_R$
- $|L(x, u)| \leq M_L$
- $|L(x, u) - L(x', u)| \leq M_L|x - x'|$.

For an arbitrary control ρ , the cost will be evaluated relatively to the following Girsanov density :

$$\Gamma_T^\rho = \exp \left(\int_0^T R(\rho_s) dW_s - \frac{1}{2} \int_0^T |R(\rho_s)|^2 ds \right).$$

We denote by \mathbb{P}_T^ρ the associated probability measure, namely : $d\mathbb{P}_T^\rho = \Gamma_T^\rho d\mathbb{P}$ on \mathcal{F}_T . Now we define the ergodic costs, relatively to a given control ρ and a starting point $x \in \mathbb{R}^d$, by :

$$I(x, \rho) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}_T^\rho \left[\int_0^T L(X_s^x, \rho_s) ds \right], \quad (6.1)$$

where E_T^ρ denotes expectation with respect to \mathbb{P}_T^ρ . We notice that the process $W_t^\rho := W_t - \int_0^t R(\rho_s) ds$ is a Wiener process on $[0, T]$ under \mathbb{P}_T^ρ . We define the Hamiltonian in the usual way :

$$\psi(x, z) = \inf_{u \in U} \{L(x, u) + zR(u)\}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^d,$$

and we remark that if, for all x, z , the infimum is attained in (6.4) then, according to Theorem 4 of [7], there exists a measurable function $\gamma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow U$ such that :

$$\psi(x, z) = L(x, \gamma(x, z)) + zR(\gamma(x, z)). \quad (6.2)$$

One can verify that γ is a Lipchitz function. Now we can prove the following theorems, exactly like in [10].

Theorem 6.1. *Assume that hypothesis of Theorem 4.4 hold true. Let (Y, Z, λ) be a solution of EBSDE (4.4) with μ fixed. Then :*

1. *For arbitrary control ρ we have $I(x, \rho) \geq \lambda$.*
2. *If $L(X_t^x, \rho_s) + Z_t^x R(\rho_t) = \psi(X_t^x, Z_t^x)$, \mathbb{P} -a.s. for almost every t then $I(x, \rho) = \lambda$.*
3. *If the infimum is attained in (6.2) then the control $\bar{\rho}_t = \gamma(X_t^x, Z_t^x)$ verifies $I(x, \bar{\rho}) = \lambda$.*

Remark 6.2. When the Neumann conditions are different from 0, we need regularity on the solution Y_t^x in order to state the same result. Again the degeneracy of the solution constructed in Theorem (4.6) or (4.8) does not allow us to conclude.

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A Proof of Lemma 2.1

Let us define $\varphi(x) = |x - a|^p$ for $p \geq 1$. We recall the following formulas for derivatives of φ , for $p \geq 2$.

$$\begin{aligned} \nabla \varphi(x) &= p(x - a)|x - a|^{p-2}. \\ \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} &= \begin{cases} p|x - a|^{p-2} + p(p-2)(x_i - a_i)^2|x - a|^{p-4} & \text{if } i = j, \\ p(p-2)(x_i - a_i)(x_j - a_j)|x - a|^{p-4} & \text{if } i \neq j. \end{cases} \end{aligned}$$

Therefore we have the following estimate

$$|\nabla^2 \varphi(x)| \leq K|x - a|^{p-2},$$

for a constant K which depends only on p and d . under the hypothesis of this Lemma, it is well known that a solution for which the explosion time is almost surely equal to infinity exists. By an Itô formula we get, for $p = 2$, for all $t \geq 0$,

$$\begin{aligned} |V_t^x - a|^2 &= |x - a|^2 + 2 \int_0^t (V_s^x - a) \cdot (f(V_s^x) ds + \sigma(V_s^x) dW_s) + \int_0^t \sum_i (\sigma(V_s^x)^t \sigma(V_s^x))_{i,i} ds \\ &\leq |x - a|^2 - 2\eta_1 \int_0^t |V_s^x - a|^2 ds + 2 \int_0^t (V_s^x - a) \sigma dW_s + \left(2\eta_2 + \sum_{i,i} (\sigma(V_s^x)^t \sigma(V_s^x))_{i,i} \right) t. \end{aligned}$$

Taking the expectation and applying a Gronwall Lemma we get :

$$\mathbb{E}|V_t^x - a|^2 \leq |x - a|^2 e^{-\eta_1 t} + \frac{(2\eta_2 + d|\sigma|_\infty^2)}{\eta_1} (1 - e^{-\eta_1 t}).$$

Therefore :

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-\eta_1 t}) \tag{A.1}$$

where C is a constant that depends only on a , η_1 , η_2 and σ but not on the time t .

For $p > 2$, an Itô formula gives us, denoting $V_t^x = (V_t^{x,1}, \dots, V_t^{x,n})$, and for a generic constant C which depends only on p , d , $|\sigma|_\infty$, η_2 , ε (defined after) :

$$\begin{aligned} |V_t^x - a|^p &\leq |x - a|^p + p \int_0^t |V_s^x - a|^{p-2} (V_s^x - a) \cdot (f(V_s^x) ds + \sigma(V_s^x) dW_s) + C \int_0^t |V_s^x - a|^{p-2} ds \\ &\leq |x|^p + p \int_0^t (-\eta_1 |V_s^x|^p + \eta_2 |V_s^x|^{p-2}) ds + p \int_0^t |V_s^x|^{p-2} V_s^x \cdot \sigma(V_s^x) dW_s + C \int_0^t |V_s^x - a|^{p-2} ds. \end{aligned}$$

Taking the expectancy we get :

$$\mathbb{E}|V_t^x - a|^p \leq |x - a|^p - p\eta_1 \int_0^t \mathbb{E}|V_s^x - a|^p ds + C \int_0^t \mathbb{E}|V_s^x - a|^{p-2} ds$$

The Young inequality $ab \leq a^p/p + b^q/q$ for $1/p + 1/q = 1$ with $p = p/(p-2)$ and $q = p/2$ applied to the last term of the inequality hereabove allows us to write :

$$|V_s^x - a|^{p-2} \leq \varepsilon |V_s^x - a|^p / p + 1/(q\varepsilon^{q/p}),$$

so,

$$\mathbb{E}|V_t^x - a|^p \leq |x|^p + C - (p\eta_1 - \varepsilon) \int_0^t \mathbb{E}|V_s^x - a|^p ds$$

Therefore the following estimates give us, by a Gronwall Lemma for ε small enough that $\mathbb{E}|V_t^x - a|^p$ is bounded with an exponential decay in front the term in $|x|^p$, namely :

$$\mathbb{E}|V_t^x - a|^p \leq C(1 + |x|^p e^{-(p\eta_1 - \varepsilon)t}).$$

This can be rewritten :

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-(p\eta_1 - \varepsilon)t}).$$

where C is constant which depends on $p, d, \|\sigma\|_\infty, \eta_1, \eta_2, \varepsilon$ and a . Please note that C does not depend on the Lipschitz constant of f but only on η_1 and η_2 which will be useful later. Now that we have this result it is classical to establish the following result, using a BDG inequality and a Hölder inequality, for all $p \geq 1$,

$$\mathbb{E} \sup_{0 \leq t \leq T} |V_t^x|^p \leq C(T)(1 + |x|^p),$$

where the constant C depends this time on the time T and other constants given before. \square

B Proof of Lemma 2.4

We adapt the proof of the Theorem 2.4 from [2]. From Remark 2.2 we can deduce, applying the Lemma 2.1 that

$$\mathbb{E}|V_t^x|^2 \leq \kappa_1(|x|^2 e^{-\eta_1 t} + 1).$$

where κ_1 is a constant which depends only on $\eta, B, |\sigma|_\infty$ and \overline{G} . Then, choosing $R = 16\kappa_1$ and T large enough we can conclude the step 1 of the proof.

The next step of the proof has to be modified too, due to the lack of linearity on f . More precisely we have to find a new coupling. Let $x, y \in \overline{G} \subset B_R$ where B_R denotes the ball of center 0 and radius R . Here we recall the main ideas of this step. The goal is to find two processes satisfying the SDE (2.1) in law, one starting from x , the other one from y , and such that an estimates of the ratio of their densities is known. We denote by μ_1 the law of $(V_t^x)_{0 \leq t \leq T}$ and by μ_2 the law of $(V_t^y)_{0 \leq t \leq T}$.

Let us introduce the process $Y_t^{x,y}$ as the solution of the the following stochastic equation, $c > 0, \forall 0 \leq t < T$:

$$\begin{cases} dY_t^{x,y} = [d(V_t^y + Y_t^{x,y}) - d(V_t^y) - \frac{c}{T-t} Y_t^{x,y}]dt + [\sigma(X_t^y + Y_t^{x,y}) - \sigma(X_t^y)] dW_t \\ Y_0^{x,y} = x - y \end{cases} \quad (\text{B.1})$$

where (V_t^x) is the strong solution of (2.1) and (V_t^y) the solution of (2.1) with x replaced by y . It is well known that the system above admit a strong solution which is strongly unique and square integrable for all $t < T$. Furthermore For all $t < T$, an Itô formula gives us :

$$|Y_t^{x,y}|^2 = |x - y|^2 + 2 \int_0^t Y_s^{x,y} (d(V_s^y + Y_s^{x,y}) - d(V_s^y)) ds - 2c \int_0^t \frac{|Y_s^{x,y}|^2}{T-s} ds \quad (\text{B.2})$$

$$\begin{aligned} &+ 2 \int_0^t Y_s^{x,y} \cdot (\sigma(X_s^y + Y_s^{x,y}) - \sigma(X_s^y)) dW_s \quad (\text{B.3}) \\ &+ \int_0^t \Sigma_{i,j} [(\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)) \cdot (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))]_{i,j} ds \end{aligned}$$

Thus, using the fact that σ is bounded, and that d is dissipative :

$$|Y_t^{x,y}|^2 + 2 \int_0^t \left(\eta + \frac{c}{T-s}\right) |Y_s^{x,y}|^2 ds \leq |x - y|^2 + CT + 2 \int_0^t Y_s^{x,y} \cdot (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)) dW_s$$

Taking the expectancy and applying a Gronwall Lemma we get :

$$\mathbb{E}(|Y_t^{x,y}|^2) \leq C(1+T) \left(\frac{T-t}{T} \right)^c$$

which shows that when $t \rightarrow T$, $Y_t^{x,y}$ converges toward 0 for the L^2 norm for c large enough. For that reason, we define $Y_T^{x,y} = 0$.

Now we consider the process $\tilde{V}_t = V_t^y + Y_t^{x,y}$. This process replace the one denoted by \tilde{X} in the step 2 of [2]. This process satisfies the following stochastic differential equation :

$$\begin{cases} d\tilde{V}_t = f(V_t^y)dt + \sigma(V_t^y)dW_t + dY_t^{x,y} \\ \tilde{V}_0 = x \end{cases} \quad (\text{B.4})$$

Thus :

$$d\tilde{V}_t = f(\tilde{V}_t)dt + \sigma(\tilde{V}_t) \left[dW_t + \sigma^{-1}(\tilde{V}_t) \left(b(V_t^y) - b(\tilde{V}_t) - \frac{cY_t^{x,y}}{T-t} \right) \right]$$

We define : $h(t) = \sigma^{-1}(\tilde{V}_t) \left(b(V_t^y) - b(\tilde{V}_t) - \frac{cY_t^{x,y}}{T-t} \right)$. Now, let us show that h satisfies the Novikov condition, namely $\mathbb{E} \exp \left(\int_0^T \frac{1}{2} |h(t)|^2 dt \right) < +\infty$. In the following of this proof, we will denote by C a generic constant which does not depend on c , on f only trough η and B . Please note that C will depend on σ trough σ_∞ and its Lipschitz constant. We recall that $\sigma^{-1}(x)$ admit an upper bound uniform on x by hypothesis. Then, we have :

$$\begin{aligned} \mathbb{E} \exp \left(\int_0^T \frac{1}{2} |h(t)|^2 dt \right) &\leq \mathbb{E} \exp \left(\int_0^T C \left(1 + \frac{c^2 |Y_t^{x,y}|^2}{(T-t)^2} \right) dt \right) \\ &\leq C \mathbb{E} \exp \int_0^T \frac{c^2 |Y_t^{x,y}|^2}{(T-t)^2} dt \\ &\leq C \mathbb{E} \sum_{k=0}^{+\infty} \frac{\int_0^T \frac{c^{2k} |Y_t^{x,y}|^{2k}}{(T-t)^{2k}} dt}{k!} \\ &\leq C \sum_{k=0}^{+\infty} c^{2k} \frac{\mathbb{E} \int_0^T \frac{|Y_t^{x,y}|^{2k}}{(T-t)^{2k}} dt}{k!}. \end{aligned}$$

We have, by an Itô formula, for all $p \in \mathbb{N}$, $p \geq 2$ and using the following notation : $\phi(y) = |y|^p$.

$$\begin{aligned} \frac{|Y_t^{x,y}|^p}{|T-t|^{p-1}} &= \frac{|x-y|^p}{|T|^{p-1}} + p \int_0^t \frac{|Y_s^{x,y}|^{p-2}}{|T-s|^{p-1}} Y_s^{x,y} dY_s^{x,y} + (p-1) \int_0^t \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds \\ &\quad + \frac{p}{2} \int_0^t \sum_i [(\sigma(\tilde{V}_s) - \sigma(V_s^y)) \cdot {}^t(\sigma(\tilde{V}_s) - \sigma(V_s^y))]_{i,i} ds \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j} p(p-2) \frac{\partial^2 \phi(Y_s^{x,y})}{\partial y_i \partial y_j} [(\sigma(\tilde{V}_s) - \sigma(V_s^y)) \cdot {}^t(\sigma(\tilde{V}_s) - \sigma(V_s^y))]_{i,j} ds. \end{aligned}$$

As $\left| p(p-2) \frac{\partial^2 \phi(Y_s^{x,y})}{\partial y_i \partial y_j} \right| \leq Cp^2 |Y_s^{x,y}|^{p-2}$ and as σ is Lipschitz and bounded, we get :

$$\begin{aligned} \frac{1}{2} \int_0^T \sum_{i,j} p(p-2) \frac{\partial^2 \phi(Y_s^{x,y})}{\partial y_i \partial y_j} [(\sigma(\tilde{V}_s) - \sigma(V_s^y)) \cdot (\sigma(\tilde{V}_s) - \sigma(V_s^y))]_{i,j} ds \\ \leq Cp^2 \int_0^T \frac{|Y_s^{x,y}|^{p-1}}{|T-s|^{p-1}} ds. \end{aligned}$$

Then, by the following Young inequality : $ab \leq \varepsilon \frac{a^m}{m} + \frac{b^n}{n\varepsilon^{n/m}}$, we get that, for $m = p/(p-1)$ and for $b = 1$:

$$Cp^2 \mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^{p-1}}{|T-s|^{p-1}} ds \leq C\varepsilon(p-1)p \mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds + \frac{CpT}{\varepsilon^{p-1}}.$$

Thus we get :

$$(pc - (p-1) - C\varepsilon(p-1)p) \mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds \leq \frac{|x-y|^p}{|T|^{p-1}} + CpT + \frac{CpT}{\varepsilon^{p-1}}.$$

It shows that, for c large enough but fixed, it is possible to find ε_p small enough so that $(pc - (p-1) - C\varepsilon_p(p-1)p) > 0$. So,

$$\mathbb{E} \int_0^T \frac{|Y_s^{x,y}|^p}{|T-s|^p} ds \leq \frac{|x-y|^p}{(pc - (p-1) - C\varepsilon_p(p-1)p)|T|^{p-1}} + \frac{CpT}{(pc - (p-1) - C\varepsilon_p(p-1)p)} \left(1 + \frac{1}{\varepsilon_p^{p-1}}\right).$$

Therefore, it is easy to see that :

$$\begin{aligned} \mathbb{E} \exp \left(\int_0^T \frac{1}{2} |h(t)|^2 dt \right) &\leq C \sum_{k=0}^{+\infty} \frac{c^{2k}}{k!} \left(\frac{|x-y|^p}{(pc - (p-1) - C\varepsilon_p(p-1)p)|T|^{p-1}} \right. \\ &\quad \left. + \frac{CpT}{(pc - (p-1) - C\varepsilon_p(p-1)p)\varepsilon_p^{p-1}} \right) \\ &:= \kappa_4 < +\infty. \end{aligned}$$

So the Novikov condition is satisfied. We stress the fact that κ_4 depends on d only through its dissipativity constant. This concludes the step 2 of the proof. The step 3 does not have to be modified. \square

C Proof of Lemma 4.2

We follow the proof of the part 3 of [8]. We need to adapt this proof because in our case, the set in which the process is reflected is not bounded. Therefore convergences are not uniform in x anymore. In our case, the dissipativity of the process is enough to avoid needing the boundedness of \bar{G} . We will use the following notation $\beta(x) = (x - \Pi(x))$. Note that $F_n(x) = -2n\beta(x)$. We recall the following properties of the penalization term :

$$(x' - x) \cdot \beta(x) \leq \beta(x') \cdot \beta(x), \quad \forall x, x' \in \mathbb{R}^n, \quad (\text{C.1})$$

$$\exists c \in \bar{G}, \quad \gamma > 0, \quad \forall x \in \mathbb{R}^n, \quad (x - c) \cdot \beta(x) \geq \gamma |\beta(x)|. \quad (\text{C.2})$$

In a first time, we show that for any $1 \leq p < \infty$,

$$\mathbb{E} \left[\left(n \int_0^T |\beta(X_t^{x,n})| ds \right)^p \right] \leq CT(1 + |x|^p), \forall n \in \mathbb{N},$$

for a constant C which does not depend on n . For $p = 2$, an Itô formula gives us :

$$\begin{aligned} |X_t^{x,n} - c|^2 &= |x - c|^2 + 2 \int_0^t (X_s^{x,n} - c) \cdot (f(X_s^{x,n}) ds + F_n(X_s^{x,n}) ds + \sigma(X_s^{x,n}) dW_s) \\ &\quad + \int_0^t \sum_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} ds. \end{aligned}$$

Using inequality (C.2), the fact that σ is bounded and Remark (2.2) we deduce :

$$4n\gamma \int_0^t |\beta(X_s^{x,n})| ds \leq |x - c|^2 + 2 \int_0^t C ds + 2 \int_0^t (X_s^{x,n} - c) \cdot \sigma(X_s^{x,n}) dW_s.$$

Using the BDG inequality :

$$\mathbb{E} \left[\left| \int_0^T b(s) dW_s \right|^p \right] \leq C(p, T) \mathbb{E} \left[\left(\int_0^T |b(s)|^2 \right)^{p/2} \right],$$

and our estimate of Lemma (2.1), it follows that :

$$\mathbb{E} \left[\left(n \int_0^T |\beta(X_t^{x,n})| ds \right)^p \right] \leq CT(1 + |x|^p), \forall n \in \mathbb{N},$$

for a constant C which does not depend on n .

Now we prove that $\forall p > 2$,

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\beta(X_t^{x,n})|^p \right) \leq \frac{CT(1 + |x|^p)}{n^{p/2-1}}.$$

We apply an Itô formula to the function $\varphi(x) = |x - \Pi(x)|^p$. We will use the following notation $\beta(x) = (x - \Pi(x))$. Note that $F_n(x) = -2n\beta(x)$. It is well known that for all $p \geq 2$, φ is \mathcal{C}^2 on \mathbb{R}^d and that $\nabla\varphi(x) = 2(x - \Pi(x))$. We recall the following formulas for Itô calculus,

$$\begin{aligned} \nabla\varphi(x) &= p|\beta(x)|^{p-2}\beta(x), \\ \nabla^2\varphi(x) &= p|\beta(x)|^{p-2}\nabla\beta(x) + p(p-2)|\beta(x)|^{p-4}(\beta(x)^t\beta(x)). \end{aligned}$$

As $\nabla\beta(x)$ is a numerical matrix one can deduce the following inequality :

$$|\nabla^2\varphi(x)| \leq C|\beta(x)|^{p-2},$$

for a constant C which depends only on p and d .

We use an Itô formula, for all $p > 2$

$$\begin{aligned} \varphi(X_t^{x,n}) &= \int_0^t \nabla\varphi(X_s^{x,n}) \cdot (d(X_s^{x,n}) + b(X_s^{x,n}) + F_n(X_s^{x,n})) ds \\ &\quad + \int_0^t \nabla\varphi(X_s^{x,n}) \cdot \sigma(X_s^{x,n}) dW_s \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j} (\nabla^2\varphi(X_s^{x,n}))_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} ds \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi(X_t^{x,n}) + 2pn \int_0^t \varphi(X_s^{x,n}) ds &\leq \int_0^t \nabla \varphi(X_s^{x,n})(d(X_s^{x,n}) + b(X_s^{x,n})) ds \\ &\quad + \int_0^t \nabla \varphi(X_s^{x,n}) \cdot \sigma(X_s^{x,n}) dW_s + C \int_0^t |\beta(X_s^{x,n})|^{p-2} ds \end{aligned} \quad (\text{C.3})$$

Since, the Young inequality is : $ab \leq a^q/q + b^{q'}/q'$ for reals q and q' such that $1/q + 1/q' = 1$, we choose $q = p/(p-2)$ and $q' = p/2$ so that, for $\alpha > 0$:

$$\begin{aligned} |\beta(X_s^{x,n})|^{p-2} &= \alpha n^{(p-2)/p} |\beta(X_s^{x,n})|^{p-2} \times \frac{1}{\alpha n^{(p-2)/p}} \\ &\leq \alpha^{p/(p-2)} \frac{p-2}{p} n |\beta(X_s^{x,n})|^p + \frac{2}{p} \left(\frac{1}{\alpha n^{(p-2)/p}} \right)^{p/2} \\ &\leq \alpha^{p/(p-2)} \frac{p-2}{p} n \varphi(X_s^{x,n}) + \frac{2}{p} \frac{1}{\alpha^{p/2} n^{(p-2)/2}} \end{aligned}$$

and another Young inequality applied with this time $q = p/(p-1)$ and $q' = p$ gives us :

$$\begin{aligned} |\nabla \varphi(X_s^{x,n}) \cdot (d(X_s^{x,n}) + b(X_s^{x,n}))| &\leq |p(|\beta(X_s^{x,n})|^{p-1})| \times |(d(X_s^{x,n}) + b(X_s^{x,n}))| \\ &\leq \alpha^{p/(p-1)} n(p-1) p^{1/p} |\beta(X_s^{x,n})|^p \\ &\quad + |d(X_s^{x,n}) + b(X_s^{x,n})|^p / (pn^{p-1} \alpha^p) \end{aligned}$$

Therefore using the second inequality of Lemma (2.1) and the two inequality hereabove we deduce, for α small enough :

$$\mathbb{E} \left(n \int_0^t \varphi(X_s^{x,n}) ds \right) \leq C \left(\frac{(1 + |x|^p)}{n^{p-1}} + \frac{1}{n^{(p-2)/2}} \right) t,$$

therefore,

$$\mathbb{E} \left(\int_0^t \varphi(X_s^{x,n}) ds \right) \leq C \left(\frac{(1 + |x|^p)}{n^{p/2}} \right) t, \quad (\text{C.4})$$

where C does not depend on n , t and x .

Now we come back to equation (C.3). Taking the supremum over time and the expectation and using a BDG inequality we get :

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \varphi(X_t^{x,n}) &\leq \mathbb{E} \int_0^T |\nabla \varphi(X_s^{x,n})| \times |d(X_s^{x,n}) + b(X_s^{x,n})| ds \\ &\quad + C \mathbb{E} \left[\left(\int_0^T |\nabla \varphi(X_s^{x,n}) \sigma|^2 ds \right)^{1/2} \right] + C \mathbb{E} \int_0^T |\beta(X_s^{x,n})|^{p-2} ds. \end{aligned} \quad (\text{C.5})$$

We call respectively I_1 , I_2 and I_3 the three quantities hereabove. We have

$$I_1 \leq \frac{CT(1 + |x|^p)}{n^{(p-1)/2}},$$

simply using an Hölder inequality, inequality (C.4) and Lemma (2.1). We have

$$\begin{aligned} I_2 &\leq C\mathbb{E} \left[\left(\int_0^T |\beta(X_s^{x,n})|^{2p-2} ds \right)^{1/2} \right] \\ &\leq \frac{1}{\sqrt{2}} \mathbb{E} \sup_{0 \leq t \leq T} |\beta(X_s^{x,n})|^p + C' \mathbb{E} \left[\int_0^T |\beta(X_s^{x,n})|^{p-2} ds \right], \end{aligned}$$

thanks to a Young inequality. Applying an Hölder inequality to the second member gives us :

$$I_2 \leq \frac{1}{\sqrt{2}} \mathbb{E} \sup_{0 \leq t \leq T} |\beta(X_s^{x,n})|^p + \frac{CT(1 + |x|^{p-2})}{n^{(p-2)/2}}.$$

Finally another Hölder inequality gives us :

$$I_3 \leq \frac{CT(1 + |x|^{p-2})}{n^{(p-2)/2}}.$$

Adding up I_1 , I_2 and I_3 gives us the following inequality :

$$\mathbb{E} \sup_{0 \leq t \leq T} \varphi(X_t^{x,n}) \leq \frac{CT(1 + |x|^p)}{n^{(p-2)/2}}.$$

Now, like in [8], we can write that for all $1 \leq p < +\infty$, $0 < 2q < p$, $n, m \in \mathbb{N}$,

$$\mathbb{E} \left[\left(m \int_0^T |\beta(X_s^{x,n}) \cdot \beta(X_s^{x,m})| ds \right)^p \right] \leq CT \frac{1 + |x|^{2p}}{n^q}.$$

Now we can study the convergence of the penalized process when n goes to ∞ . We will need the following result when we will study the invariant measure of the reflected process, an Itô formula gives us, for all $0 \leq t \leq T < +\infty$ and for $q < 1$

$$\begin{aligned} |X_t^{x,n} - X_t^{x,m}|^2 &= 2 \int_0^t (X_s^{x,n} - X_s^{x,m})((d+b)(X_s^{x,n}) - (d+b)(X_s^{x,m})) ds \\ &\quad - 4n \int_0^t (X_s^{x,n} - X_s^{x,m}) \beta(X_s^{x,n}) ds + 4m \int_0^t (X_s^{x,n} - X_s^{x,m}) \beta(X_s^{x,m}) ds \\ &\quad + 2 \int_0^t (X_s^{x,n} - X_s^{x,m}) (\sigma(X_s^{x,n}) - \sigma(X_s^{x,m})) dW_s \\ &\quad + \int_0^t \sum_{i,j} [(\sigma(X_s^{x,n}) - \sigma(X_s^{x,m}))^i (\sigma(X_s^{x,n}) - \sigma(X_s^{x,m}))^j]_{i,j} ds. \end{aligned}$$

We use the hypothesis on d , b and σ and then we take the expectation :

$$\begin{aligned} \mathbb{E}|X_t^{x,n} - X_t^{x,m}|^2 &\leq -2\eta \int_0^t \mathbb{E}|X_s^{x,n} - X_s^{x,m}|^2 ds + 2B \int_0^t \mathbb{E}|X_s^{x,n} - X_s^{x,m}| ds \\ &\quad + 4\mathbb{E}n \int_0^t \beta(X_s^{x,n}) \cdot \beta(X_s^{x,m}) ds + 4\mathbb{E}m \int_0^t \beta(X_s^{x,n}) \cdot \beta(X_s^{x,m}) ds \\ &\quad + C \int_0^t \mathbb{E}|X_s^{x,n} - X_s^{x,m}| ds. \end{aligned}$$

Then we write that for all $\varepsilon > 0$:

$$(2B + C)|X_t^{x,n} - X_t^{x,m}| \leq \varepsilon \frac{(2B + C)^2}{2} |X_t^{x,n} - X_t^{x,m}|^2 + \frac{1}{2\varepsilon},$$

and we take ε small enough such that $\delta := 2\eta - \varepsilon \frac{(2B+C)^2}{2} > 0$. Now we have, for $0 < q < 1/2$:

$$\mathbb{E}|X_t^{x,n} - X_t^{x,m}|^2 \leq -\delta \int_0^t \mathbb{E}|X_s^{x,n} - X_s^{x,m}|^2 ds + \frac{1}{2\varepsilon} + Ct(1 + |x|^2) \left(\frac{1}{n^q} + \frac{1}{m^q} \right).$$

A Gronwall Lemma gives us

$$\mathbb{E}|X_t^{x,n} - X_t^{x,m}|^2 \leq Ce^{-\delta t} + C(1 + |x|^2) \left(\frac{1}{n^q} + \frac{1}{m^q} \right) (1 - e^{-\delta t}). \quad (\text{C.6})$$

Previous estimates allow us to write that (following [8])

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_s^{x,n} - X_s^{x,m}|^p \leq C(T)(1 + |x|^p) \left(\frac{1}{n^q} + \frac{1}{m^q} \right),$$

for any $1 < q < p/2$. We remark that this convergence is not uniform in x because \overline{G} is not bounded. \square

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