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# A sharp Abelian theorem for the Laplace transform

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## Abstract

This paper states asymptotic equivalents for the three first moments of the Eescher transform of a distribution on  $\mathbb{R}$  with smooth density in the upper tail.

**Key words.** Abelian theorem. Laplace transform. Eescher transform.

## 1 Introduction

Let  $X$  denote a real valued random variable with support  $\mathbb{R}$  and absolutely continuous distribution  $P_X$  with density  $p$ .

The moment generating function of  $X$

$$\Phi(t) := \mathbb{E}[\exp(tX_1)] \quad (1.1)$$

is supposed to be finite in a non void neighborhood  $\mathcal{N}$  of 0. This hypothesis is usually referred to as a Cramer type condition.

The tilted density of  $X$  (or Eescher transform of its distribution) with parameter  $t$  in  $\mathcal{N}$  is defined on  $\mathbb{R}$  by

$$\pi_t(x) := \frac{\exp(tx)}{\Phi(t)} p(x) \quad (1.2)$$

For  $t \in \mathcal{N}$  the functions

$$t \rightarrow m(t) := \frac{d}{dt} \log \Phi(t), \quad (1.3)$$

$$t \rightarrow s^2(t) := \frac{d^2}{dt^2} \log \Phi(t), \quad (1.4)$$

$$t \rightarrow \mu_j(t) := \frac{d^j}{dt^j} \log \Phi(t), j = 3, 4. \quad (1.5)$$

are respectively the expectation and the three first centered moments of a random variable with density  $\pi_t$ .

When  $\Phi$  is steep, meaning that

$$\lim_{t \rightarrow t^+} m(t) = \infty \quad (1.6)$$

and

$$\lim_{t \rightarrow t^-} m(t) = -\infty$$

where  $t^+ := \text{ess sup } \mathcal{N}$ , and  $t^- := \text{ess inf } \mathcal{N}$  then  $m$  parametrizes  $\mathbb{R}$  (this is a special case of steepness, see Barndorff-Nielsen [1]). We will only require (1.6) to hold.

This paper presents sharp approximations for the first three moments of the tilted density  $\pi_t$  under conditions pertaining to the shape of  $p$  in its upper tail, when  $t$  tends to the upper bound of  $\mathcal{N}$ .

## 2 Notation and hypotheses

The density  $p$  is assumed to satisfy the following requirements. It is a bounded density function of the form

$$p(x) = c \exp(-(g(x) - q(x))) \quad x \in \mathbb{R}_+, \quad (2.1)$$

where  $c$  is some positive normalized constant. For the sake of this paper, only the form of  $p$  for positive  $x$  matters. The function  $g$  is positive, convex, twice differentiable, and satisfies

$$\frac{g(x)}{x} \xrightarrow{x \rightarrow \infty} \infty. \quad (2.2)$$

Define  $h(x) := g'(x)$ . We assume that there exists some positive constant  $\vartheta$ , such that

$$\sup_{|v-x| < \vartheta x} |q(v)| \underset{x \rightarrow \infty}{=} o\left(\frac{1}{\sqrt{xh(x)}}\right), \quad h \in R_\beta, \quad \sup_{|v-x| < \vartheta x} |q(v)| \underset{x \rightarrow \infty}{=} o\left(\frac{\sqrt{x}}{h(x)^2}\right), \quad h \in R_\infty, \quad (2.3)$$

where  $R_\beta$  and  $R_\infty$  are defined below.

Not all positive  $g$ 's satisfying (2.2) are adapted to our purpose. We follow the line of Juszczak and Nagaev [5] to describe the assumed regularity conditions of  $h$ .

Two cases will be considered. The first one is adapted to regularly functions  $g$ , whose smoothness is described through the following condition pertaining to  $h$ .

**Case 1:** The case when  $g$  is a Regularly varying function. It will be assumed that  $h$  belongs to the class  $R_\beta$ ,  $\beta > 0$ , which is the class of all measurable real values functions  $h$  defined on a neighborhood of  $+\infty$ , which can be represented by

$$h(x) = x^\beta l(x),$$

where  $l(x)$  is a slowly varying function with Karamata representation

$$l(x) = \exp\left(\int_1^x \frac{\epsilon(u)}{u} du\right), \quad x \geq 1, \quad (2.4)$$

and  $\epsilon(x)$  is twice differentiable and  $\epsilon(x) \xrightarrow{x \rightarrow \infty} 0$ , together with

satisfies

$$\limsup_{x \rightarrow \infty} x|\epsilon'(x)| < \infty, \quad \limsup_{x \rightarrow \infty} x^2|\epsilon''(x)| < \infty \quad (2.5)$$

**Case 2:** In this case  $g$  is a rapidly varying function. The behavior of  $h$  at infinity is described through its inverse function

$$\psi(u) := h^{\leftarrow}(u) := \inf\{x : h(x) \geq u\} \quad (2.6)$$

which is a slowly varying function with some regularity, to be stated through the following description.

The class  $\widetilde{R}_0$  is the subclass of the slowly varying functions with  $\epsilon(x)$  twice differentiable and  $\epsilon(x) \xrightarrow{x \rightarrow \infty} 0$ , which satisfy further  $l(x) \xrightarrow{x \rightarrow \infty} \infty$  and

$$\lim_{x \rightarrow \infty} \frac{x\epsilon'(x)}{\epsilon(x)} = 0, \quad \lim_{x \rightarrow \infty} \frac{x^2\epsilon''(x)}{\epsilon(x)} = 0 \quad (2.7)$$

and for some  $\eta \in (0, 1/8)$

$$\liminf_{x \rightarrow \infty} x^\eta \epsilon(x) > 0. \quad (2.8)$$

We say that  $h \in R_\infty$  if  $h$  is increasing and strictly monotone and its inverse function  $\psi$  belongs to  $\widehat{R}_0$ .

Denote  $\mathcal{R} := \left( \bigcup_{B>0} R_B \right) \cup R_\infty$  which covers a large class of functions, with either regular or rapid variation at infinity.

We will assume that

$h$  belongs to  $\mathcal{R}$ .

**Remark 2.1.** *The role of (2.4) is to make  $h(x)$  smooth enough. Under (2.4) the third order derivative of  $h(x)$  exists, which is needed in order to use a Laplace method for the asymptotic evaluation of the moment generating function  $\Phi(t)$  as  $t \rightarrow \infty$ , where*

$$\Phi(t) = \int_0^\infty e^{tx} p(x) dx = c \int_0^\infty \exp(K(x, t) + q(x)) dx, \quad t \in (0, \infty)$$

in which

$$K(x, t) = tx - g(x). \quad (2.9)$$

If  $h \in \mathcal{R}$ , then for large  $t$ ,  $K(x, t)$  takes its maximum at  $\hat{x} = h^\leftarrow(t)$  and is concave with respect to  $x$ .

The evaluation of  $\Phi(t)$  for large  $t$  follows from an expansion of  $K(x, t)$  in a neighborhood of  $\hat{x}$ , using Laplace's method. This expansion yields

$$\begin{aligned} K(x, t) &= K(\hat{x}, t) + K'(\hat{x}, t)(x - \hat{x}) + \frac{K''(\hat{x}, t)}{2}(x - \hat{x})^2 + \frac{K^{(3)}(\hat{x}, t)}{6}(x - \hat{x})^3 + \varepsilon(x, t) \\ &= K(\hat{x}, t) - \frac{1}{2}h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6}h''(\hat{x})(x - \hat{x})^3 + \varepsilon(x, t), \end{aligned} \quad (2.10)$$

where  $\varepsilon(x, t)$  is some error term. Conditions (2.5), (2.7) and (2.8) guarantee that  $\varepsilon(x, t)$  goes to 0 when  $t$  tends to  $\infty$  and  $x$  belongs to some neighborhood of  $\hat{x}$ .

**Example 2.1. Weibull density.** Let  $p$  be a Weibull density with shape parameter  $k > 1$  and scale parameter 1, namely

$$\begin{aligned} p(x) &= kx^{k-1} \exp(-x^k), x \geq 0 \\ &= k \exp(-(x^k - (k-1) \log x)). \end{aligned}$$

Take  $g(x) = x^k - (k-1) \log x$  and  $q(x) = 0$ . Then it holds

$$h(x) = kx^{k-1} - \frac{k-1}{x} = x^{k-1} \left( k - \frac{k-1}{x^k} \right).$$

Set  $l(x) = k - (k-1)/x^k$ ,  $x \geq 1$ , then (2.4) holds, namely,

$$l(x) = \exp \left( \int_1^x \frac{\epsilon(u)}{u} du \right), x \geq 1,$$

with

$$\epsilon(x) = \frac{k(k-1)}{kx^k - (k-1)}.$$

The function  $\epsilon$  is twice differentiable and goes to 0 as  $x \rightarrow \infty$ . Additionally,  $\epsilon$  satisfies condition (2.5). Hence we have shown that  $h \in R_{k-1}$ .

**Example 2.2. A rapidly varying density.** Define  $p$  through

$$p(x) = c \exp(-e^{x-1}), x \geq 0.$$

Then  $g(x) = h(x) = e^{x-1}$  and  $q(x) = 0$  for all non negative  $x$ . We show that  $h \in R_\infty$ . It holds  $\psi(x) = \log x + 1$ . Since  $h(x)$  is increasing and monotone, it remains to show that  $\psi(x) \in \widetilde{R}_0$ . When  $x \geq 1$ ,  $\psi(x)$  admits the representation (2.4) with  $\epsilon(x) = 1/(\log x + 1)$ . Also conditions (2.7) and (2.8) are satisfied. Thus  $h \in R_\infty$ .

### 3 An Abelian-type theorem

We have

**Theorem 3.1.** Let  $p(x)$  be defined as in (2.1) and  $h(x) \in \mathcal{R}$ . Denote by  $m(t)$ ,  $s^2(t)$  and  $\mu_3(t)$  the functions defined in (1.3), (1.4) and (1.5). Then with  $\psi$  defined as in (2.6), it holds

$$m(t) \underset{t \rightarrow \infty}{=} \psi(t)(1 + o(1)), \quad (3.1)$$

$$s^2(t) \underset{t \rightarrow \infty}{=} \psi'(t)(1 + o(1)), \quad (3.2)$$

$$\mu_3(t) \underset{t \rightarrow \infty}{=} \psi''(t)(1 + o(1)), \quad (3.3)$$

The proof of this result relies on Lemmas 4.2, 4.3, 4.4 and 4.5. Lemma 4.1 is instrumental for Lemma 4.5. The proof of these Lemmas and of Theorem 3.1 and Corollary 3.1 are left to Section 4.

**Remark 3.1.** As a by-product of Theorem 3.1, we obtain the following Abel type result

$$\Phi(t) = c\sqrt{2\pi}\sigma e^{K(\hat{x},t)}(1 + o(1)),$$

where  $K(\hat{x}, t)$  is defined as in (4.11) and  $\sigma^2$  is defined in the beginning of Section 4. It is easily verified that this result is in accordance with Theorem 4.12.11 of [2], Theorem 3 of [3] and Theorem 4.2 of [5].

**Corollary 3.1.** Let  $p(x)$  be defined as in (2.1) and  $h(x) \in \mathcal{R}$ . Then it holds

$$\frac{\mu_3(t)}{s^3(t)} \underset{t \rightarrow \infty}{\longrightarrow} 0 \quad (3.4)$$

Proof: The proof relies on Theorem 3.13.1 and is put in Section 4.

### 4 Appendix: Proofs

For density functions  $p(x)$  defined in (2.1) satisfying also  $h(x) \in \mathcal{R}$ , denote by  $\psi(x)$  the inverse function of  $h(x)$  and  $\sigma^2(v) = h'^{-1}(v)$ ,  $v \in \mathbb{R}_+$ . For brevity, we write  $\hat{x}$ ,  $\sigma$  and  $l$  instead of  $\hat{x}(t)$ ,  $\sigma(\psi(t))$  and  $l(t)$ .

When  $t$  is given,  $K(x, t)$  attains its maximum at  $\hat{x} = \psi(t)$  as seen in Remark 2.1. The fourth order Taylor-Lagrange expansion of  $K(x, t)$  on  $x \in [\hat{x} - \sigma l, \hat{x} + \sigma l]$  yields

$$K(x, t) = K(\hat{x}, t) - \frac{1}{2}h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6}h''(\hat{x})(x - \hat{x})^3 + \varepsilon(x, t), \quad (4.1)$$

for some  $\theta \in (0, 1)$

$$\varepsilon(x, t) = -\frac{1}{24}h^{(3)}(\hat{x} + \theta(x - \hat{x}))(x - \hat{x})^4. \quad (4.2)$$

#### 4.1 Some useful Lemmas

The following Lemmas will be useful for proving Theorem 3.1 and Corollary 3.1.

**Lemma 4.1.** *It holds*

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u) du} \xrightarrow{t \rightarrow \infty} 0. \quad (4.3)$$

Proof:

**Case 1:** If  $h \in R_\beta$ , by Theorem (1.5.15) of [2], there exists some slowly varying function  $l_1$  such that  $\psi(x) \sim x^{1/\beta} l_1(x)$  holds. Hence (see Theorem 1 of [4], Chapter 8.9 or Proposition 1.5.8 of [2])

$$\int_1^t \psi(u) du \underset{t \rightarrow \infty}{\sim} \frac{t^{1+\frac{1}{\beta}} l_1(t)}{\frac{1}{\beta} + 1} = \frac{\beta t \psi(t)}{1 + \beta}. \quad (4.4)$$

On the other hand,  $h'(x) = x^{\beta-1} l(x) (\beta + \epsilon(x))$ . Thus we have as  $x \rightarrow \infty$

$$\begin{aligned} |\log \sigma(x)| &= \left| \log(h'(x))^{-\frac{1}{2}} \right| \\ &= \left| \frac{1}{2} ((\beta - 1) \log x + \log l(x) + \log(\beta + \epsilon(x))) \right| \end{aligned}$$

Firstly,  $l(x)$  is a slowly varying function thus  $\log l(x) \underset{x \rightarrow \infty}{=} o(x^\alpha)$ , for  $\alpha > 0$ . Hence for  $x$  large enough and for all  $\delta > 0$ , we have  $|\log l(x)| \leq \delta x^\alpha$ . Secondly,  $|\log x| \leq \theta x^\gamma$ , for  $\gamma > 0$ , and  $x$  large enough, and for all  $\theta > 0$ . Thirdly,  $\epsilon(x) \xrightarrow{x \rightarrow \infty} 0$  which implies that for  $x$  large enough,  $|\epsilon(x)| \leq C$ , for some positive constant  $C$ . Hence  $|\log(\beta + \epsilon(x))| \leq \vartheta(\beta + \epsilon(x))^\eta \leq \vartheta(\beta + C)^\eta$ , for  $\eta > 0$ ,  $x$  large enough and for all  $\vartheta > 0$ . Consider the special case  $\alpha = \delta = \gamma = \theta = \vartheta = \eta = 1$ . We obtain

$$|\log \sigma(x)| \leq \frac{1}{2}(\beta x + (\beta + C)).$$

Set  $x = \psi(t)$ ; we get for  $t$  large enough

$$|\log \sigma(\psi(t))| \leq \frac{1}{2}(\beta \psi(t) + (\beta + C)),$$

which, together with (4.4), gives

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u) du} \leq \frac{\beta \psi(t) + (\beta + C)}{2\beta t \psi(t)} (1 + \beta) = (1 + \beta) \left( \frac{1}{2t} + \frac{\beta + C}{2\beta t \psi(t)} \right) \xrightarrow{t \rightarrow \infty} 0,$$

which yields (4.3).

**Case 2:** If  $h \in R_\infty$ , then since  $\psi(x) \in \tilde{R}_0$  is slowly varying as  $x \rightarrow \infty$  it holds, as  $t \rightarrow \infty$

$$\int_1^t \psi(u) du \sim t \psi(t) \quad (4.5)$$

(see Theorem 1 of [4], Chapter 8.9 or Proposition 1.5.8 of [2]).

Additionally, we have  $h'(x) = (\psi^{-1}(x))' = 1/(\psi'(\psi^{-1}(x))) = 1/\psi'(t)$  with  $x = \psi(t)$ . It follows

$$|\log \sigma(x)| = |\log(h'(x))^{-\frac{1}{2}}| = \frac{1}{2} |\log \psi'(t)|.$$

Since  $\psi(t) \in \tilde{R}_0$ , it holds

$$\begin{aligned} |\log \sigma(\psi(t))| &= \frac{1}{2} |\log \psi'(t)| = \frac{1}{2} |\log(\psi(t) \frac{\epsilon(t)}{t})| \\ &\leq \frac{1}{2} (|\log \psi(t)| + |\log \epsilon(t)| + |\log(t)|) \\ &\leq t + C_1, \end{aligned} \quad (4.6)$$

for some positive constant  $C_1$ . Indeed, since  $\psi(t)$  is a slowly varying function and  $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ , we have for  $t$  large enough, as previously,  $|\log \psi(t)| \leq t$ ,  $|\log t| \leq t$  and  $|\log \epsilon(t)| \leq C$  for  $C = 2C_1$ . (4.5) and (4.6) imply (4.3)

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u) du} \leq \frac{t + C_1}{t\psi(t)} = \frac{1}{\psi(t)} + \frac{C_1}{t\psi(t)} \xrightarrow{t \rightarrow \infty} 0.$$

This completes the proof.

**Lemma 4.2.** *For  $p(x)$  in (2.1),  $h \in \mathcal{R}$  and for any slowly varying function  $l$  such that  $l(t) \xrightarrow{t \rightarrow \infty} \infty$ , it holds*

$$\sup_{|x| \leq \sigma l} h^{(3)}(\hat{x} + x) \sigma^4 l^4 \xrightarrow{t \rightarrow \infty} 0. \quad (4.7)$$

where  $l := l(t)$ .

Proof:

**Case 1:** If  $h \in R_\beta$ , then we have  $h(x) = x^\beta l_0(x)$ ,  $l_0(x) \in R_0$ ,  $\beta > 0$ , for  $x$  large enough. The function  $h$  is three times differentiable

$$h'(x) = \beta x^{\beta-1} l_0(x) + x^\beta l_0'(x), \quad (4.8)$$

$$h''(x) = \beta(\beta-1)x^{\beta-2} l_0(x) + 2\beta x^{\beta-1} l_0'(x) + x^\beta l_0''(x), \quad (4.9)$$

$$h^{(3)}(x) = \beta(\beta-1)(\beta-2)x^{\beta-3} l_0(x) + 3\beta(\beta-1)x^{\beta-2} l_0'(x) + 3\beta x^{\beta-1} l_0''(x) + x^\beta l_0^{(3)}(x). \quad (4.10)$$

Since  $l_0 \in R_0$ , it is easy to obtain

$$l_0'(x) = \frac{l_0(x)}{x} \epsilon(x), \quad (4.11)$$

$$l_0''(x) = \frac{l_0(x)}{x^2} (\epsilon^2(x) + x\epsilon'(x) - \epsilon(x)), \quad (4.12)$$

$$l_0^{(3)}(x) = \frac{l_0(x)}{x^3} (\epsilon^3(x) + 3x\epsilon'(x)\epsilon(x) - 3\epsilon^2(x) - 2x\epsilon'(x) + 2\epsilon(x) + x^2\epsilon''(x)). \quad (4.13)$$

Under condition (2.4), there exists some positive constant  $C_1$  such that  $|\epsilon(x)| \leq C_1$  for  $x$  large enough. Under conditions (2.5), there exists some positive constants  $C_2$  and  $C_3$  such that  $x|\epsilon'(x)| \leq C_2$  and  $x^2|\epsilon''(x)| \leq C_3$ , for  $x$  large enough. Together with (4.12) and (4.13), there exists some positive constants  $Q_1$  and  $Q_2$  such that, for  $x$  large enough,

$$\begin{aligned} |l_0''(x)| &\leq Q_1 \frac{l_0(x)}{x^2}, \\ |l_0^{(3)}(x)| &\leq Q_2 \frac{l_0(x)}{x^3}. \end{aligned}$$

Including these results in (4.10) yields with some positive constant  $Q_3$  and for  $x$  large enough,

$$|h^{(3)}(x)| \leq Q_3 \frac{h(x)}{x^3}. \quad (4.14)$$

By definition, we have  $\sigma^2(x) = 1/h'(x)$  with  $h'(x) = \beta x^{\beta-1}l_0(x) + x^\beta l'_0(x) = x^{\beta-1}l_0(x)(\beta + \epsilon(x)) = h(x)(\beta + \epsilon(x))/x$ , thus  $\sigma^2(x) = x/(h(x)(\beta + \epsilon(x)))$ . Hence it follows that

$$\sigma^2 = \sigma^2(\hat{x}) = \frac{\hat{x}}{h(\hat{x})(\beta + \epsilon(\hat{x}))} = \frac{\psi(t)}{t(\beta + \epsilon(\psi(t)))} \xrightarrow[t \rightarrow \infty]{} \frac{\psi(t)}{\beta t} (1 + o(1)), \quad (4.15)$$

where the last equality follows from  $\psi(t) \xrightarrow[t \rightarrow \infty]{} \infty$  and  $\epsilon(x) \xrightarrow[x \rightarrow \infty]{} 0$ , which implies, by composition,  $\epsilon(\psi(t)) \xrightarrow[t \rightarrow \infty]{} 0$ .

Result (4.15) implies

$$\frac{\sigma l}{\psi(t)} \underset{t \rightarrow \infty}{\sim} \frac{l}{\sqrt{\beta t \psi(t)}} \xrightarrow[t \rightarrow \infty]{} 0,$$

since  $l$  slowly varying then negligible compared to any power of  $t$ . This means that

$$\sigma l \underset{t \rightarrow \infty}{=} o(\psi(t)) = o(\hat{x}). \quad (4.16)$$

By (4.14) we have

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \leq \sup_{|x| \leq \sigma l} Q_3 \frac{h(\hat{x} + x)}{(\hat{x} + x)^3}. \quad (4.17)$$

Since  $x \underset{t \rightarrow \infty}{=} o(\hat{x})$  on  $\{x : |x| \leq \sigma l\}$ , then  $\hat{x} + x \underset{t \rightarrow \infty}{\sim} \hat{x}$  and  $(\hat{x} + x)^3 \underset{t \rightarrow \infty}{\sim} \hat{x}^3$ . Moreover, since  $h(x) = x^\beta l_0(x)$  for  $x$  large enough and  $l_0(x) \in R_0$ , then  $h(\hat{x} + x) = (\hat{x} + x)^\beta l_0(\hat{x} + x) \underset{t \rightarrow \infty}{\sim} \hat{x}^\beta l_0(\hat{x} + x)$ . By (2.4),

$$\begin{aligned} \frac{l_0(\hat{x} + x)}{l_0(\hat{x})} &= \exp \left( \int_1^{\hat{x}+x} \frac{\epsilon(u)}{u} du - \int_1^{\hat{x}} \frac{\epsilon(u)}{u} du \right) \\ &= \exp \left( \int_{\hat{x}}^{\hat{x}+x} \frac{\epsilon(u)}{u} du \right) \underset{t \rightarrow \infty}{\sim} \exp \left( \frac{\epsilon(\hat{x})}{\hat{x}} \right) \xrightarrow[t \rightarrow \infty]{} 1, \end{aligned}$$

seeing that  $\hat{x} \xrightarrow[t \rightarrow \infty]{} \infty$  and  $\epsilon(\hat{x}) \xrightarrow[t \rightarrow \infty]{} 0$ . Thus we obtain  $h(\hat{x} + x) \underset{t \rightarrow \infty}{\sim} h(\hat{x})$ . Together with (4.17), we have

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \leq Q_3 \frac{h(\hat{x})}{\hat{x}^3} = Q_3 \frac{t}{\psi^3(t)}, \quad (4.18)$$

which, combined with (4.15) yields

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \sigma^4 l^4 \leq Q_3 \frac{t}{\psi^3(t)} \sigma^4 l^4 \underset{t \rightarrow \infty}{\sim} \frac{Q_3 l^4}{\beta^2 t \psi(t)} \xrightarrow[t \rightarrow \infty]{} 0,$$

since  $l$  varies slowly. This proves (4.7).

**Case 2:** If  $h \in R_\infty$ , we have  $h'(x) = 1/\psi'(h(x))$ . Since  $\hat{x} = \psi(t)$ , we have  $h(\hat{x}) = t$ . Thus it holds

$$h'(\hat{x}) = \frac{1}{\psi'(t)}, \quad (4.19)$$

$$h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^2}, \quad (4.20)$$

$$h^{(3)}(\hat{x}) = -\frac{\psi^{(3)}(t)\psi'(t) - 2(\psi''(t))^2}{(\psi'(t))^3}. \quad (4.21)$$



Notice if  $h(\hat{x}) \in R_\infty$ , then  $\psi(t) \in \tilde{R}_0$ . Therefore we obtain

$$\psi'(t) = \frac{\psi(t)}{t} \epsilon(t), \quad (4.22)$$

$$\begin{aligned} \psi''(t) &= \frac{\psi(t)}{t^2} (\epsilon^2(t) + t\epsilon'(t) - \epsilon(t)) \\ &= -\frac{\psi(t)}{t^2} \epsilon(t) (1 - \epsilon(t) - t\frac{\epsilon'(t)}{\epsilon(t)}) \end{aligned} \quad (4.23)$$

$$\begin{aligned} \psi^{(3)}(t) &= \frac{\psi(t)}{t^3} (\epsilon^3(t) - 3\epsilon^2(t) + 2\epsilon(t) - 2t\epsilon'(t) + 3t\epsilon'(t)\epsilon(t) + t^2\epsilon''(t)) \\ &= \frac{\psi(t)}{t^3} \epsilon(t) (2 + \epsilon^2(t) + 3t\epsilon'(t) - 3\epsilon(t) - \frac{2t\epsilon'(t)}{\epsilon(t)} + \frac{t^2\epsilon''(t)}{\epsilon(t)}) \end{aligned} \quad (4.24)$$

Putting (4.22), (4.23) and (4.24) into (4.21), we get

$$\begin{aligned} h^{(3)}(\hat{x}) &= -\frac{\frac{\psi^2(t)\epsilon^2(t)}{t^4} [\epsilon(t) - \epsilon^2(t) - t\epsilon'(t) + 2t\frac{\epsilon'(t)}{\epsilon(t)} + t^2\frac{\epsilon''(t)}{\epsilon(t)} - 2t^2\frac{(\epsilon'(t))^2}{\epsilon^2(t)}]}{\frac{\psi^3(t)\epsilon^3(t)}{t^3}} \\ &= \frac{\epsilon^2(t) - \epsilon(t) + t\epsilon'(t) - 2t\frac{\epsilon'(t)}{\epsilon(t)} - t^2\frac{\epsilon''(t)}{\epsilon(t)} + 2t^2\frac{(\epsilon'(t))^2}{\epsilon^2(t)}}{t\psi(t)\epsilon(t)} \\ &\stackrel{t \rightarrow \infty}{=} o\left(\frac{1}{t\psi(t)\epsilon(t)}\right), \end{aligned} \quad (4.25)$$

since  $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ ,  $t\epsilon'(t)/\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ ,  $t^2\epsilon''(t)/\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$  by (2.7) and  $t\epsilon'(t) = \epsilon(t) \times t\epsilon'(t)/\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ . As previously  $\sigma^2 = \psi'(t) = \psi(t)\epsilon(t)/t \stackrel{t \rightarrow \infty}{=} o(\psi(t)/t)$ , then  $\sigma l/\psi(t) \leq l/\sqrt{t\psi(t)} \xrightarrow{t \rightarrow \infty} 0$ . By (2.8),  $x^\eta \epsilon(x) > C$  with  $C$  some positive constant and  $\eta \in (0, 1/8)$ . Thus we obtain

$$h^{(3)}(\hat{x}) \leq \frac{Q_4}{t^{1-\eta}\psi(t)} = \frac{Q_4}{(h(\hat{x}))^{1-\eta}\hat{x}}.$$

Notice that  $\sigma l \stackrel{t \rightarrow \infty}{=} o(\psi(t)) = o(\hat{x})$ ; therefore

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \leq \frac{Q_4}{(h(\hat{x} - \sigma l))^{1+\eta}(\hat{x} - \sigma l)} \stackrel{t \rightarrow \infty}{\sim} \frac{Q_4}{(h(\hat{x} - \sigma l))^{1+\eta}\hat{x}} \stackrel{t \rightarrow \infty}{=} o\left(\frac{1}{\hat{x}}\right), \quad (4.26)$$

since  $h(x)$  is increasing. Moreover,  $h(x) \xrightarrow{x \rightarrow \infty} \infty$  and  $\hat{x} - \sigma l \xrightarrow{t \rightarrow \infty} \infty$  and  $1 - \eta > 0$  imply  $1/(h(\hat{x} - \sigma l))^{1-\eta} \xrightarrow{t \rightarrow \infty} 0$ . Thus it follows that

$$\sup_{|x| \leq \sigma l} |h^{(3)}(\hat{x} + x)| \sigma^4 l^4 \leq \frac{Q_5}{\psi(t)} \frac{(\psi(t))^2}{t^2} l^4 = Q_5 \frac{\psi(t)}{t^2} l^4 \xrightarrow{t \rightarrow \infty} 0,$$

for some positive constant  $Q_5$  and where  $l$  and  $\psi$  are slowly varying. This completes the proof.

**Lemma 4.3.** For  $p(x)$  in (2.1),  $h \in \mathcal{R}$  and for any slowly varying function  $l(t) \xrightarrow{t \rightarrow \infty} \infty$ , it holds

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l \right| \xrightarrow{t \rightarrow \infty} 0 \quad (4.27)$$

and

$$h''(\hat{x}) \sigma^3 l \xrightarrow{t \rightarrow \infty} 0, \quad h''(\hat{x}) \sigma^4 \xrightarrow{t \rightarrow \infty} 0. \quad (4.28)$$

where  $l := l(t)$ .

Proof:

**Case 1:**  $h \in R_\beta$ . Using (4.9), (4.11) and (4.12), we get

$$\begin{aligned} h''(x) &= x^{\beta-2} l_0(x) (\beta(\beta-1) + (2\beta-1)\epsilon(x) + \epsilon^2(x) + x\epsilon'(x)) \\ &\underset{x \rightarrow \infty}{\sim} x^{\beta-2} l_0(x) (\beta(\beta-1) + x\epsilon'(x)), \end{aligned} \quad (4.29)$$

By (2.5), for some real  $M$ , we have  $x|\epsilon'(x)| \leq M$  which implies  $x\epsilon'(x) \geq -M$ . Let  $N := \beta(\beta-1) - M$ . Since  $\hat{x} = \psi(t)$ , for  $t$  large enough it holds

$$h''(\hat{x}) \geq N \frac{t}{\psi^2(t)}, \quad (4.30)$$

which, together with (4.15) and (4.18), yields

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l \right| \leq \frac{Q_3}{N} \frac{t}{\psi^3(t)} \frac{\psi^2(t)}{t} \sqrt{\frac{\psi(t)}{\beta t}} l = \frac{Q_3 l}{N \sqrt{\beta t \psi(t)}} \xrightarrow{y \rightarrow \infty} 0 \quad (4.31)$$

since  $l$  varies slowly. Hence we get (4.26).

From (4.15) and (4.18), we obtain for  $t$  large enough and some positive constant  $Q_6$

$$h''(\hat{x}) \sigma^3 l \leq Q_6 \frac{t}{\psi^2(t)} \frac{\psi^{\frac{3}{2}}(t)}{\beta^{\frac{3}{2}} t^{\frac{3}{2}}} l = \frac{Q_6}{\beta^{\frac{3}{2}}} \frac{l}{\sqrt{t \psi(t)}} \xrightarrow{t \rightarrow \infty} 0, \quad (4.32)$$

as sought previously. This implies that the first limit in (4.28) holds. Likewise we get the second limit in (4.28) since

$$h''(\hat{x}) \sigma^4 \leq Q_7 \frac{t}{\psi^2(t)} \frac{\psi^2(t)}{\beta^2 t^2} = \frac{Q_7}{\beta^2 t} \xrightarrow{t \rightarrow \infty} 0 \quad (4.33)$$

for some constant  $Q_7$ .

**Case 2:**  $h \in R_\infty$ . Using (4.20), (4.22) and (4.23) we obtain

$$h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^2} \underset{t \rightarrow \infty}{=} \frac{\psi(t)\epsilon(t)}{t^2} \frac{t^2}{\psi^2(t)\epsilon^2(t)} (1 + o(1)) \underset{t \rightarrow \infty}{=} \frac{1}{\psi(t)\epsilon(t)} (1 + o(1)). \quad (4.34)$$

Combining (4.26) and (4.34), using  $\sigma = (h'(\hat{x}))^{-\frac{1}{2}}$ , we have as  $t \rightarrow \infty$  and for some positive constant  $Q_8$

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l \right| \leq \frac{Q_8}{\psi(t)} \psi(t)\epsilon(t) \sqrt{\frac{\psi(t)\epsilon(t)}{t}} l \xrightarrow{t \rightarrow \infty} 0,$$

where  $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ ,  $l$  and  $\psi$  vary slowly.  $h^{(3)}(\hat{x}) \xrightarrow{t \rightarrow \infty} 0$  according to (4.25) and  $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$ . For some positive constant  $Q_9$ , we can write for large  $t$

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l \right| \leq Q_9 \frac{\psi^{\frac{3}{2}}(t) l}{\sqrt{t}},$$

where  $l$  and  $\psi$  vary slowly. This implies (4.27).

Using (4.34), we have

$$h''(\hat{x}) \sigma^3 l \underset{t \rightarrow \infty}{\sim} \frac{1}{\psi(t)\epsilon(t)} \left( \frac{\psi(t)\epsilon(t)}{t} \right)^{\frac{3}{2}} l = \frac{\sqrt{\psi(t)\epsilon(t)}}{t^{\frac{3}{2}}} l \xrightarrow{t \rightarrow \infty} 0,$$

where  $\psi$  and  $l$  vary slowly and  $\epsilon(t) \xrightarrow[t \rightarrow \infty]{} 0$ . This implies that the first limit in (4.28) holds. Similarly we have

$$h''(\hat{x})\sigma^4 \underset{t \rightarrow \infty}{\sim} \frac{\psi(t)\epsilon(t)}{t^2} \xrightarrow[t \rightarrow \infty]{} 0,$$

hence the proof of the second limit in (4.28).

**Lemma 4.4.** *For any slowly varying function  $l$  such that  $l(t) \xrightarrow[t \rightarrow \infty]{} \infty$ , it holds*

$$\sup_{y \in [-l, l]} \frac{|\xi(\sigma y + \hat{x}, t)|}{h''(\hat{x})\sigma^3} \xrightarrow[t \rightarrow \infty]{} \infty, \quad (4.35)$$

where  $\xi(x, t) = \varepsilon(x, t) + q(x)$  and  $l := l(t)$ .

Proof:

A close look to the proof of Lemma 4.3 shows that (4.27) can be slightly modified into

$$\sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l^4 \right| \xrightarrow[t \rightarrow \infty]{} 0$$

Hence for  $y \in [-l, l]$ , by (4.2), it holds

$$\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \left| \frac{h^{(3)}(\hat{x} + \theta \sigma y)(\sigma l)^4}{h''(\hat{x})\sigma^3} \right| = \left| \frac{h^{(3)}(\hat{x} + \theta \sigma y)\sigma l^4}{h''(\hat{x})} \right|,$$

with  $\theta \in (0, 1)$ . Let  $x = \theta \sigma y$ . It then holds  $|x| \leq \sigma l$ . Therefore

$$\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \left| \frac{h^{(3)}(\hat{x} + x)\sigma l^4}{h''(\hat{x})} \right|.$$

Hence

$$\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \sup_{|x| \leq \sigma l} \left| \frac{h^{(3)}(\hat{x} + x)}{h''(\hat{x})} \sigma l^4 \right| \xrightarrow[t \rightarrow \infty]{} 0, \quad (4.36)$$

by Lemma 4.3.

Under condition (2.3), setting  $x = \psi(t)$ , we get

$$\sup_{|v - \psi(t)| \leq \vartheta \psi(t)} |q(v)| \underset{t \rightarrow \infty}{=} o\left(\frac{1}{\sqrt{t\psi(t)}}\right),$$

for  $h \in R_\beta$  and

$$\sup_{|v - \psi(t)| \leq \vartheta \psi(t)} |q(v)| \underset{t \rightarrow \infty}{=} o\left(\frac{\sqrt{\psi(t)}}{t^2}\right),$$

for  $h \in R_\infty$ .

We have to prove that

$$\left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \xrightarrow[t \rightarrow \infty]{} 0. \quad (4.37)$$

**Case 1:**  $h \in R_\beta$ . We have  $h(x) = x^\beta l_0(x)$ ,  $l_0(x) \in R_0$ ,  $\beta > 0$ . Hence

$$h'(x) = x^{\beta-1} l_0(x)(\beta + \epsilon(x)).$$

Notice that  $\sigma^2 = \psi'(t) = 1/h'(\psi(t))$  It holds

$$\frac{\sigma l}{\vartheta\psi(t)} = \frac{l}{\vartheta\psi(t)\sqrt{h'(\psi(t))}} = \frac{l}{\vartheta(\psi(t))^{\frac{\beta+1}{2}}l_0(\psi(t))^{\frac{1}{2}}(\beta + \epsilon(\psi(t)))^{\frac{1}{2}}} \xrightarrow{t \rightarrow \infty} 0, \quad (4.38)$$

since  $\psi(t) \underset{t \rightarrow \infty}{\sim} t^{\frac{1}{\beta}}l_1(t)$  for some slowly varying function  $l_1$ ,  $l_0$  varies slowly and  $\epsilon(\psi(t)) \xrightarrow{t \rightarrow \infty} 0$ . We can write (4.38) as

$$\sigma l \underset{t \rightarrow \infty}{=} o(\vartheta\psi(t)).$$

Thus  $\{v : |v - \psi(t)| \leq \sigma l\} \subset \{v : |v - \psi(t)| \leq \vartheta\psi(t)\}$  which implies

$$\sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \leq \sup_{|v - \psi(t)| \leq \vartheta\psi(t)} |q(v)| \underset{t \rightarrow \infty}{=} o\left(\frac{1}{\sqrt{t\psi(t)}}\right). \quad (4.39)$$

By (4.15) and (4.30), it holds for  $t$  large enough

$$\begin{aligned} \left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| &\leq |q(\sigma y + \hat{x})| \frac{1}{N} \frac{\psi^2(t)}{t} \left( \frac{\beta t}{\psi(t)} \right)^{\frac{3}{2}} \\ &= |q(\sigma y + \hat{x})| \frac{\beta^{\frac{3}{2}}}{N} \sqrt{t\psi(t)}. \end{aligned}$$

Let  $v = \sigma y + \hat{x}$ , then  $|v - \psi(t)| \leq |\sigma y| \leq \sigma l$ . Thus we obtain

$$\left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \leq \sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \frac{\beta^{\frac{3}{2}}}{N} \sqrt{t\psi(t)} \xrightarrow{t \rightarrow \infty} 0, \quad (4.40)$$

where last step follows from (4.39).

**Case 2:**  $h \in R_\infty$ . For any slowly varying function  $l(t)$  it holds

$$\frac{\sigma l}{\vartheta\psi(t)} = \frac{\sqrt{\psi'(t)}l}{\vartheta\psi(t)} = \sqrt{\frac{\epsilon(t)}{t\psi(t)}} \frac{l}{\vartheta} \xrightarrow{t \rightarrow \infty} 0,$$

hence we have

$$\sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \underset{t \rightarrow \infty}{=} o\left(\frac{\sqrt{\psi(t)}}{t^2}\right). \quad (4.41)$$

Using this relation and (4.34), when  $y \in [-l, l]$ , it holds for  $t$  large enough

$$\begin{aligned} \left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| &\underset{t \rightarrow \infty}{\sim} |q(\sigma y + \hat{x})| \frac{t^{\frac{3}{2}}}{\sqrt{\psi(t)\epsilon(t)}} \\ &\leq \sup_{|v - \psi(t)| \leq \sigma l} |q(v)| \frac{t^{\frac{3}{2}}}{\sqrt{\psi(t)\epsilon(t)}} \\ &\leq Q \frac{\sqrt{\psi(t)}}{t^2} \frac{t^{\frac{3+\eta}{2}}}{\sqrt{\psi(t)}} = \frac{Q}{t^{\frac{1-\eta}{2}}} \xrightarrow{t \rightarrow \infty} 0, \end{aligned} \quad (4.42)$$

for some constant  $Q$ , since (2.8) implies  $\epsilon(t) \geq C/t^\eta$ , for  $\eta \in (0, 1/8)$  and some positive constant  $C$ .

Combining (4.40) and (4.42) completes the proof.

**Lemma 4.5.** For  $\alpha \in \mathbb{N}$ , denote

$$\Psi(t, \alpha) := \int_0^\infty (x - \hat{x})^\alpha e^{tx} p(x) dx. \quad (4.43)$$

Then

$$\Psi(t, \alpha) \underset{t \rightarrow \infty}{=} c\sigma^{\alpha+1} e^{K(\hat{x}, t)} T_1(t, \alpha) (1 + o(1)), \quad (4.44)$$

where

$$T_1(t, \alpha) = \int_{-\frac{t^{1/3}}{\sqrt{2}}}^{\frac{t^{1/3}}{\sqrt{2}}} y^\alpha \exp(-\frac{y^2}{2}) dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{t^{1/3}}{\sqrt{2}}}^{\frac{t^{1/3}}{\sqrt{2}}} y^{3+\alpha} \exp(-\frac{y^2}{2}) dy. \quad (4.45)$$

Proof:

Given a slowly varying function  $l(t)$  with  $l(t) \xrightarrow{t \rightarrow \infty} \infty$ , we define the interval  $I_t$  as follows

$$I_t := \left( -\frac{l^{\frac{1}{3}}\sigma}{\sqrt{2}}, \frac{l^{\frac{1}{3}}\sigma}{\sqrt{2}} \right).$$

For large enough  $\tau$ , when  $t \rightarrow \infty$  we can partition  $\mathbb{R}_+$  as

$$\mathbb{R}_+ = \{x : 0 < x < \tau\} \cup \{x : x \in \hat{x} + I_t\} \cup \{x : x \geq \tau, x \notin \hat{x} + I_t\},$$

where  $\tau$  is large enough so that for  $x > \tau$ ,  $q(x) < \log 2$ . Thus we have

$$p(x) < 2ce^{-g(x)}. \quad (4.46)$$

Obviously, for fixed  $\tau$ ,  $\{x : 0 < x < \tau\} \cap \{x : x \in \hat{x} + I_t\} = \emptyset$  since for large  $t$  we have  $\min(x : x \in \hat{x} + I_t) \xrightarrow{t \rightarrow \infty} \infty$ . Therefore  $\tau < \hat{x} - \frac{l^{\frac{1}{3}}\sigma}{\sqrt{2}} \leq \hat{x}$ . Hence it holds

$$\begin{aligned} \Psi(t, \alpha) &= \int_0^\tau (x - \hat{x})^\alpha e^{tx} p(x) dx + \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha e^{tx} p(x) dx + \int_{x \notin \hat{x} + I_t, x > \tau} (x - \hat{x})^\alpha e^{tx} p(x) dx \\ &:= \Psi_1(t, \alpha) + \Psi_2(t, \alpha) + \Psi_3(t, \alpha). \end{aligned} \quad (4.47)$$

We estimate  $\Psi_1(t, \alpha)$ ,  $\Psi_2(t, \alpha)$  and  $\Psi_3(t, \alpha)$  in Step 1, Step 2 and Step 3.

**Step 1:** Using (4.46), for  $\tau$  large enough, we have

$$|\Psi_1(t, \alpha)| \leq \int_0^\tau |x - \hat{x}|^\alpha e^{tx} p(x) dx \leq 2c \int_0^\tau |x - \hat{x}|^\alpha e^{tx-g(x)} dx \leq 2c \int_0^\tau \hat{x}^\alpha e^{tx} dx,$$

since when  $0 < x < \tau < \hat{x}$  then  $|x - \hat{x}| = \hat{x} - x < \hat{x}$  and  $g$  is positive.

Since

$$\int_0^\tau \hat{x}^\alpha e^{tx} dx = \hat{x}^\alpha \frac{e^{t\tau}}{t} - \frac{\hat{x}^\alpha}{t} \leq \hat{x}^\alpha \frac{e^{t\tau}}{t},$$

we obtain

$$|\Psi_1(t, \alpha)| \leq 2c\hat{x}^\alpha \frac{e^{t\tau}}{t}, \quad (4.48)$$

for  $\tau$  large enough.

We now show that for  $h \in \mathcal{R}$ , it holds

$$\hat{x}^\alpha \frac{e^{t\tau}}{t} \underset{t \rightarrow \infty}{=} o(\sigma^{\alpha+1} e^{K(\hat{x}, t)} h''(\hat{x}) \sigma^3), \quad (4.49)$$

with  $K(x, t)$  defined in (2.9). This is equivalent to

$$\frac{\hat{x}^\alpha e^{t\tau}}{t\sigma^{\alpha+4}h''(\hat{x})} \underset{t \rightarrow \infty}{=} o(e^{K(\hat{x}, t)}),$$

which is implied by

$$\exp(-(\alpha + 4) \log \sigma - \log t + \alpha \log \hat{x} + \tau t - \log h''(\hat{x})) \underset{t \rightarrow \infty}{=} o(e^{K(\hat{x}, t)}).$$

Since  $\hat{x} = \psi(t)$ , it holds

$$K(\hat{x}, t) = t\psi(t) - g(\psi(t)). \quad (4.50)$$

Setting  $u = h(v)$  in  $\int_1^t \psi(u) du$ ,

$$\begin{aligned} \int_1^t \psi(u) du &= \int_{\psi(1)}^{\psi(t)} \psi(h(v)) h'(v) dv \\ &= \int_{\psi(1)}^{\psi(t)} v h'(v) dv \\ &= [vh(v)]_{\psi(1)}^{\psi(t)} - \int_{\psi(1)}^{\psi(t)} h(v) dv \\ &= t\psi(t) - \psi(1) - g(\psi(t)) + g(\psi(1)) \end{aligned} \quad (4.51)$$

by integration by parts, and then using  $h(x) = g'(x)$ .

Finally (4.50) and (4.51) produce

$$K(\hat{x}, t) = \int_1^t \psi(u) du + \psi(1) - g(\psi(1)). \quad (4.52)$$

By Lemma 4.1,  $\log \sigma \underset{t \rightarrow \infty}{=} o(\int_1^t \psi(u) du)$ . Furthermore  $K(\hat{x}, t) \underset{t \rightarrow \infty}{\sim} \int_1^t \psi(u) du$ . Hence  $\log \sigma \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ . It remains to show that  $t \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ ,  $\log \hat{x} \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$  and  $\log h''(\hat{x}) \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ .

- If  $h \in R_\beta$ , by Theorem (1.5.12) of [2], it holds  $\psi(x) \underset{x \rightarrow \infty}{\sim} x^{\frac{1}{\beta}} l_1(x)$  for some slowly varying function  $l_1(x)$ . Together with (4.4) and (4.52), this yields

$$K(\hat{x}, t) \underset{t \rightarrow \infty}{\sim} \int_1^t \psi(u) du \underset{t \rightarrow \infty}{\sim} \frac{\beta}{1 + \beta} t\psi(t).$$

Consequently,

$$\frac{t}{K(\hat{x}, t)} \underset{t \rightarrow \infty}{\sim} \frac{1 + \beta}{\beta} \frac{1}{\psi(t)} \underset{t \rightarrow \infty}{\longrightarrow} 0,$$

which is equivalent to  $t \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ . In addition,  $\log \hat{x} = \log \psi(t) \underset{t \rightarrow \infty}{\sim} (\frac{1}{\beta}) \log t + \log l_1(t) \underset{t \rightarrow \infty}{=} o(t) \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ , using the slow variation of  $l_1$  and  $\log t \underset{t \rightarrow \infty}{=} o(t)$ . For the same reason and using (4.29), we have  $\log h''(\hat{x})/t \leq Q(\log t - 2 \log \psi(t))/t \underset{t \rightarrow \infty}{\longrightarrow} 0$  which implies  $\log h''(\hat{x}) \underset{t \rightarrow \infty}{=} o(t) \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ . Since  $K(\hat{x}, t) \underset{t \rightarrow \infty}{\longrightarrow} \infty$  and using previous results, then we have that (4.49) holds.

- If  $h \in R_\infty$ ,  $\psi(x) \in \tilde{R}_0$  is slowly varying. Thus  $\log \hat{x} = \log \psi(t) \underset{t \rightarrow \infty}{=} o(t)$ . Therefore, by (4.5) and (4.52), it holds  $t \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ ; using the same method as previously. Consequently,  $\log \hat{x} \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ . (4.34) yields  $\log h''(\hat{x}) \leq Q h''(\hat{x}) \underset{t \rightarrow \infty}{\sim} Q/\psi(t)\epsilon(t)$ , for some positive constant  $Q$ . Thus we have  $\log h''(\hat{x})/t \leq Q/(t\psi(t)\epsilon(t)) \leq Q_1/(t^{1-\eta}\psi(t))$ , for  $Q_1 = Q/C$ , since  $\epsilon(t) \geq C/t^\eta$ , for some positive constant  $C$  and  $\eta \in (0, 1/8)$ . Since  $Q_1/(t^{1-\eta}\psi(t)) \xrightarrow[t \rightarrow \infty]{} 0$ , then  $\log h''(\hat{x}) \underset{t \rightarrow \infty}{=} o(t) \underset{t \rightarrow \infty}{=} o(K(\hat{x}, t))$ . We get (4.49).

(4.48) and (4.49) yield together

$$|\Psi_1(t, \alpha)| \underset{t \rightarrow \infty}{=} o(\sigma^{\alpha+1} e^{K(\hat{x}, t)} h''(\hat{x}) \sigma^3). \quad (4.53)$$

**Step 2:** By (2.1) and (4.1), as  $t \rightarrow \infty$

$$\begin{aligned} \Psi_2(t, \alpha) &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c e^{K(x, t) + q(x)} dx \\ &= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c \exp \left( (K(\hat{x}, t) - \frac{1}{2} h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6} h''(\hat{x})(x - \hat{x})^3 + \xi(x, t)) \right) dx, \end{aligned}$$

where  $\xi(x, t) = \varepsilon(x, t) + q(x)$ . Making the substitution  $y = x - \hat{x}/\sigma$ , it holds

$$\begin{aligned} \Psi_2(t, \alpha) &= c \sigma^\alpha e^{K(\hat{x}, t)} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp \left( -\frac{\sigma^2 y^2}{2} h'(\hat{x}) - \frac{\sigma^3 y^3}{6} h''(\hat{x}) + \xi(\sigma y + \hat{x}, t) \right) \sigma dy \\ &= c \sigma^{\alpha+1} e^{K(\hat{x}, t)} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp \left( -\frac{y^2}{2} - \frac{\sigma^3 y^3}{6} h''(\hat{x}) + \xi(\sigma y + \hat{x}, t) \right) dy \end{aligned} \quad (4.54)$$

where the last equality follows from  $h'(\hat{x}) = 1/\sigma^2$ .

On  $\{y : y \in (-l^{\frac{1}{3}}/\sqrt{2}, l^{\frac{1}{3}}/\sqrt{2})\}$ , by (4.28),  $|h''(\hat{x})\sigma^3 y^3| \leq |h''(\hat{x})\sigma^3 l|/2^{\frac{3}{2}} \xrightarrow[t \rightarrow \infty]{} 0$ . Perform the first order Taylor expansion

$$\exp \left( -\frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t) \right) \underset{t \rightarrow \infty}{=} 1 - \frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t) + o_1(t, y),$$

where

$$o_1(t, y) = o \left( -\frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t) \right). \quad (4.55)$$

We obtain

$$\begin{aligned} &\int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp \left( -\frac{y^2}{2} - \frac{\sigma^3 y^3}{6} h''(\hat{x}) + \xi(\sigma y + \hat{x}, t) \right) dy \\ &= \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} \left( 1 - \frac{h''(\hat{x})\sigma^3}{6} y^3 + \xi(\sigma y + \hat{x}, t) + o_1(t, y) \right) y^\alpha \exp \left( -\frac{y^2}{2} \right) dy = T_1(t, \alpha) + T_2(t, \alpha), \end{aligned}$$

where  $T_1(t, \alpha)$  is defined in (4.45) and

$$T_2(t, \alpha) := \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} (\xi(\sigma y + \hat{x}, t) + o_1(t, y)) y^\alpha \exp \left( -\frac{y^2}{2} \right) dy.$$

For  $T_2(t, \alpha)$ , using (4.55) we have for  $t$  large enough

$$|T_2(t, \alpha)| \leq \sup_{y \in [-\frac{l^{\frac{1}{3}}}{\sqrt{2}}, \frac{l^{\frac{1}{3}}}{\sqrt{2}}]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy \\ + \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} \left( \left| o\left(\frac{h''(\hat{x})\sigma^3}{6} h^3\right) \right| + |o(\xi(\sigma y + \hat{x}, t))| \right) |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy,$$

where  $\sup_{y \in [-l^{\frac{1}{3}}/\sqrt{2}, l^{\frac{1}{3}}/\sqrt{2}]} |\xi(\sigma y + \hat{x}, t)| \leq \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)|$  since  $l^{\frac{1}{3}}/\sqrt{2} \leq l$ , which is equivalent to  $l \geq 1/8$ , is true for  $t$  large enough. This implies  $[-l^{\frac{1}{3}}/\sqrt{2}, l^{\frac{1}{3}}/\sqrt{2}] \subset [-l, l]$ . Thus

$$|T_2(t, \alpha)| \leq 2 \sup_{y \in [-l, l]} |\xi(\sigma y + \hat{x}, t)| \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy + \left| o\left(\frac{h''(\hat{x})\sigma^3}{6}\right) \right| \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \\ = \left| o\left(\frac{h''(\hat{x})\sigma^3}{6}\right) \right| \left( \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{y^2}{2}\right) dy + \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \right),$$

where the last equality holds from Lemma 4.4. Since the integrals in the last equality are both bounded, it holds

$$T_2(t, \alpha) \xrightarrow{t \rightarrow \infty} o(h''(\hat{x})\sigma^3).$$

When  $\alpha$  is even, we have  $T_1(t, \alpha) = \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} y^\alpha \exp\left(-\frac{y^2}{2}\right) dy \xrightarrow{t \rightarrow \infty} \sqrt{2\pi} M_\alpha$ , where  $M_\alpha = \alpha!/(2^{\alpha/2}(\alpha/2)!)$  is the moment with order  $\alpha$  of a standard normal distribution. It follows by (4.28) in Lemma 4.3

$$\frac{T_2(t, \alpha)}{T_1(t, \alpha)} \leq \frac{h''(\hat{x})\sigma^3}{\sqrt{2\pi} M_\alpha} \xrightarrow{t \rightarrow \infty} 0. \quad (4.56)$$

When  $\alpha$  is odd, we have  $T_1(t, \alpha) = -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{\frac{1}{3}}}{\sqrt{2}}}^{\frac{l^{\frac{1}{3}}}{\sqrt{2}}} |y|^{3+\alpha} \exp\left(-\frac{y^2}{2}\right) dy \xrightarrow{t \rightarrow \infty} -\frac{h''(\hat{x})\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}$ ,

where  $M_{\alpha+3} = (\alpha+3)!/(2^{(\alpha+3)/2}((\alpha+3)/2)!)$  is the moment with order  $\alpha+3$  of a standard normal distribution. It follows

$$\frac{T_2(t, \alpha)}{T_1(t, \alpha)} \xrightarrow{t \rightarrow \infty} -\frac{6}{\sqrt{2\pi} M_{\alpha+3}} o(1) \xrightarrow{t \rightarrow \infty} 0. \quad (4.57)$$

(4.56) and (4.57) imply that, for  $\alpha \in \mathbb{N}$ ,

$$T_2(t, \alpha) \xrightarrow{t \rightarrow \infty} o(T_1(t, \alpha)). \quad (4.58)$$

Using (4.55) and (4.58), we get

$$\Psi_2(t, \alpha) = c\sigma^{\alpha+1} \exp(K(\hat{x}, t)) T_1(t, \alpha)(1 + o(1)). \quad (4.59)$$

**Step 3:** For any  $t$ ,  $K(x, t)$  as a function of  $x$  is concave.

The Three Chords Lemma implies, for  $x \mapsto K(x, t)$  concave and  $(x, y, z) \in \mathbb{R}_+$  such that  $x < y < z$

$$\frac{K(y, t) - K(z, t)}{y - z} \leq \frac{K(x, t) - K(z, t)}{x - z} \leq \frac{K(x, t) - K(y, t)}{x - y}. \quad (4.60)$$



Since  $x \mapsto K(x, t)$  is concave and attains its maximum in  $\hat{x}$ , we consider two cases:  $x < \hat{x}$  and  $x \geq \hat{x}$ . Consider a positive real  $x$  such that  $x \notin \hat{x} + I_t$  and  $x > \tau$ . This implies that  $x > \sup(\tau, \hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}}) = \hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}} > \hat{x}$  or  $\tau < x < \hat{x} - \frac{l^{1/3}\sigma}{\sqrt{2}} < \hat{x}$ . For  $x < \hat{x}$ , by (4.60), we have

$$\frac{K(x, t) - K(\hat{x}, t)}{x - \hat{x}} \leq \frac{K(\hat{x} - \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{-\frac{l^{1/3}\sigma}{\sqrt{2}}}. \quad (4.61)$$

Similarly for  $x \geq \hat{x}$ ,

$$\frac{K(x, t) - K(\hat{x}, t)}{x - \hat{x}} \leq \frac{K(\hat{x} + \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{\frac{l^{1/3}\sigma}{\sqrt{2}}}. \quad (4.62)$$

(4.61) and (4.62) are summed up as

$$\frac{K(x, t) - K(\hat{x}, t)}{x - \hat{x}} \leq \frac{K(\hat{x} + \operatorname{sgn}(x - \hat{x}) \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t)}{\operatorname{sgn}(x - \hat{x}) \frac{l^{1/3}\sigma}{\sqrt{2}}}, \quad (4.63)$$

where

$$\operatorname{sgn}(x - \hat{x}) = \begin{cases} 1 & \text{if } x \geq \hat{x} \\ -1 & \text{if } x < \hat{x} \end{cases}$$

Using (4.1), we get

$$K(\hat{x} + \operatorname{sgn}(x - \hat{x}) \frac{l^{1/3}\sigma}{\sqrt{2}}) - K(\hat{x}, t) \leq -\frac{1}{4}h'(\hat{x})l^{2/3}\sigma^2 = -\frac{1}{4}l^{2/3},$$

which, combined with (4.63), yields

$$K(x, t) - K(\hat{x}, t) \leq -\frac{\sqrt{2}}{4} \frac{l^{1/3}}{\sigma} |x - \hat{x}|.$$

We obtain

$$\begin{aligned} |\Psi_3(t, \alpha)| &\leq 2c \int_{x \notin \hat{x} + I_t, x > \tau} |x - \hat{x}|^\alpha e^{K(x, t)} dx \\ &\leq 2ce^{K(\hat{x}, t)} \int_{|x - \hat{x}| > \frac{l^{1/3}\sigma}{\sqrt{2}}, x > \tau} |x - \hat{x}|^\alpha \exp\left(-\frac{\sqrt{2}}{4} \frac{l^{1/3}}{\sigma} |x - \hat{x}|\right) dx \\ &= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} \int_{|y| > \frac{l^{1/3}}{\sqrt{2}}, y > \frac{\tau - \hat{x}}{\sigma}} |y|^\alpha \exp\left(-\frac{\sqrt{2}}{4} l^{1/3} |y|\right) dy \\ &= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} \left[ \int_{\frac{l^{1/3}}{\sqrt{2}}}^{+\infty} y^\alpha \exp\left(-\frac{\sqrt{2}}{4} l^{1/2} y\right) dy + \int_{\frac{\tau - \hat{x}}{\sigma}}^{-\frac{l^{1/3}}{\sqrt{2}}} (-y)^\alpha \exp\left(\frac{\sqrt{2}}{4} l^{1/3} y\right) dy \right] \\ &:= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} (I_\alpha + J_\alpha). \end{aligned}$$

It is easy but a bit tedious to show that

$$\begin{aligned} I_\alpha &= \int_{\frac{l^{1/3}}{\sqrt{2}}}^{+\infty} y^\alpha \exp\left(-\frac{\sqrt{2}}{4} l^{1/2} y\right) dy \\ &= \exp\left(-\frac{1}{4} l^{2/3}\right) \sum_{i=0}^{\alpha} 2^{\frac{4i+3-\alpha}{2}} l^{\frac{\alpha-(2i+1)}{3}} \frac{\alpha!}{(\alpha-i)!} \\ &\sim_{t \rightarrow \infty} 2^{\frac{3-\alpha}{2}} \exp\left(-\frac{1}{4} l^{2/3}\right) l^{\frac{\alpha-1}{3}} \end{aligned}$$

and

$$\begin{aligned}
J_\alpha &= \int_{\frac{\tau-\hat{x}}{\sigma}}^{-\frac{l^{1/3}}{\sqrt{2}}} (-y)^\alpha \exp\left(\frac{\sqrt{2}}{4}l^{1/3}y\right) dy \\
&= I_\alpha + \exp\left(\frac{\sqrt{2}}{4}l^{1/3}\frac{\tau-\hat{x}}{\sigma}\right) \sum_{i=0}^{\sigma} \left(\frac{\hat{x}-\tau}{\sigma}\right)^{\alpha-i} l^{-\frac{i+1}{2}} 2^{\frac{3i}{2}} \frac{\alpha!}{(\alpha-i)!} \\
&\underset{t \rightarrow \infty}{\sim} I_\alpha + \exp\left(\frac{\sqrt{2}}{4}l^{1/3}\frac{\tau-\hat{x}}{\sigma}\right) \left(\frac{\hat{x}-\tau}{\sigma}\right)^\alpha l^{-1/3} \\
&\leq I_\alpha + \exp\left(\frac{\sqrt{2}}{4}l^{1/3}\frac{\tau-\hat{x}}{\sigma}\right) \frac{\hat{x}^\alpha l^{-1/3}}{\sigma^\alpha} \\
&= I_\alpha + M(t),
\end{aligned}$$

with  $\hat{x}/\sigma \xrightarrow[t \rightarrow \infty]{} \infty$  when  $h \in \mathcal{R}$ . Furthermore, since  $M(t) \xrightarrow[t \rightarrow \infty]{} 0$ , we have for some positive constant  $Q$

$$|\Psi_3(t, \alpha)| \leq Q c e^{K(\hat{x}, t)} \sigma^{\alpha+1} \exp(-\frac{1}{4}l^{2/3}) l^{\frac{\alpha-1}{3}}.$$

With (4.59), we obtain

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{Q \exp(-\frac{1}{4}l^{2/3}) l^{\frac{\alpha-1}{3}}}{|T_1(t, \alpha)|}.$$

In Step 2, we saw that  $T_1(t, \alpha) \underset{t \rightarrow \infty}{\sim} \sqrt{2\pi} M_\alpha$ , for  $\alpha$  even and  $T_1(t, \alpha) \underset{t \rightarrow \infty}{\sim} -\frac{h''(x)\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}$ , for  $\alpha$  odd. Hence for  $\alpha$  even and  $t$  large enough

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{\exp(-\frac{1}{4}l^{2/3}) l^{\frac{\alpha-1}{3}}}{\sqrt{2\pi} M_\alpha} \xrightarrow[t \rightarrow \infty]{} 0. \quad (4.64)$$

And for  $\alpha$  odd and  $t$  large enough

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{\exp(-\frac{1}{4}l^{2/3}) l^{\frac{\alpha-1}{3}}}{\frac{h''(x)\sigma^3}{6} \sqrt{2\pi} M_{\alpha+3}}.$$

For example, we can take  $l(t) = (\log t)^3$ .

If  $h \in R_\beta$ , using (4.30), we have for  $t$  large enough

$$h''(\hat{x})\sigma^3 \geq \frac{C}{t^{\frac{1}{2} + \frac{1}{2\beta}}} \quad (4.65)$$

since  $\psi(t) \underset{t \rightarrow \infty}{\sim} t^{1/\beta} l_1(t)$  for some slowly varying function  $l_1$ .

With the choice of  $l_2$  as above, we have

$$\begin{aligned}
\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| &\leq \frac{1}{C} \exp\left(-\frac{1}{4}l^{2/3} + \left(\frac{1}{2} + \frac{1}{2\beta}\right) \log t\right) \frac{l^{\frac{\alpha-1}{3}}}{\sqrt{2\pi} M_{\alpha+3}} \\
&\underset{t \rightarrow \infty}{\sim} \frac{1}{C} \exp\left(-\frac{1}{4}(\log t)^2\right) \frac{(\log t)^{\alpha-1}}{\sqrt{2\pi} M_{\alpha+3}} \xrightarrow[t \rightarrow \infty]{} 0.
\end{aligned} \quad (4.66)$$

If  $h \in R_\infty$ , using (4.34), then we have

$$h''(\hat{x})\sigma^3 \underset{t \rightarrow \infty}{\sim} \frac{\sqrt{\psi(t)\epsilon(t)}}{t^{3/2}},$$

with  $\psi(t)$  a slowly varying function and  $\epsilon(t) > D/t^\eta$ , for  $\eta \in (0, 1/8)$  and some constant  $D$  by (2.8). Thus it holds as  $t \rightarrow \infty$  and with the choice of  $l$  as above

$$\begin{aligned} \left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| &\leq \exp \left( -\frac{1}{4}l^{2/3} + \frac{3}{2} \log t - \frac{1}{2} \log \psi(t) - \frac{1}{2} \log \epsilon(t) \right) \frac{l^{\frac{\alpha-1}{3}} \sqrt{2\pi} M_{\alpha+3}}{\sqrt{2\pi} M_{\alpha+3}} \\ &\underset{t \rightarrow \infty}{\sim} \exp \left( -\frac{1}{4}(\log t)^2 \right) \frac{(\log t)^{\alpha-1}}{\sqrt{2\pi} M_{\alpha+3}} \xrightarrow{t \rightarrow \infty} 0. \end{aligned} \quad (4.67)$$

As a consequence of (4.64), (4.66) and (4.67), for  $h \in \mathcal{R}$ , we obtain

$$\Psi_3(t, \alpha) \underset{t \rightarrow \infty}{=} o(\Psi_2(t, \alpha)). \quad (4.68)$$

The proof is completed by combining (4.47), (4.53), (4.59) and (4.68).

## 4.2 Proof of Theorem 3.1

By Lemma 4.5, if  $\alpha = 0$ , it holds

$$\begin{aligned} T_1(t, 0) &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} e^{-\frac{y^2}{2}} dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^3 e^{-\frac{y^2}{2}} dy \\ &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} e^{-\frac{y^2}{2}} dy \\ &\xrightarrow{t \rightarrow \infty} \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}, \end{aligned}$$

since  $l(t) \xrightarrow{t \rightarrow \infty} \infty$ . Hence we can approximate the moment generating function of  $X$  as follows.

$$\begin{aligned} \Phi(t) &= \int_0^\infty e^{tx} p(x) dx = \Psi(t, 0) \\ &\underset{t \rightarrow \infty}{=} c\sigma e^{K(\hat{x}, t)} T_1(t, 0) (1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} c\sqrt{2\pi}\sigma e^{K(\hat{x}, t)} (1 + o(1)). \end{aligned} \quad (4.69)$$

If  $\alpha = 1$ , it holds

$$\begin{aligned} T_1(t, 1) &= \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y e^{-\frac{y^2}{2}} dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} y^4 e^{-\frac{y^2}{2}} dy \\ &= \frac{h''(\hat{x})\sigma^3}{6} \left( e^{-\frac{l^{2/3}}{4}} \left( \frac{l}{\sqrt{2}} + 6 \frac{l^{1/3}}{\sqrt{2}} \right) - 3 \int_{-\frac{l^{1/3}}{\sqrt{2}}}^{\frac{l^{1/3}}{\sqrt{2}}} e^{-\frac{y^2}{2}} dy \right), \end{aligned}$$

where the first term goes to 0 as  $t \rightarrow \infty$  and the second term goes to  $3\sqrt{2\pi}$  as  $t \rightarrow \infty$ . Consequently, we obtain

$$T_1(t, 1) \underset{t \rightarrow \infty}{=} -\frac{\sqrt{2\pi} h''(\hat{x})\sigma^3}{2} (1 + o(1)),$$

and

$$\Psi(t, 1) \underset{t \rightarrow \infty}{=} -c\sqrt{2\pi}\sigma^2 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{2} (1 + o(1)) \underset{t \rightarrow \infty}{=} -\Phi(t) \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)), \quad (4.70)$$

which, together with the definition of  $\Psi(t, \alpha)$ , yields

$$\int_0^\infty x e^{tx} p(x) dx = \Psi(t, 1) + \hat{x} \Phi(t) = \left( \hat{x} - \frac{h''(\hat{x})\sigma^4}{2}(1 + o(1)) \right) \Phi(t). \quad (4.71)$$

Hence we get

$$m(t) = \frac{d \log \Phi(t)}{dt} = \frac{\Phi'(t)}{\Phi(t)} = \frac{\int_0^\infty x e^{tx} p(x) dx}{\Phi(t)} = \hat{x} - \frac{h''(\hat{x})\sigma^4}{2}(1 + o(1)). \quad (4.72)$$

By (4.28), we obtain

$$m(t) \underset{t \rightarrow \infty}{\sim} \hat{x} = \psi(t). \quad (4.73)$$

If  $\alpha = 2$ , it follows

$$\begin{aligned} T_1(t, 2) &= \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} y^2 e^{-\frac{y^2}{2}} dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} y^5 e^{-\frac{y^2}{2}} dy \\ &= \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} y^2 e^{-\frac{y^2}{2}} dy \\ &= 2 \frac{l\frac{1}{3}}{\sqrt{2}} e^{\frac{l^2}{2}} + \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} e^{-\frac{y^2}{2}} dy \\ &\underset{t \rightarrow \infty}{=} \sqrt{2\pi}(1 + o(1)). \end{aligned} \quad (4.74)$$

Thus we have

$$\Psi(t, 2) \underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)). \quad (4.75)$$

Using (4.70), (4.72) and (4.75), it follows

$$\begin{aligned} \int_0^\infty (x - m(t))^2 e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^2 e^{tx} p(x) dx \\ &= \int_0^\infty (x - \hat{x})^2 e^{tx} p(x) dx + 2(\hat{x} - m(t)) \int_0^\infty (x - \hat{x}) e^{tx} p(x) dx + (\hat{x} - m(t))^2 \Phi(t) \\ &= \Psi(t, 2) + 2(\hat{x} - m(t))\Psi(t, 1) + (\hat{x} - m(t))^2 \Phi(t) \\ &\underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)) - h''(\hat{x})\sigma^4 \left( \Phi(t) \frac{h''(\hat{x})\sigma^4}{2} \right) (1 + o(1)) + \left( \frac{h''(\hat{x})\sigma^4}{2} \right)^2 \Phi(t)(1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)) - \frac{(h''(\hat{x})\sigma^4)^2}{4} \Phi(t)(1 + o(1)) \underset{t \rightarrow \infty}{=} \sigma^2 \Phi(t)(1 + o(1)), \end{aligned}$$

where the last equality holds since  $h''(\hat{x})\sigma^3$  goes to 0 by (4.28). Hence

$$\begin{aligned} s^2(t) &= \frac{d^2 \log \Phi(t)}{dt^2} = \frac{\Phi'(t)}{\Phi(t)} - \left( \frac{\Phi'(t)}{\Phi(t)} \right)^2 \\ &= \frac{\int_0^\infty x^2 e^{tx} p(x) dx}{\Phi(t)} - (m(t))^2. \end{aligned}$$

Since

$$\begin{aligned} \frac{\int_0^\infty (x - m(t))^2 e^{tx} p(x) dx}{\Phi(t)} &= \frac{\int_0^\infty x^2 e^{tx} p(x) dx}{\Phi(t)} - \frac{2m(t) \int_0^\infty x e^{tx} p(x) dx}{\Phi(t)} + \frac{(m(t))^2 \int_0^\infty e^{tx} p(x) dx}{\Phi(t)} \\ &= \frac{\int_0^\infty x^2 e^{tx} p(x) dx}{\Phi(t)} - (m(t))^2, \end{aligned}$$

we obtain

$$s^2(t) = \frac{\int_0^\infty (x - m(t))^2 e^{tx} p(x) dx}{\Phi(t)} \underset{t \rightarrow \infty}{\sim} \sigma^2 = \psi'(t). \quad (4.76)$$

If  $\alpha = 3$ , it holds

$$\begin{aligned} T_1(t, 3) &= \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} y^3 e^{-\frac{y^2}{2}} dy - \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} y^6 e^{-\frac{y^2}{2}} dy \\ &= -\frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} y^6 e^{-\frac{y^2}{2}} dy \end{aligned}$$

Thus we have

$$\begin{aligned} \Psi(t, 3) &= -c\sqrt{2\pi}\sigma^4 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{6} \int_{-\frac{l\frac{1}{3}}{\sqrt{2}}}^{\frac{l\frac{1}{3}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} y^6 e^{-\frac{y^2}{2}} dy \\ &\underset{t \rightarrow \infty}{=} -cM_6 \sqrt{2\pi} e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^7}{6} (1 + o(1)) \\ &= -M_6 \frac{h''(\hat{x})\sigma^6}{6} \Phi(t) (1 + o(1)), \end{aligned} \quad (4.77)$$

where  $M_6 = 15$  denotes the sixth order moment of standard normal distribution. Using (4.70), (4.72), (4.75) and (4.77), we have

$$\begin{aligned} \int_0^\infty (x - m(t))^3 e^{tx} p(x) dx &= \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^3 e^{tx} p(x) dx \\ &= \int_0^\infty ((x - \hat{x})^3 + 3(x - \hat{x})^2(\hat{x} - m(t)) + 3(x - \hat{x})(\hat{x} - m(t))^2 + (\hat{x} - m(t))^3) e^{tx} p(x) dx \\ &= \Psi(t, 3) + 3(\hat{x} - m(t))\Psi(t, 2) + 3(\hat{x} - m(t))^2\Psi(t, 1) + (\hat{x} - m(t))^3\Phi(t) \\ &\underset{t \rightarrow \infty}{=} -5\frac{h''(\hat{x})\sigma^6}{2}\Phi(t)(1 + o(1)) + 3\frac{h''(\hat{x})\sigma^4}{2}\sigma^2\Phi(t)(1 + o(1)) \\ &\quad - 3\left(\frac{h''(\hat{x})\sigma^4}{2}\right)^2\Phi(t)\frac{h''(\hat{x})\sigma^4}{2}(1 + o(1)) + \left(\frac{h''(\hat{x})\sigma^4}{2}\right)^3\Phi(t)(1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} -h''(\hat{x})\sigma^6\Phi(t)(1 + o(1)) - h''(\hat{x})\sigma^6\Phi(t)\frac{(h''(\hat{x})\sigma^3)^2}{4}(1 + o(1)) \\ &\underset{t \rightarrow \infty}{=} -h''(\hat{x})\sigma^6\Phi(t)(1 + o(1)), \end{aligned}$$

where the last equality holds since  $h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0$  by (4.28). Hence we get

$$\begin{aligned} \mu_3(t) &= \frac{d^3 \log \Phi(t)}{dt^3} = \frac{\int_0^\infty (x - m(t))^3 e^{tx} p(x) dx}{\Phi(t)} \\ &\underset{t \rightarrow \infty}{\sim} -h''(\hat{x})\sigma^6 = \frac{\psi''(t)}{(\psi'(t))^3} (\psi'(t))^3 = \psi''(t). \end{aligned} \quad (4.78)$$

The proof is completed by combining (4.73), (4.76) and (4.78).

### 4.3 Proof of Corollary 3.1

The proof follows from (4.28), from which  $h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0$  since  $l(t) \xrightarrow[t \rightarrow \infty]{} \infty$ . By (4.76) and (4.78), it holds

$$\frac{\mu_3}{s^3} \underset{t \rightarrow \infty}{\sim} -h''(\hat{x})\sigma^3 \xrightarrow[t \rightarrow \infty]{} 0. \quad (4.79)$$

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