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Simultaneous controllability and discrimination of collections of perturbed bilinear control systems on the Lie group SU(N)

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Abstract

The controllability of bilinear systems is well understood for isolated systems where the control can be implemented exactly. However when perturbations are present some interesting theoretical questions are raised. We consider in this paper a control system whose control cannot be implemented exactly but is shifted by a time independent constant in a given list of possibilities. We prove under general hypothesis that the collection of possible systems (one for each possible perturbation) is simultaneously controllable with a common control. The result is extended to the situations where the perturbations are constant over a common, long enough, time frame. We apply the result to the controllability of quantum systems.

Keywords:
quantum control, Lie group controllability, bilinear system, perturbations

1. Introduction

The fundamental importance of addressing the controllability of bilinear systems has long been recognized in engineering control applications (see [1–9]). Among recent applications one may cite the field of quantum control with optical or magnetic external fields (see [5, 9–19]).

Although the controllability is well understood when the system is isolated and the control can be implemented exactly new theoretical and numerical questions are raised when perturbations are present.

The question that is addressed in this paper is related to the simultaneous controllability of bilinear systems. Consider general systems $\frac{dX_k(t)}{dt} = (A_k + u(t)B_k)X_k$ on some finite dimensional Lie group G. Simultaneous controllability is the question of whether all states X_k can be controlled with the same control $u(t)$.

Problems of simultaneous control of multiple systems have been addressed recently in applications related to quantum control [20–31]. In such circumstances the system is a collection of molecules or atoms or spin systems and the control is a magnetic field (in NMR) or a laser. The assessment of whether a single control pulse can drive independent (i.e., distinct) quantum systems to their respective target states was addressed theoretically in [20] for general A_k, B_k and applied to the optimal dynamic discrimination of separate quantum systems in [21]. The particular case of identical molecules with $A_k = A$ (constant) and $B_k = \xi_k B, \xi_k \in \mathbb{R}$, was treated in [22, 23] where Turinici and coworkers proved that all members of an ensemble of randomly oriented molecules subjected to a single ultra-fast laser control pulse can be simultaneously controlled. An independent work [30] treats the circumstance when $A_k = \epsilon_i A, \epsilon_i \neq \pm 1$ and $B_k = B$ (constant) and was used to show controllability for ensembles N-level of quantum systems having different Larmor dispersion. This last result generalizes the findings of [25] for ensembles of spin $1/2$ systems.

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In this paper we extend the result in [30] to the new circumstance when $A_k = A + \alpha_k B, \alpha_k \in \mathbb{R}$ and $B_k = B$ (constant) or, equivalently, the simultaneous controllability of systems submitted to time independent perturbations $\frac{dX_k(t)}{dt} = [A+(u(t)+\alpha_k)B]X_k$. As the result in [30] does not apply to this situation we exploit techniques used previously in [22, 23] and prove positive controllability results.

The balance of the paper is as follows: in Section 2 we introduce the general framework and the main notations and in Section 3 we present our main result. In Section 4, we apply our results to the controllability of quantum systems. Finally, some conclusions and perspectives of future work are given in Section 5.

2. Problem formulation

Given a matrix M, we denote by $\mathrm{Tr}(M)$ its trace.

As a fundamental aspect of bilinear control theory, controllability has been widely studied for isolated (unperturbed) systems where the control can be implemented exactly. However when perturbations are present a legitimate question arises: what states can still be attained? Consider the following control systems on $U(N)$

$$\frac{dX(t)}{dt} = (A + u(t)B)X(t), \quad X(0) = Id. \tag{1}$$

Here A and B are skew-Hermitian matrices in $u(N)$. The matrix $X(t)$ evolves in the Lie group of unitary matrices $U(N)$, or, if both the matrices A and B have zero trace, in the Lie group of special unitary matrices $SU(N)$. We will assume without loss of generality (see [7]) that $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$ from now on, i.e., $A, B \in su(N)$ and then $X(t) \in SU(N)$.

The controllability of a system on Lie groups such as (1) is a well-studied problem [4–9]. The literature on the subject of bilinear control relies essentially on the following Theorem (originally due to [32]):

Theorem 1. *Let $A, B \in su(N)$ and denote by $\mathbb{L}_{A,B}$ the Lie subalgebra of $su(N)$ generated by A and B. The system (1) on the Lie group $SU(N)$ is controllable if and only if $\mathbb{L}_{A,B} = su(N)$ or equivalently if $\dim_{\mathbb{R}}\mathbb{L}_{A,B} = N^2 - 1$. Moreover there exists $T_{A,B} > 0$ such that any target can be reached in time $t \geq T_{A,B}$ with controls u such that $|u(s)| \leq 1, \forall s \in [0,t]$. Here $\dim_{\mathbb{R}}\mathbb{L}_{A,B}$ stands for the dimension of $\mathbb{L}_{A,B}$ as linear vector space over $\mathbb{R}$.*

An important question is what happens if the control $u(t)$ in (1) is submitted to some perturbations in a predefined (discrete) list $\{\alpha_k, k = 1, \cdots, K\}$?

$$\frac{dX_k(t)}{dt} = AX_k(t) + [u(t) + \alpha_k]BX_k(t), \;\; X_k(0) = Id. \tag{2}$$

Can one still control the systems simultaneously? The real perturbation α_k for a given system is not known beforehand, therefore in order to be certain that the system is controlled one has to find a control $u(t)$ that simultaneously control all states $X_k(t)$, i.e., find $u(t)$ such that $X_k(T) = V$ for $k = 1, \cdots, K$ (here V is the target state).

Yet a distinct circumstance is when α_k are not arbitrary perturbations but unknown characteristics of the system to be identified. Here the goal is to find $u(t)$ such that, given *distinct* V_k one has $X_k(T) = V_k$. By measuring the state of the system at the final time T one knows what α_k was effective during $[0, T]$.

In conclusion, our problem can be formalized as follows: let $V_k \in SU(N), k = 1, \cdots, K$ be arbitrary. Is it possible to find $T > 0$ and a measurable $u : [0, T] \to \mathbb{R}$ such that the system given by (2) satisfies $X_k(T) = V_k \; \forall \; k = 1, \cdots, K$? If the answer to this question is positive then the system in (2) will be called *simultaneously controllable*.

3. Simultaneous controllability for perturbations

3.1. Tools for simultaneous controllability

We recall in this section some known results on simultaneous controllability that will be necessary in the following sections. Consider K bilinear systems on $SU(N)$:

$$\frac{dX_k(t)}{dt} = (A_k + u(t)B_k)X_k(t), \;\; X_k(0) = Id, \tag{3}$$

where $A_k, B_k \in su(N), k = 1, \cdots, K$. We denote by $diag\{M_1, \cdots, M_P\}$ the block diagonal matrix $\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_P \end{pmatrix}$ obtained by setting the square matrices $M_1, \cdots, M_P$ on its diagonal. This definition allows for introducing $\mathcal{A} = diag\{A_1, \cdots, A_K\}$ as a $KN \times KN$ matrix constructed from $A_k, k = 1, \cdots, K$ and $\mathcal{B} = diag\{B_1, ..., B_K\}$. By assembling the K bilinear systems (3), the evolution of this collection of states can be written as a bilinear system (with block diagonal entries) on $(SU(N))^K$:

$$\frac{d\mathbf{X}(t)}{dt} = \mathcal{A}\mathbf{X}(t) + u(t)\mathcal{B}\mathbf{X}(t),\ \mathbf{X}(0) = \mathbf{Id} \in (SU(N))^K. \tag{4}$$

Denote by $\mathbb{L}_{\mathcal{A},\mathcal{B}}$ the Lie algebra generated by the matrices $\mathcal{A}$ and $\mathcal{B}$. Then we have the following result (see [20, 21]):

Theorem 2. *The K bilinear systems (3) are simultaneously controllable if and only if* $\mathbb{L}_{\mathcal{A},\mathcal{B}} = (su(N))^K$ *or equivalently*

$$\dim_{\mathbb{R}} \mathbb{L}_{\mathcal{A},\mathcal{B}} = K(N^2 - 1).$$

Here $\dim_{\mathbb{R}} \mathbb{L}_{\mathcal{A},\mathcal{B}}$ *stands for the dimension of* $\mathbb{L}_{\mathcal{A},\mathcal{B}}$ *as linear vector space over* $\mathbb{R}$*. Moreover there exists* $T_{\mathcal{A},\mathcal{B}} > 0$ *such that any collection of targets* $(V_k)_{k=1}^K \in (SU(N))^K$ *can be reached in time* $t \geq T_{\mathcal{A},\mathcal{B}}$ *with controls* $u(t)$ *such that* $|u(s)| \leq 1, \forall s \in [0, t]$.

A specific result has been proved in [22] for particular choices $A_k = A$ and $B_k = \alpha_k B$.

Theorem 3 ([22])**.** *Let* $A, B \in su(N)$ *and consider a basis where A is diagonal, denote* $A = (-i)diag(\lambda_1, ..., \lambda_N)$*,* $\lambda_i \in \mathbb{R}$ *being the eigenvalues of iA. In this basis consider the following non-oriented graph* $\mathcal{G}_B = (\mathcal{V}_B, \mathcal{E}_B)$*,* $\mathcal{V}_B = \{1, 2, \cdots, N\}$*,* $\mathcal{E}_B = \{(k, l) \mid B_{kl} \neq 0\}$*.*
Assume that the graph $\mathcal{G}_B$ *is connected and that* $\forall (k_1, l_1), (k_2, l_2) \in \mathcal{E}_B$*,* $(k_1, l_1) \neq (k_2, l_2) : \lambda_{k_1} - \lambda_{l_1} \neq \lambda_{k_2} - \lambda_{l_2}$*. Consider also* $\alpha_1, \cdots, \alpha_K \in \mathbb{R}$ *such that* $|\alpha_k| \neq |\alpha_l|$*,* $\forall k \neq l$*. Then the system on* $(SU(N))^K$

$$\frac{dX_k(t)}{dt} = (A + u(t)\alpha_k B)X_k,\ X_k(0) = Id, \tag{5}$$

is simultaneously controllable and $\mathbb{L}_{\mathcal{A},\mathcal{B}} = (SU(N))^K$*. Moreover there exists* $T_{A,B,\alpha_1,\cdots,\alpha_K}$ *such that any collection of targets* $(V_k)_{k=1}^K \in (SU(N))^K$ *can be reached in a time smaller than* $T_{A,B,\alpha_1,\cdots,\alpha_K}$ *with controls u such that* $|u(s)| \leq 1, \forall s \in [0, t]$*.*

3.2. Main result

Using the previous results we can now attack the situation when the control seen by the k-th system is $u(t) + \alpha_k$ and not $u(t)\alpha_k$ as in [22].

Theorem 4. *Consider the bilinear system on* $SU(N)$ *in equation (2), where* $A, B \in su(N)$*. Suppose that* $\mathbb{L}_{[A,B],B} = su(N)$*.*
Then for any distinct $\alpha_k \in \mathbb{R}, k = 1, .., K$*, the collection of systems (2) is simultaneously controllable in the sense described above. Moreover there exists* $T_{A,B,\alpha_1,\cdots,\alpha_K} > 0$ *such that the system is controllable in any time* $t \geq T_{A,B,\alpha_1,\cdots,\alpha_K}$ *with controls u such that* $|u(s)| \leq 1, \forall s \in [0, t]$*.*

Proof. To assess controllability of (2), we consider it as a system on $(SU(N))^K$ given by matrices $\mathcal{A} = diag(A + \alpha_1 B, \cdots, A + \alpha_K B)$ and $\mathcal{B} = diag(B, \cdots, B)$. Consider also the Lie algebra $\mathbb{L} = \mathbb{L}_{\mathcal{A},\mathcal{B}}$ spanned by $\mathcal{A}$ and $\mathcal{B}$. Note that $\mathbb{L}_{[A,B],B} = su(N)$ implies $\mathbb{L}_{A,B} = su(N)$. Since $[\mathcal{A}, \mathcal{B}] = diag([A, B], \cdots, [A, B])$ and since $\mathbb{L}_{[A,B],B} = su(N)$ it follows that $\mathbb{L}$ contains any matrix of the form $diag(X, \cdots, X), X \in su(N)$. Thus $\mathbb{L}$ contains $Lie\{\mathcal{A}, diag(X, \cdots, X), X \in su(N)\}$ which contains $diag(A, \cdots, A)$ thus contains $\mathcal{A} - diag(A, \cdots, A) = diag(\alpha_1 B, \cdots, \alpha_K B)$. Consequently $\mathbb{L}$ contains $Lie\{diag(\alpha_1 B, \cdots, \alpha_K B), diag(X, \cdots, X), X \in su(N)\}$. Consider now a particular basis, i.e., the one that diagonalizes the Hermitian matrix $i[A, B]$. Since $\mathbb{L}_{[A,B],B} = su(N)$ the graph $\mathcal{G}_B$ of B (see Theorem 3 for its definition) has to be connected in this basis [15]. Let us take now $\bar{X}$ a matrix such that $i\bar{X}$ satisfies condition of degenerate transitions. Then, $\mathbb{L}$ contains $Lie\{diag((\alpha_1 + \eta)B), \cdots, (\alpha_K + \eta)B)], diag(\bar{X}, \cdots, \bar{X}\}\ \forall\ \eta \in \mathbb{R}$. In particular there exists $\bar{\eta} \in \mathbb{R}$ such that $|\alpha_k + \bar{\eta}| \neq |\alpha_j + \bar{\eta}|\ \forall\ j \neq k$. Then by Theorem 3 it follows that $Lie\{diag((\alpha_1 + \bar{\eta})B), \cdots, (\alpha_K + \bar{\eta})B))], diag(\bar{X}, \cdots, \bar{X})\} = (su(N))^K$. Thus the system (5) is controllable. The assertions on $T_{A,B,\alpha_1,\cdots,\alpha_K}$ are consequences of Theorem 3, Q.E.D.

Remark 1. It is important to mention that the condition $\mathbb{L}_{[A,B],B} = su(N)$ is *sufficient but not necessary*. In order to illustrate this remark, we consider $K = 2$ bilinear systems in (2) and choose $A = \frac{1}{i}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$, $B = \frac{1}{i}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\alpha_1 = 1, \alpha_2 = -1$. Using the online calculator available at [33] we obtain that $\dim_{\mathbb{R}} \mathbb{L}_{A,B} = 8$ thus $\mathbb{L}_{A,B} = su(3)$ and $\dim_{\mathbb{R}} \mathbb{L}_{[A,B],B} = 4$ thus $\mathbb{L}_{[A,B],B} \neq su(3)$. However $\dim_{\mathbb{R}} \mathbb{L}_{diag(A+\alpha_1 B, A+\alpha_2 B), diag(B,B)} = 16$ thus $\mathbb{L}_{diag(A+\alpha_1 B, A+\alpha_2 B), diag(B,B)} = (su(3))^2$ and the bilinear systems are simultaneously controllable by the Theorem 2 despite $\mathbb{L}_{[A,B],B} \neq su(3)$.

Remark 2. Having proved the result above for the *bilinear setting*, it is interesting to compare with the analogous result in the *linear case*. For this we consider the following linear systems:

$$\frac{d}{dt}x_1 = Ax_1 + Bu(t),\ x_1(0) = 0,\ \frac{d}{dt}x_2 = Ax_2 + B[u(t) + \alpha],\ x_2(0) = 0.$$

The dynamics of $x_2(t) - x_1(t)$ is not influenced by the control since $\frac{d}{dt}(x_2 - x_1) = A(x_2 - x_1) + B\alpha,\ x_2(0) - x_1(0) = 0$. Hence this collection of systems is not simultaneously controllable. One can conclude that simultaneous control is a nonlinear phenomena.

The result in Theorem 4 can be extended to the situation where the perturbations of the control can depend on time. We will require however that the perturbations be constant on a common, long enough, time interval.

Theorem 5. *Consider the collection of control systems on $SU(N)$ where $A, B \in su(N)$:*

$$\frac{dY_k(t)}{dt} = \Big\{A + (u(t) + \delta_k u(t))B\Big\}Y_k(t), Y_k(0) = Y_{k,0} \in SU(N). \tag{6}$$

Suppose that $\mathbb{L}_{[A,B],B} = su(N)$ and there exists $0 < t_1 < t_2 < \infty$ such that $\delta_k u(t) = \alpha_k$ (constant) $\forall t \in [t_1, t_2]$. Then there exists $T_{A,B,\alpha_1,\cdots,\alpha_K}$ such that if $t_2 - t_1 \geq T_{A,B,\alpha_1,\cdots,\alpha_K}$ the collection of systems (6) is simultaneously controllable at any time $T \geq t_2$.

Proof. Let V_k be given targets for the systems (6) at time $T \geq t_2$. Define $u(t)$ to be zero on $[0, t_1] \cup [t_2, T]$ and $V_k^- = Y_k^-(t_1)$ where $Y_k^-(t)$ is the solution of $\frac{dY_k^-(t)}{dt} = (A + \delta_k u(t)B)Y_k^-(t),\ Y_k^-(0) = Y_{k,0}$ and $V_k^+ = Y_k^+(T)$ where $Y_k^+(t)$ satisfies $\frac{dY_k^+(t)}{dt} = (A + \delta_k u(t)B)Y_k^+(t),\ Y_k^+(t_2) = Id$. Set targets $W_k = (V_k^+)^{-1} V_k (V_k^-)^{-1}$ for the system (2) on $[0, t_2 - t_1]$ and initial states $X_k(0) = Id$ and denote $\tilde{u}(t)$ be the control that drives X_k from $X_k(0) = Id$ to $X_k(t_2 - t_1) = W_k, \forall k = 1, \cdots, K$. Then the control $u(s)$ with $u(s) = 0$, for $s \in [0, t_1[\cup]t_2, T]$ and $u(s) = \tilde{u}(s - t_1)$, for $s \in [t_1, t_2]$ is such that $Y_k(T) = V_k^+ W_k V_k^- = V_k$, Q.E.D.

3.3. *Further results on related models*

Note that the model in equation (1) implies that the perturbation α_k is present even when the control $u(t)$ is null. In practice it may sometimes be possible to eliminate the perturbations when the control field is not used and in this situation the controller can switch between a free, unperturbed dynamics and a controlled, perturbed one. This circumstance is modeled as

$$\frac{dZ_k(t)}{dt} = AZ_k(t) + [u(t) + \alpha_k]\xi(t)BZ_k(t), Z_k(0) = Z_{k,0} \in SU(N), \tag{7}$$

where the controls are $u(t)$ and $\xi(t)$, but $\xi(t) \in \{0, 1\} \forall t \geq 0$ (ξ being a measurable function). We obtain the following

Theorem 6. *The system (7) is simultaneously controllable if and only if $\mathbb{L}_{A,B} = su(N)$.*

Proof. With $\xi(t)$ as a new control the system (7) is controllable if and only if $\mathbb{L}_{diag(A,\cdots,A), diag(A+\alpha_1 B,\cdots,A+\alpha_K B), diag(B,\cdots,B)} = (su(N))^K$ or, equivalently $\mathbb{L}_{diag(A,\cdots,A), diag(B,\cdots,B), diag(\alpha_1 B,\cdots,\alpha_K B)} = (su(N))^K$. Denote $\mathbb{L}_1 = \mathbb{L}_{diag(A,\cdots,A), diag(B,\cdots,B), diag(\alpha_1 B,\cdots,\alpha_K B)}$. Note that the control $\xi(t)$ allows to add $diag(A, \cdots, A)$ to the Lie algebra. Suppose now $\mathbb{L}_{A,B} = su(N)$. As A, B span the whole $su(N)$ then $\mathbb{L}_1$ contains any matrix of the form $diag(X, \cdots, X),\ X \in su(N)$. From this point the proof is similar as the one of Theorem 4. Of course $\mathbb{L}_{A,B} = su(N)$ is a necessary condition for controllability, which proves the reverse implication, Q.E.D.

Remark 3. For the situation (7) a result analogous to the Theorem 5 can be proved. We leave the proof as an exercise to the reader.

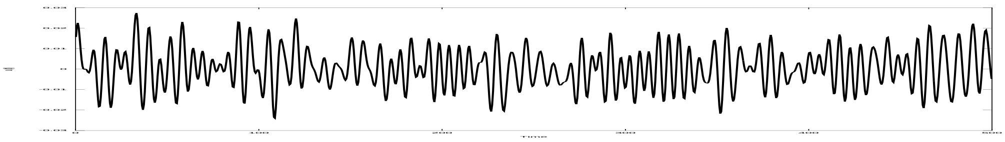

Figure 1: The control that drives ψ_0 to ψ_T (cf. equation (8)) irrespective of the perturbation $\alpha_k \in \{-0.1, 0, 0.1\}$. The quality of the control is over 99% for any perturbation. However the trajectories $\psi(t)$ corresponding to $u(t) - 0.1$, $u(t)$ and $u(t) + 0.1$ are all different.

4. Application to the control of a quantum system

Consider now a quantum bilinear system (cf. [5, 9, 14]):

$$i\frac{d}{dt}\psi = [H_0 + u(t)\mu]\psi(t),\ H_0 = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 0 & 2.15 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{8}$$

controlled by the control $u(t)$ and with $\psi(0) = (1/\sqrt{2}, 0, 0, 1/\sqrt{2}, 0)^T$ and target $\psi_T = (0, 1/\sqrt{2}, 0, 0, 1/\sqrt{2})^T$. Define $B = \mu/i$ and, for simplicity, $A = [H_0 - 0.2\text{Tr}(H_0).Id]/i$ such that both A and B belong to $su(5)$. Using the tool in [33] we obtain $\dim_{\mathbb{R}} \mathbb{L}_{A,B} = \dim_{\mathbb{R}} \mathbb{L}_{[A,B],B} = 24 = \dim_{\mathbb{R}} su(5)$ and since $\mathbb{L}_{[A,B],B} \subset \mathbb{L}_{A,B} \subset su(5)$ it follows that $\mathbb{L}_{[A,B],B} = \mathbb{L}_{A,B} = su(5)$. Consider the perturbations $\alpha_1 = -0.1, \alpha_2 = 0, \alpha_3 = 0.1$. Therefore theorems 4, 5 and 6 of the previous section apply. Since $SU(5)$ is transitive on the unit sphere of $\mathbb{C}^N$(cf. [7]) there exists U_T such that $U_T\psi_0 = \psi_T$ and by the Theorem 4 there exists a time T and a control $u : [0, T] \to \mathbb{R}$ such that $u(t), u(t) - 0.1$ and $u(t) + 0.1$ all drive Id to U_T in equation (1) thus all drive the initial state ψ_0 to the final state ψ_T in equation (8). We searched numerically the control $u(t)$ using a so-called monotonic procedure, see [34–39] for details. For $T = 500$ we obtain the control presented in Figure 1. The quality of the control, i.e. the quantity $\frac{|\langle\psi(T),\psi(0)\rangle|}{\|\psi(0)\|}$ is over 99% for all perturbations $\alpha_k, k = 1, 2, 3$. We also tested different pairs of initial and target states (ψ_0, ψ_T) and in all cases high quality controls were found.

5. Conclusion and perspectives

Using Lie-algebraic methods, sufficient conditions have been derived for the simultaneous controllability of a finite-dimensional system, in the case where the control is submitted to constant or partially constant perturbations.

This work studied the controllability for possibly large final times. A related question is whether small time local controllability (called STLC) is also true. A further question is whether the result extends to more general, time dependent, perturbations.

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