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# Symmetries and conserved quantities for minimal surfaces

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## Abstract

We describe here a general method for finding symmetries of minimal surfaces in  $\mathbb{R}^3$ , namely transformations sending a minimal immersion to another minimal immersion. More specifically we will be looking for infinitesimal symmetries, i.e. vector fields tangent to a Lie group acting on the set of minimal surfaces. Using Noether's theorem, we derive conserved quantities, i.e. cohomology classes on  $H_1$ , that permit us to write so called *balancing formulas*. Some examples of applications of such balancing formulas are quoted below. Others may be found in [6].

## 1 Jet bundles and differential equations

We follow the formalism in [5] transposed in the fiber bundle setting. Though we are mostly interested in maps from a surface  $M$  to  $\mathbb{R}^3$ , we will write them as sections of the trivial bundle  $M \times \mathbb{R}^3$ . This method follows the more general construction (see [9]) and explains the type of vector fields and transformations we will encounter.

### 1.1 Jet bundles and prolongation

Let  $N$  be a trivial bundle with base  $M^m$  and fiber  $F^n$ . Coordinates on  $M$  and  $F$  will be noted respectively  $x = (x^i)_{1 \leq i \leq m}$  and  $u = (u^\alpha)_{1 \leq \alpha \leq n}$  (called the *independent* and *dependent* variables). Two local sections  $s_1, s_2$  around  $p \in M$  are equivalent at  $p$  if they coincide up to first order, i.e.  $\frac{\partial(u^\alpha \circ s_1)}{\partial x^i}(p) = \frac{\partial(u^\alpha \circ s_2)}{\partial x^i}(p)$  for all  $i, \alpha$ . The equivalence class is called the 1-jet of  $s_1$  at  $p$ ; the collection of all 1-jets is a manifold  $N_1$ , called the first jet bundle, endowed with two bundle maps  $\pi_1 : N_1 \rightarrow M$  and  $\pi_{1,0} : N_1 \rightarrow N$ . It is easy to describe  $N_1$  using the coordinate  $u^{(1)} = (x^i, u^\alpha, u_i^\alpha)$  with  $u_i^\alpha = \frac{\partial(u^\alpha \circ s)}{\partial x^i}(p)$ ; projections  $\pi_1$  and  $\pi_{1,0}$  are obvious. One defines then a prolongation operator from  $\Gamma(N)$  to  $\Gamma(N_1)$  by

$$\text{pr}^1 s(x) = (x^i, s^\alpha, \frac{\partial s^\alpha}{\partial x^i})_{i,\alpha}$$

In the same way, we can define the  $k$ -jet bundle  $N_k \rightarrow N \rightarrow M$  and the  $k$ -th prolongation  $\text{pr}^k : \Gamma(N) \rightarrow \Gamma(N_k)$ . We need adopt a multi-index notation<sup>1</sup>:  $I = (i_1, \dots, i_\ell)$  will denote a  $\ell$ -tuple with  $1 \leq i_1 \leq \dots \leq i_\ell \leq m$  and

$$\partial_I = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_\ell}}$$

Furthermore  $|I| = \ell$  is called the length of  $I$ ; if  $|I| = 0$  then  $I = \emptyset$  and  $\partial_I f = f$ . The  $k$ -jet at  $p$  of a local section is its equivalence class for the relation  $s_1 \sim_p s_2$  if  $\partial_I s_1(p) = \partial_I s_2(p)$  for all  $|I| \leq k$ . A coordinate on  $N_k$  is  $u^{(k)} = (x^i, u_I^\alpha)$  for  $|I| \leq k$  and any  $\alpha$ , and the  $k$ -th prolongation is

$$\text{pr}^k s(x) = (x^i, \partial_I s^\alpha)_{i, \alpha, |I| \leq k}$$

We have a sequence of fiber bundles

$$\begin{array}{ccccccc} \cdots & \rightarrow & N_k & \rightarrow & \cdots & \rightarrow & N_1 & \rightarrow & N \\ & & \downarrow & & & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{\text{id}} & M & \xrightarrow{\text{id}} & \cdots & \xrightarrow{\text{id}} & M & \xrightarrow{\text{id}} & M \end{array}$$

and we define  $N_\infty \rightarrow M$  to be its inverse limit. The infinite jet at  $p$  of a local section is its equivalence class for the relation  $s_1 \sim_p s_2$  if  $\partial_I s_1(p) = \partial_I s_2(p)$  for all  $I$ .  $N_\infty$  is an infinite dimensional bundle with coordinate  $u^{(\infty)} = (x^i, u_I^\alpha)_{\alpha, I}$ . The infinite prolongation  $\text{pr}^\infty$  is defined in a similar way.

Vector fields on  $N_\infty$  are sections of the tangent bundle  $TN_\infty$ ; they are written as linear combinations of the  $\frac{\partial}{\partial x^i}$  and  $\frac{\partial}{\partial u_I^\alpha}$  without limitation on the number of terms. Vector fields in  $\mathfrak{X}(N) = \Gamma(TN)$  can also be prolonged like the sections. However the formula is more delicate to establish and write; it involves the *total derivatives*  $D_i \in \Gamma(TN_\infty)$ , defined in coordinates by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha, I} u_{I \cup i}^\alpha \frac{\partial}{\partial u_I^\alpha}$$

and  $D_I = D_{i_1} \cdots D_{i_\ell}$  (when working in  $N_k$  one simply writes a finite sum with  $|I| \leq k$ ). Then if  $X = \sum_i X^i \frac{\partial}{\partial x^i} + \sum_\alpha U^\alpha \frac{\partial}{\partial u^\alpha}$ ,

$$\text{pr}^k X = \sum_i X^i \frac{\partial}{\partial x^i} + \sum_\alpha \sum_{0 \leq |I| \leq k} U^{\alpha, I} \frac{\partial}{\partial u_I^\alpha} \quad (k \in \mathbb{N} \cup \infty)$$

where

$$U^{\alpha, I} = \sum_i X^i u_{I \cup i}^\alpha + D_I \left( U^\alpha - \sum_i X^i u_i^\alpha \right)$$

One frequently uses a specific function algebra on  $N_\infty$ , namely  $\mathfrak{F} = \varinjlim \mathfrak{F}(N_k) = \{f \in \mathfrak{F}(N_\infty); \exists k, f \in \mathfrak{F}(N_k)\}$ .

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<sup>1</sup>Warning: this notation (as found in [5]) is standard in PDE theory but not in jet bundle theory (see [9]).

## 1.2 Differential equations

A differential equation of order  $k$  in this context is just a closed subbundle  $\mathcal{R}$  in  $N_k$ ; locally it is given by the equations  $R(u^{(k)}) = 0$  where  $R = (R^1, \dots, R^p)$  is a  $p$ -tuple of functions in  $\mathfrak{F}(N_k)$ . A *solution* is a section  $s \in \Gamma(E)$  such that  $\text{pr}^k s \in \mathcal{R}$ . We can also embed the problem in some superior jet bundle or even  $N_\infty$ ; we need then prolong  $\mathcal{R}$  in the subbundle  $\mathcal{R}_\infty$  locally defined by the set of equations

$$D_K R^j(u^{(\infty)}) = 0 \quad \forall K$$

In particular a variational principle is given by a lagrangian density<sup>2</sup>  $L \in \mathfrak{F}(N_k)$ . For any  $s \in \Gamma(N_k)$  set

$$\mathcal{L}(s) = \int_M L(\text{pr}^k s(x)) dx^1 \wedge \dots \wedge dx^m$$

The critical points of  $\mathcal{L}$  are the sections  $s \in \Gamma(N)$  whose prolongation  $\text{pr} s$  stay in the subbundle determined by the Euler-Lagrange equations  $E_\alpha(L)(u) = 0$  for  $1 \leq \alpha \leq n$  ( $E_\alpha(L)$  generally belongs in  $\mathfrak{F}(N_{2k})$ ):

$$E_\alpha(L) = \sum_I (-D)_I \frac{\partial L}{\partial u_I^\alpha}$$

where  $(-D)_I = (-1)^{|I|} D_I$ .

There eventually remains the problem of recognizing the immersions; that is not a priori implied by the previous study. Either one accepts solutions with singularities, or one restricts to the sections of the open set in  $N_k$  given by:  $\text{rank}(u_1, \dots, u_m) = m$ .

## 1.3 Symmetries and variational symmetries

Let  $G$  be a Lie group acting on the bundle  $N$ . For instance,  $G$  may be a group of transformations for the base space  $M$ , or fiber transformations; typically  $(x, u) \mapsto (\tilde{x}(x), \tilde{u}(u))$ . These are called *classical* transformations [2]. Elements of  $G$  may also be *gauge transformations*  $(x, u) \mapsto (\tilde{x}(x), \tilde{u}(x, u))$ ; or even more generally *point transformations*  $(x, u) \mapsto (\tilde{x}(x, u), \tilde{u}(x, u))$ . Given a group  $G$  one computes its  $k$ -th prolongation,  $\text{pr}^k G$  which acts on  $N_k$  ( $k$  possibly infinite). Of course  $G$  may be only a local group of transformations. We may also consider the infinitesimal action of  $G$ , i.e. vector fields in the Lie algebra  $\mathfrak{g}$ , and we will always write  $\mathfrak{g}$  as a subalgebra of  $\Gamma(TN)$ .

Let  $R = (R^1, \dots, R^p)$  be a partial differential equation of order  $k$ , i.e. a subbundle  $\mathcal{R}$  in  $N_k$ , satisfying some technical hypotheses<sup>3</sup> (clear for minimal surfaces). A group  $G$  leaves  $R$  invariant iff  $\text{pr} G$  maps  $\mathcal{R}$  into itself. A vector field  $X \in \Gamma(TN)$  is an infinitesimal symmetry of  $R$  if  $e^{t \text{pr} X}$  maps  $\mathcal{R}$  into itself, that is  $(\text{pr}^k X)R = 0$

<sup>2</sup>if  $N$  is not trivial,  $L$  is rather defined as a section of the pull-back bundle  $\pi_k^*(\bigwedge^m T^*M)$ , where  $\pi_k$  is the projection  $N_k \rightarrow M$ .

<sup>3</sup>we ask for local solvability, which is implied by analyticity and normality of the Euler-Lagrange equations.

on  $\mathcal{R}$  (taking the Lie derivative). To check for such a symmetry we need only write the  $k$ -th prolongation and apply it to the functions  $R^1, \dots, R^p$ .

If  $R$  is the Euler-Lagrange equation for a Lagrangian  $\mathcal{L} = \int_M L$ , we define a *variational symmetry* to be a vector field  $X = \sum_i X^i \frac{\partial}{\partial x^i} + \sum_\alpha U^\alpha \frac{\partial}{\partial u^\alpha}$  such that

$$(\text{pr}X)L + L\text{Div}(X^1, \dots, X^m) = 0$$

for all  $u$ , where  $\text{Div}(X^1, \dots, X^m) = D_1X^1 + \dots + D_mX^m$  is called the *total divergence* of  $(X^1, \dots, X^m)$ . Such an infinitesimal symmetry preserves critical points of  $\mathcal{L}$  hence  $(\text{pr}X)\mathbf{E}_\alpha(L) = 0$  for all  $\alpha$ . However this condition is too restrictive. Indeed it is enough for  $X$  to be a *divergence symmetry* :

$$\exists P = (P_1, \dots, P_m), \quad (\text{pr}X)L + L\text{Div}(X^1, \dots, X^m) = \text{Div}(P_1, \dots, P_m)$$

This ensures the invariance of the Euler-Lagrange<sup>4</sup> equations under  $X$ .

**Generalizations:** It is possible to look for more general type of infinitesimal symmetries and indeed we will meet some of these in the following. A first approach (due to Nøther herself) consists in taking fields in  $\mathfrak{F} \otimes \Gamma(TN)$  :

$$X = \sum_i X^i \frac{\partial}{\partial x^i} + \sum_\alpha U^\alpha \frac{\partial}{\partial u^\alpha}$$

with  $X^i$  and  $U^\alpha$  in  $\mathfrak{F}$  (more generally  $X$  is a section of the pull back bundle  $\pi_{\ell,0}^*(TN)$  for some finite  $\ell$ , where  $\pi_{\ell,0} : N_\ell \rightarrow N$ ; see [9] for details). Another approach uses *contact* transformations and vector fields: we take  $X$  in  $\Gamma(TN_\ell)$  instead of  $\Gamma(TN)$ ; however we have to restrict to those which map solutions to solutions; there is then a compatibility condition written in terms of connection or differential ideal (see [2, 10]). This might however take us too far from the classical transformations. Finally we may consider vector fields  $X$  defined only on  $\mathcal{R}$ ; that enlarges our scope (e.g. the conjugate family for minimal surfaces).

## 1.4 Nøther's theorem

A *conservation law* is a  $(m-1)$ -form  $\omega$  in  $\Omega^{m-1}(M) \otimes \mathfrak{F}$ , whose restriction to  $\mathcal{R}_\infty$  is closed for the total exterior derivative  $D$ . In other words, let  $s \in \Gamma(N)$  be a solution with prolongation  $\text{pr}s$ , then

$$(D\omega)(\text{pr}s) = d(\omega(\text{pr}s)) = 0$$

Let  $\gamma$  be a  $(m-1)$ -chain in  $M$ ; then  $\int_\gamma \omega(\text{pr}s)$  only depends on the homology class of  $\gamma$ , for any fixed solution  $s$ . We call this integral a *conserved quantity*.

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<sup>4</sup>The Euler-Lagrange equations usually have higher order than  $L$ ; hence this condition is a simpler one; however there may be symmetries of the Euler-Lagrange equation which are not variational symmetries (even not divergence symmetries).

Noether's theorem states the equivalence between variational symmetries and conservation laws. We will use it in the simple case where  $L$  is an order 1 lagrangian, and the variational symmetry is *vertical*, i.e. only changes the fiber  $N$ :  $X = \sum X^\alpha \frac{\partial}{\partial u^\alpha}$ . Then using integration by parts:

$$\begin{aligned} (\text{pr}^1 X)L &= \sum_\alpha \left[ X^\alpha \frac{\partial L}{\partial u^\alpha} + \sum_i D_i X^\alpha \frac{\partial L}{\partial u_i^\alpha} \right] \\ &= \sum_\alpha \left[ X^\alpha \frac{\partial L}{\partial u^\alpha} - \sum_i X^\alpha D_i \frac{\partial L}{\partial u_i^\alpha} \right] + \sum_i D_i \left( \sum_\alpha X^\alpha \frac{\partial L}{\partial u_i^\alpha} \right) \\ &= XE(L) + \sum_i D_i C_i \quad \text{with } C_i = \sum_\alpha X^\alpha \frac{\partial L}{\partial u_i^\alpha} \end{aligned}$$

We define the conservation law

$$\omega = \sum_i (-1)^{i+1} C_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m$$

so that  $D\omega = (\sum_i D_i C_i) dx^1 \wedge \cdots \wedge dx^m$  which vanishes when evaluated at a prolonged solution  $u^{(\infty)} = \text{pr}s$ . If  $X$  is a divergence symmetry, so that  $(\text{pr}^1 X)L = \text{Div}(P_1, \dots, P_m)$ , we need only replace  $C_i$  by  $C_i - P_i$ .

## 2 Minimal surfaces

Let  $M$  be a simply-connected bounded domain of  $\mathbb{R}^2$  and  $N$  be the trivial bundle  $M \times \mathbb{R}^3$ . We will see later how and when symmetries and conserved quantities extend to non-simply connected surfaces. For the sake of simplicity, we assume that  $M$  is the domain of a coordinate chart  $(x^1, x^2)$ . Minimal surfaces  $M \rightarrow \mathbb{R}^3$  are the critical points of the order 1 lagrangian

$$\mathcal{L} = \int_M L\left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}\right) dx^1 dx^2$$

with  $L \in \mathfrak{F}(N_1)$  is the map :  $u^{(1)} \mapsto \|u_1 \times u_2\|$ ;  $\times$  is the cross product in  $\mathbb{R}^3$  and  $\|\cdot\|$  the Euclidean norm associated to the scalar product  $\langle \cdot | \cdot \rangle$ . As the target space is a vector space, we use simpler index-free notations, like  $\nabla_{u_i} L$  for the vector  $(\frac{\partial L}{\partial u_i^\alpha})_{1 \leq \alpha \leq 3}$ . Then the Euler-Lagrange equation in the second order jet bundle is:

$$\begin{aligned} E_\alpha(L)(u^{(2)}) &= \sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha}(u^{(2)}) \\ &= -\sum_\beta \left( \left( u_{11}^\beta \frac{\partial}{\partial u_1^\beta} + u_{12}^\beta \frac{\partial}{\partial u_2^\beta} \right) \frac{\partial L}{\partial u_1^\alpha} + \left( u_{12}^\beta \frac{\partial}{\partial u_1^\beta} + u_{22}^\beta \frac{\partial}{\partial u_2^\beta} \right) \frac{\partial L}{\partial u_2^\alpha} \right) \end{aligned}$$

or in vector notation, calling  $\nu$  the unit normal vector field,  $\nu = (u_1 \times u_2)/L$  :

$$\begin{aligned} E(L)(u^{(2)}) &= \sum_J (-D)_J \nabla_{u_J} L(u^{(2)}) = -D_1(u_2 \times \nu) - D_2(\nu \times u_1) \\ &= -u_2 \times D_1 \nu - u_{12} \times \nu - \nu \times u_{12} - D_2 \nu \times u_1 \\ &= -u_2 \times D_1 \nu - D_2 \nu \times u_1 \end{aligned}$$

One checks that  $\langle \mathbf{E}(L)|u_1 \rangle = \langle \mathbf{E}(L)|u_2 \rangle = 0$  thus  $\mathbf{E}(L) = \langle \mathbf{E}(L)|\nu \rangle \nu = -2LH\nu = -2H(u_1 \times u_2)$  with  $H$  the mean curvature:

$$H = \frac{\langle \nu | \|u_2\|^2 u_{11} + \|u_1\|^2 u_{22} - 2\langle u_1 | u_2 \rangle u_{12} \rangle}{2\|u_1 \times u_2\|^2}$$

If the immersion is conformal then  $H = \frac{1}{2L} \langle \Delta u | \nu \rangle$ . Furthermore  $\Delta u$  is parallel to  $\nu$ , hence  $\mathbf{E}(L) = -\Delta u$ .

## 2.1 Symmetries

Notice that  $L$  does not depend on the  $x$  variable, but actually we know more : a change in parametrisation does not change the minimality<sup>5</sup>. This can be easily verified : let  $X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2}$ ; its first prolongation is

$$\begin{aligned} (\text{pr}^1 X)L(u) &= -\sum_{\alpha} [(X_1^1 u_1^\alpha + X_1^2 u_2^\alpha) \frac{\partial L}{\partial u_1^\alpha}(u) + (X_2^1 u_1^\alpha + X_2^2 u_2^\alpha) \frac{\partial L}{\partial u_2^\alpha}(u)] \\ &= -\langle X_1^1 u_1 + X_1^2 u_2 | \nabla_{u_1} L \rangle - \langle X_2^1 u_1 + X_2^2 u_2 | \nabla_{u_2} L \rangle \\ &= -\langle X_1^1 u_1 + X_1^2 u_2 | u_2 \times \nu \rangle - \langle X_2^1 u_1 + X_2^2 u_2 | \nu \times u_1 \rangle \\ &= -(X_1^1 + X_2^2) \langle \nu | u_1 \times u_2 \rangle = -L \text{Div}(X^1, X^2) \end{aligned}$$

This equation tells us that a change of variable  $\tilde{x} = \phi(x)$  is a symmetry; indeed  $L$  may change, but the area  $\mathcal{L}$  does not.

From now on we restrict our attention to infinitesimal symmetries keeping  $x$  fixed (so-called *vertical* generalized vector fields)<sup>6</sup>, so the term  $L \text{Div}(X^1, \dots, X^m)$  will always vanish, so we write:

$$X = \sum_{\alpha=1}^3 X^\alpha \frac{\partial}{\partial u^\alpha}$$

and  $X^\alpha \in \mathfrak{F}$ . The first prolongation is simply

$$\text{pr}^1 X = X + \sum_{\alpha} [(D_1 X^\alpha) \frac{\partial}{\partial u_1^\alpha} + (D_2 X^\alpha) \frac{\partial}{\partial u_2^\alpha}]$$

and noting  $D_i X = (D_i X^1, D_i X^2, D_i X^3)$ ,  $X$  is a variational symmetry if

$$(\text{pr}^1 X)L(u^{(1)}) = \langle D_1 X | u_2 \times \nu \rangle + \langle D_2 X | \nu \times u_1 \rangle = 0 \quad (1)$$

Let us analyse what are the known symmetries.

<sup>5</sup>for that very reason we often think of the minimal surface independently of its parametrisation.

<sup>6</sup>as proven in [5], we can do so without loss of generality.

**Classical transformations:** Classical vector fields<sup>7</sup> leaving the metric in  $\mathbb{R}^3$  invariant will not change  $L$ , let alone  $H$ . These are simply translations  $X = \text{constant}$  (hence  $D_1X = D_2X = 0$ ) and rotations in  $\mathbb{R}^3$  for which  $X(u) = A \times u$  for some axis vector  $A$ .

$$(\text{pr}^1 X)L(u^{(1)}) = \langle A \times u_1 | u_2 \times \nu \rangle + \langle A \times u_2 | \nu \times u_1 \rangle = \langle \nu | (A \times u_1) \times u_2 + u_1 \times (A \times u_2) \rangle$$

Jacobi's equality for the cross product implies:

$$(A \times u_1) \times u_2 + u_1 \times (A \times u_2) = (A \times u_1) \times u_2 + (u_2 \times A) \times u_1 = -(u_1 \times u_2) \times A$$

hence  $(\text{pr}^1 X)L(u^{(1)}) = \langle \nu | A \times (u_1 \times u_2) \rangle = 0$ . Dilations are *not* variational symmetries, however it will be useful to compute  $(\text{pr}^1 X)L(u^{(1)})$  for  $X = \sum u^\alpha \frac{\partial}{\partial u^\alpha}$ :

$$(\text{pr}^1 X)L(u^{(1)}) = \langle u_1 | u_2 \times \nu \rangle + \langle u_2 | \nu \times u_1 \rangle = \text{Div}(P_1, P_2)$$

with  $P_1 = \langle u | u_2 \times \nu \rangle$  and  $P_2 = \langle u | \nu \times u_1 \rangle$ ; indeed

$$\begin{aligned} \text{Div}(P_1, P_2) &= \langle u_1 | u_2 \times \nu \rangle + \langle u | u_{12} \times \nu + u_2 \times D_1 \nu \rangle \\ &\quad + \langle u_2 | \nu \times u_1 \rangle + \langle u | \nu \times u_{12} + D_2 \nu \times u_1 \rangle \\ &= (\text{pr}^1 X)L(u^{(1)}) + \langle u | u_2 \times D_1 \nu + D_2 \nu \times u_1 \rangle \\ &= (\text{pr}^1 X)L(u^{(1)}) - \langle u | \mathbf{E}(L)(u^{(1)}) \rangle \end{aligned}$$

which vanishes for minimal surfaces only.

**PROPOSITION 1** *The classical variational vertical symmetries correspond exactly to the translations and affine rotations of  $\mathbb{R}^3$ .*

**PROOF:** Let  $X \in \mathfrak{X}(\mathbb{R}^3)$  be a variational symmetry; then (1) is true for any prolonged solution  $u = \text{pr}s$ ; in particular for conformal immersions, satisfying  $\langle u_1 | u_2 \rangle = 0$  and  $\|u_1\| = \|u_2\|$ . We get the simpler equation:

$$(\text{pr}^1 X)L(u^{(1)}) = \langle dX(u) u_1 | u_1 \rangle + \langle dX(u) u_2 | u_2 \rangle \quad (2)$$

For any rotation  $R$ ,  $R_s$  still is minimal and conformal. Fix any point  $p \in M$ , and choose an affine rotation keeping  $s(p)$  fixed (so  $u^\alpha(p)$  does not change); only  $u_1$  and  $u_2$  vary, and take actually arbitrary values (provided  $\langle u_1 | u_2 \rangle = 0$  and  $\|u_1\| = \|u_2\|$ ). Clearly for any  $u \in \mathbb{R}^3$ ,  $dX(u)$  is a linear map satisfying

$$\forall \xi, \eta \in \mathbb{R}^3 \text{ such that } \langle \xi | \eta \rangle = 0, \|\xi\| = \|\eta\|, \quad \langle dX(u) \xi | \xi \rangle + \langle dX(u) \eta | \eta \rangle = 0 \quad (3)$$

Then  $dX(u)$  is skew-symmetric (see lemma 2 below). That is not possible, except for constant  $X$ . Indeed the second order differential  $d^2X(u) = d(dX)(u)$  is a symmetric vector valued form and we have both properties:

$$\begin{cases} \langle d^2X(u) (\zeta, \eta) | \xi \rangle = \langle d^2X(u) (\eta, \zeta) | \xi \rangle \text{ symmetry} \\ \langle d^2X(u) (\zeta, \eta) | \xi \rangle = \langle d^2X(u) (\zeta, \xi) | \eta \rangle \text{ skew-symmetry of } d^2X(u)\zeta \end{cases}$$

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<sup>7</sup>in the sense defined above: vector fields on  $\mathbb{R}^3$  only.

We get a contradiction by writing

$$\langle d^2X(u)(\zeta, \eta)|\xi \rangle = -\langle d^2X(u)(\zeta, \xi)|\eta \rangle = -\langle d^2X(u)(\xi, \zeta)|\eta \rangle$$

so that the expression is both symmetric and skew-symmetric in  $\zeta$  and  $\eta$  ! We conclude that  $d^2X$  vanishes identically, so  $X(u) = dX(0)u + \text{constant}$  with  $dX(0) \in \mathfrak{so}(3)$ , which is indeed an affine rotation (or simply a translation if  $dX(0) = 0$ ). ■

LEMMA 2 *Let  $A \in \mathcal{L}(\mathbb{R}^3)$  be such that*

$$\forall \xi, \eta \in \mathbb{R}^3 \text{ such that } \langle \xi|\eta \rangle = 0, \|\xi\| = \|\eta\|, \quad \langle A\xi|\xi \rangle + \langle A\eta|\eta \rangle = 0$$

*then  $A$  is skew-symmetric.*

PROOF: Obviously skew-symmetry is a sufficient condition for that property to hold. Now let  $\xi$  be a unit real eigenvector of  $A$  with associated eigenvalue  $\lambda$  (there has to be at least one such). Then for any unit vector  $\eta \in \xi^\perp$ ,  $\langle A\eta|\eta \rangle = -\lambda = -\lambda\langle \eta|\eta \rangle$ . Thus on  $\xi^\perp$ ,  $\zeta \mapsto \langle A\zeta|\zeta \rangle$  is just the quadratic form  $-\lambda\|\zeta\|^2$ . Let  $(\eta_1, \eta_2)$  be an orthonormal basis of  $\xi^\perp$ ; then applying the hypothesis yields:

$$0 = \langle A\eta_1|\eta_1 \rangle + \langle A\eta_2|\eta_2 \rangle = -2\lambda$$

So in the orthonormal basis  $(\xi, \eta_1, \eta_2)$  the matrix of the bilinear form  $\langle A \cdot | \cdot \rangle$  is written

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}$$

But reapplying the hypothesis to  $(\eta_2, \frac{\xi + \eta_1}{\sqrt{2}})$  yields:

$$0 = \langle A\eta_2|\eta_2 \rangle + \frac{1}{2}\langle A(\xi + \eta_1)|(\xi + \eta_1) \rangle = \frac{1}{2}\langle A\eta_1|\xi \rangle = a$$

The same computation gives  $b = 0$ . We conclude that  $A$  is skew-symmetric. ■

Let us now describe generalized symmetries.

**The conjugate family:** We define an action of  $\mathbb{S}^1$  on the solutions by  $\theta \mapsto u^\theta = \cos \theta u + \sin \theta v$ , where  $v$  is “the” conjugate immersion (since  $v$  is only defined up to translation, assume a point  $p \in M$  is fixed and  $u(p) = v(p) = 0$ ). If  $u$  is harmonic,  $v$  is just a conjugate harmonic map; otherwise there exists a change of coordinates  $\phi$  such that  $\hat{u} = u \circ \phi$  is harmonic; then set  $v = \hat{v} \circ \phi^{-1}$ . It is straightforward to check that the definition does not

depend on the choice of  $\phi$ . Thus we define:  $X = \frac{d}{d\theta}|_{\theta=0} u^\theta = v$ . A (somewhat long) calculation, eliminating  $\phi$ , shows that

$$D_1 X = u_1 \times \nu \quad D_2 X = u_2 \times \nu$$

which corresponds to  $D_1 X = v_1 = -u_2$  and  $D_2 X = v_2 = u_1$  in the conformal case. Clearly condition (1) is satisfied. Although  $D_i X$  exists for any  $u \in TN_\infty$  it is locally integrable if and only if  $D_2 D_1 X = D_1 D_2 X$  but  $D_1(u_2 \times \nu) - D_2(u_1 \times \nu) = -E(L)$  vanishes exactly on minimal maps. So this is truly a generalized variational symmetry tangent only to the solution subbundle. Moreover,  $X = \int^x u_1 \times \nu dx^1 + u_2 \times \nu dx^2$  cannot truly be described as a function on any  $k$ -th order jet space; but – considering analyticity – as a function of all derivatives, i.e.  $X \in \mathfrak{F}(N_\infty) \otimes \mathfrak{X}(N)$ .

**The imaginary rotations:** Start with the Weierstrass (hence conformal) representation for minimal surfaces, so  $u(x) = \text{Re} \int_p^x (g^{-1} - g, i(g^{-1} + g), 2)\eta$ ; following Lopez and Ros ([7]), we define an action of  $\mathbb{R}_+^*$ :  $(g, \eta) \mapsto (\lambda g, \eta)$ , so  $u^\lambda(x) = \text{Re} \int_p^x (\lambda^{-1} g^{-1} - \lambda g, i(\lambda^{-1} g^{-1} + \lambda g), 2)\eta$  Thus

$$X = \frac{du^\lambda}{d\lambda}|_{\lambda=1} = \text{Re} \int^x (-(g^{-1} + g), i(-g^{-1} + g), 0)\eta$$

Comparing it with the conjugate surface parametrisation  $v = \text{Re} \int_p^x (i(g^{-1} - g), -(g^{-1} + g), 2i)\eta$  we infer that  $X = (v^2, -v^1, 0) = v \times e_3$ . Obviously we can generalize to the field  $X = A \times v$  for some fixed vector  $A$  (here again  $X$  is only defined on the solution subbundle). We get

$$\begin{aligned} (\text{pr}^1 X)L(u^{(1)}) &= \langle A \times (u_1 \times \nu) | u_2 \times \nu \rangle + \langle A \times (u_2 \times \nu) | \nu \times u_1 \rangle \\ &= \langle A | (u_1 \times \nu) \times (u_2 \times \nu) + (u_2 \times \nu) \times (\nu \times u_1) \rangle \\ &= 2\langle u_1 \times u_2 | A \rangle \end{aligned}$$

Hence  $(\text{pr}^1 X)L(u^{(1)}) = \text{Div}(P_1, P_2)$  with  $P_1(u^{(1)}) = \langle u \times u_2 | A \rangle$  and  $P_2(u^{(1)}) = \langle u_1 \times u | A \rangle$ . We see that its form is very similar to rotations (but for taking  $v$  instead of  $u$ ); meaning of the name "imaginary rotations" will become more obvious later.

## 3 Applications of Noether's theorem

### 3.1 Formulation

For minimal surfaces it is very easy to define a Noether 1-form  $\omega$ . Let  $X$  be a variational (possibly divergence) symmetry:  $(\text{pr}^1 X)L = D_1 P_1 + D_2 P_2$ . Then set

$$\omega_X = (\langle X | u_1 \times \nu \rangle + P_2) dx^1 + (\langle X | u_2 \times \nu \rangle - P_1) dx^2$$

and by construction  $\omega_X$  is closed, hence locally integrable on the solution subbundle into a function  $f_X$ .

**Translations and flux:** Take  $X$  any constant vector  $A \in \mathbb{R}^3$  generating a translation subgroup, then there is no divergence term and

$$\omega = \langle A | (u_1 \times \nu) dx^1 + (u_2 \times \nu) dx^2 \rangle$$

Clearly one can put together these results for various  $A$  by defining a  $\mathbb{R}^3$ -valued 1-form

$$\vec{\omega} = (u_1 \times \nu) dx^1 + (u_2 \times \nu) dx^2$$

then  $\int_\gamma \vec{\omega}$  is a homological invariant, called the *flux* or *force*  $\mathcal{F}$  along the curve  $\gamma$ . We note that the primitive of  $\vec{\omega}$  is nothing else than the conjugate map  $v$  (up to a constant); indeed the flux of  $u$  along  $\gamma$  is the period of  $v$  along that same curve.

**Rotations and torque:** For rotation with vector axis  $A$ , we get

$$\omega = \langle A \times u | (u_1 \times \nu) dx^1 + (u_2 \times \nu) dx^2 \rangle$$

Here again we define a vector-valued 1-form  $\vec{\omega} = u \times (u_1 \times \nu) dx^1 + u \times (u_2 \times \nu) dx^2$ . The integral of  $\vec{\omega}$  along a curve is called the *torque*  $\mathcal{T}$  of  $\gamma$ . It must be noted that unlike the flux,  $\mathcal{T}(\gamma)$  changes when the immersion  $u$  is translated (because of the Lie bracket of the translation with the rotation); so if  $u \rightarrow u + c$  then  $\mathcal{T}(\gamma) \rightarrow \mathcal{T}(\gamma) + c \times \mathcal{F}(\gamma)$ . A neat result can be proved using the torque:

**PROPOSITION 3 :** *Let  $M$  be a minimal surface with two (or three) vertical catenoidal ends, with axes labelled  $D_1, D_2 (D_3)$ . If there are two ends then both axes are identical. In there are three ends, then all three axes lie on the same vertical plane.*

Proof: Let us first compute the flux and torque around a vertical catenoidal end<sup>8</sup> centered around the third coordinate axis. The end can be parametrized as an analytic graph :

$$x^3(x^1, x^2) = c \log R + b + O(r^{-2})$$

where  $c, b$  are constants and  $r = \|x\| = \sqrt{(x^1)^2 + (x^2)^2}$ . Then up to an additive factor  $O(r^{-3})$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ cx^1/r^2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ cx^2/r^2 \end{pmatrix}, \quad \nu = \begin{pmatrix} -cx^1/r^2 \\ -cx^2/r^2 \\ 1 - c^2/2r^2 \end{pmatrix}$$

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<sup>8</sup>meaning by that along a positively oriented curve representing the homology of the end.

$$u_1 \times \nu = \begin{pmatrix} c^2 x^1 x^2 / r^4 \\ -1 + c^2((x^2)^2 - (x^1)^2) / 2r^4 \\ -cx^2 / r^2 \end{pmatrix}, \quad u_2 \times \nu = \begin{pmatrix} 1 + c^2((x^2)^2 - (x^1)^2) / 2r^4 \\ -c^2 x^1 x^2 / r^4 \\ cx^1 / r^2 \end{pmatrix}$$

so

$$\int_{\|x\|=r} \vec{\omega} = 2\pi c e_3 + O(r^{-2})$$

Since the integral is independent of  $r$  for homological reasons,  $\mathcal{F} = 2\pi c e_3$ . For the torque note that it is hence invariant by vertical translation; we may as well assume  $b = 0$ . Then up to  $O(r^{-2})$

$$u \times (u_1 \times \nu) = \begin{pmatrix} c \log r - c^3 \log r((x^2)^2 - (x^1)^2) / 2r^4 - c(x^2)^2 / r^2 \\ c^3 \log r x^1 x^2 / r^4 + cx^1 x^2 / r^2 \\ -x^1 + c^2((x^2)^2 - (x^1)^2)x^1 / 2r^4 - c^2 x^1 (x^2)^2 / r^4 \end{pmatrix}$$

Hence the torque vanishes (taking the limit as  $r \rightarrow +\infty$ ). Now if the catenoidal end is not centered on the  $x_3$ -axis, but on an axis  $D$ , such that  $D - c$  is the  $x_3$ -axis, then the torque around the end is  $\mathcal{T} = c \times \mathcal{F}$ . If we have many catenoidal ends with respective fluxes  $\mathcal{F}_i = f_i e_3$  and torques  $\mathcal{T}_i = c_i \times \mathcal{F}_i = f_i c_i \times e_3$  (taking all  $c_i$  to be horizontal vectors). We choose oriented representatives  $\gamma_i$  of all ends, and they bound a compact piece of the surface, hence the conserved quantity vanish. For instance, the sum of fluxes and the sum of torques vanish. For two ends,  $f_1 = -f_2 \neq 0$  and

$$0 = \mathcal{T}_1 + \mathcal{T}_2 = f_1(c_1 - c_2) \times e_3$$

thus  $c_1 = c_2$ . For three ends, the sum of torques is

$$\left( \sum_1^3 f_i c_i \right) \times e_3 = 0$$

However we may translate the surface so that  $c_3 = 0$ ; we then get  $f_1 c_1 + f_2 c_2 = 0$ , a non trivial linear relation. Of course similar relationship hold for a greater number of ends, though without giving sufficient information. This kind of relation is called a *balancing formula*.  $\blacksquare$

**Dilations:** For dilations the conserved quantity may be computed (though dilations are no generalized variational symmetries), but it is simply a constant, indeed if  $X(u) = u$ ,  $\omega_X = 0$ .

**Conjugate family:** We denote here  $v$  a conjugate surface; notice however that  $v$  is *not* well-defined, since periods may be non-zero; so that  $v$  is defined on some cover of  $M$ ; furthermore  $v$  is only defined up to translation. For any

closed curve  $\gamma$ , the invariant associated with the infinitesimal conjugate action is

$$\mathcal{C}(\gamma) = \int_{\gamma} \langle v | dv \rangle$$

A different choice of origin for  $v$  changes this invariant; if  $v \rightarrow v + c$ ,  $\mathcal{C}(\gamma) \rightarrow \mathcal{C}(\gamma) + \langle c | \mathcal{F}(\gamma) \rangle$ . which makes the invariant geometrically meaningful only for curves with zero flux<sup>9</sup>. Notice however that this integral is trivial for zero flux, yielding  $\int_{\gamma} d \|v\|^2 = 0$ .

**Lopez-Ros transformation:** The infinitesimal symmetry is  $X = A \times v$  for a chosen vector  $A$ ; taking all direction in account we define here a vector valued 1-form  $\vec{\omega}$  by

$$\vec{\omega} = (v \times v_1 - u \times u_1)dx^1 + (v \times v_2 - u \times u_2)dx^2 = v \times dv - u \times du$$

and we set  $\mathcal{P}(\gamma) = \int_{\gamma} \vec{\omega}$ . Clearly  $\mathcal{P}$  is independent of translations of  $u$ , but if  $v \rightarrow v + c$ , then  $\mathcal{P} \rightarrow \mathcal{P} + c \times \mathcal{F}(\gamma)$ .

## 4 Flat ends

We study in details a flat embedded annular end  $M$ . Assume for simplicity that  $M$  is horizontal. Let  $u(z) = \text{Re} \int^z (1 - g^2, i(1 + g^2), 2g) f dz$  be the standard Weierstrass representation, defined on a open neighbourhood of zero in  $\mathbb{C}$ . Up to a change of orientation, we may assume  $g(0) = 0$  with multiplicity  $k \geq 2$ . Embeddedness forces  $f$  to have a double pole. We may also choose the local coordinate around zero and rotate the surface so that  $g(z) = az^k + O(z^{k+1})$  for some real number  $a > 0$  and  $f(z) = e^{i\alpha} z^{-2}$  for some other real number  $\alpha$ . Then up to translation and noting  $z = re^{i\theta}$

$$u(z) = \begin{pmatrix} -r^{-1} \cos(\alpha - \theta) + O(r^{2k-1}) \\ r^{-1} \sin(\alpha - \theta) + O(r^{2k-1}) \\ \frac{2a}{k-1} \cos(\alpha + (k-1)\theta)r^{k-1} + O(r^k) \end{pmatrix}$$

and the conjugate map is well-defined

$$v(z) = \begin{pmatrix} -r^{-1} \sin(\alpha - \theta) + O(r^{2k-1}) \\ -r^{-1} \cos(\alpha - \theta) + O(r^{2k-1}) \\ \frac{2a}{k-1} \sin(\alpha + (k-1)\theta)r^{k-1} + O(r^k) \end{pmatrix}$$

hence there is no flux around that end. Notice that the integration constants are chosen so that both ends are ‘‘centered’’. To imagine the end, we study its intersections with the asymptotic plane  $x^3 = 0$ , and that gives us  $\theta \equiv$

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<sup>9</sup>In this latter case only, the starting point in the integral does not matter.

$(\frac{\pi}{2}-\alpha)/(k-1)+O(r) \bmod \pi/(k-1)$ . So the intersection stays within bounded distance from  $2(k-1)$  rays from the origin given by  $r \mapsto (-r^{-1} \cos(\alpha - \theta), r^{-1} \sin(\alpha - \theta), 0)$  for any  $\theta$  satisfying the previous relation. In particular for multiplicity 2 ends, the intersection stays close to the line  $\mathbb{R}(\sin 2\alpha, \cos 2\alpha, 0)$ .

The positively oriented curve  $\gamma$  will be for constant  $r$   $\theta \mapsto r e^{-i\theta}$  (coherent with the Gauss map pointing towards the south pole). Let us compute the torque  $\mathcal{T} = \int_{\gamma} u \times dv$ .

$$\begin{aligned} u \times \frac{\partial v}{\partial \theta} &= \begin{pmatrix} -r^{-1} \cos(\alpha + \theta) + O(r^{2k-1}) \\ r^{-1} \sin(\alpha + \theta) + O(r^{2k-1}) \\ \frac{2a}{k-1} \cos(\alpha - (k-1)\theta) r^{k-1} + O(r^k) \end{pmatrix} \times \begin{pmatrix} -r^{-1} \cos(\alpha + \theta) + O(r^{2k-1}) \\ r^{-1} \sin(\alpha + \theta) + O(r^{2k-1}) \\ -2ar^{k-1} \cos(\alpha - (k-1)\theta) + O(r^k) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2ka}{k-1} r^{k-2} \sin(\alpha + \theta) \cos(\alpha - (k-1)\theta) + O(r^{k-1}) \\ -\frac{2ka}{k-1} r^{k-2} \cos(\alpha + \theta) \cos(\alpha - (k-1)\theta) + O(r^{k-1}) \\ O(r^{2k-2}) \end{pmatrix} \end{aligned}$$

When we integrate as  $r \rightarrow 0$ , all positive powers of  $r$  vanish and only constant terms may remain, that is in  $r^{k-2}$  iff  $k = 2$ ; otherwise the torque simply vanishes<sup>10</sup>. Let us compute  $\mathcal{T}$  in the  $k = 2$  case. Then

$$\mathcal{T} = -4a \int_{-\pi}^{\pi} \begin{pmatrix} \sin(\alpha + \theta) \cos(\alpha - \theta) \\ \cos(\alpha + \theta) \cos(\alpha - \theta) \\ 0 \end{pmatrix} d\theta = -4\pi a \begin{pmatrix} \sin 2\alpha \\ \cos 2\alpha \\ 0 \end{pmatrix}$$

That is a horizontal vector, depicting the asymptotic direction characterizing the end. Indeed the intersection with the asymptotic horizontal plane is parametrised by  $r \mapsto \pm r^{-1}(\sin 2\alpha, \cos 2\alpha) + O(1)$ ; hence there is a well-defined asymptotic direction, with vector  $\mathcal{T}$ . The sign of  $\mathcal{T}$  determines on which side of this asymptotic line is the surface above (or under) the asymptotic plane (see figure ??).

[[ Image ]]

That stays true, even under less restrictive conditions as those used above for the computations. In the general case, the end is still horizontal but we do not assume that the leading coefficients in  $g$  and  $f$  expansion are respectively real and unitary. Then

$$\mathcal{T} = 4\pi \begin{pmatrix} \operatorname{Im} c \\ \operatorname{Re} c \\ 0 \end{pmatrix} \quad \text{where } c = \operatorname{Res}(\sqrt{g} f dz; 0)^2$$

Indeed it is possible to choose locally a square root  $h$  of  $g$  defined up to sign. So  $c$  is a well defined invariant.

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<sup>10</sup>A few simple consequences may be drawn from this and the previous proposition, such as: surfaces with two catenoidal ends and flat ends of multiplicities greater than 2 have the same axis for both catenoidal ends (e.g. Costa's surface).

The conserved quantity for the infinitesimal conjugation is  $\mathcal{C} = \int_{\gamma} \langle v | dv \rangle$

$$\begin{aligned} \langle v | \frac{\partial v}{\partial \theta} \rangle &= \left\langle \left( \begin{array}{c} -r^{-1} \sin(\alpha + \theta) + O(r^{2k-1}) \\ -r^{-1} \cos(\alpha + \theta) + O(r^{2k-1}) \\ \frac{2a}{k-1} \sin(\alpha - (k-1)\theta)r^{k-1} + O(r^k) \end{array} \right) \mid \left( \begin{array}{c} -r^{-1} \cos(\alpha + \theta) + O(r^{2k-1}) \\ r^{-1} \sin(\alpha + \theta) + O(r^{2k-1}) \\ -2ar^{k-1} \cos(\alpha + (1-k)\theta) + O(r^k) \end{array} \right) \right\rangle \\ &= O(r^{2k-2}) \end{aligned}$$

which vanishes when integrated along  $\gamma$ .

For the Lopez-Ros transformation,  $\mathcal{P} = \int_{\gamma} v \times dv - u \times du$ . Here again the only interesting terms are those in  $r^0$ , requiring  $k = 2$ ; let us assume that and single out the relevant terms

$$\begin{aligned} v \times \frac{\partial v}{\partial \theta} &= \begin{pmatrix} -r^{-1} \sin(\alpha + \theta) + O(r^3) \\ -r^{-1} \cos(\alpha + \theta) + O(r^3) \\ 2ar \sin(\alpha - \theta) + O(r^2) \end{pmatrix} \times \begin{pmatrix} -r^{-1} \cos(\alpha + \theta) + O(r^3) \\ r^{-1} \sin(\alpha + \theta) + O(r^3) \\ -2ar \cos(\alpha - \theta) + O(r^2) \end{pmatrix} \\ &= 2a \begin{pmatrix} \cos(\alpha + \theta) \cos(\alpha - \theta) - \sin(\alpha + \theta) \sin(\alpha - \theta) \\ -\cos(\alpha + \theta) \sin(\alpha - \theta) - \sin(\alpha + \theta) \cos(\alpha - \theta) \\ 0 \end{pmatrix} + \text{vanishing terms} \\ &= 2a \begin{pmatrix} \cos 2\alpha \\ -\sin 2\alpha \\ 0 \end{pmatrix} + \text{vanishing terms} \end{aligned}$$

and  $\int_{\gamma} v \times dv = -4\pi a e_1$ . Also

$$\begin{aligned} u \times \frac{\partial u}{\partial \theta} &= \begin{pmatrix} -r^{-1} \cos(\alpha + \theta) + O(r^3) \\ r^{-1} \sin(\alpha + \theta) + O(r^3) \\ 2ar \cos(\alpha - \theta) + O(r^2) \end{pmatrix} \times \begin{pmatrix} r^{-1} \sin(\alpha + \theta) + O(r^3) \\ r^{-1} \cos(\alpha + \theta) + O(r^3) \\ 2ar \sin(\alpha - \theta) + O(r^2) \end{pmatrix} \\ &= 2a \begin{pmatrix} \sin(\alpha + \theta) \sin(\alpha - \theta) - \cos(\alpha + \theta) \cos(\alpha - \theta) \\ \cos(\alpha + \theta) \sin(\alpha - \theta) + \sin(\alpha + \theta) \cos(\alpha - \theta) \\ 0 \end{pmatrix} + \text{vanishing terms} \\ &= 2a \begin{pmatrix} -\cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix} + \text{vanishing terms} \end{aligned}$$

So the integral yields

$$\mathcal{P} = 8\pi a \begin{pmatrix} \cos 2\alpha \\ -\sin 2\alpha \\ 0 \end{pmatrix} = 2 \begin{pmatrix} & -1 \\ 1 & \\ & 0 \end{pmatrix} \mathcal{T}$$

In the general case, let  $\nu$  the unit normal at the end, then

$$\mathcal{P} = 2\nu \times \mathcal{T}$$

PROPOSITION 4 *Let  $M$  be a singly-periodic complete minimal cylinder, with two flat ends in the quotient, and suppose  $M$  is symmetric with respect to a vertical plane  $P$ . Then both ends are asymptotically identical.*

PROOF: Let  $T$  be the period and  $\gamma$  a generator of the homology of the cylinder. Then both  $T$  and  $\mathcal{F}(\gamma)$  belong in  $P$  the plane of symmetry. Let  $\gamma + T$  be the curve  $\gamma$  translated by  $T$ ; then  $\mathcal{T}(\gamma + T) = \mathcal{T}(\gamma) + T \times \mathcal{F}(\gamma)$ . So that if  $\mathcal{T}_i$  denotes the torque around the  $i$ -th end

$$\mathcal{T}(\gamma + T) - \mathcal{T}(\gamma) = T \times \mathcal{F}(\gamma) = \mathcal{T}_1 + \mathcal{T}_2$$

And by symmetry, both torques  $\mathcal{T}_i$  are perpendicular to  $P$ . Consider now the Lopez-Ros invariant  $\mathcal{P}$ . Since the conjugate map is not well defined (there may be flux along  $\gamma$ ),  $\mathcal{P}$  is not well-defined; however  $\mathcal{P}' = \langle \mathcal{P} | \mathcal{F}(\gamma) \rangle$  is. Translation in  $u$  leaves  $\mathcal{P}$  invariant, thus  $\mathcal{P}'(\gamma + T) = \mathcal{P}'(\gamma)$ . So we must have at the ends

$$0 = \mathcal{P}'_1 + \mathcal{P}'_2 = 2\nu_1 \times \mathcal{T}_1 + 2\nu_2 \times \mathcal{T}_2 = 2\nu_1 \times (\mathcal{T}_1 - \mathcal{T}_2)$$

where  $\nu_i$  is the normal at then end, and belongs in  $P$ . Hence  $\mathcal{T}_1 = \mathcal{T}_2$ . Both ends are indeed asymptotically equal. ■

If there is no plane of symmetry, we can however say something. Suppose  $T \times \mathcal{F}(\gamma) \neq 0$  (can it vanish ?). Define then the plane  $P = \text{Vect}(T, \mathcal{F}(\gamma))$ . This plane is vertical. Indeed  $\mathcal{T}(\gamma + T) - \mathcal{T}(\gamma) = T \times \mathcal{F}(\gamma) = \mathcal{T}_1 + \mathcal{T}_2$  which is horizontal. Use the splitting  $\mathbb{R}^3 = P \oplus P^\perp$  to write  $\mathcal{T}_i = \mathcal{T}_i^P \oplus \mathcal{T}_i^\perp$ . If  $\mathcal{F}(\gamma)$  is not vertical we can follow the same reasoning as above, and conclude that  $\mathcal{T}_1^\perp = \mathcal{T}_2^\perp$ , and then  $\mathcal{T}_1^P = \mathcal{T}_2^P$ . What remains is  $\mathcal{P}_1 + \mathcal{P}_2 = 2\nu_1 \times \mathcal{T}_1^P$  which is equal to  $\pm \mathcal{F}(\gamma) \times \mathcal{F}(\delta)$  where  $\delta$  is the other generator of the torus (in the quotient).

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## 5 Other symmetries

**The harmonic+conformal subbundle:** For simplicity reasons, we may consider instead of the minimal surface equation, the harmonic+conformal equation, defined by

$$\begin{cases} u_{11} + u_{22} = 0 \\ \|u_1\|^2 = \|u_2\|^2 \\ \langle u_1 | u_2 \rangle = 0 \end{cases} \quad (4)$$

The corresponding subbundle  $\mathcal{H} \subset N_2$  is obtained through (4) and its derivatives, up to order 2. This adds in theory 4 equations

$$\begin{cases} \langle u_{11} | u_1 \rangle = \langle u_{12} | u_2 \rangle \\ \langle u_{12} | u_1 \rangle = \langle u_{22} | u_2 \rangle \\ \langle u_{11} | u_2 \rangle + \langle u_{12} | u_1 \rangle = 0 \\ \langle u_{12} | u_2 \rangle + \langle u_{22} | u_1 \rangle = 0 \end{cases} \quad (5)$$

Using the condition  $u_{11} + u_{22} = 0$ , we see that two of the previous conditions are redundant, hence we get a rank 7 system

$$\begin{cases} u_{11} + u_{22} = 0 \\ \|u_1\|^2 = \|u_2\|^2 \\ \langle u_1 | u_2 \rangle = 0 \\ \langle u_{11} | u_1 \rangle = \langle u_{12} | u_2 \rangle \\ \langle u_{12} | u_1 \rangle = \langle u_{11} | u_2 \rangle \end{cases} \quad (6)$$

Hence  $\mathcal{H}$  is a dimension 13 subbundle of  $N_2$ .

### 5.1 Harmonic conformal analysis

Let us now consider only minimal immersions  $f$  which are conformal (hence harmonic). The equation reduces to two (non-variational) PDEs, namely the Cauchy-Riemann equations for  $\phi = \partial_1 s - i\partial_2 s$

$$F(\phi) = \partial_1 \phi + i\partial_2 \phi = 0$$

and the conformality condition  $G(\phi) = \langle \phi | \phi \rangle = \sum_{\alpha} (\phi^{\alpha})^2 = 0$ . Notice that by using this formulation one avoids the problem of choosing an integration constant; two solutions differing by a translation are identified. Also first-order symmetries in  $u$  become zero-order in  $\phi$  (as will be illustrated below). However it must be remembered that  $\phi$  does not actually define the minimal immersion, only  $\phi dz$  does.

Using complex notations as a shortcut, let  $Z = X + iY$  be an infinitesimal symmetry acting on the prolonged  $\phi$ -space. Then  $Z$  preserves  $F$  iff  $Z$  is holomorphic in its variables:

$$(D_1 + iD_2)Z(\phi) = 0$$

for any prolonged solution  $\phi$ ; and the condition for keeping conformality is simply  $\langle Z|\phi \rangle = 0$ . The simplest possibility is for linear and zero-order vector fields:  $Z(\phi) = A\phi$  for constant matrix  $A$ . Then conformality imposes that  $A$  belong in  $\mathbb{C}I_3 \oplus \mathfrak{so}(3, \mathbb{C})$ . These are the already known transformations:  $I_3$  is the dilation,  $iI_3$  is the conjugation,  $\mathfrak{so}(3, \mathbb{R})$  corresponds to rotations, and  $i\mathfrak{so}(3, \mathbb{R})$  to the 3-dimensional family of imaginary rotations (Lopez-Ros transformations).

**How to get the corresponding real field:** Let  $Z$  be a infinitesimal symmetry. The corresponding variational symmetry  $\hat{Z}$  is simply  $\hat{Z}(u) = \text{Re} \int_{x_0}^x Z(\phi) dx$ ; of course it is defined up to translation, and is by construction well defined locally (there may be period problems). The symmetry  $\hat{Z}$  may be a divergence symmetry, indeed

$$\begin{aligned} (\text{pr } \hat{Z})L(u) &= \langle \mathbf{D}_1 \hat{Z}|u_1 \rangle + \langle \mathbf{D}_2 \hat{Z}|u_2 \rangle = \langle X|u_1 \rangle - \langle Y|u_2 \rangle \\ &= \mathbf{D}_1 \langle X|u \rangle - \langle \mathbf{D}_1 X|u \rangle - \mathbf{D}_2 \langle Y|u \rangle + \langle \mathbf{D}_2 Y|u \rangle \\ &= \mathbf{D}_1 P_1 + \mathbf{D}_2 P_2 + \langle u|\mathbf{D}_2 Y - \mathbf{D}_1 X \rangle = \text{Div } P \end{aligned}$$

with  $P_1 = \langle X|u \rangle$ ,  $P_2 = -\langle Y|u \rangle$ , since holomorphicity implies  $\mathbf{D}_2 Y = \mathbf{D}_1 X$ . On the other hand

$$(\text{pr } \hat{Z})L(u) = \langle \text{Re} [Z(\phi)]|\text{Re} [\phi] \rangle + \langle \text{Im} [Z(\phi)]|\text{Im} [\phi] \rangle = \text{Re} \langle Z(\phi)|\bar{\phi} \rangle = \frac{1}{2} Z \cdot \langle \phi|\bar{\phi} \rangle$$

Hence  $\hat{Z}$  is a (non-divergence) variational symmetry iff  $Z$  keeps the metric; that is obvious: indeed for conformal maps, fixing the metric is equivalent to fixing the area. If  $Z$  fixes the metric and depends only on  $\phi$ , then it falls under the classification in [4]. It is clear that a metric-preserving vector field in  $\mathbb{C}[3]$  has to be linear (??). The dilation and the imaginary rotations however are truly divergence symmetries.

The Noether 1-form can be written as

$$\omega = -(\langle \hat{Z}|u_2 \rangle + \langle Y|u \rangle)dx^1 + (\langle \hat{Z}|u_1 \rangle - \langle X|u \rangle)dx^2$$

However in this definition there may integration constants. So the conserved quantity is not always well-defined. If  $\hat{Z} \rightarrow \hat{Z} + c$  then  $\omega \rightarrow \omega - \langle c|u_2 \rangle dx^1 + \langle c|u_1 \rangle dx^2$  so

$$\int_{\gamma} \omega \rightarrow \int_{\gamma} \omega + \langle c|\mathcal{F}(\gamma) \rangle$$

If  $u \rightarrow u + c$  then  $\omega \rightarrow \omega + \langle \mathbf{D}_2 \hat{Z}|c \rangle dx^1 - \langle \mathbf{D}_1 \hat{Z}|c \rangle dx^2$  so

$$\int_{\gamma} \omega \rightarrow \int_{\gamma} \omega + \langle c|\int_{\gamma} \mathbf{D}_2 \hat{Z} dx^1 - \mathbf{D}_1 \hat{Z} dx^2 \rangle$$

We may write  $\omega$  in terms of the complex variable  $z$ . Fixing an arbitrary integration starting point, and denoting  $W = \int Z$  and  $\Phi = \int \phi$

$$\begin{aligned}
\omega &= -\left(\langle \operatorname{Re} \int Z | -\operatorname{Im} \phi \rangle + \langle -\operatorname{Im} Z | \operatorname{Re} \int \phi \rangle\right) \frac{dz + d\bar{z}}{2} \\
&\quad + \left(\langle \operatorname{Re} \int Z | \operatorname{Re} \phi \rangle - \langle \operatorname{Re} Z | \operatorname{Re} \int \phi \rangle\right) \frac{dz - d\bar{z}}{2i} \\
&= \frac{1}{8i} \left(\langle W + \bar{W} | \phi - \bar{\phi} \rangle + \langle Z - \bar{Z} | \Phi + \bar{\Phi} \rangle\right) (dz + d\bar{z}) \\
&\quad + \frac{1}{8i} \left(\langle W + \bar{W} | \phi + \bar{\phi} \rangle - \langle Z + \bar{Z} | \Phi + \bar{\Phi} \rangle\right) (dz - d\bar{z}) \\
&= \frac{1}{4i} \left(\langle W + \bar{W} | \phi \rangle - \langle \bar{Z} | \Phi + \bar{\Phi} \rangle\right) dz + \frac{1}{4i} \left(-\langle W + \bar{W} | \bar{\phi} \rangle + \langle Z | \Phi + \bar{\Phi} \rangle\right) d\bar{z}
\end{aligned}$$

So  $\omega = \frac{1}{2} \operatorname{Im} \alpha$  with  $\alpha = (\langle W + \bar{W} | \phi \rangle - \langle \bar{Z} | \Phi + \bar{\Phi} \rangle) dz$ . Since  $\int \omega$  is a homological invariant, we may integrate along a curve  $|z| = r = c^{te}$  in the  $z$ -plane, and the result has to be independent from  $r$ .

**Examples** of non linear fields:  $Z(\phi) = h\phi$  for any  $\mathbb{C}$ -valued holomorphic map  $h$  (for instance  $h = \phi^\alpha$ ). More interesting: for any non zero integer  $k$  the generalized field  $Z = \mathbf{D}_k \phi \times \phi = \phi_{k.1} \times \phi$ . Clearly  $\langle \phi_{k.1} \times \phi | \phi \rangle = 0$ . For instance  $k = 1$  and

$$(\operatorname{pr} \hat{Z})L = \langle u_{11} \times u_1 - u_{12} \times u_2 | u_1 \rangle + \langle u_{11} \times u_2 + u_{12} \times u_1 | u_2 \rangle = 2\langle u_{12} | u_1 \times u_2 \rangle$$

which does not vanish in general, so area is not fixed by  $\hat{Z}$ . Let us analyse this case in detail. For instance for a catenoidal end at  $z = 0$  let  $\phi = (1 - g^2, i(1 + g^2), 2g)f$  and

$$\begin{cases} g(z) = z(a + bz + cz^2 + O(z^3)) \\ f(z) = z^{-2}(\alpha + \gamma z^2 + O(z^3)) \end{cases}$$

So  $g^2 = z^2(a^2 + 2abz + (b^2 + 2ac)z^2 + O(z^3))$ . Then

$$\begin{aligned}
\phi &= z^{-2} \begin{pmatrix} \alpha + (\gamma - a^2)z^2 + O(z^3) \\ i(\alpha + (\gamma + a^2)z^2 + O(z^3)) \\ 2z(a\alpha + \alpha bz + (a\gamma + \alpha c)z^2 + O(z^3)) \end{pmatrix} \\
\phi' &= \begin{pmatrix} -2\alpha z^{-3} + O(1) \\ -2i\alpha z^{-3} + O(1) \\ -2a\alpha z^{-2} + 2(\gamma + \alpha c) + O(z) \end{pmatrix} \\
\Phi &= \begin{pmatrix} -\alpha z^{-1} + (\gamma - a^2)z + O(z^2) \\ -i\alpha z^{-1} + i(\gamma + a^2)z + O(z^2) \\ 2a\alpha \log z + 2\alpha bz + (a\gamma + \alpha c)z^2 + O(z^3) \end{pmatrix}
\end{aligned}$$

$$Z = \phi' \times \phi = \alpha z^{-2} \begin{pmatrix} 2ia\alpha z^{-2} + 4ib\alpha z^{-1} + 2i(a\gamma + 3\alpha c - a^3 + \gamma) + O(z) \\ -2a\alpha z^{-2} - 4b\alpha z^{-1} - 2(a\gamma + 3\alpha c + a^3 + \gamma) + O(z) \\ 4ia^2 z^{-1} + O(1) \end{pmatrix}$$

$$W = \alpha \begin{pmatrix} -\frac{2i}{3}a\alpha z^{-3} - 2ib\alpha z^{-2} - 2i(a\gamma + 3\alpha c - a^3 + \gamma)z^{-1} + O(\log z) \\ \frac{2}{3}a\alpha z^{-3} + 2b\alpha z^{-2} + 2(a\gamma + 3\alpha c + a^3 + \gamma)z^{-1} + O(\log z) \\ -2ia^2 z^{-2} + O(z^{-1}) \end{pmatrix}$$

et

$$\langle W | \phi \rangle =$$

## 6 Élucubrations

La définition d'Olver pour les symétries variationnelles (divergente ou non) réclame que l'équation soit vraie pour tout  $u$ , et pas seulement pour les prolongations des solutions. Ce n'est pas le cas par contre des symétries des EDP, qui doivent simplement annuler l'expression *pour toute solution* ! Il semble cependant que pour appliquer Noether, il suffise d'avoir une symétrie divergente restreinte aux solutions (soit  $\text{pr}X(L) + L\text{Div} \xi = \text{Div} P$  seulement sur les solutions d'Euler-Lagrange) !! Alors la dilatation serait bien génératrice d'une loi de conservation (quoique triviale).

## 7 Lie brackets (unfinished)

The Lie bracket between generalized vertical symmetries is defined by (see [5])

$$[[X, Y]] = \text{pr}X(Y) - \text{pr}Y(X) = \sum_I \sum_{\alpha, \beta} \left( (D_I X^\beta) \frac{\partial Y^\alpha}{\partial u_I^\beta} - (D_I Y^\beta) \frac{\partial X^\alpha}{\partial u_I^\beta} \right) \frac{\partial}{\partial u^\alpha}$$

In vector notations

$$[[X, Y]] = \sum_I (d_{u_I} Y D_I X - d_{u_I} X D_I Y)$$

For classical vector fields,  $[[X, Y]] = [X, Y]$ .

## 8 Finding the conformal parametrization

### 8.1 The space of diffeomorphisms

Let  $\Delta$  be the unit disk and  $\mathcal{D}$  (resp  $\mathcal{C}$ ) be the groups of orientation preserving (conformal) diffeomorphisms. Notice that  $\mathcal{C}$  is a 3-dimensional Lie group consisting in maps

$$z \mapsto \frac{\lambda z + \alpha}{\bar{\alpha} z + \bar{\lambda}} \quad |\lambda| = 1, |\alpha| < 1$$

where  $(\alpha, \lambda)$  are defined up to sign. For any  $f : \Delta \rightarrow \mathbb{R}^3$  there exists  $\phi \in \mathcal{D}$  such that  $f \circ \phi$  is conformal;  $\phi$  is unique up to left composition by  $\mathcal{C}$ ; if one imposes the 3-dim following conditions

$$\phi(0) = 0 \quad \partial_1 \phi^2(0) = 0$$

then  $\phi$  is unique. We call  $\mathcal{D}_+$  the group of such diffeomorphisms.

## 8.2 First order approximation

Let  $u(\varepsilon) = u + \varepsilon v$  be a local deformation (dropping here and afterwards all higher order terms); there exists diffeomorphisms  $\phi(\varepsilon)$  (unique if chosen in  $\mathcal{D}_+$ ) such that  $u(\varepsilon) \circ \phi(\varepsilon)$  is conformal. Let  $Y = \frac{d}{d\varepsilon} \phi(\varepsilon)|_{\varepsilon=0}$

$$u(\varepsilon) \circ \phi(\varepsilon) = (u + \varepsilon v) \circ (\phi + \varepsilon Y) = u \circ \phi + \varepsilon(v \circ \phi + Du \circ \phi.Y)$$

$$\partial_1(u(\varepsilon) \circ \phi(\varepsilon)) = (u \circ \phi)_1 + \varepsilon\{(v \circ \phi)_1 + (Du \circ \phi)_1.Y + Du \circ \phi.Y_1\}$$

$$\partial_2(u(\varepsilon) \circ \phi(\varepsilon)) = (u \circ \phi)_2 + \varepsilon\{(v \circ \phi)_2 + (Du \circ \phi)_2.Y + Du \circ \phi.Y_2\}$$

First conformality condition

$$\frac{1}{2\varepsilon}(\|\partial_1(u(\varepsilon) \circ \phi(\varepsilon))\|^2 - \|\partial_2(u(\varepsilon) \circ \phi(\varepsilon))\|^2) = \langle (u \circ \phi)_1 | (v \circ \phi)_1 + (Du \circ \phi)_1.Y + Du \circ \phi.Y_1 \rangle$$

$$-\langle (u \circ \phi)_2 | (v \circ \phi)_2 + (Du \circ \phi)_2.Y + Du \circ \phi.Y_2 \rangle = 0$$

Second conformality condition

$$\varepsilon^{-1} \langle \partial_1(u(\varepsilon) \circ \phi(\varepsilon)) | \partial_2(u(\varepsilon) \circ \phi(\varepsilon)) \rangle = \langle (u \circ \phi)_1 | (v \circ \phi)_2 + (Du \circ \phi)_2.Y + Du \circ \phi.Y_2 \rangle$$

$$+\langle (u \circ \phi)_2 | (v \circ \phi)_1 + (Du \circ \phi)_1.Y + Du \circ \phi.Y_1 \rangle = 0$$

which tells us that the basis  $((v \circ \phi + Du \circ \phi.Y)_1, (v \circ \phi + Du \circ \phi.Y)_2)$  is (almost) conformal (and direct):

$$\|(v \circ \phi + Du \circ \phi.Y)_1\|^2 - \|(v \circ \phi + Du \circ \phi.Y)_2\|^2 = 0$$

$$\langle (v \circ \phi + Du \circ \phi.Y)_1 | (v \circ \phi + Du \circ \phi.Y)_2 \rangle = 0$$

+ directness !