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A SHARP ABELIAN THEOREM FOR THE LAPLACE TRANSFORM

MAEVA BIRET⁽¹⁾, MICHEL BRONIATOWSKI^(1,*), AND ZANSHENG CAO⁽¹⁾

ABSTRACT. This paper states asymptotic equivalents for the moments of the Esscher transform of a distribution on \mathbb{R} with smooth density in the upper tail.

1. INTRODUCTION

Let X denote a real-valued random variable with support \mathbb{R} and absolutely continuous distribution P_X with density p .

The moment generating function of X

$$\Phi(t) := \mathbb{E}[\exp(tX)]$$

is supposed to be finite in a non void neighborhood \mathcal{N} of 0. This hypothesis is usually referred to as a Cramer type condition.

The tilted density of X (or Esscher transform of its distribution) with parameter t in \mathcal{N} is defined on \mathbb{R} by

$$\pi_t(x) := \frac{\exp(tx)}{\Phi(t)} p(x).$$

For $t \in \mathcal{N}$, the functions

$$(1.1) \quad t \rightarrow m(t) := \frac{d}{dt} \log \Phi(t),$$

$$(1.2) \quad t \rightarrow s^2(t) := \frac{d^2}{dt^2} \log \Phi(t),$$

$$(1.3) \quad t \rightarrow \mu_j(t) := \frac{d^j}{dt^j} \log \Phi(t), j \in (2, \infty).$$

are respectively the expectation and the centered moments of a random variable with density π_t .

When Φ is steep, meaning that

$$(1.4) \quad \lim_{t \rightarrow t^+} m(t) = \infty$$

and

$$\lim_{t \rightarrow t^-} m(t) = -\infty$$

where $t^+ := \text{ess sup } \mathcal{N}$ and $t^- := \text{ess inf } \mathcal{N}$ then m parametrizes \mathbb{R} (this is steepness, see Barndorff-Nielsen [1]). We will only require (1.4) to hold.

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This paper presents sharp approximations for the moments of the tilted density π_t under conditions pertaining to the shape of p in its upper tail, when t tends to the upper bound of \mathcal{N} .

2. NOTATION AND HYPOTHESES

The density p is assumed to satisfy the following requirements. It is a density function of the form

$$(2.1) \quad p(x) = c \exp(-(g(x) - q(x))) \quad x \in \mathbb{R}_+,$$

where c is some positive normalizing constant. For the sake of this paper, only the form of p for positive x matters. The function g is positive, convex, twice differentiable and satisfies

$$(2.2) \quad \frac{g(x)}{x} \xrightarrow{x \rightarrow \infty} \infty.$$

Define $h(x) := g'(x)$. We assume that there exists some positive constant ϑ , such that

$$(2.3) \quad \sup_{|v-x| < \vartheta x} |q(v)| \underset{x \rightarrow \infty}{=} o\left(\frac{1}{\sqrt{xh(x)}}\right),$$

Not all positive convex g 's satisfying (2.2) are adapted to our purpose. We follow the line of Juszcak and Nagaev [5] to describe the assumed regularity conditions of h .

Two cases will be considered. The first one is adapted to regularly varying functions g , whose smoothness is described through the following condition pertaining to h .

Case 1: The case when g is a regularly varying function. It will be assumed that h belongs to the class R_β , $\beta > 0$, which is the class of all measurable real-valued functions h defined on a neighborhood of $+\infty$, which can be represented by

$$h(x) = x^\beta l(x),$$

where $l(x)$ is a slowly varying function with Karamata representation

$$(2.4) \quad l(x) = \exp\left(\int_1^x \frac{\epsilon(u)}{u} du\right), \quad x \geq 1,$$

with $\epsilon(x)$ twice differentiable, $\epsilon(x) \xrightarrow{x \rightarrow \infty} 0$, and

$$(2.5) \quad \limsup_{x \rightarrow \infty} x|\epsilon'(x)| < \infty, \quad \limsup_{x \rightarrow \infty} x^2|\epsilon''(x)| < \infty$$

Case 2: In this case, g is a rapidly varying function. The behavior of h at infinity is described through its inverse function

$$(2.6) \quad \psi(u) := h^{\leftarrow}(u)$$

which is a slowly varying function with some regularity, to be stated through the following description.

The class \widetilde{R}_0 consists in all slowly varying functions $x \mapsto l(x)$, with $l(x) \xrightarrow{x \rightarrow \infty} \infty$, where, in (2.4), $\epsilon(x)$ is twice differentiable, $\epsilon(x) \xrightarrow{x \rightarrow \infty} 0$ and

$$(2.7) \quad \lim_{x \rightarrow \infty} \frac{x\epsilon'(x)}{\epsilon(x)} = 0, \quad \lim_{x \rightarrow \infty} \frac{x^2\epsilon''(x)}{\epsilon(x)} = 0$$

and for some $\eta \in (0, 1/6)$

$$(2.8) \quad \liminf_{x \rightarrow \infty} x^\eta \epsilon(x) > 0.$$

We say that $h \in R_\infty$ if h is increasing and strictly monotone and its inverse function ψ belongs to \widetilde{R}_0 .

Denote $\mathcal{R} := (R_\beta) \cup R_\infty$ which covers a large class of functions, with either regular or rapid variation at infinity.

We will assume that

h belongs to \mathcal{R} .

Remark 1. *The role of (2.4) is to make $h(x)$ smooth enough. Under (2.4) the third order derivative of $h(x)$ exists, which is needed in order to use a Laplace method for the asymptotic evaluation of the moment generating function $\Phi(t)$ as $t \rightarrow \infty$, where*

$$\Phi(t) = \int_0^\infty e^{tx} p(x) dx = c \int_0^\infty \exp(K(x, t) + q(x)) dx, \quad t \in (0, \infty)$$

in which

$$(2.9) \quad K(x, t) = tx - g(x).$$

If $h \in \mathcal{R}$, then for large t , $K(x, t)$ takes its maximum at $\hat{x} = h^\leftarrow(t)$ and is concave with respect to x .

The evaluation of $\Phi(t)$ for large t follows from an expansion of $K(x, t)$ in a neighborhood of \hat{x} , using Laplace's method. This expansion yields

$$\begin{aligned} K(x, t) &= K(\hat{x}, t) + K'(\hat{x}, t)(x - \hat{x}) + \frac{K''(\hat{x}, t)}{2}(x - \hat{x})^2 + \frac{K^{(3)}(\hat{x}, t)}{6}(x - \hat{x})^3 + \varepsilon(x, t) \\ &= K(\hat{x}, t) - \frac{1}{2}h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6}h''(\hat{x})(x - \hat{x})^3 + \varepsilon(x, t), \end{aligned}$$

where $\varepsilon(x, t)$ is some error term. Conditions (2.5), (2.7) and (2.8) guarantee that $\varepsilon(x, t)$ goes to 0 when t tends to ∞ and x belongs to some neighborhood of \hat{x} .

Example 1. Weibull density. Let p be a Weibull density with shape parameter $k > 1$ and scale parameter 1, namely

$$\begin{aligned} p(x) &= kx^{k-1} \exp(-x^k), \quad x \geq 0 \\ &= k \exp(-(x^k - (k-1) \log x)). \end{aligned}$$

Take $g(x) = x^k - (k-1) \log x$ and $q(x) = 0$. Then it holds

$$h(x) = kx^{k-1} - \frac{k-1}{x} = x^{k-1} \left(k - \frac{k-1}{x^k} \right).$$

Set $l(x) = k - (k-1)/x^k, x \geq 1$, then (2.4) holds, namely,

$$l(x) = \exp\left(\int_1^x \frac{\epsilon(u)}{u} du\right), x \geq 1,$$

with

$$\epsilon(x) = \frac{k(k-1)}{kx^k - (k-1)}.$$

The function ϵ is twice differentiable and goes to 0 as $x \rightarrow \infty$. Additionally, ϵ satisfies condition (2.5). Hence we have shown that $h \in R_{k-1}$.

Example 2. A rapidly varying density. Define p through

$$p(x) = c \exp(-e^{x-1}), x \geq 0.$$

Then $g(x) = h(x) = e^{x-1}$ and $q(x) = 0$ for all non negative x . We show that $h \in R_\infty$. It holds $\psi(x) = \log x + 1$. Since $h(x)$ is increasing and monotone, it remains to show that $\psi(x) \in \widetilde{R}_0$. When $x \geq 1$, $\psi(x)$ admits the representation (2.4) with $\epsilon(x) = 1/(\log x + 1)$. Also conditions (2.7) and (2.8) are satisfied. Thus $h \in R_\infty$.

3. AN ABELIAN-TYPE THEOREM

We have

Theorem 1. Let $p(x)$ be defined as in (2.1) and $h \in \mathcal{R}$. Denote by $m(t)$, $s^2(t)$ and $\mu_j(t)$ for $j \in (2, \infty)$ the functions defined in (1.1), (1.2) and (1.3). Then with ψ defined as in (2.6), it holds

$$\begin{aligned} m(t) &\underset{t \rightarrow \infty}{=} \psi(t)(1 + o(1)), \\ s^2(t) &\underset{t \rightarrow \infty}{=} \psi'(t)(1 + o(1)), \\ \mu_3(t) &\underset{t \rightarrow \infty}{=} \psi''(t)(1 + o(1)), \\ \mu_j(t) &\underset{t \rightarrow \infty}{=} \begin{cases} M_j s^j(t)(1 + o(1)), & \text{for even } j > 3 \\ \frac{M_{j+3} - 3jM_{j-1}}{6} \mu_3(t) s^{j-3}(t)(1 + o(1)), & \text{for odd } j > 3 \end{cases} \end{aligned}$$

where $M_i, i > 0$, denotes the i th order moment of standard normal distribution.

As a direct by product of Theorem 1 we obtain the following Abel type result.

Define

$$\sigma^2(v) = (h'(v))^{-1}, v \in \mathbb{R}_+.$$

It holds

Theorem 2. Under the same hypotheses as in Theorem 1,

$$\Phi(t) = c\sqrt{2\pi}\sigma e^{K(\hat{x}, t)}(1 + o(1)),$$

where $K(\hat{x}, t)$ is defined as in (2.9) and $\sigma := \sigma(\hat{x})$.

Remark 2. It is easily verified that this result is in accordance with Theorem 4.12.11 of [2], Theorem 3 of [3] and Theorem 4.2 of [5].

Corollary 1. Let $p(x)$ be defined as in (2.1) and $h(x) \in \mathcal{R}$. Then it holds

$$\frac{\mu_3(t)}{s^3(t)} \underset{t \rightarrow \infty}{\longrightarrow} 0.$$

The proofs of these results rely on Lemmas 2, 3, 4 and 5. Lemma 1 is instrumental for Lemma 5. The proof of these Lemmas, Theorem 1, Theorem 2 and Corollary 1 are left to Section 4.

Thereafter we will use indifferently the notation $f(t) \underset{t \rightarrow \infty}{\sim} g(t)$ and $f(t) \underset{t \rightarrow \infty}{=} g(t)(1 + o(1))$ to specify that f and g are asymptotically equivalent functions.

4. APPENDIX: PROOFS

For density functions $p(x)$ defined as in (2.1) satisfying also $h(x) \in \mathcal{R}$, denote by $\psi(x)$ the inverse function of $h(x)$. For brevity, we write \hat{x} , σ and l instead of $\hat{x}(t)$, $\sigma(\psi(t))$ and $l(t)$.

4.1. Some useful Lemmas. The following Lemmas will be useful for proving Theorem 1 and Corollary 1.

Lemma 1. *It holds*

$$(4.1) \quad \frac{\log \sigma(\psi(t))}{\int_1^t \psi(u) du} \underset{t \rightarrow \infty}{\rightarrow} 0.$$

Proof:

Case 1: If $h \in R_\beta$, by Theorem (1.5.15) of [2], there exists some slowly varying function l_1 such that $\psi(x) \sim x^{1/\beta} l_1(x)$ holds. Hence (see Theorem 1 of [4], Chapter 8.9 or Proposition 1.5.8 of [2])

$$(4.2) \quad \int_1^t \psi(u) du \underset{t \rightarrow \infty}{\sim} \frac{t^{1+\frac{1}{\beta}} l_1(t)}{\frac{1}{\beta} + 1} = \frac{\beta t \psi(t)}{1 + \beta}.$$

On the other hand, $h'(x) = x^{\beta-1} l(x)(\beta + \epsilon(x))$. Thus we have as $x \rightarrow \infty$

$$\begin{aligned} |\log \sigma(x)| &= \left| \log(h'(x))^{-\frac{1}{2}} \right| \\ &= \left| \frac{1}{2} ((\beta - 1) \log x + \log l(x) + \log(\beta + \epsilon(x))) \right|. \end{aligned}$$

Firstly, $l(x)$ is a slowly varying function thus $\log l(x) \underset{x \rightarrow \infty}{=} o(x^\alpha)$, for $\alpha > 0$. Hence for x large enough and for all $\delta > 0$, we have $|\log l(x)| \leq \delta x^\alpha$. Secondly, $|\log x| \leq \theta x^\gamma$, for $\gamma > 0$, and x large enough, and for all $\theta > 0$. Thirdly, $\epsilon(x) \underset{x \rightarrow \infty}{\rightarrow} 0$ which implies that for x large enough, $|\epsilon(x)| \leq C$, for some positive constant C . Hence $|\log(\beta + \epsilon(x))| \leq \vartheta(\beta + \epsilon(x))^\eta \leq \vartheta(\beta + C)^\eta$, for $\eta > 0$, x large enough and for all $\vartheta > 0$. Consider the special case $\alpha = \delta = \gamma = \theta = \vartheta = \eta = 1$. We obtain

$$|\log \sigma(x)| \leq \frac{1}{2}(\beta x + (\beta + C)).$$

Set $x = \psi(t)$; we get for t large enough

$$|\log \sigma(\psi(t))| \leq \frac{1}{2}(\beta \psi(t) + (\beta + C)),$$

which, together with (4.2), gives

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u) du} \leq \frac{\beta\psi(t) + (\beta + C)}{2\beta t\psi(t)}(1 + \beta) = (1 + \beta) \left(\frac{1}{2t} + \frac{\beta + C}{2\beta t\psi(t)} \right) \xrightarrow{t \rightarrow \infty} 0,$$

which yields (4.1).

Case 2: If $h \in R_\infty$, then since $\psi(x) \in \tilde{R}_0$ is slowly varying as $x \rightarrow \infty$ it holds

$$(4.3) \quad \int_1^t \psi(u) du \underset{t \rightarrow \infty}{\sim} t\psi(t)$$

(see Theorem 1 of [4], Chapter 8.9 or Proposition 1.5.8 of [2]).

Additionally, we have $h'(x) = (\psi^{-1}(x))' = 1/(\psi'(\psi^{-1}(x))) = 1/\psi'(t)$ with $x = \psi(t)$. It follows

$$|\log \sigma(x)| = |\log(h'(x))^{-\frac{1}{2}}| = \frac{1}{2} |\log \psi'(t)|.$$

Since $\psi(t) \in \tilde{R}_0$, it holds

$$(4.4) \quad |\log \sigma(\psi(t))| = \frac{1}{2} \left| \log \left(\psi(t) \frac{\epsilon(t)}{t} \right) \right| \leq C_1 t,$$

for some positive constant C_1 , using (2.8).

(4.3) and (4.4) imply (4.1)

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u) du} \leq \frac{C_1}{\psi(t)} \xrightarrow{t \rightarrow \infty} 0.$$

This completes the proof.

Lemma 2. For $p(x)$ as in (2.1), $h \in \mathcal{R}$ and for any slowly varying function l such that

$l(t) \xrightarrow{t \rightarrow \infty} \infty$, it holds

$$(4.5) \quad \sup_{|x| \leq \sigma t} h^{(3)}(\hat{x} + x) \sigma^4 l^4 \xrightarrow{t \rightarrow \infty} 0.$$

where $l := l(t)$.

Proof:

Case 1: If $h \in R_\beta$, then we have $h(x) = x^\beta l_0(x)$, $l_0(x) \in R_0$, $\beta > 0$, for x large enough. The function h is three times differentiable

$$(4.6) \quad \begin{aligned} h'(x) &= \beta x^{\beta-1} l_0(x) + x^\beta l_0'(x), \\ h''(x) &= \beta(\beta-1)x^{\beta-2} l_0(x) + 2\beta x^{\beta-1} l_0'(x) + x^\beta l_0''(x), \\ h^{(3)}(x) &= \beta(\beta-1)(\beta-2)x^{\beta-3} l_0(x) + 3\beta(\beta-1)x^{\beta-2} l_0'(x) + 3\beta x^{\beta-1} l_0''(x) + x^\beta l_0^{(3)}(x). \end{aligned}$$

Since $l_0 \in R_0$, it is easy to obtain

$$(4.7) \quad \begin{aligned} l_0'(x) &= \frac{l_0(x)}{x} \epsilon(x), \\ l_0''(x) &= \frac{l_0(x)}{x^2} (\epsilon^2(x) + x\epsilon'(x) - \epsilon(x)), \\ l_0^{(3)}(x) &= \frac{l_0(x)}{x^3} (\epsilon^3(x) + 3x\epsilon'(x)\epsilon(x) - 3\epsilon^2(x) - 2x\epsilon''(x) + 2\epsilon(x) + x^2\epsilon''(x)). \end{aligned}$$

Since $\epsilon(t) \xrightarrow[t \rightarrow \infty]{} 0$, and making use of (2.5) together with the above formulas there exists some positive constants Q_1 and Q_2 such that, for x large enough,

$$|l_0''(x)| \leq Q_1 \frac{l_0(x)}{x^2} \quad \text{and} \quad |l_0^{(3)}(x)| \leq Q_2 \frac{l_0(x)}{x^3}.$$

Including these results in (4.6) and followings yields with some positive constant Q_3 and for x large enough,

$$(4.8) \quad |h^{(3)}(x)| \leq Q_3 \frac{h(x)}{x^3}.$$

By definition, we have $\sigma^2(x) = 1/h'(x)$ with $h'(x) = \beta x^{\beta-1} l_0(x) + x^\beta l_0'(x) = x^{\beta-1} l_0(x)(\beta + \epsilon(x)) = h(x)(\beta + \epsilon(x))/x$, thus $\sigma^2(x) = x/(h(x)(\beta + \epsilon(x)))$. Hence it follows that

$$(4.9) \quad \sigma^2 = \frac{\hat{x}}{h(\hat{x})(\beta + \epsilon(\hat{x}))} = \frac{\psi(t)}{t(\beta + \epsilon(\psi(t)))} \underset{t \rightarrow \infty}{=} \frac{\psi(t)}{\beta t} (1 + o(1)),$$

where the last equality follows from $\psi(t) \xrightarrow[t \rightarrow \infty]{} \infty$ and $\epsilon(x) \xrightarrow[x \rightarrow \infty]{} 0$, which implies, by composition, $\epsilon(\psi(t)) \xrightarrow[t \rightarrow \infty]{} 0$.

Result (4.9) implies

$$\frac{\sigma l}{\psi(t)} \underset{t \rightarrow \infty}{\sim} \frac{l}{\sqrt{\beta t \psi(t)}} \xrightarrow[t \rightarrow \infty]{} 0,$$

since l is slowly varying, hence negligible compared to any power of t . This means that

$$\sigma l \underset{t \rightarrow \infty}{=} o(\psi(t)) = o(\hat{x}).$$

By (4.8) we have

$$(4.10) \quad \sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \leq \sup_{|x| \leq \sigma l} Q_3 \frac{h(\hat{x} + x)}{(\hat{x} + x)^3}.$$

Since $x \underset{t \rightarrow \infty}{=} o(\hat{x})$ on $\{x : |x| \leq \sigma l\}$, then $(\hat{x} + x)^3 \underset{t \rightarrow \infty}{\sim} \hat{x}^3$. Moreover, since $h(x) = x^\beta l_0(x)$ for x large enough and $l_0(x) \in R_0$, then $h(\hat{x} + x) = (\hat{x} + x)^\beta l_0(\hat{x} + x) \underset{t \rightarrow \infty}{\sim} \hat{x}^\beta l_0(\hat{x} + x)$. By (2.4),

$$\frac{l_0(\hat{x} + x)}{l_0(\hat{x})} = \exp \left(\int_{\hat{x}}^{\hat{x}+x} \frac{\epsilon(u)}{u} du \right) \xrightarrow[t \rightarrow \infty]{} 1.$$

Thus we obtain $h(\hat{x} + x) \underset{t \rightarrow \infty}{\sim} h(\hat{x})$. Together with (4.10), we have

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \leq Q_3 \frac{h(\hat{x})}{\hat{x}^3} = Q_3 \frac{t}{\psi^3(t)},$$

which, combined with (4.9) yields

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \sigma^4 l^4 \leq Q_3 \frac{t}{\psi^3(t)} \times \frac{\psi^2(t)}{\beta^2 t^2} l^4 = \frac{Q_3 l^4}{\beta^2 t \psi(t)} \xrightarrow{t \rightarrow \infty} 0,$$

since l varies slowly. This proves (4.5).

Case 2: If $h \in R_\infty$, we have $h'(x) = 1/\psi'(h(x))$. Since $\hat{x} = \psi(t)$, we have $h(\hat{x}) = t$. Thus it holds

$$(4.11) \quad \begin{aligned} h'(\hat{x}) &= \frac{1}{\psi'(t)}, \\ h''(\hat{x}) &= -\frac{h'(\hat{x})\psi''(h(\hat{x}))}{(\psi'(h(\hat{x})))^2} = -\frac{\psi''(t)}{(\psi'(t))^3}, \\ h^{(3)}(\hat{x}) &= -\frac{\psi^{(3)}(t)\psi'(t) - 3(\psi''(t))^2}{(\psi'(t))^5}. \end{aligned}$$

Also

$$(4.12) \quad \begin{aligned} \psi'(t) &= \frac{\psi(t)}{t} \epsilon(t), \\ \psi''(t) &= -\frac{\psi(t)}{t^2} \epsilon(t) \left(1 - \epsilon(t) - t \frac{\epsilon'(t)}{\epsilon(t)} \right) \\ \psi^{(3)}(t) &= \frac{\psi(t)}{t^3} \epsilon(t) \left(2 + \epsilon^2(t) + 3t\epsilon'(t) - 3\epsilon(t) - \frac{2t\epsilon'(t)}{\epsilon(t)} + \frac{t^2\epsilon''(t)}{\epsilon(t)} \right). \end{aligned}$$

Putting the above formulas into (4.11), we get

$$\begin{aligned} h^{(3)}(\hat{x}) &= t \frac{1 + 2\epsilon^2(t) - 3\epsilon(t) + 3t\epsilon'(t) - 4t \frac{\epsilon'(t)}{\epsilon(t)} - t^2 \frac{\epsilon''(t)}{\epsilon(t)} + 3t^2 \frac{(\epsilon'(t))^2}{\epsilon^2(t)}}{\psi^3(t)\epsilon^3(t)} \\ &\underset{t \rightarrow \infty}{\sim} \frac{t}{\psi^3(t)\epsilon^3(t)}, \end{aligned}$$

using (2.7).

By (2.8), $t^\eta \epsilon(t) > C$ with C some positive constant and $\eta \in (0, 1/6)$. Thus we obtain

$$(4.13) \quad \sup_{|v| \leq t/4} h^{(3)}(\psi(t+v)) \underset{t \rightarrow \infty}{\sim} \sup_{|v| \leq t/4} \frac{t+v}{\psi^3(t+v)\epsilon^3(t+v)} \leq \sup_{|v| \leq t/4} \frac{(t+v)^{1+3\eta}}{C^3 \psi^3(t+v)} \leq Q_4 \frac{t^{1+3\eta}}{\psi^3(t)},$$

where $Q_4 = (5/4)^{1+3\eta}/C^3$ is a positive constant. The last inequality holds from the slowly varying property of $\psi(t)$ which implies $\psi(t+v) \underset{t \rightarrow \infty}{\sim} \psi(t)$. Using $\sigma^2 = \psi'(t)$, it holds

$$\sup_{|v| \leq t/4} h^{(3)}(\psi(t+v)) \sigma^4 l^4 \leq Q_4 \frac{t^{1+3\eta}}{\psi^3(t)} \frac{\psi^2(t)\epsilon^2(t)}{t^2} l^4 = Q_4 \frac{\epsilon^2(t)l^4}{\psi(t)t^{1-3\eta}} \xrightarrow{t \rightarrow \infty} 0,$$

since $\eta \in (0, 1/6)$ and l is a slowly varying function. Setting $\psi(t+v) = \hat{x} + x$, we obtain the following equality

$$\sup_{|v| \leq t/4} h^{(3)}(\psi(t+v)) = \sup_{x \in [\xi_1, \xi_2]} h^{(3)}(\hat{x} + x),$$

where

$$\xi_1 = \psi\left(\frac{3t}{4}\right) - \hat{x}, \quad \xi_2 = \psi\left(\frac{5t}{4}\right) - \hat{x}.$$

Thus we obtain

$$\sup_{x \in [\xi_1, \xi_2]} h^{(3)}(\hat{x} + x) \sigma^4 l^4 \xrightarrow{t \rightarrow \infty} 0.$$

For completing the proof, it remains to show that $[-\sigma l, \sigma l] \subset [\xi_1, \xi_2]$ as $t \rightarrow \infty$, namely for t large enough

$$(4.14) \quad \sigma l \leq \min(-\xi_1, \xi_2).$$

The first order Taylor-Lagrange expansion of $\psi(3t/4)$ and $\psi(5t/4)$ yields for some $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$

$$\begin{aligned} \xi_1 &= \psi\left(\frac{3t}{4}\right) - \psi(t) = -\psi'\left(t - \frac{\alpha_1 t}{4}\right) \frac{t}{4} = -\frac{\psi\left(t - \frac{\alpha_1 t}{4}\right)}{4 - \alpha_1} \epsilon\left(t - \frac{\alpha_1 t}{4}\right), \\ \xi_2 &= \psi\left(\frac{5t}{4}\right) - \psi(t) = \psi'\left(t + \frac{\alpha_2 t}{4}\right) \frac{t}{4} = \frac{\psi\left(t + \frac{\alpha_2 t}{4}\right)}{4 + \alpha_2} \epsilon\left(t + \frac{\alpha_2 t}{4}\right). \end{aligned}$$

By (2.8), for large t , $\epsilon(t) \geq C/t^\eta$, for C some positive constant and $\eta \in (0, 1/6)$, and slowly varying property of $\psi(t)$, it holds as $t \rightarrow \infty$

$$\begin{aligned} -\xi_1 &\geq \frac{\psi\left(t - \frac{\alpha_1 t}{4}\right)}{4} \epsilon\left(t - \frac{\alpha_1 t}{4}\right) \underset{t \rightarrow \infty}{\sim} \frac{\psi(t)}{4} \epsilon\left(t - \frac{\alpha_1 t}{4}\right) \geq \frac{\psi(t)}{4} \frac{C}{\left(t - \frac{\alpha_1 t}{4}\right)^\eta} \geq \frac{C}{4} \frac{\psi(t)}{t^\eta}, \\ \xi_2 &\geq \frac{C\psi\left(t + \frac{\alpha_2 t}{4}\right)}{5\left(t + \frac{\alpha_2 t}{4}\right)^\eta} \underset{t \rightarrow \infty}{\sim} \frac{C\psi(t)}{5\left(t + \frac{\alpha_2 t}{4}\right)^\eta} \geq \frac{4^\eta C\psi(t)}{5^{\eta+1} t^\eta}, \end{aligned}$$

which, together with $\sigma = \sqrt{\psi'(t)} = \sqrt{\psi(t)\epsilon(t)/t}$, yields

$$\begin{aligned} \frac{\sigma l}{-\xi_1} &\leq \frac{4t^{\eta-\frac{1}{2}} \sqrt{\epsilon(t)} l}{C \sqrt{\psi(t)}} \xrightarrow{t \rightarrow \infty} 0, \\ \frac{\sigma l}{\xi_2} &\leq \frac{5^{\eta+1} t^{\eta-\frac{1}{2}} \sqrt{\epsilon(t)} l}{4^\eta C \sqrt{\psi(t)}} \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

since $\eta < 1/6$, $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ and l is a slowly varying function. Hence $\sigma l \underset{t \rightarrow \infty}{=} o(-\xi_1)$ and $\sigma l \underset{t \rightarrow \infty}{=} o(\xi_2)$, which imply (4.14). Then

$$\sup_{|x| \leq \sigma l} h^{(3)}(\hat{x} + x) \sigma^4 l^4 \leq \sup_{x \in [\xi_1, \xi_2]} h^{(3)}(\hat{x} + x) \sigma^4 l^4 \xrightarrow{t \rightarrow \infty} 0,$$

which completes the proof.

Lemma 3. For $p(x)$ in (2.1), $h \in \mathcal{R}$ and for some slowly varying function $l(t) \xrightarrow{t \rightarrow \infty} \infty$, it holds

$$(4.15) \quad \sup_{|x| \leq \sigma l} \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l \xrightarrow{t \rightarrow \infty} 0$$

and

$$(4.16) \quad h''(\hat{x}) \sigma^3 l \xrightarrow{t \rightarrow \infty} 0, \quad h''(\hat{x}) \sigma^4 \xrightarrow{t \rightarrow \infty} 0.$$

where $l := l(t)$. We may assume $l(t) := (\log t)^3$.

Proof:

Case 1: $h \in R_\beta$. We get

$$h''(x) \underset{x \rightarrow \infty}{\sim} x^{\beta-2} l_0(x) (\beta(\beta-1) + x\epsilon'(x)).$$

By (2.5), for some real M , we have $x|\epsilon'(x)| \leq M$ which implies $x\epsilon'(x) \geq -M$. Let $N := \beta(\beta-1) - M$. Since $\hat{x} = \psi(t)$, for t large enough it holds

$$(4.17) \quad h''(\hat{x}) \geq N \frac{h(\hat{x})}{\hat{x}^2} = N \frac{t}{\psi^2(t)},$$

which, together with (4.9), yields

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l \right| \leq \frac{Q_3}{N} \frac{t}{\psi^3(t)} \frac{\psi^2(t)}{t} \sqrt{\frac{\psi(t)}{\beta t}} l = \frac{Q_3 l}{N \sqrt{\beta t \psi(t)}} \xrightarrow{t \rightarrow \infty} 0$$

since l varies slowly. Hence we get (4.15).

Using above equivalents obtained in Lemma 1 we obtain for t large enough and some positive constant Q_5

$$h''(\hat{x}) \sigma^3 l \leq Q_5 \frac{t}{\psi^2(t)} \frac{\psi^{\frac{3}{2}}(t)}{\beta^{\frac{3}{2}} t^{\frac{3}{2}}} l = \frac{Q_5}{\beta^{\frac{3}{2}}} \frac{l}{\sqrt{t \psi(t)}} \xrightarrow{t \rightarrow \infty} 0,$$

as sought previously. This implies that the first limit in (4.16) holds. Likewise we get the second limit in (4.16) since

$$h''(\hat{x}) \sigma^4 \leq Q_5 \frac{t}{\psi^2(t)} \frac{\psi^2(t)}{\beta^2 t^2} = \frac{Q_5}{\beta^2 t} \xrightarrow{t \rightarrow \infty} 0$$

Case 2: $h \in R_\infty$. We obtain

$$(4.18) \quad h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^3} \underset{t \rightarrow \infty}{=} \frac{1}{8} \frac{t}{\psi^2(t) \epsilon^2(t)} (1 + o(1)).$$

Combining (4.13) and (4.18), using $\sigma = (h'(\hat{x}))^{-\frac{1}{2}}$, we have as $t \rightarrow \infty$

$$\sup_{|v| \leq t/4} \frac{h^{(3)}(\psi(t+v))}{h''(\hat{x})} \sigma l \leq 8Q_4 \frac{\epsilon^{\frac{5}{2}}(t) l}{\sqrt{\psi(t) t^{\frac{1}{2}-3\eta}}} \xrightarrow{t \rightarrow \infty} 0,$$

since $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$, l is some slowly varying function and by (2.8), $\eta \in (0, 1/6)$. In the same way as case 2 in Lemma 2, we get (4.15).

Using (4.18), we have

$$h''(\hat{x}) \sigma^3 l \underset{t \rightarrow \infty}{\sim} \frac{l}{8 \sqrt{t \psi(t) \epsilon(t)}} \leq \frac{l}{8 \sqrt{C \psi(t) t^{1-\eta/2}}} \xrightarrow{t \rightarrow \infty} 0,$$

where ψ and l vary slowly and $\epsilon(t) > C/t^\eta$, for $\eta \in (0, 1/6)$ by (2.8). This implies that the first limit in (4.16) holds.

Likewise we have

$$h''(\hat{x}) \sigma^4 \underset{t \rightarrow \infty}{\sim} \frac{1}{8t} \xrightarrow{t \rightarrow \infty} 0,$$

hence the proof of the second limit in (4.16).

Lemma 4. For some slowly varying function l such that $l(t) \xrightarrow[t \rightarrow \infty]{} \infty$, it holds

$$\sup_{y \in [-l, l]} \frac{|\xi(\sigma y + \hat{x}, t)|}{h''(\hat{x})\sigma^3} \xrightarrow[t \rightarrow \infty]{} 0,$$

where $\xi(x, t) = \varepsilon(x, t) + q(x)$ and $l := l(t)$. We may assume $l(t) := (\log t)^3$.

Proof:

A close look to the proof of Lemma 3 shows that (4.15) can be slightly modified into

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l^4 \right| \xrightarrow[t \rightarrow \infty]{} 0.$$

Hence for $y \in [-l, l]$, it holds

$$\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \left| \frac{h^{(3)}(\hat{x} + \theta \sigma y)(\sigma y)^4}{h''(\hat{x})\sigma^3} \right| \leq \left| \frac{h^{(3)}(\hat{x} + \theta \sigma y)\sigma l^4}{h''(\hat{x})} \right|,$$

with $\theta \in (0, 1)$. Let $x = \theta \sigma y$. It then holds $|x| \leq \sigma l$. Therefore

$$\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \left| \frac{h^{(3)}(\hat{x} + x)\sigma l^4}{h''(\hat{x})} \right|.$$

Hence by Lemma 3

$$\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l^4 \right| \xrightarrow[t \rightarrow \infty]{} 0.$$

Under condition (2.3), setting $x = \psi(t)$, we get

$$\sup_{|v - \psi(t)| \leq \vartheta \psi(t)} |q(v)| \xrightarrow[t \rightarrow \infty]{} o\left(\frac{1}{\sqrt{t\psi(t)}}\right),$$

We have to prove that

$$\left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \xrightarrow[t \rightarrow \infty]{} 0.$$

Case 1: $h \in R_\beta$. We have $h(x) = x^\beta l_0(x)$, $l_0(x) \in R_0$, $\beta > 0$. Hence

$$h'(x) = x^{\beta-1} l_0(x) (\beta + \epsilon(x)).$$

By (4.9), it holds

$$(4.19) \quad \frac{\sigma l}{\vartheta \psi(t)} \xrightarrow[t \rightarrow \infty]{} \sqrt{\frac{\psi(t)}{\beta t}} \frac{l}{\vartheta \psi(t)} = \frac{1}{\vartheta \sqrt{\beta}} \frac{l}{\sqrt{t\psi(t)}} \xrightarrow[t \rightarrow \infty]{} 0,$$

since $\psi(t) \xrightarrow[t \rightarrow \infty]{} t^{\frac{1}{\beta}} l_1(t) \xrightarrow[t \rightarrow \infty]{} \infty$ for some slowly varying function l_1 and l varies slowly. We can write (4.19) as

$$\sigma l \xrightarrow[t \rightarrow \infty]{} o(\vartheta \psi(t)).$$

Thus $\{v : |v - \psi(t)| \leq \sigma l\} \subset \{v : |v - \psi(t)| \leq \vartheta \psi(t)\}$ which implies

$$(4.20) \quad \sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \leq \sup_{|v - \psi(t)| \leq \vartheta \psi(t)} |q(v)| \xrightarrow[t \rightarrow \infty]{} o\left(\frac{1}{\sqrt{t\psi(t)}}\right).$$

By (4.9) and (4.17), it holds for t large enough

$$\left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \leq |q(\sigma y + \hat{x})| \frac{\beta^{\frac{3}{2}}}{N} \sqrt{t\psi(t)}.$$

Let $v = \sigma y + \hat{x}$, then $|v - \psi(t)| = |\sigma y| \leq \sigma l$. Thus we obtain

$$(4.21) \quad \left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \leq \sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \frac{\beta^{\frac{3}{2}}}{N} \sqrt{t\psi(t)} \xrightarrow{t \rightarrow \infty} 0,$$

where the last step follows from (4.20).

Case 2: $h \in R_\infty$. For any slowly varying function $l(t)$ it holds

$$\frac{\sigma l}{\vartheta\psi(t)} = \sqrt{\frac{\epsilon(t)}{t\psi(t)}} \frac{l}{\vartheta} \xrightarrow{t \rightarrow \infty} 0,$$

hence we have, by (2.3)

$$\sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \xrightarrow{t \rightarrow \infty} o\left(\frac{1}{\sqrt{t\psi(t)}}\right).$$

Using this relation and (4.18), when $y \in [-l, l]$, it holds for t large enough

$$(4.22) \quad \begin{aligned} \left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| &\xrightarrow{t \rightarrow \infty} \sim |q(\sigma y + \hat{x})| 8\sqrt{t\psi(t)\epsilon(t)} \\ &\leq \sup_{|v - \psi(t)| \leq \sigma l} |q(v)| 8\sqrt{t\psi(t)\epsilon(t)} \\ &\leq Q_6 \sqrt{\epsilon(t)} \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

for some constant Q_6 , since $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$.

Combining (4.21) and (4.22) completes the proof.

Lemma 5. For $\alpha \in \mathbb{N}$, denote

$$\Psi(t, \alpha) := \int_0^\infty (x - \hat{x})^\alpha e^{tx} p(x) dx.$$

Then

$$\Psi(t, \alpha) \xrightarrow{t \rightarrow \infty} c\sigma^{\alpha+1} e^{K(\hat{x}, t)} T_1(t, \alpha) (1 + o(1)),$$

where

$$(4.23) \quad T_1(t, \alpha) = \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2}\right) dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy,$$

for some $l(t) \xrightarrow{t \rightarrow \infty} \infty$; we may choose $l(t) = (\log t)^3$.

Proof:

Given a slowly varying function $l(t)$ with $l(t) \xrightarrow{t \rightarrow \infty} \infty$, we define the interval I_t as follows

$$I_t := \left(-\frac{l^{\frac{1}{3}}\sigma}{\sqrt{2}}, \frac{l^{\frac{1}{3}}\sigma}{\sqrt{2}} \right),$$

where $l := l(t)$.

For large enough τ , when $t \rightarrow \infty$ we can partition \mathbb{R}_+ :

$$\mathbb{R}_+ = \{x : 0 < x < \tau\} \cup \{x : x \in \hat{x} + I_t\} \cup \{x : x \geq \tau, x \notin \hat{x} + I_t\},$$

where τ is large enough so that for $x > \tau$, $q(x) < \log 2$. Thus we have

$$(4.24) \quad p(x) < 2ce^{-g(x)}.$$

Obviously, for fixed τ , $\{x : 0 < x < \tau\} \cap \{x : x \in \hat{x} + I_t\} = \emptyset$ since for large t we have $\min(x : x \in \hat{x} + I_t) \xrightarrow[t \rightarrow \infty]{} \infty$. Therefore $\tau < \hat{x} - \frac{l\frac{3}{2}g}{\sqrt{2}} \leq \hat{x}$ for t large enough. Hence it holds

$$(4.25) \quad \Psi(t, \alpha) =: \Psi_1(t, \alpha) + \Psi_2(t, \alpha) + \Psi_3(t, \alpha),$$

where

$$\begin{aligned} \Psi_1(t, \alpha) &= \int_0^\tau (x - \hat{x})^\alpha e^{tx} p(x) dx \\ \Psi_2(t, \alpha) &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha e^{tx} p(x) dx \\ \Psi_3(t, \alpha) &= \int_{x \notin \hat{x} + I_t, x \geq \tau} (x - \hat{x})^\alpha e^{tx} p(x) dx \end{aligned}$$

We estimate $\Psi_1(t, \alpha)$, $\Psi_2(t, \alpha)$ and $\Psi_3(t, \alpha)$ in Step 1, Step 2 and Step 3.

Step 1: Using (4.24), for τ and t large enough, we have

$$|\Psi_1(t, \alpha)| \leq \int_0^\tau |x - \hat{x}|^\alpha e^{tx} p(x) dx \leq 2c \int_0^\tau |x - \hat{x}|^\alpha e^{tx-g(x)} dx \leq 2c \int_0^\tau \hat{x}^\alpha e^{tx} dx,$$

since when $0 < x < \tau < \hat{x}$ then $|x - \hat{x}| = \hat{x} - x < \hat{x}$ for t large enough and g is positive.

Since for t large enough

$$\int_0^\tau \hat{x}^\alpha e^{tx} dx = \hat{x}^\alpha \frac{e^{t\tau}}{t} - \frac{\hat{x}^\alpha}{t} \leq \hat{x}^\alpha \frac{e^{t\tau}}{t},$$

we obtain

$$(4.26) \quad |\Psi_1(t, \alpha)| \leq 2c \hat{x}^\alpha \frac{e^{t\tau}}{t},$$

for τ and t large enough.

We now show that for $h \in \mathcal{R}$, it holds

$$(4.27) \quad \hat{x}^\alpha \frac{e^{t\tau}}{t} \underset{t \rightarrow \infty}{=} o(\sigma^{\alpha+1} e^{K(\hat{x}, t)} h''(\hat{x}) \sigma^3),$$

with $K(x, t)$ defined in (2.9). This is equivalent to

$$\frac{\hat{x}^\alpha e^{t\tau}}{t \sigma^{\alpha+4} h''(\hat{x})} \underset{t \rightarrow \infty}{=} o(e^{K(\hat{x}, t)}),$$

which is implied by

$$\exp(-(\alpha + 4) \log \sigma - \log t + \alpha \log \hat{x} + \tau t - \log h''(\hat{x})) \underset{t \rightarrow \infty}{=} o(e^{K(\hat{x}, t)}).$$

Since $\hat{x} = \psi(t)$, it holds

$$(4.28) \quad K(\hat{x}, t) = t\psi(t) - g(\psi(t)).$$

Setting $u = h(v)$ in $\int_1^t \psi(u)du$,

$$(4.29) \quad \int_1^t \psi(u)du = t\psi(t) - \psi(1) - g(\psi(t)) + g(\psi(1))$$

by integration by parts, and then using $h(x) = g'(x)$.

Finally (4.28) and (4.29) produce

$$(4.30) \quad K(\hat{x}, t) = \int_1^t \psi(u)du + \psi(1) - g(\psi(1)).$$

By Lemma 1, $\log \sigma \stackrel{t \rightarrow \infty}{=} o(\int_1^t \psi(u)du)$. Furthermore $K(\hat{x}, t) \stackrel{t \rightarrow \infty}{\sim} \int_1^t \psi(u)du$. Hence $\log \sigma \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$. It remains to show that $t \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$, $\log \hat{x} \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ and $\log h''(\hat{x}) \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$.

- If $h \in R_\beta$, by Theorem (1.5.12) of [2], it holds $\psi(x) \stackrel{x \rightarrow \infty}{\sim} x^{\frac{1}{\beta}} l_1(x)$ for some slowly varying function $l_1(x)$. Together with (4.2) and (4.30), this yields

$$K(\hat{x}, t) \stackrel{t \rightarrow \infty}{\sim} \int_1^t \psi(u)du \stackrel{t \rightarrow \infty}{\sim} \frac{\beta}{1+\beta} t\psi(t).$$

Consequently,

$$\frac{t}{K(\hat{x}, t)} \stackrel{t \rightarrow \infty}{\sim} \frac{1+\beta}{\beta} \frac{1}{\psi(t)} \xrightarrow{t \rightarrow \infty} 0,$$

which is equivalent to $t \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$. In addition, $\log \hat{x} = \log \psi(t) \stackrel{t \rightarrow \infty}{\sim} (\frac{1}{\beta}) \log t + \log l_1(t) \stackrel{t \rightarrow \infty}{=} o(t) \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$, using the slow variation of l_1 and $\log t \stackrel{t \rightarrow \infty}{=} o(t)$. For the same reason we have $\log h''(\hat{x})/t \leq Q(\log t - 2 \log \psi(t))/t \xrightarrow{t \rightarrow \infty} 0$ which implies $\log h''(\hat{x}) \stackrel{t \rightarrow \infty}{=} o(t) \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$. Since $K(\hat{x}, t) \xrightarrow{t \rightarrow \infty} \infty$ then $e^{o(K(\hat{x}, t))} = o(e^{K(\hat{x}, t)})$ and using previous results, we have that (4.27) holds.

- Assume now that $h \in R_\infty$, $\psi(x) \in \tilde{R}_0$ is slowly varying. Thus $\log \hat{x} = \log \psi(t) \stackrel{t \rightarrow \infty}{=} o(t)$. Therefore, by (4.3) and (4.30), it holds $t \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$; using the same method as previously. Consequently, $\log \hat{x} \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$. (4.18) yields $\log h''(\hat{x}) \stackrel{t \rightarrow \infty}{\sim} -\log 8 + \log t - 2 \log \psi(t) - 2 \log \epsilon(t)$. By (2.8), $-2 \log \epsilon(t) < 2\eta \log t - 2 \log C$, for $\eta \in (0, 1/6)$. Thus we have $\log h''(\hat{x}) < -\log 8 + (1 + 2\eta) \log t - 2 \log \psi(t) - 2 \log C$. Hence $\log h''(\hat{x}) \stackrel{t \rightarrow \infty}{=} o(t) \stackrel{t \rightarrow \infty}{=} o(K(\hat{x}, t))$. We get (4.27).

(4.26) and (4.27) yield together

$$(4.31) \quad |\Psi_1(t, \alpha)| \stackrel{t \rightarrow \infty}{=} o(c\sigma^{\alpha+1} e^{K(\hat{x}, t)} h''(\hat{x}) \sigma^3).$$

When α is even, $T_1(t, \alpha) = \int_{-\frac{t^{1/3}}{\sqrt{2}}}^{\frac{t^{1/3}}{\sqrt{2}}} y^\alpha \exp(-\frac{y^2}{2}) dy \stackrel{t \rightarrow \infty}{\sim} \sqrt{2\pi} M_\alpha$, where M_α is the moment of order α of a standard normal distribution. Thus by (4.16) in Lemma 3

we have

$$\frac{h''(\hat{x})\sigma^3}{T_1(t, \alpha)} \xrightarrow{t \rightarrow \infty} 0.$$

When α is odd, $T_1(t, \alpha) = -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \underset{t \rightarrow \infty}{\sim} -\frac{h''(\hat{x})\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}$,

where $M_{\alpha+3}$ is the moment with order $\alpha+3$ of a standard normal distribution. Thus we have

$$\frac{h''(\hat{x})\sigma^3}{T_1(t, \alpha)} \underset{t \rightarrow \infty}{\sim} -\frac{6}{\sqrt{2\pi} M_{\alpha+3}}$$

Combined with (4.31), these results imply for $\alpha \in \mathbb{N}$

$$(4.32) \quad |\Psi_1(t, \alpha)| \underset{t \rightarrow \infty}{=} o(c\sigma^{\alpha+1} e^{K(\hat{x}, t)} T_1(t, \alpha)).$$

Step 2: By (2.1) as $t \rightarrow \infty$

$$\begin{aligned} \Psi_2(t, \alpha) &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c e^{K(x, t) + q(x)} dx \\ &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c \exp\left(K(\hat{x}, t) - \frac{1}{2} h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6} h''(\hat{x})(x - \hat{x})^3 + \xi(x, t)\right) dx, \end{aligned}$$

where $\xi(x, t) = \varepsilon(x, t) + q(x)$. Making the substitution $y = x - \hat{x}/\sigma$, it holds

$$\begin{aligned} \Psi_2(t, \alpha) &= c\sigma^\alpha e^{K(\hat{x}, t)} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{\sigma^2 y^2}{2} h'(\hat{x}) - \frac{\sigma^3 y^3}{6} h''(\hat{x}) + \xi(\sigma y + \hat{x}, t)\right) \sigma dy \\ (4.33) \quad &= c\sigma^{\alpha+1} e^{K(\hat{x}, t)} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2} - \frac{\sigma^3 y^3}{6} h''(\hat{x}) + \xi(\sigma y + \hat{x}, t)\right) dy \end{aligned}$$

where the last equality follows from $h'(\hat{x}) = 1/\sigma^2$.

On $\{y : y \in (-l^{\frac{1}{3}}/\sqrt{2}, l^{\frac{1}{3}}/\sqrt{2})\}$, by (4.16), $|h''(\hat{x})\sigma^3 y^3| \leq |h''(\hat{x})\sigma^3 l| / 2^{\frac{3}{2}} \xrightarrow{t \rightarrow \infty} 0$. Perform the first order Taylor expansion

$$\exp\left(-\frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t)\right) \underset{t \rightarrow \infty}{=} 1 - \frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t) + o_1(t, y),$$

where

$$(4.34) \quad o_1(t, y) = o\left(-\frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t)\right).$$

We obtain

$$\int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2} - \frac{\sigma^3 y^3}{6} h''(\hat{x}) + \xi(\sigma y + \hat{x}, t)\right) dy =: T_1(t, \alpha) + T_2(t, \alpha),$$

where $T_1(t, \alpha)$ is defined in (4.23) and

$$T_2(t, \alpha) := \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} (\xi(\sigma y + \hat{x}, t) + o_1(t, y)) y^\alpha \exp\left(-\frac{y^2}{2}\right) dy.$$

For $T_2(t, \alpha)$, using (4.34) we have for t large enough

$$\begin{aligned} |T_2(t, \alpha)| \leq & \sup_{y \in [-\frac{l^{\frac{1}{3}}}{\sqrt{2}}, \frac{l^{\frac{1}{3}}}{\sqrt{2}}]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\ & + \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} \left(\left| o\left(\frac{h''(\hat{x})\sigma^3}{6} h^3\right) \right| + |o(\xi(\sigma y + \hat{x}, t))| \right) |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy, \end{aligned}$$

where $\sup_{y \in [-l^{\frac{1}{3}}/\sqrt{2}, l^{\frac{1}{3}}/\sqrt{2}]} |\xi(\sigma y + \hat{x}, t)| \leq \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)|$ since $l^{\frac{1}{3}}/\sqrt{2} \leq l$, which is equivalent to $l \geq 1/8$, is true for t large enough. This implies $[-l^{\frac{1}{3}}/\sqrt{2}, l^{\frac{1}{3}}/\sqrt{2}] \subset [-l, l]$. Thus

$$\begin{aligned} |T_2(t, \alpha)| \leq & 2 \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\ & + \left| o\left(\frac{h''(\hat{x})\sigma^3}{6}\right) \right| \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \\ = & \left| o\left(\frac{h''(\hat{x})\sigma^3}{6}\right) \right| \left(\int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy + \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \right), \end{aligned}$$

where the last equality holds from Lemma 4. Since the integrals in the last equality are both bounded, it holds

$$T_2(t, \alpha) \underset{t \rightarrow \infty}{=} o(h''(\hat{x})\sigma^3).$$

When α is even, we have $T_1(t, \alpha) \underset{t \rightarrow \infty}{\sim} \sqrt{2\pi} M_\alpha$, where $M_\alpha = \alpha! / (2^{\alpha/2} (\alpha/2)!)$ is the moment with order α of a standard normal distribution. It follows by (4.16) in Lemma 3

$$(4.35) \quad \frac{T_2(t, \alpha)}{T_1(t, \alpha)} \leq \frac{h''(\hat{x})\sigma^3}{\sqrt{2\pi} M_\alpha} \underset{t \rightarrow \infty}{\rightarrow} 0.$$

When α is odd, we have $T_1(t, \alpha) \underset{t \rightarrow \infty}{\sim} -\frac{h''(\hat{x})\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}$, where $M_{\alpha+3} = (\alpha + 3)! / (2^{(\alpha+3)/2} ((\alpha + 3)/2)!)$ is the moment with order $\alpha + 3$ of a standard normal distribution. It follows

$$(4.36) \quad \frac{T_2(t, \alpha)}{T_1(t, \alpha)} \underset{t \rightarrow \infty}{=} -\frac{6}{\sqrt{2\pi} M_{\alpha+3}} o(1) \underset{t \rightarrow \infty}{\rightarrow} 0.$$

(4.35) and (4.36) imply that, for $\alpha \in \mathbb{N}$,

$$(4.37) \quad T_2(t, \alpha) \underset{t \rightarrow \infty}{=} o(T_1(t, \alpha)).$$

Using (4.33) and (4.37), we get

$$(4.38) \quad \Psi_2(t, \alpha) = c\sigma^{\alpha+1} \exp(K(\hat{x}, t)) T_1(t, \alpha) (1 + o(1)).$$

Step 3: For any t , $K(x, t)$ as a function of x is concave.

The Three Chords Lemma implies, for $x \mapsto K(x, t)$ concave and $(x, y, z) \in \mathbb{R}_+^3$ such that $x < y < z$

$$(4.39) \quad \frac{K(y, t) - K(z, t)}{y - z} \leq \frac{K(x, t) - K(z, t)}{x - z} \leq \frac{K(x, t) - K(y, t)}{x - y}.$$

Since $x \mapsto K(x, t)$ is concave and attains its maximum in \hat{x} , we consider two cases: $x < \hat{x}$ and $x \geq \hat{x}$. Consider a positive real x such that $x \notin \hat{x} + I_t$ and $x \geq \tau$. This implies that $x > \sup(\tau, \hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}}) = \hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}} > \hat{x}$ or $\tau \leq x < \hat{x} - \frac{l^{1/3}\sigma}{\sqrt{2}} < \hat{x}$. For $x < \hat{x}$, by (4.39), we have

$$\frac{K(x, t) - K(\hat{x}, t)}{x - \hat{x}} \geq \frac{K(\hat{x} - \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{-\frac{l^{1/3}\sigma}{\sqrt{2}}}.$$

Since $x - \hat{x} < 0$, we have

$$(4.40) \quad K(x, t) - K(\hat{x}, t) \leq \frac{K(\hat{x} - \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{-\frac{l^{1/3}\sigma}{\sqrt{2}}}(x - \hat{x})$$

Similarly for $x \geq \hat{x}$,

$$(4.41) \quad K(x, t) - K(\hat{x}, t) \leq \frac{K(\hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{\frac{l^{1/3}\sigma}{\sqrt{2}}}(x - \hat{x}).$$

(4.40) and (4.41) are summed up as

$$(4.42) \quad K(x, t) - K(\hat{x}, t) \leq \frac{K(\hat{x} + \text{sgn}(x - \hat{x})\frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{\text{sgn}(x - \hat{x})\frac{l^{1/3}\sigma}{\sqrt{2}}}(x - \hat{x}),$$

where

$$\text{sgn}(x - \hat{x}) = \begin{cases} 1 & \text{if } x \geq \hat{x} \\ -1 & \text{if } x < \hat{x} \end{cases}$$

A third-order Taylor expansion in the numerator of (4.41) gives

$$K(\hat{x} + \text{sgn}(x - \hat{x})\frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t) \leq -\frac{1}{4}h'(\hat{x})l^{2/3}\sigma^2 = -\frac{1}{4}l^{2/3},$$

which, combined with (4.42), yields

$$K(x, t) - K(\hat{x}, t) \leq -\frac{\sqrt{2}}{4} \frac{l^{1/3}}{\sigma} |x - \hat{x}|.$$

We obtain

$$\begin{aligned} |\Psi_3(t, \alpha)| &\leq 2c \int_{x \notin \hat{x} + I_t, x > \tau} |x - \hat{x}|^\alpha e^{K(x, t)} dx \\ &\leq 2ce^{K(\hat{x}, t)} \int_{|x - \hat{x}| > \frac{l^{1/3}\sigma}{\sqrt{2}}, x > \tau} |x - \hat{x}|^\alpha \exp\left(-\frac{\sqrt{2}}{4} \frac{l^{1/3}}{\sigma} |x - \hat{x}|\right) dx \\ &= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} \left[\int_{\frac{l^{1/3}}{\sqrt{2}}}^{+\infty} y^\alpha \exp\left(-\frac{\sqrt{2}}{4} l^{1/3} y\right) dy + \int_{\frac{\tau - \hat{x}}{\sigma}}^{-\frac{l^{1/3}}{\sqrt{2}}} (-y)^\alpha \exp\left(\frac{\sqrt{2}}{4} l^{1/3} y\right) dy \right] \\ &:= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} (I_\alpha + J_\alpha). \end{aligned}$$

It is easy but a bit tedious to show by recursion that

$$\begin{aligned} I_\alpha &= \int_{\frac{l^{1/3}}{\sqrt{2}}}^{+\infty} y^\alpha \exp\left(-\frac{\sqrt{2}}{4}l^{1/3}y\right) dy \\ &= \exp\left(-\frac{1}{4}l^{2/3}\right) \sum_{i=0}^{\alpha} 2^{\frac{4i+3-\alpha}{2}} l^{\frac{\alpha-(2i+1)}{3}} \frac{\alpha!}{(\alpha-i)!} \\ &\underset{t \rightarrow \infty}{\sim} 2^{\frac{3-\alpha}{2}} \exp\left(-\frac{1}{4}l^{2/3}\right) l^{\frac{\alpha-1}{3}} \end{aligned}$$

and

$$\begin{aligned} J_\alpha &= \int_{\frac{\tau-\hat{x}}{\sigma}}^{-\frac{l^{1/3}}{\sqrt{2}}} (-y)^\alpha \exp\left(\frac{\sqrt{2}}{4}l^{1/3}y\right) dy \\ &= I_\alpha - \exp\left(\frac{\sqrt{2}}{4}l^{1/3}\frac{\tau-\hat{x}}{\sigma}\right) \sum_{i=0}^{\alpha} \left(\frac{\hat{x}-\tau}{\sigma}\right)^{\alpha-i} l^{-\frac{i+1}{3}} 2^{\frac{3i+3}{2}} \frac{\alpha!}{(\alpha-i)!} \\ &= I_\alpha + M(t), \end{aligned}$$

with $\hat{x}/\sigma \xrightarrow[t \rightarrow \infty]{} \infty$ when $h \in \mathcal{R}$ and $\tau - \hat{x} < 0$. Thus $M(t) \xrightarrow[t \rightarrow \infty]{} 0$ and we have for some positive constant Q

$$|\Psi_3(t, \alpha)| \leq Q c e^{K(\hat{x}, t)} \sigma^{\alpha+1} \exp\left(-\frac{1}{4}l^{2/3}\right) l^{\frac{\alpha-1}{3}}.$$

With (4.38), we obtain

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{Q \exp\left(-\frac{1}{4}l^{2/3}\right) l^{\frac{\alpha-1}{3}}}{|T_1(t, \alpha)|}.$$

In Step 2, we saw that $T_1(t, \alpha) \underset{t \rightarrow \infty}{\sim} \sqrt{2\pi} M_\alpha$, for α even and $T_1(t, \alpha) \underset{t \rightarrow \infty}{\sim} -\frac{h''(x)\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}$, for α odd. Hence for α even and t large enough

$$(4.43) \quad \left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq Q \frac{\exp\left(-\frac{1}{4}l^{2/3}\right) l^{\frac{\alpha-1}{3}}}{\sqrt{2\pi} M_\alpha} \xrightarrow[t \rightarrow \infty]{} 0,$$

and for α odd and t large enough

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq Q \frac{\exp\left(-\frac{1}{4}l^{2/3}\right) l^{\frac{\alpha-1}{3}}}{\frac{h''(x)\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}}.$$

Use now $l(t) = (\log t)^3$.

If $h \in R_\beta$, using (4.17), we have for t large enough

$$h''(\hat{x})\sigma^3 \geq \frac{N}{\beta^{3/2}} \frac{1}{\sqrt{t\psi(t)}} = \frac{N}{\beta^{3/2}} \frac{1}{t^{\frac{1}{2} + \frac{1}{2\beta}} \sqrt{l_1(t)}},$$

since $\psi(t) \underset{t \rightarrow \infty}{\sim} t^{1/\beta} l_1(t)$ for some slowly varying function l_1 .

With the same choice of $l(t)$ as above, we have

$$(4.44) \quad \left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{Q\beta^{3/2}}{N} \exp\left(-\frac{1}{4}l^{2/3} + \left(\frac{1}{2} + \frac{1}{2\beta}\right) \log t + \frac{1}{2} \log l_1(t)\right) \frac{l^{\frac{\alpha-1}{3}}}{\sqrt{2\pi} M_{\alpha+3}} \xrightarrow[t \rightarrow \infty]{} 0,$$

since $-(\log t)^2/4 + (1/2 + 1/(2\beta)) \log t + 1/2 \log l_1(t) \underset{t \rightarrow \infty}{\sim} -(\log t)^2/4 \xrightarrow{t \rightarrow \infty} -\infty$.
If $h \in R_\infty$, using (4.18), then we have

$$h''(\hat{x})\sigma^3 \underset{t \rightarrow \infty}{\sim} \frac{1}{8 \sqrt{t\psi(t)\epsilon(t)}},$$

with $\psi(t)$ a slowly varying function and $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$. Thus it holds as $t \rightarrow \infty$ and with the choice of l as above

$$(4.45) \quad \left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \exp \left(-\frac{1}{4} l^{2/3} + \frac{1}{2} \log t + \frac{1}{2} \log \psi(t) + \frac{1}{2} \log \epsilon(t) \right) \frac{l^{\frac{\alpha-1}{3}}}{\sqrt{2\pi} M_{\alpha+3}} \xrightarrow{t \rightarrow \infty} 0.$$

As a consequence of (4.43), (4.44) and (4.45), for $h \in \mathcal{R}$, we obtain

$$(4.46) \quad \Psi_3(t, \alpha) \underset{t \rightarrow \infty}{=} o(\Psi_2(t, \alpha)).$$

The proof is completed by combining (4.25), (4.32), (4.38) and (4.46).

4.2. Proof of Theorem 1. By Lemma 5, if $\alpha = 0$, it holds

$$T_1(t, 0) \xrightarrow{t \rightarrow \infty} \sqrt{2\pi},$$

since $l(t) \xrightarrow{t \rightarrow \infty} \infty$. Hence we can approximate the moment generating function of X as follows

$$(4.47) \quad \begin{aligned} \Phi(t) &= \int_0^\infty e^{tx} p(x) dx = \Psi(t, 0) \\ &\underset{t \rightarrow \infty}{=} c\sigma e^{K(\hat{x}, t)} T_1(t, 0) (1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} c\sqrt{2\pi}\sigma e^{K(\hat{x}, t)} (1 + o(1)). \end{aligned}$$

If $\alpha = 1$, it holds

$$T_1(t, 1) \underset{t \rightarrow \infty}{=} -\frac{h''(\hat{x})\sigma^3}{6} M_4 \sqrt{2\pi} (1 + o(1)),$$

where $M_4 = 3$ denotes the fourth order moment of the standard normal distribution. Consequently, we obtain

$$T_1(t, 1) \underset{t \rightarrow \infty}{=} -\frac{\sqrt{2\pi} h''(\hat{x})\sigma^3}{2} (1 + o(1)),$$

and

$$(4.48) \quad \Psi(t, 1) \underset{t \rightarrow \infty}{=} -c\sqrt{2\pi}\sigma^2 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{2} (1 + o(1)) \underset{t \rightarrow \infty}{=} -\Phi(t) \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)),$$

which, together with the definition of $\Psi(t, \alpha)$, yields

$$\int_0^\infty x e^{tx} p(x) dx = \Psi(t, 1) + \hat{x}\Phi(t) \underset{t \rightarrow \infty}{=} \left(\hat{x} - \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)) \right) \Phi(t).$$

Hence we get

$$(4.49) \quad m(t) = \frac{d \log \Phi(t)}{dt} = \hat{x} - \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)).$$

By (4.16), we obtain

$$(4.50) \quad m(t) \underset{t \rightarrow \infty}{\sim} \hat{x} = \psi(t).$$

If $\alpha = 2$, it follows

$$T_1(t, 2) \underset{t \rightarrow \infty}{=} \sqrt{2\pi}(1 + o(1)).$$

Thus we have

$$(4.51) \quad \Psi(t, 2) \underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)).$$

Using (4.48), (4.49) and (4.51), it follows

$$\begin{aligned} \int_0^\infty (x - m(t))^2 e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^2 e^{tx} p(x) dx \\ &= \int_0^\infty (x - \hat{x})^2 e^{tx} p(x) dx + 2(\hat{x} - m(t)) \int_0^\infty (x - \hat{x}) e^{tx} p(x) dx + (\hat{x} - m(t))^2 \Phi(t) \\ &= \Psi(t, 2) + 2(\hat{x} - m(t))\Psi(t, 1) + (\hat{x} - m(t))^2 \Phi(t) \\ &\underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)) - \sigma^2 \Phi(t) \frac{(h''(\hat{x})\sigma^3)^2}{4} (1 + o(1)) \underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)), \end{aligned}$$

where the last equality holds since $h''(\hat{x})\sigma^3$ goes to 0 by (4.16).

Since

$$\frac{\int_0^\infty (x - m(t))^2 e^{tx} p(x) dx}{\Phi(t)} = \frac{\int_0^\infty x^2 e^{tx} p(x) dx}{\Phi(t)} - (m(t))^2,$$

we obtain

$$(4.52) \quad s^2(t) = \frac{\int_0^\infty (x - m(t))^2 e^{tx} p(x) dx}{\Phi(t)} \underset{t \rightarrow \infty}{\sim} \sigma^2 = \psi'(t).$$

If $\alpha = 3$, it holds

$$T_1(t, 3) = -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} y^6 e^{-\frac{y^2}{2}} dy.$$

Thus we have

$$(4.53) \quad \begin{aligned} \Psi(t, 3) &= -c\sqrt{2\pi}\sigma^4 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} y^6 e^{-\frac{y^2}{2}} dy \\ &\underset{t \rightarrow \infty}{=} -M_6 \frac{h''(\hat{x})\sigma^6}{6} \Phi(t)(1 + o(1)), \end{aligned}$$

where $M_6 = 15$ denotes the sixth order moment of standard normal distribution.

Using (4.48), (4.49), (4.51) and (4.53), we have

$$\begin{aligned} \int_0^\infty (x - m(t))^3 e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^3 e^{tx} p(x) dx \\ &= \Psi(t, 3) + 3(\hat{x} - m(t))\Psi(t, 2) + 3(\hat{x} - m(t))^2 \Psi(t, 1) + (\hat{x} - m(t))^3 \Phi(t) \\ &\underset{t \rightarrow \infty}{=} -5 \frac{h''(\hat{x})\sigma^6}{2} \Phi(t)(1 + o(1)) + 3 \frac{h''(\hat{x})\sigma^4}{2} \sigma^2 \Phi(t)(1 + o(1)) \\ &\quad - 3 \left(\frac{h''(\hat{x})\sigma^4}{2} \right)^2 \Phi(t) \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)) + \left(\frac{h''(\hat{x})\sigma^4}{2} \right)^3 \Phi(t)(1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} -h''(\hat{x})\sigma^6 \Phi(t)(1 + o(1)) - h''(\hat{x})\sigma^6 \Phi(t) \frac{(h''(\hat{x})\sigma^3)^2}{4} (1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} -h''(\hat{x})\sigma^6 \Phi(t)(1 + o(1)), \end{aligned}$$

where the last equality holds since $h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0$ by (4.16). Hence we get

$$(4.54) \quad \begin{aligned} \mu_3(t) &= \frac{d^3 \log \Phi(t)}{dt^3} = \frac{\int_0^\infty (x - m(t))^3 e^{tx} p(x) dx}{\Phi(t)} \\ &\underset{t \rightarrow \infty}{\sim} -h''(\hat{x})\sigma^6 = \frac{\psi''(t)}{(\psi'(t))^3} (\psi'(t))^3 = \psi''(t). \end{aligned}$$

Now we consider $\alpha = j > 2$ for even j . Using (4.49) and Lemma 5, we have

$$(4.55) \quad \begin{aligned} \int_0^\infty (x - m(t))^j e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^j e^{tx} p(x) dx \\ &= \sum_{i=0}^j \binom{j}{i} \left(\frac{h''(\hat{x})\sigma^4}{2} \right)^i c\sigma^{j-i+1} e^{K(\hat{x},t)} T_1(t, j-i) (1 + o(1)), \end{aligned}$$

with

$$T_1(t, j-i) = \begin{cases} \int_{-\frac{i\frac{1}{3}}{\sqrt{2}}}^{\frac{i\frac{1}{3}}{\sqrt{2}}} y^{j-i} e^{-\frac{y^2}{2}} dy & \text{for even } i \\ -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{i\frac{1}{3}}{\sqrt{2}}}^{\frac{i\frac{1}{3}}{\sqrt{2}}} y^{3+j-i} e^{-\frac{y^2}{2}} dy & \text{for odd } i \end{cases}$$

$$\underset{t \rightarrow \infty}{=} \begin{cases} \sqrt{2\pi} M_{j-i} (1 + o(1)) & \text{if } i \text{ is even} \\ -\sqrt{2\pi} \frac{h''(\hat{x})\sigma^3}{6} M_{3+j-i} & \text{if } i \text{ is odd} \end{cases}$$

Since $\Phi(t) \underset{t \rightarrow \infty}{=} c\sqrt{2\pi}\sigma e^{K(\hat{x},t)} (1 + o(1))$, we obtain

$$\begin{aligned} &\int_0^\infty (x - m(t))^j e^{tx} p(x) dx \\ &\underset{t \rightarrow \infty}{=} \sum_{i=0}^j \binom{j}{i} \left(\frac{h''(\hat{x})\sigma^4}{2} \right)^i \Phi(t) \times \\ &\left[\sigma^{j-i} M_{j-i} (1 + o(1)) \mathbb{I}_{\text{even } i} - \frac{h''(\hat{x})\sigma^4}{2} \sigma^{j-i-1} \frac{M_{3+j-i}}{3} (1 + o(1)) \mathbb{I}_{\text{odd } i} \right] \\ &\underset{t \rightarrow \infty}{=} \sum_{k=0}^{j/2} \binom{j}{2k} \left(\frac{h''(\hat{x})\sigma^4}{2} \right)^{2k} \Phi(t) \sigma^{j-2k} M_{j-2k} (1 + o(1)) \\ &\quad - \sum_{k=0}^{j/2-1} \binom{j}{2k+1} \left(\frac{h''(\hat{x})\sigma^4}{2} \right)^{2(k+1)} \Phi(t) \sigma^{j-2k-2} \frac{M_{3+j-2k-1}}{3} (1 + o(1)) \\ &\underset{t \rightarrow \infty}{\sim} \sigma^j \Phi(t) \times \\ &\left(M_j + \sum_{k=1}^{j/2} \binom{j}{2k} (h''(\hat{x})\sigma^3)^{2k} \frac{M_{j-2k}}{2^{2k}} - \sum_{k=0}^{j/2-1} \binom{j}{2k+1} (h''(\hat{x})\sigma^3)^{2(k+1)} \frac{M_{3+j-2k-1}}{3 \times 2^{2(k+1)}} \right) \\ &\underset{t \rightarrow \infty}{=} M_j \sigma^j \Phi(t) (1 + o(1)), \end{aligned}$$

since $h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0$ by (4.16). Hence we get for even j

$$(4.56) \quad \begin{aligned} \mu_j(t) &= \frac{d^j \log \Phi(t)}{dt^j} = \frac{\int_0^\infty (x - m(t))^j e^{tx} p(x) dx}{\Phi(t)} \\ &\underset{t \rightarrow \infty}{\sim} M_j \sigma^j \underset{t \rightarrow \infty}{\sim} M_j s^j(t), \end{aligned}$$

by (4.52).

To finish, we consider $\alpha = j > 2$ for odd j . (4.55) stays true but

$$T_1(t, j - i) = \begin{cases} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^{j-i} e^{-\frac{y^2}{2}} dy & \text{for odd } i \\ -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^{3+j-i} e^{-\frac{y^2}{2}} dy & \text{for even } i \end{cases}$$

$$\underset{t \rightarrow \infty}{=} \begin{cases} \sqrt{2\pi} M_{j-i} (1 + o(1)) & \text{if } i \text{ is odd} \\ -\sqrt{2\pi} \frac{h''(\hat{x})\sigma^3}{6} M_{3+j-i} & \text{if } i \text{ is even} \end{cases}$$

Thus, with the same tools as above, some calculus and making use of (4.16),

$$\int_0^\infty (x - m(t))^j e^{tx} p(x) dx \underset{t \rightarrow \infty}{=} \frac{M_{j+3} - 3jM_{j-1}}{6} \times (-h''(\hat{x})\sigma^{j+3})\Phi(t).$$

Hence we get for odd j

$$(4.57) \quad \begin{aligned} \mu_j(t) &= \frac{d^j \log \Phi(t)}{dt^j} = \frac{\int_0^\infty (x - m(t))^j e^{tx} p(x) dx}{\Phi(t)} \\ &\underset{t \rightarrow \infty}{\sim} \frac{M_{j+3} - 3jM_{j-1}}{6} \times (-h''(\hat{x})\sigma^{j+3}) \\ &\underset{t \rightarrow \infty}{\sim} \frac{M_{j+3} - 3jM_{j-1}}{6} \mu_3(t) s^{j-3}(t), \end{aligned}$$

by (4.52) and (4.54).

The proof is complete by combining (4.50), (4.52), (4.54), (4.56) and (4.57).

4.3. Proof of Theorem 2. It is proved incidentally in (4.47).

4.4. Proof of Corollary 3.1. The proof follows from (4.16), from which $h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0$ since $l(t) \xrightarrow[t \rightarrow \infty]{} \infty$. By (4.52) and (4.54), it holds

$$\frac{\mu_3}{s^3} \underset{t \rightarrow \infty}{\sim} -h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0.$$

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