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WEAK SOLUTIONS AND NUMERICAL APPROXIMATION OF A COUPLED SYSTEM MODELING CARDIAC ELECTROMECHANICS

BORIS ANDREIANOV*, MOSTAFA BENDAHMANE†, ALFIO QUARTERONI‡, AND RICARDO RUIZ-BAIER§

Abstract. This paper is concerned with the mathematical analysis of a coupled elliptic-parabolic system modeling the interaction between the propagation of electric potential and subsequent deformation of the cardiac tissue. The problem consists in a reaction-diffusion system governing the dynamics of ionic quantities, intra and extra-cellular potentials, and the elasticity equations are adopted to describe the motion of an incompressible material. The coupling between muscle contraction, biochemical reactions and electric activity is introduced with a so-called active strain decomposition framework, where the material gradient of deformation is split into an active (electrophysiology-dependent) part and an elastic (passive) one. Under the assumption of linearized elastic behavior, we prove existence of weak solutions to the underlying coupled reaction-diffusion system and uniqueness of regular solutions. The proof of existence is based on a combination of parabolic regularization, the Faedo-Galerkin method, and the monotonicity-compactness method of J.L. Lions. A finite element formulation is also introduced, for which we establish existence of discrete solutions and show convergence to a weak solution of the original problem. We close with a numerical example illustrating the behavior of the method and some features of the model.

Key words. Electro-mechanical coupling, Bidomain equations, Active deformation, Weak solutions, Weak compactness method, Weak-strong uniqueness, Finite element approximation, Convergence of approximations.

AMS subject classifications. 74F99,35K57,92C10,65M60

1. Introduction. We are interested in the mathematical study of the interaction between the propagation of the electrical potential through the cardiac tissue, on the one side, and the related elastic mechanical response, on the other side. The propagation of the electrical potential through the heart can be described by the so-called bidomain equations (see e.g. [36]). These are obtained by writing the conservation of electrical fluxes between the extra and intracellular domains separated by a membrane acting as a capacitor. Within these domains, the conductivities are of different magnitudes, and they also change depending on the particular orientation of the cardiac tissue fibers. Muscle deformation can be described by the equations of motion for a hyperelastic material, written in the reference configuration. The medium itself is active, in the sense that it is able to contract without the need of external loads, but rather influenced by intrinsic mechanisms taking place at the microscale. In order to incorporate these effects we follow an active strain formulation (see e.g. [24, 9]), for which a decomposition of the deformation gradient into an active and a passive factors is assumed. Such a decomposition implies, in particular, that the fiber contraction driving the depolarization of the cardiomyocytes rewrites in the mechanical balance of forces as a prescribed active deformation, rather than as an additive contribution to the stress. Moreover, this approach directly incorporates the micro-level information on the fiber contraction, and fiber directions in the kinematics (through the active part of the deformation gradient), without the intermediate transcription of their role in terms of stress [1]. In this paper, the coupling is achieved by considering that the evolution of the electrical potentials governed by the bidomain equations depend on the deformation gradient, which enters into the bidomain equations after a transformation of coordinates from Eulerian to Lagrangian, and by virtue of the Piola identity. The coupling in the opposite direction is

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modeled by assuming that the active part of the deformation incorporates the influence of the electrical potential into the balance equations for the structural mechanics.

The mathematical analysis of macroscopic cardiac models has been mainly related with the study of solutions to the bidomain equations and related formulations (see e.g. [5, 7, 8, 11, 21, 36, 41] and the references therein). Existence theorems of general nonlinear elasticity can be found in [10, 3], whereas applications of those theories to the particular case of hyperelastic materials and cardiac mechanics and their discretizations include e.g. [25, 18, 15, 33, 37, 4]. However, even if the literature related to numerical methods and models for cardiac electromechanics is quite large (see, for instance, [14, 19, 26, 28, 39]), rigorous studies about solvability and stability of solutions are still not well established. To the authors' knowledge, the only available existence results devoted specifically to cardiac electromechanics correspond to those by Pathmanathan et al. [29, 30], who analyzed a general model where the activation depends on the local stretch rate, and derived constraints on the initial data. Here, we also assume linearized elasticity equations, but we follow a somewhat different model (our active strain description depends only on ionic quantities and we do not consider viscous effects).

We establish existence of weak solutions by means of the Faedo-Galerkin method and compactness arguments. The fully discrete counterpart employs classical finite element method with backward Euler discretization in time; moreover, we linearize the equations in order to get an efficient fully practical numerical scheme. The linearization makes the convergence proof more delicate than in the continuous case (see § 3 and § 5). We also prove a uniqueness result for the continuous problem in a weak-strong comparison setting. While some classical estimates and arguments are sketched, we concentrate on the main analytical aspects of the proofs. In our presentation, modeling, numerical and implementation details will be reduced as much as possible, and we refer to [27, 32] for further specifications.

We have organized the contents of this paper as follows. § 2 collects the main aspects of the cardiac electromechanical model we analyze, presenting the equations of passive nonlinear mechanics, the bidomain system, and the active-strain-based coupling strategy. We also list the basic assumptions of the model and provide a definition of weak solution. In § 3 we state and prove the solvability of the continuous problem employing Galerkin approximations and classical compactness theory. Then, in § 4 we develop a strong-weak uniqueness argument and discuss regularity of solutions needed to apply it. The fully discrete linearized finite element formulation, along with additional analytic arguments developed for the proof of convergence, and a numerical test are finally presented in § 5.

2. Governing equations for the electromechanical coupling.

2.1. A general nonlinear elasticity problem. Let us consider a homogeneous continuous material occupying in the initial undeformed configuration a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 3$) with Lipschitz continuous boundary $\partial\Omega$. We look for the deformation field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ that maps a material particle that originally occupied the position x to its current position $\hat{x}(x) = \mathbf{u}$. The tensor gradient of deformation is $\mathbf{D}\mathbf{u}$, where \mathbf{D} denotes the gradient operator with respect to the material coordinates x . The cardiac tissue is assumed hyperelastic, and there exists a strain stored energy function $\mathcal{W} = \mathcal{W}(\mathbf{D}\mathbf{u})$ from which constitutive relations between strain and stresses are obtained. We further assume incompressibility of the material, that is, the total elastic energy is minimized subject to the usual local constraint $\det(\mathbf{D}\mathbf{u}) = 1$, which is enforced via a scalar Lagrange multiplier p , interpreted as pressure. As a measure of stresses we use the first Piola-Kirchhoff tensor obtained from \mathcal{W} by direct differentiation:

$$\mathbf{P} = \frac{\partial \mathcal{W}}{\partial (\mathbf{D}\mathbf{u})} - p \text{Cof}(\mathbf{D}\mathbf{u}),$$

where $\text{Cof}(\cdot)$ is the cofactor matrix. The balance equations for deformations and pressure read as: Find \mathbf{u}, p such that

$$\begin{aligned} \nabla \cdot \mathbf{P}(\mathbf{D}\mathbf{u}, p) + \mathbf{f} &= \mathbf{0} & \text{in } \Omega, \\ \det(\mathbf{D}\mathbf{u}) &= 1 & \text{in } \Omega, \end{aligned} \tag{2.1}$$

completed with the Robin boundary data

$$\mathbf{P}\mathbf{n} = \alpha\mathbf{u} \quad \text{on } \partial\Omega, \quad (2.2)$$

where \mathbf{f} is a prescribed body force, \mathbf{n} stands for the unit outward normal vector to $\partial\Omega$, and α is a constant parameter. Evidently, the precise form of the first equation in (2.1) depends on the particular constitutive relation defining \mathcal{W} . For sake of clarity we restrict ourselves to the case of Neo-Hookean materials, that is, $\mathcal{W} = \frac{1}{2}\mu\text{tr}[(\mathbf{D}\mathbf{u})^2 - \mathbf{I}]$, which gives $\mathbf{P} = \mu\mathbf{D}\mathbf{u} - p\text{Cof}(\mathbf{D}\mathbf{u})$, where μ is an elastic modulus. Even if simplified, such a description of the passive response of the muscle already features a nonlinear strain-stress relationship.

2.2. The bidomain equations. The description of the electrophysiology in the cardiac tissue is incorporated in the model in the form of the so-called bidomain equations [40]. The quantities of interest are the intra and extracellular electric potentials ($u_i = u_i(x, t)$, $u_e = u_e(x, t)$ respectively), the transmembrane potential $v = v(x, t) := u_i - u_e$, and the so-called gating or recovery variable $w = w(x, t)$ at $(x, t) \in \Omega_T := \Omega \times (0, T)$, where T is the final time instant. The electrical conductivity of the tissue is represented by the orthotropic tensors

$$\mathbf{D}_k(x) = \sigma_k^l \mathbf{d}_l \otimes \mathbf{d}_l + \sigma_k^t \mathbf{d}_t \otimes \mathbf{d}_t + \sigma_k^n \mathbf{d}_n \otimes \mathbf{d}_n, \quad k \in \{e, i\},$$

where $\sigma_k^s = \sigma_k^s(x) \in C^1(\mathbb{R}^3)$, $k \in \{e, i\}$, $s \in \{l, t, n\}$, are the intra- and extracellular conductivities along, transversal, and normal to the direction of the fibers, respectively. The fibers direction being a local quantity, we have $\mathbf{d}_s = \mathbf{d}_s(x)$, $s \in \{l, t, n\}$. The stimulation current externally applied to the intra- and extracellular spaces is represented by the functions $I_{\text{app}}^i, I_{\text{app}}^e$. The system corresponds to

$$\begin{aligned} \chi c_m \partial_t v - \nabla \cdot (\mathbf{D}_i \nabla u_i) + \chi I_{\text{ion}}(v, w) &= I_{\text{app}}^i, \\ \chi c_m \partial_t v + \nabla \cdot (\mathbf{D}_e \nabla u_e) + \chi I_{\text{ion}}(v, w) &= I_{\text{app}}^e, \\ \partial_t w - H(v, w) &= 0, \quad (x, t) \in \Omega_T, \end{aligned} \quad (2.3)$$

where we recall that $v = u_i - u_e$. Here c_m and χ are model parameters. Problem (2.3) is provided with homogeneous Neumann boundary conditions for all fields. The choice of the membrane model to be used is reflected in the functions $H(v, w)$ and $I_{\text{ion}}(v, w)$. For a phenomenological description of the action potential, it suffices to consider the FitzHugh-Nagumo model [13, 23], given as in assumption (E.6) below.

2.3. The active strain model for the coupling of elasticity and bidomain equations. In the so-called active strain model for cardiac modeling (see [1, 9]), the deformation gradient $\mathbf{D}\mathbf{u}$ is factorized into a *passive* part acting at a macroscale (tissue level), and an *active* factor operating at the microscale (cellular level), $\mathbf{D}\mathbf{u} = \mathbf{F}_p \mathbf{F}_a$. This implies that an intermediate configuration exists between the reference and the current frames. In that configuration, we re-write the strain energy in such a way that the stress tensor is given by

$$\mathbf{P} = \mu \mathbf{D}\mathbf{u} \mathbf{C}_a^{-1} - p \text{Cof}(\mathbf{D}\mathbf{u}),$$

where $\mathbf{C}_a^{-1} := \det(\mathbf{F}_a) \mathbf{F}_a^{-1} \mathbf{F}_a^{-T}$. In order to cover the *electrical-to-mechanical* coupling, the active deformation is assumed to depend directly on the electrophysiology through the relation

$$\mathbf{F}_a = \mathbf{I} + \gamma_l \mathbf{d}_l \otimes \mathbf{d}_l + \gamma_t \mathbf{d}_t \otimes \mathbf{d}_t + \gamma_n \mathbf{d}_n \otimes \mathbf{d}_n,$$

where for $s = l, t, n$, γ_s are quantities whose evolution depends, non-locally in time, on that of the gating variable w . This is because of the well known property that the onset of mechanical activation follows closely the dynamics of calcium release rather than repolarization waves [6]. By assuming transverse isotropy and incompressibility at the fiber level, we can write γ_l , γ_t and γ_n as functions of a parameter γ :

$$\gamma_{l,t,n} = \gamma_{l,t,n}(\gamma), \quad (2.4)$$

where $\gamma_{l,t,n} : \mathbb{R} \mapsto [-\Gamma_{l,t,n}, 0]$ are Lipschitz continuous monotone functions. The values $\Gamma_{l,t,n}$ should be small enough, so that to ensure that $\det \mathbf{F}_a$ stays uniformly far from zero, for $\gamma \in \mathbb{R}$. The scalar field γ is the solution of the following ODE associated to the solution v of the bidomain system (2.3):

$$\partial_t \gamma - G(\gamma, w) = 0, \quad (x, t) \in \Omega_T,$$

where $G(\gamma, w) = -Aw - B\gamma$, for positive parameters A, B (see [35]). It remains to fix the form of the functions $\gamma_{l,t,n}$, which we assume to be

$$\gamma_{l,t,n} = -\Gamma_{l,t,n} \frac{2}{\pi} \arctan(\gamma^+ / \gamma_R), \quad \text{where } \gamma_R \text{ is a reference value.}$$

The *mechanical-to-electrical* coupling is achieved by pulling back the bidomain equations to the reference configuration, which, by virtue of the Piola identity, leads to a diffusion term depending on the deformation gradient $\mathbf{D}\mathbf{u}$.

Summarizing, the active strain formulation for the electromechanical activity in the heart is written as follows (see [27]):

$$\begin{aligned} -\nabla \cdot (a(x, \gamma, \mathbf{D}\mathbf{u}, p)) &= \mathbf{f} && \text{in } \Omega, \\ \chi_{C_m} \partial_t v - \nabla \cdot (\mathbf{M}_e(\mathbf{D}\mathbf{u}) \nabla u_e) + \chi I_{\text{ion}} &= I_{\text{app}}^e && \text{in } \Omega_T, \\ \chi_{C_m} \partial_t v + \nabla \cdot (\mathbf{M}_i(\mathbf{D}\mathbf{u}) \nabla u_i) + \chi I_{\text{ion}} &= I_{\text{app}}^i && \text{in } \Omega_T, \\ u_i - u_e &= v && \text{in } \Omega_T, \\ \partial_t w - H(v, w) &= 0 && \text{in } \Omega_T, \\ \partial_t \gamma - G(w, \gamma) &= 0 && \text{in } \Omega_T. \end{aligned} \tag{2.5}$$

Here, according to the above discussion, we should take

$$a(x, \gamma, \mathbf{D}\mathbf{u}, p) := \mu \mathbf{D}\mathbf{u} \mathbf{C}_a^{-1}(x, \gamma) - p \text{Cof}(\mathbf{D}\mathbf{u}), \tag{2.6}$$

and

$$\mathbf{M}_k(\mathbf{D}\mathbf{u}) := (\mathbf{D}\mathbf{u})^{-1} \mathbf{D}_k(\mathbf{D}\mathbf{u})^{-T}, \quad k \in \{i, e\} \tag{2.7}$$

Moreover, we have the incompressibility constraint under the form

$$\det(\mathbf{D}\mathbf{u}) = 1 \quad \text{in } \Omega, \quad \text{for a.e. } t \in (0, T). \tag{2.8}$$

The system of equations (2.5) has to be completed with suitable initial data for v, w, γ and with boundary data on $u_{i,e}$ and on the elastic flux $a(\cdot, \cdot, \cdot, \cdot)$.

2.4. Linearizing the elasticity equations. In the remaining part of this paper, for the sake of simplicity of both numerical and mathematical analysis of the problem we introduce two modifications into (2.5),(2.7),(2.8). Firstly, we linearize the incompressibility constraint by imposing

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad \text{for a.e. } t \in (0, T); \tag{2.9}$$

we also linearize the flux in (2.6) with respect to $\mathbf{D}\mathbf{u}$ by replacing it with

$$a(x, \gamma, \mathbf{D}\mathbf{u}, p) := \mu \mathbf{D}\mathbf{u} \mathbf{C}_a^{-1}(x, \gamma) - p \mathbf{I}. \tag{2.10}$$

Introducing the notation $\sigma(x, \gamma)$ for $\mu \mathbf{C}_a^{-1}(x, \gamma)$, we rewrite the first equation of (2.5) as

$$-\nabla \cdot (\sigma(x, \gamma) \mathbf{D}\mathbf{u}) + \nabla p = \mathbf{f}.$$

The linearization results, however, in the fact that the matrices \mathbf{M}_k , $k = i, e$ in (2.7) may become ill-defined. We proceed to a linearization and to a truncation, so that to ensure the boundedness and coercivity (see (E.3) below) of $\mathbf{M}_i, \mathbf{M}_e$:

$$\mathbf{M}_k(\mathbf{D}\mathbf{u}) := \mathbf{D}_k - T_\delta(\Delta_k(\mathbf{D}\mathbf{u}) + \Delta_k(\mathbf{D}\mathbf{u})^T), \quad \text{where } \Delta_k(\mathbf{D}\mathbf{u}) := (\mathbf{D}\mathbf{u} - \mathbf{I})\mathbf{D}_k, \quad k \in \{i, e\}, \quad (2.11)$$

and T_δ is a suitable truncation function that coincides with the identity map in a neighborhood of the origin. For the sake of being definite, we set

$$T_\delta(\mathbf{M}) := \begin{cases} \mathbf{M} & , \quad \|\mathbf{M}\| \leq \delta \\ \mathbf{M} \frac{\delta}{\|\mathbf{M}\|} & , \quad \|\mathbf{M}\| \geq \delta \end{cases}$$

with some $\delta < 1$.

REMARK 2.1. *The amplitude of contractions in the heart tissue is rather large, which makes the validity of the linearized model questionable, so we cannot claim that in its present form it is able to represent physiological regimes. However, we consider the subsequent analysis as a baseline for future developments including more accurate and more general models. In any case, we will briefly address the influence of the linearization and of the truncation on the solutions of (2.5) from a numerical viewpoint, in § 5.*

2.5. The problem to be solved and its weak formulation. Let us consider the following class of problems:

$$-\nabla \cdot (\sigma(x, \gamma)\mathbf{D}\mathbf{u}) + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \text{ for a.e. } t \in (0, T), \quad (2.12)$$

$$c_m \chi \partial_t v - \nabla \cdot (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}) \nabla u_i) + \chi I_{\text{ion}}(v, w) = I_{\text{app}}^i(t, x) \quad \text{in } \Omega_T, \quad (2.13)$$

$$c_m \chi \partial_t v + \nabla \cdot (\mathbf{M}_e(x, \mathbf{D}\mathbf{u}) \nabla u_e) + \chi I_{\text{ion}}(v, w) = I_{\text{app}}^e(t, x) \quad \text{in } \Omega_T, \quad (2.14)$$

$$v = u_i - u_e \quad \text{in } \Omega_T, \quad (2.15)$$

$$\partial_t w - H(v, w) = 0 \quad \text{in } \Omega_T, \quad (2.16)$$

$$\partial_t \gamma - G(w, \gamma) = 0 \quad \text{in } \Omega_T. \quad (2.17)$$

Equations (2.12),(2.13),(2.14) are complemented with the boundary data (including the linearization of (2.2)):

$$\sigma(x, \gamma)\mathbf{D}\mathbf{u} \mathbf{n} - p \mathbf{n} = \alpha \mathbf{u} \quad \text{on } \partial\Omega, \text{ for a.e. } t \in (0, T) \quad (2.18)$$

for some $\alpha > 0$ and

$$\mathbf{M}_k(x, \nabla u_k) \cdot \mathbf{n} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad k = i, e \quad (2.19)$$

(different boundary conditions can be imposed on $u_{i,e}$; the choice of Neumann conditions (2.19) results in the compatibility constraint (2.22) below). The initial data are:

$$v(0, \cdot) = v_0, \quad w(0, \cdot) = w_0, \quad \gamma(0, \cdot) = \gamma_0 \quad \text{in } \Omega. \quad (2.20)$$

The following properties of the model (2.12)–(2.17) and (2.18)–(2.20) are instrumental for the subsequent analysis:

(E.1) $(\sigma(x, \gamma))_{x \in \Omega, \gamma \in \mathbb{R}}$ is a family of symmetric tensors, uniformly bounded and positive definite:

$$\exists c > 0 : \text{ for a.e. } x \in \Omega, \forall \gamma \in \mathbb{R} \quad \forall \mathbf{M} \in \mathbb{M}_{3 \times 3} \quad \frac{1}{c} |\mathbf{M}|^2 \leq (\sigma(x, \gamma)\mathbf{M}) : \mathbf{M} \leq c |\mathbf{M}|^2;$$

(E.2) the map $\gamma \mapsto \sigma(\cdot, \gamma)$ is uniformly Lipschitz continuous;

(E.3) $(\mathbf{M}_{i,e}(x, \mathbf{M}))_{x \in \Omega, \mathbf{M} \in \mathbb{M}_{3 \times 3}}$ is a family of symmetric matrices, uniformly bounded and positive definite:

$$\exists c > 0 : \text{ for a.e. } x \in \Omega, \forall \mathbf{M} \in \mathbb{M}_{3 \times 3} \quad \forall \xi \in \mathbb{R}^3 \quad \frac{1}{c} |\xi|^2 \leq (\mathbf{M}_{i,e}(x, \mathbf{M})\xi) \cdot \xi \leq c |\xi|^2;$$

- (E.4) the maps $\mathbf{M} \mapsto \mathbf{M}_{i,e}(\cdot, \mathbf{M})$ are uniformly Lipschitz continuous;
(E.5) the function G is given by $G(\gamma, w) = \eta_1(\beta w - \eta_2 \gamma)$ with $\beta, \eta_1, \eta_2 > 0$;
(E.6) the functions H and I_{ion} are given by the FitzHugh-Nagumo kinetics

$$H(v, w) = Av - Bw, \quad I_{\text{ion}}(v, w) = j(v) + Cw,$$

where $j \in \mathcal{C}^1(\mathbb{R})$ and A, B, C are positive parameters. Moreover, we assume that there exist constants $A_1, A_2, A_3 > 0$ such that

$$\begin{aligned} j(0) &= 0, & \frac{j(v_1) - j(v_2)}{v_1 - v_2} &> -A_1, \quad \forall v_1 \neq v_2, \\ 0 < A_2 &\leq \liminf_{|v| \rightarrow \infty} \frac{j(v)}{v^3} \leq \limsup_{|v| \rightarrow \infty} \frac{j(v)}{v^3} \leq A_3. \end{aligned} \quad (2.21)$$

In particular, $\mathcal{B} : v \mapsto j(v)/v + A_1$ is a non-negative function that satisfies, for $|v|$ large enough, $\frac{1}{2}A_2|v|^2 \leq \mathcal{B}(v) \leq 2A_3|v|^2$; and the function $v \mapsto v\mathcal{B}(v) = j(v) + A_1v$ is monotone increasing;

- (E.7) the following compatibility condition holds

$$\int_{\Omega} I_{\text{app}}^i = \int_{\Omega} I_{\text{app}}^e \quad \text{and} \quad \int_{\Omega} u_e(x, t) \, dx = 0 \quad \text{for a.e. } t \in (0, T); \quad (2.22)$$

- (E.8) the data v_0, w_0, γ_0 lie in $L^2(\Omega)$ whereas $\mathbf{f} \in L^2(\Omega_T)^3$ and $I_{\text{app}}^{i,e} \in L^2(\Omega_T)$.

Note that, in practice, one starts with an undeformed configuration, i.e., with $\gamma \equiv 0$.

Observe also that the above system (2.5), (2.9) with $a(\cdot, \cdot, \cdot, \cdot)$ and $\mathbf{M}_{i,e}(\cdot, \cdot)$ given by (2.10), (2.11) falls within the framework described by (2.12)–(2.20) and (E.1)–(E.8). Indeed, it is enough to check that assumptions (E.1)–(E.4) are satisfied (assumptions (E.5)–(E.8) are already enforced). Let us stress that due to the assumption (2.4), the properties (E.1), (E.2) hold. Similarly, the definition (2.11) along with the truncation T_{δ} , with δ small enough with respect to the eigenvalues of matrices \mathbf{D}_k , guarantees (E.3), (E.4).

Due to the properties (E.1)–(E.8), the following weak formulation makes sense.

DEFINITION 2.2. *A weak solution of problem (2.12)–(2.20) is $U = (\mathbf{u}, p, u_i, u_e, v, w, \gamma)$ such that:*

- (i) $\mathbf{u} \in L^2(0, T; H^1(\Omega)^3)$, $p \in L^2(\Omega_T)$, $u_{i,e} \in L^2(0, T; H^1(\Omega))$;
 $v \in E := L^2(0, T; H^1(\Omega)) \cap L^4(\Omega_T)$ with $\partial_t v \in E' := L^2(0, T; (H^1(\Omega))') + L^{4/3}(\Omega_T)$;
and $\gamma, w \in \mathcal{C}^1(0, T; L^2(\Omega))$;
(ii) For a.e. $t \in (0, T)$ for all $\mathbf{v} \in H^1(\Omega)^3$ there holds

$$\int_{\Omega} \left(\sigma(x, \gamma) \mathbf{D}\mathbf{u} : \mathbf{D}\mathbf{v} + Dp \cdot \mathbf{v} \right) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\partial\Omega} \alpha \mathbf{u} \cdot \mathbf{v} \quad (2.23)$$

(in the last integral, \mathbf{u}, \mathbf{v} are short-cuts for the traces of \mathbf{u}, \mathbf{v} on $\partial\Omega$)
and moreover $\nabla \cdot \mathbf{u} = 0$ a.e. in Ω_T , which can be expressed as

$$\forall q \in H_0^1(\Omega) \quad \int_{\Omega} \mathbf{u} \cdot \nabla q = 0. \quad (2.24)$$

- (iii) The distributional derivative $\partial_t v$ can be identified with an element $b \in E'$ such that for all $\xi \in E$ with $\xi_t \in L^\infty(\Omega_T)$ and $\xi(0, \cdot) = 0$, there holds

$$\int_0^T c_m \chi \langle b, \xi \rangle + \int_0^T \int_{\Omega} \left(\mathbf{M}_i(x, \mathbf{D}\mathbf{u}) \nabla u_i \cdot \nabla \xi + \chi I_{\text{ion}}(v, w) \xi \right) = \int_0^T \int_{\Omega} I_{\text{app}}^i \xi, \quad (2.25)$$

$$\int_0^T c_m \chi \langle b, \xi \rangle - \int_0^T \int_{\Omega} \left(\mathbf{M}_e(x, \mathbf{D}\mathbf{u}) \nabla u_e \cdot \nabla \xi + \chi I_{\text{ion}}(v, w) \xi \right) = \int_0^T \int_{\Omega} I_{\text{app}}^e \xi, \quad (2.26)$$

with

$$\int_0^T \langle b, \xi \rangle = - \int_0^T \int_{\Omega} v \partial_t \xi - \int_{\Omega} v_0 \xi(0, \cdot); \quad (2.27)$$

in addition, $v = u_i - u_e$ a.e. on Ω_T ;

- (iv) For a.e. $t \in (0, T)$ the equations (2.16),(2.17) are fulfilled in $L^2(\Omega)$, and $w(0, \cdot) = w_0$, $\gamma(0, \cdot) = \gamma_0$ a.e. in Ω .

REMARK 2.3. Although the variational formulation (2.25)–(2.27) is standard in the context of degenerate parabolic problems, here we have a delicate point. Because $u_{i,e}$ do not necessarily belong to $L^4(\Omega_T)$, it is not allowed to take u_i for the test function in (2.25) nor u_e , for the test function in (2.25). Yet, thanks to the regularization approach presented in [2, Lemma 2.3], it is possible to sum up the weak formulation (2.25) with $\xi = u_i$ and the weak formulation (2.25) with $\xi = u_e$. Indeed, our assumptions include $v \in L^4(\Omega_T)$ and $I_{\text{ion}}(v, w) \in L^{4/3}(\Omega_T)$, thus the term

$$I_{\text{ion}}(v, w)v = I_{\text{ion}}(v, w)(u_i - u_e) = I_{\text{ion}}(v, w)u_i - I_{\text{ion}}(v, w)u_e,$$

belongs to $L^1(\Omega_T)$ even if the two terms $I_{\text{ion}}(v, w)u_{i,e}$ can be non-integrable.

3. Existence proof by convergence of Galerkin approximations for a regularized problem. In this section, we prove the main existence result of this paper and prepare the ground for proving convergence of numerical approximations of the system.

THEOREM 3.1. Assume that conditions (E.1)–(E.8) hold. If $v_0, w_0, \gamma_0 \in L^2(\Omega)$ and $\mathbf{f} \in L^2(\Omega_T)^3$, $I_{\text{app}}^{i,e} \in L^2(\Omega_T)$, then there exists a weak solution $U = (\mathbf{u}, p, u_i, u_e, v, w, \gamma)$ to (2.12)–(2.17) with the boundary and initial data specified as in (2.18)–(2.20).

Although the fixed-point techniques are also well suited for the proof of existence, here we prefer to treat it with the Galerkin method in space. A parabolic approximation similar to the one proposed in [5] is used to ensure existence of approximate solutions. The proof of convergence of the finite element approximation of § 5 will follow the same guidelines, but the parabolic regularization will be replaced by the linearized time-implicit discretization. In particular, except for the L^4 bound, all the estimates that we establish in this section on the sequence of approximate solutions will also be valid for the numerical approximate solutions constructed in § 5.

3.1. Construction of approximate solutions. We start by fixing some increasing, as $h \downarrow 0$, families of linear finite-dimensional subspaces $(\mathbf{V}^h)_h$ and $(V^h)_h$ of $H^1(\Omega)^3$, $H^1(\Omega)$, respectively, such that $\cup_{h>0} \mathbf{V}^h$ is dense in $H^1(\Omega)^3$ and $\cup_{h>0} V^h$ is dense both in $L^2(\Omega)$ and in $H^1(\Omega)$. In view of the structure of the FitzHugh-Nagumo nonlinearity I_{ion} (see assumption (E.6)), it is convenient to ask for the inclusion $V^h \subset L^4(\Omega)$. As usual, we require the Ladyzhenskaya-Babuška-Brezzi (or inf-sup) condition

$$\exists \beta > 0 : \forall h > 0, \quad \min_{q \in V^h, q \neq 0} \max_{\mathbf{v} \in \mathbf{V}^h, \mathbf{v} \neq 0} \frac{\int_{\Omega} q \nabla \cdot \mathbf{v}}{\|q\|_{L^2} \|\mathbf{v}\|_{H^1}} \geq \beta > 0. \quad (3.1)$$

Such sequences of finite-dimensional subspaces do exist, see in particular § 5.

Then we look for a discrete solution $U^h = (\mathbf{u}^h, p^h, u_i^h, u_e^h, v^h, w^h, \gamma^h)$ with $\mathbf{u}^h \in L^2(0, T; \mathbf{V}^h)$ and for $p^h \in L^2(0, T; V^h)$ and $u_i^h, u_e^h, v^h, w^h, \gamma^h \in C^1(0, T; V^h)$ that satisfy the Galerkin formulation of (2.12)–(2.20). This means that each of the equations of the system is recast into a weak formulation on Ω_T where time derivatives and initial conditions (projected on V^h by means of the L^2 -hilbertian projection \mathbb{P}_{V^h}) are included in a strong sense. E.g., the discrete analogue of (2.25),(2.27) writes

$$v^h(0) = \mathbb{P}_{V^h}(v_0) \quad \text{and,} \quad \forall \xi \in V^h, \quad c_m \chi \frac{d}{dt} \int_{\Omega} v^h(t) \xi$$

$$+ \int_{\Omega} \left(\mathbf{M}_i(x, \mathbf{D}\mathbf{u}^h(t)) \nabla u_i^h(t) \cdot \nabla \xi + \chi I_{\text{ion}}(v^h(t), w^h(t)) \xi \right) = \int_{\Omega} I_{\text{app}}^i(t) \xi. \quad (3.2)$$

If necessary, $I_{\text{app}}^{i,\varepsilon}$ can be regularized in time. Further, the discrete analogue of (2.23),(2.24) reads, pointwise in t ,

$$\begin{aligned} \forall \mathbf{v} \in \mathcal{V}^h \quad \int_{\Omega} \sigma(x, \gamma^h(t)) \mathbf{D}\mathbf{u}^h(t) : \mathbf{D}\mathbf{v} + Dp^h(t) \cdot \mathbf{v} &= \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} + \int_{\partial\Omega} \alpha \mathbf{u}^h(t) \cdot \mathbf{v}, \quad (3.3) \\ \forall q \in V^h \quad \int_{\Omega} \mathbf{u}^h(t) \cdot \nabla q &= 0. \end{aligned}$$

Notice that after discretization, we have a system of ODEs coupled to a system of algebraic equations to be solved at every time t . While the ODE part of the system obeys the conditions of the Cauchy-Lipschitz theorem, because of the coupling with the algebraic part existence of a discrete solution is not obvious. To prove existence of U^h , we regularize the Galerkin discretization in the spirit of [5]. Namely, in the left-hand side of the ODE in (3.2) we add the term $\varepsilon \frac{d}{dt} \int_{\Omega} u_i^h(t) \xi$ (the term $-\varepsilon \frac{d}{dt} \int_{\Omega} u_e^h(t) \xi$ is added into the analogous equation written for u_e^h) and we add the term $\varepsilon \frac{d}{dt} \int_{\Omega} \mathbf{u}^h(t) \mathbf{v}$ into the left-hand side of (3.3). As for $I_{\text{app}}^{i,\varepsilon}$, the source term \mathbf{f} can be regularized in t . The initial data can be fixed to $\mathbf{u}^{h,\varepsilon}(0) = 0$, $u_i^{\varepsilon,h} = \mathbb{P}_{V^h}(v_0)/2$, $u_e^{\varepsilon,h} = -\mathbb{P}_{V^h}(v_0)/2$. Then the system on the new unknown $U^{h,\varepsilon}$ becomes a well-posed ODE problem; solutions are defined globally on $[0, T]$ because estimates preclude finite-time explosion. Further, the estimates we establish in §3.2 below are actually valid also for $\varepsilon > 0$ and moreover, they are independent of the parabolic regularization parameter ε . Since our functions take values into a finite-dimensional space, from these bounds it is easy to deduce strong compactness of the family $(v^{h,\varepsilon}, w^{h,\varepsilon}, \gamma^{h,\varepsilon})_{\varepsilon}$ in $\mathbf{C}^0([0, T]; V^h)^3$ and the weak compactness of the family $(u_i^{h,\varepsilon}, u_e^{h,\varepsilon}, p^{h,\varepsilon})_{\varepsilon}$ in $L^2(0, T; V^h)^3$ as well as the weak compactness of $(\mathbf{u}^{h,\varepsilon})_{\varepsilon}$ in $L^2(0, T; \mathcal{V}^h)$. The strong compactness of $(\mathbf{D}\mathbf{u}^{h,\varepsilon})_{\varepsilon}$ needed to conclude the passage to the limit in the parabolic regularization of (3.2), as $\varepsilon \rightarrow 0$, is obtained using the Minty-Browder argument, as in §3.3 below. Thus, passing to the limit $\varepsilon \rightarrow 0$ we prove existence of U^h solving the Galerkin discretization of problem (2.12)–(2.20).

3.2. A priori estimates. Assuming that there exists a solution to the above problems, we derive estimates that are uniform in $h > 0$ (addition of a parabolic penalization with $\varepsilon > 0$ in the equations for $\mathbf{u}^h, u_i^h, u_e^h$ leads to the same estimates, provided $\mathbf{u}^h(0), u_i^h(0), u_e^h(0)$ remain bounded).

First, according to the definition of Galerkin approximations we are authorized to take \mathbf{u}^h and p^h for the test functions in the weak formulations of the first and the second equations of (2.12), respectively. Summing up the resulting identities, we deduce that

$$\int_0^T \int_{\Omega} (\sigma(x, \gamma^h) \mathbf{D}\mathbf{u}^h) : \mathbf{D}\mathbf{u}^h + \int_0^T \int_{\partial\Omega} \alpha \mathbf{u}^h \cdot \mathbf{u}^h = \int_0^T \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^h.$$

By Cauchy-Schwarz and Poincaré inequalities, we deduce the uniform bound

$$\|\mathbf{u}^h\|_{L^2(0,T;H^1(\Omega)^3)} \leq \mathcal{C};$$

here and until the end of the proof, \mathcal{C} is a generic constant that may possibly depend on the L^2 norms of the initial data and source terms of the system and on the constants appearing in assumptions (E.1)–(E.6), but not on h . Then, from the Galerkin formulation and the inf-sup condition (3.1) we derive the uniform L^2 estimate on p^h :

$$\|p^h\|_{L^2(\Omega_T)} \leq \mathcal{C}.$$

Next, we impose the relation (2.15): $v^h = u_e^h - u_i^h$, and we look at the part (2.14),(2.13),(2.16) of the system. Due to the assumption $V^h \subset L^4(\Omega)$ we are allowed to take u_e^h for the test function in (2.14) and u_i^h for the test function in (2.13) and make the difference; to this, we add

(2.16) with the test function $\chi C/A w^h$. Notice that due to the time-regularity of $u_{i,e}^h$ and w^h , we do have $(\partial_t v^h)v^h = \partial_t (v^h)^2/2$. In time, we integrate the equations on $(0, s)$ for every $s < T$, which yields the equality

$$\begin{aligned} \int_0^s \int_{\Omega} \left(\chi(j(v^h)v^h + \frac{BC}{A}(w^h)^2) + (\mathbf{M}_e(x, \mathbf{D}\mathbf{u}^h)\nabla u_e^h) \cdot \nabla u_e^h + (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}^h)\nabla u_i^h) \cdot \nabla u_i^h \right) \\ + \int_{\Omega} \frac{c_m \chi}{2} (v^h)^2(s, \cdot) = \int_{\Omega} \frac{c_m \chi}{2} (v_0^h)^2 + \int_0^s \int_{\Omega} I_{\text{app}}^i u_i^h - I_{\text{app}}^e u_e^h. \end{aligned} \quad (3.4)$$

The properties (2.21) of j along with the Gronwall lemma (see, e.g., [31]) yield the following uniform in h estimates:

$$\|u_{i,e}^h\|_{L^2(0,T;H^1(\Omega))} + \|v^h\|_{L^4(\Omega_T)} + \|w^h\|_{L^2(\Omega_T)} \leq \mathcal{C}.$$

Finally, from the L^2 estimate on v^h and from the Galerkin approximation of equation (2.17) satisfied by γ^h we deduce a uniform $L^2(\Omega_T)$ estimate on γ^h .

Further, we consider the time translates $(T^\tau v^h)(t, \cdot) := v^h(t + \tau, \cdot) - v^h(t, \cdot)$, $(T^\tau w^h)(t, \cdot) := w^h(t + \tau, \cdot) - w^h(t, \cdot)$ and $(T^\tau \gamma^h)(t, \cdot) := \gamma^h(t + \tau, \cdot) - \gamma^h(t, \cdot)$, and integrate the Galerkin approximations of (2.14), (2.16), (2.17), respectively, between t and $t + \tau$ (with $0 < \tau < T$). Taking for the test functions, the corresponding translates $T^\tau v^h$, $T^\tau w^h$ and $T^\tau \gamma^h$, respectively, using the previous estimates, the form of the FitzHugh-Nagumo nonlinearity, the Fubini theorem and the Hölder inequality, we find

$$\int_0^{T-\tau} \int_{\Omega} \left(|v^h(t + \tau, \cdot) - v^h(t, \cdot)|^2 + |w^h(t + \tau, \cdot) - w^h(t, \cdot)|^2 + |\gamma^h(t + \tau, \cdot) - \gamma^h(t, \cdot)|^2 \right) \leq \mathcal{C}\tau.$$

In addition, we get a uniform estimate of space translates of w^h and of γ^h from the uniform L^2 estimate of ∇v^h . Indeed, notice that the equations (2.16),(2.17) are linear and x is merely a parameter of the ODEs with respect to t ; moreover, all the terms in their Galerkin approximations of (2.16),(2.17) take values in the same space V^h . It follows that the equations are satisfied pointwise. Therefore, we inherit analogous ODEs on the space translates $(J^r w^h)(\cdot, x) = w^h(\cdot, x + r) - w^h(\cdot, x)$ and $(J^r \gamma^h)(\cdot, x) = \gamma^h(\cdot, x + r) - \gamma^h(\cdot, x)$ (the term v^h should be replaced by $(J^r v^h)(\cdot, x) = v^h(\cdot, x + r) - v^h(\cdot, x)$):

$$\partial_t(J^r w^h) - H(J^r v^h, J^r w^h) = 0, \quad \partial_t(J^r \gamma^h) - G(J^r w^h, J^r \gamma^h) = 0.$$

This system is satisfied pointwise; therefore we can apply $J^r w^h$ and $J^r \gamma^h$ as the test functions (notice that here we have ODEs, and no boundary condition is needed). Because from the $L^2(0, T; H^1(\Omega))$ estimate of v^h we get $\int_0^T \int_{\Omega_r} |J^r v^h|^2 \leq \mathcal{C}|r|^2$ (here $r \in \mathbb{R}^3$ and $\Omega_r := \{x \in \Omega \mid x_r \in \Omega\}$), the above system of ODEs readily yields the estimate

$$\sup_{r \neq 0, |r| \leq \delta} \int_0^T \int_{\Omega_r} \left(|J^r w^h|^2 + |J^r \gamma^h|^2 \right) \leq \mathcal{C}|r|^2 + T \sup_{r \neq 0, |r| \leq \delta} \int_{\Omega_r} \left(|J^r w_0^h|^2 + |J^r \gamma_0^h|^2 \right), \quad (3.5)$$

whose right-hand side vanishes as $r \rightarrow 0$, uniformly in h .

3.3. Compactness properties and passage to the limit. In order to pass to the limit in the Galerkin formulation for U^h , we proceed as follows. Firstly, the passage to the (weak) limit in the ODEs is straightforward, since they are linear. In order to pass to the limit in the PDEs involved in the system, we notice that we also have the strong convergence of γ^h that implies a strong convergence of $\sigma(x, \gamma^h)$. In the next step, we pass to the (weak) limit in the Galerkin formulation of (2.12). Further, because the limit \mathbf{u} solves the limit equation (2.12), using the Minty-Browder trick (we can also use the technique of Young measures, see [16] and §5) we are able to assert that $\mathbf{D}\mathbf{u}^h$ actually converges to $\mathbf{D}\mathbf{u}$ strongly in $L^2(\Omega_T)$. Finally, the strong compactness of $\mathbf{D}\mathbf{u}^h$ proved above implies strong convergence of $\mathbf{M}_{i,e}(x, \mathbf{D}\mathbf{u}^h)$ to the limit $\mathbf{M}_{i,e}(x, \mathbf{D}\mathbf{u})$. Then it is easy to pass to the limit in the corresponding equations (2.14),(2.13), following e.g. the work [2].

4. Weak-strong comparison and uniqueness of regular solutions. The strategy for proving uniqueness relies on straightforward estimates on the difference $U - \hat{U}$, in particular, the Lipschitz continuity properties (E.2),(E.4) are instrumental. However, in order to obtain exploitable estimates on $U - \hat{U}$, higher regularity solutions should be considered.

4.1. Strong-weak uniqueness argument. Let us first introduce a suitable notion of strong solution, in relation with the uniqueness argument developed in Theorem 4.2.

DEFINITION 4.1. *We will say that $U = (\mathbf{u}, p, u_i, u_e, v, w, \gamma)$ is a bounded-gradient weak solution¹ of problem (2.12)–(2.20) if it is a weak solution in the sense of Definition 2.2 and moreover, one has the following regularity:*

$$\mathbf{D}\mathbf{u} \in L^\infty(\Omega_T)^{3 \times 3}, \nabla u_{i,e} \in L^\infty(\Omega_T)^3.$$

Such regularity of weak solutions is not possible without additional assumptions on the data. We postpone an heuristic discussion of regularity to §4.2.

We have the following weak-strong uniqueness and continuous dependence result.

THEOREM 4.2. *Assume that $U = (\mathbf{u}, p, u_i, u_e, v, w, \gamma)$ is a bounded-gradient weak solution of problem (2.12)–(2.20) in the sense of the above definition, and $\hat{U} = (\hat{\mathbf{u}}, \hat{p}, \hat{u}_i, \hat{u}_e, \hat{v}, \hat{w}, \hat{\gamma})$ is a weak solution in the sense of Definition 2.2 corresponding to the data $\hat{\mathbf{f}}, \widehat{I_{\text{app}}^i}, \widehat{I_{\text{app}}^e}, \hat{v}_0, \hat{w}_0, \hat{\gamma}_0$. Then there exists $\mathcal{K} = \mathcal{K}(\|\mathbf{D}\mathbf{u}\|_\infty, \|\nabla u_{i,e}\|_\infty)$ (depending also on the different constants involved in assumptions (E.1)–(E.6) and on the initial and source data for both U and \hat{U}) such that*

$$\begin{aligned} & \|(v, w, \gamma) - (\hat{v}, \hat{w}, \hat{\gamma})\|_{L^\infty(0,T;L^2(\Omega))^3} \\ & \quad + \|\mathbf{u} - \hat{\mathbf{u}}\|_{L^2(0,T;H^1(\Omega)^3)} + \|p - \hat{p}\|_{L^2(\Omega_T)} + \|(u_i, u_e) - (\hat{u}_i, \hat{u}_e)\|_{L^2(0,T;H^1(\Omega))^2} \\ & \leq \mathcal{K}(\|\mathbf{D}\mathbf{u}\|_\infty, \|\nabla u_{i,e}\|_\infty) \left(\|\mathbf{f} - \hat{\mathbf{f}}\|_{L^2(\Omega_T)^3} + \|(I_{\text{app}}^i, I_{\text{app}}^e) - (\widehat{I_{\text{app}}^i}, \widehat{I_{\text{app}}^e})\|_{L^2(\Omega_T)^2} \right. \\ & \quad \left. + \|(v_0, w_0, \gamma_0) - (\hat{v}_0, \hat{w}_0, \hat{\gamma}_0)\|_{L^2(\Omega)^3} \right). \end{aligned}$$

In particular, if a bounded-gradient weak solution exists for data $\mathbf{f}, I_{\text{app}}^i, I_{\text{app}}^e, v_0, w_0, \gamma_0$, then it is a unique weak solution to the problem.

Proof. Consider equations (2.12) written for \mathbf{u} and $\hat{\mathbf{u}}$; using $\mathbf{u} - \hat{\mathbf{u}}$ as test function, we obtain

$$\begin{aligned} & \left(\sigma(x, \gamma) \mathbf{D}\mathbf{u} - \sigma(x, \hat{\gamma}) \mathbf{D}\hat{\mathbf{u}} \right) : (\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}) \\ & = \left(\sigma(x, \hat{\gamma}) (\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}) \right) : (\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}) + \left((\sigma(x, \gamma) - \sigma(x, \hat{\gamma})) \mathbf{D}\mathbf{u} \right) : (\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}). \end{aligned} \quad (4.1)$$

Then, thanks to Young inequality, the coercivity assumption (E.1), and the Lipschitz continuity (E.2), we obtain for all $t \in (0, T)$

$$\int_{\partial\Omega} \frac{\alpha}{2} |\mathbf{u} - \hat{\mathbf{u}}|^2(t) + \int_{\Omega} \frac{1}{2c} |\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}|^2(t) \leq \mathcal{C} \left(\int_{\Omega} |\mathbf{f} - \hat{\mathbf{f}}|^2(t) + \|\mathbf{D}\mathbf{u}\|_{L^\infty(\Omega_T)} \int_{\Omega} |\gamma - \hat{\gamma}|^2(t) \right). \quad (4.2)$$

Here and in the sequel, \mathcal{C} is a generic constant depending on the data of both systems (for U and \hat{U}) and on the constants in (E.1)–(E.6). Similarly, we use the ODEs (2.17) for both γ and $\hat{\gamma}$ with test function $\gamma - \hat{\gamma}$ and deduce

$$\int_{\Omega} |\gamma - \hat{\gamma}|^2(t) \leq \mathcal{C} \left(\int_{\Omega} |\gamma_0 - \hat{\gamma}_0|^2 + \int_0^t \int_{\Omega} |w - \hat{w}|^2(s) ds \right). \quad (4.3)$$

Finally, we combine equations (2.13)–(2.16) written for U and \hat{U} in the same way as for the proof of (3.4); we use [2, Lemma 2.3] in order to justify the possibility to take test functions

¹A related concept of bounded-gradient solutions can be found in e.g. [20].

$u_{i,e} - \hat{u}_{i,e}$ in (2.13),(2.14), respectively. Rearranging the terms as in (4.1), using in addition the fact that

$$(v\mathcal{B}(v) - \hat{v}\mathcal{B}(\hat{v}))(v - \hat{v}) \geq 0,$$

we find for a.e. $t \in (0, T)$ the following bound:

$$\begin{aligned} & \int_{\Omega} \left(\frac{c_m \chi}{2} |v - \hat{v}|^2(t) + \frac{BC}{A} |w - \hat{w}|^2(t) \right) + \frac{1}{2c} \int_0^t \int_{\Omega} \left(|\nabla u_i - \nabla \hat{u}_i|^2(s) + |\nabla u_e - \nabla \hat{u}_e|^2(s) \right) ds \\ & \leq \mathcal{C} \left(\int_{\Omega} (|v_0 - \hat{v}_0|^2 + |w_0 - \hat{w}_0|^2) + \int_0^t \int_{\Omega} |(I_{\text{app}}^i, I_{\text{app}}^e) - (\widehat{I}_{\text{app}}^i, \widehat{I}_{\text{app}}^e)|^2(s) ds \right. \\ & \quad + \|(\nabla u_i, \nabla u_e)\|_{L^\infty(\Omega_T)^{2 \times 3}} \int_0^t \int_{\Omega} |\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}|^2(s) ds \\ & \quad \left. + \int_0^t \int_{\Omega} (|v - \hat{v}|^2(s) + |w - \hat{w}|^2(s)) ds \right). \quad (4.4) \end{aligned}$$

Then it is easy to combine (4.2)–(4.4) and obtain a Gronwall-type inequality that bounds $U - \hat{U}$ in the way stated in the theorem. Indeed, setting

$$\begin{aligned} U(t) &:= \|\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}\|_{L^2(\Omega)^{3 \times 3}}^2, \quad \Gamma(t) := \|\gamma - \hat{\gamma}\|_{L^2(\Omega)}(t), \\ R(t) &= \frac{c_m \chi}{2} \|v - \hat{v}\|_{L^2(\Omega)}^2(t) + \frac{BC}{A} \|w - \hat{w}\|_{L^2(\Omega)}^2(t), \\ D(t) &:= \|\mathbf{f} - \hat{\mathbf{f}}\|_{L^2(\Omega)^3}^2(t) + \|(I_{\text{app}}^i, I_{\text{app}}^e) - (\widehat{I}_{\text{app}}^i, \widehat{I}_{\text{app}}^e)\|_{L^2(\Omega)^2}^2(t) \\ & \quad + \|(v_0, w_0, \gamma_0) - (\hat{v}_0, \hat{w}_0, \hat{\gamma}_0)\|_{L^2(\Omega)^3}^2, \end{aligned}$$

substituting (4.3) into the right-hand side of (4.2) then substituting the obtained inequality into the right-hand side of (4.4), dropping nonnegative terms we get

$$R(t) \leq \mathcal{K}(D(t) + \int_0^t R(s) ds), \quad R(0) \leq \mathcal{C}(\|\hat{v}_0 - v_0\|_{L^2(\Omega)}^2 + \|\hat{w}_0 - w_0\|_{L^2(\Omega)}^2)$$

with \mathcal{K} as in the statement of the theorem. This permits to bound $\|(v, w) - (\hat{v}, \hat{w})\|_{L^2(\Omega)^2}(t)$ thanks to the Gronwall lemma. Then with the help of (4.3) we also bound $\|\gamma - \hat{\gamma}\|_{L^2(\Omega)}(t)$. Next, with (4.2) we bound $\|\mathbf{D}\mathbf{u} - \mathbf{D}\hat{\mathbf{u}}\|_{L^2(\Omega)^{3 \times 3}}(t)$ and $\|\mathbf{u} - \hat{\mathbf{u}}\|_{L^2(\partial\Omega)^3}(t)$, which implies the $H^1(\Omega)^3$ bound on $(\mathbf{u} - \hat{\mathbf{u}})(t)$, thanks to Poincaré inequality. Now, the bound on $\|p - \hat{p}\|_{L^2(\Omega)}(t)$ follows from (2.12) written for $\mathbf{u}(t)$ and $\hat{\mathbf{u}}(t)$, using a test function $\mathbf{v} \in H^1(\Omega)$ such that $\nabla \cdot \mathbf{v} = p(t) - \hat{p}(t)$. Such a function can be constructed for each t by taking $\mathbf{v} = \nabla \theta$ using the auxiliary Dirichlet problem $-\Delta \theta = (p - \hat{p})$ in $\tilde{\Omega}$, $\theta = 0$ on $\partial\tilde{\Omega}$, where $\tilde{\Omega}$ is a regular domain containing Ω ; we have $\theta \in H^2(\tilde{\Omega})$ by the classical elliptic regularity results. Finally, with (4.4) we also achieve the desired bounds on $\|\nabla u_{i,e} - \nabla \hat{u}_{i,e}\|_{L^2(\Omega)}(t)$. \square

4.2. On the regularity of weak solutions. While we cannot prove regularity of weak solutions to problem (2.12)–(2.20) even for regular data, let us give some arguments that might help establishing the boundedness of $\mathbf{D}\mathbf{u}$ and $\nabla u_{i,e}$, in the case of sufficiently regular initial data and for matrices $\sigma(x, \gamma)$ with weak enough anisotropy.

Indeed, since $v \in L^2(0, T; H^1(\Omega))$, from the two ODEs on w and γ and an H^1 regularity assumption on w_0, γ_0 we deduce that $w, \gamma \in H^1(0, T; H^1(\Omega))$. Now we can look at equation (2.12) and attempt for establishing a Hölder $C^{1,\alpha}$ regularity or a Calderón-Zygmund regularity. Should this be the case, taking $\mathbf{f}(t) \in L^r(\Omega)$ with $r > 3$, using Morrey's embedding inequality we would find a $C^{1,\alpha}(\Omega)$ bound on $\mathbf{D}\mathbf{u}$. Notice that it seems difficult to justify the continuity of $x \mapsto \sigma(x, \gamma(t, x))$ before we justify the continuity of $x \mapsto v(t, x)$; therefore in the subsequent analysis we assume that the coefficients of the elliptic equation (2.12) with incompressibility constraint satisfy only an L^∞ bound, without additional regularity required for the classical Calderón-Zygmund approach. We have not found in the literature any results that apply in this

situation². But, if one replaces the incompressible elasticity model by a nearly incompressible one, then Calderón-Zygmund regularity results are available (see, e.g., [22]) either for **VMO** regularity of coefficients (*VMO* is the space of functions of vanishing mean oscillation, see, e.g., [17]), or under the Cordès condition which strongly restricts the spread of eigenvalues of a merely L^∞ diffusion matrix (see [22]). Namely, the Cordès condition (see [38, 22]) requires that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the 3×3 positive definite diffusion matrix satisfy

$$2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) < (\lambda_1 + \lambda_2 + \lambda_3)^2. \quad (\text{4.5})$$

The **VMO** regularity of diffusion coefficients is almost as difficult to obtain as the $C^{1,\alpha}$ regularity: it would require, e.g., an additional $H^{3/2}(\Omega)$ regularity of $\gamma(t, \cdot)$ which in turn would require the same regularity of $v(t, \cdot)$. On the contrary, the Cordès assumption is an *a priori* hypothesis that is realistic for eigenvalues of the family of matrices $(\sigma(x, \gamma))_{x \in \Omega, \gamma \in \mathbb{R}}$.

In order to continue, let us assume that one can justify an $L^\infty(\Omega_T)$ bound on v :

$$v \in L^\infty(\Omega_T) \quad (\text{H.1})$$

in this case, for bounded initial data we also get $L^\infty(\Omega_T)$ bound on w, γ and on $\partial_t w, \partial_t \gamma$. Then under an *ad hoc* assumption on $\partial_t \mathbf{f}$ it is easy to obtain the $L^\infty(\Omega_T)$ bound on $\partial_t \mathbf{D}\mathbf{u}$. With this bound in hand, using the technique of [5] we would deduce $L^2(\Omega_T)$ regularity for $\partial_t v$ valid under assumption (H.1) and for an H^1 datum v_0 . Notice that the estimate of $\|\partial_t v\|_{L^2(\Omega_T)}$ is obtained in [5] for the solution constructed at the limit of a parabolic regularization (cf. §3 above); but in fact the estimate is valid for any solution, because the uniqueness of a weak solution for (2.3) holds true (one can use the same arguments as in [2, Th. 2.4]). Then we have to analyze the regularity of u_i, u_e : e.g., for u_i we have for a.e. $t \in (0, T)$,

$$-\nabla \cdot (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}) \nabla u_i) = s^i(t, x) := -c_m \chi \partial_t v - \chi I_{\text{ion}}(v, w) + I_{\text{app}}^i(t, x), \quad (\text{4.6})$$

in Ω with the appropriate boundary conditions. It should be stressed that under assumption (H.1), we have $s^i \in L^2(\Omega)$ for a.e. t . Further, Calderon-Zygmund regularity for (4.6) holds true. Indeed, while the Cordès condition is not realistic for the family of matrices $\mathbf{M}_{i,e}(x, \mathbf{D}\mathbf{u}(t, x))$, we can use the already justified Hölder continuity of $\mathbf{D}\mathbf{u}(t, \cdot)$; in this case, we need to require the regularity of $\mathbf{M}_{i,e}$ in x . Unfortunately, this does not ensure the L^∞ bound on $\nabla u_{i,e}$, but only an $L^2(0, T; H^1(\Omega))$ bound. In order to prove that U is a bounded-gradient solution, one has to require some time regularity of the source terms and to assume in addition that

$$\partial_t v \in L^\infty(\Omega_T). \quad (\text{H.2})$$

In conclusion, existence on bounded-gradient solutions can be analyzed rigorously if we replace the incompressible elasticity model by the more accurate nearly incompressible one; if we impose the Cordès restriction (4.5) on the spread of eigenvalues of the matrices in assumption (E.1), a space regularity of matrices in assumption (E.3) and a time regularity of source terms; and if we assume or justify the bounds (H.1), (H.2). Let us stress that using the same analytical and numerical techniques as those we used in this paper, one can treat the coupled electromechanical system using a nearly incompressible elasticity model. We refer to [32] for the precise formulation and numerical results on such model.

5. Numerical approximation. Here, we present the finite element method for approximation of the problem studied in the previous section, the associated numerical results and a critical analysis of validity of the approximations made in §2.4.

² In presence of the incompressibility constraint, Calderón-Zygmund regularity results seem to be known only in the Laplacian case (see in particular [12] for 3D Stokes equations), and generalization of these results would be a huge work on its own.

5.1. A finite element method. Let \mathcal{T}_h be a regular partition of Ω into tetrahedra K with boundary ∂K and diameter h_K . We define the mesh parameter $h = \max_{K \in \mathcal{T}_h} \{h_K\}$ and the associated finite element spaces V^h (respectively, \mathbf{V}^h), for the approximation of pressure, electrical potentials and ionic variables (respectively, of displacements). In order to satisfy the discrete Ladyzhenskaya-Babuška-Brezzi stability condition (3.1), piecewise quadratic finite elements are used to approximate the displacements field, while for pressure, electrical potentials and ionic variables, we use piecewise linear elements. That is, the involved spaces are defined as

$$\begin{aligned} V^h &= \{s \in C^0(\bar{\Omega}) : v|_K \in \mathbb{P}_1(K) \text{ for all } K \in \mathcal{T}_h\}, \\ \mathbf{V}^h &= \{\mathbf{v} \in \mathbf{C}^0(\bar{\Omega}) : \mathbf{v}|_K \in \mathbb{P}_2(K)^3 \text{ for all } K \in \mathcal{T}_h\}. \end{aligned}$$

In order to lighten the notation, in this section we put $\chi = c_m = 1$ (the general case is completely analogous). The semidiscrete Galerkin finite element formulation used in §3.1 then reads: For $t > 0$, find $\mathbf{u}^h \in \mathbf{V}^h$, $u_i^h(t), u_e^h(t), v^h(t), w^h(t), \gamma^h(t), p^h(t) \in V^h$ such that (with the standard finite element notation for L^2 scalar products) one has

$$\begin{aligned} (\boldsymbol{\sigma}(x, \gamma^h(t)) \mathbf{D}\mathbf{u}^h(t), \mathbf{D}\psi^h)_\Omega + (p^h(t), \nabla \cdot \psi^h)_\Omega &= (\mathbf{f}^h, \psi^h)_\Omega + (\alpha \mathbf{u}^h(t), \psi^h)_{\partial\Omega} \\ (\mathbf{u}^h(t), \nabla \phi^h)_\Omega &= 0 \\ \frac{d}{dt}(v^h(t), \phi^h)_\Omega + (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}^h(t)) \nabla u_i^h(t), \nabla \phi^h)_\Omega &= (I_{\text{app}}^i - I_{\text{ion}}(v^h(t), w^h(t)), \phi^h)_\Omega \\ \frac{d}{dt}(v^h(t), \phi^h)_\Omega - (\mathbf{M}_e(x, \mathbf{D}\mathbf{u}^h(t)) \nabla u_e^h(t), \nabla \phi^h)_\Omega &= (I_{\text{app}}^e - I_{\text{ion}}(v^h(t), w^h(t)), \phi^h)_\Omega \quad (5.1) \\ \frac{d}{dt}(w^h(t), \phi^h)_\Omega &= (H(v^h(t), w^h(t)), \phi^h)_\Omega, \\ \frac{d}{dt}(\gamma^h(t), \phi^h)_\Omega &= (G(v^h(t), \gamma^h(t)), \phi^h)_\Omega, \end{aligned}$$

for all $\psi^h \in \mathbf{V}^h$ and all $\phi^h \in V^h$; one also sets $v^h(0) = \mathbb{P}_{V^h}(v_0)$ (analogous initialization is used for w^h and γ^h). A classical backward Euler integration method is employed for the time discretization of (5.1) with time step $\delta t = T/N$, moreover, we linearize and decouple the resulting time-implicit scheme by employing $\gamma^{h,n-1}$ (respectively, $\mathbf{D}\mathbf{u}^{h,n-1}$) in the equation on $\mathbf{u}^{h,n}$ (respectively, in the equations on $u_{i,e}^{h,n}$) and by linearizing I_{ion} with the help of the new function

$$I_{\text{ion}}^{\text{lin}} : (v, \tilde{v}, w) \mapsto v\mathcal{B}(\tilde{v}) - A_1 v + Cw \quad \text{where } \mathcal{B} : z \mapsto \frac{j(z)}{z} + A_1 \geq 0.$$

This results in the following fully discrete method: find $u_i^h(t), u_e^h(t), v^h(t), w^h(t), \gamma^h(t), p^h(t) \in V^h$ and $\mathbf{u}^h \in \mathbf{V}^h$ such that

$$\begin{aligned} \mathbf{u}^h(t, x) &= \sum_{n=1}^N \mathbf{u}^{h,n}(x) \mathbb{1}_{((n-1)\delta t, n\delta t]}(t), \\ (u_i^h, u_e^h, v^h, w^h, \gamma^h, p^h)(t, x) &= \sum_{n=1}^N (u_i^{h,n}, u_e^{h,n}, v^{h,n}, w^{h,n}, \gamma^{h,n}, p^{h,n})(x) \mathbb{1}_{((n-1)\delta t, n\delta t]}(t), \end{aligned}$$

satisfy the algebraic system of equations

$$\begin{aligned} (\boldsymbol{\sigma}(x, \gamma^{h,n-1}) \mathbf{D}\mathbf{u}^{h,n}, \mathbf{D}\psi^h)_\Omega + (p^{h,n}, \nabla \cdot \psi^h)_\Omega &= (\mathbf{f}^{h,n}, \psi^h)_\Omega + (\alpha \mathbf{u}^{h,n}, \psi^h)_{\partial\Omega} \\ (\mathbf{u}^{h,n}, \nabla \phi^h)_\Omega &= 0 \\ \left(\frac{v^{h,n} - v^{h,n-1}}{\delta t}, \phi^h\right)_\Omega + (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}^{h,n-1}) \nabla u_i^{h,n}, \nabla \phi^h)_\Omega &= (I_{\text{app}}^i - I_{\text{ion}}^{\text{lin}}(v^{h,n}, v^{h,n-1}, w^{h,n}), \phi^h)_\Omega \\ \left(\frac{v^{h,n} - v^{h,n-1}}{\delta t}, \phi^h\right)_\Omega - (\mathbf{M}_e(x, \mathbf{D}\mathbf{u}^{h,n-1}) \nabla u_e^{h,n}, \nabla \phi^h)_\Omega &= (I_{\text{app}}^e - I_{\text{ion}}^{\text{lin}}(v^{h,n}, v^{h,n-1}, w^{h,n}), \phi^h)_\Omega \end{aligned}$$

$$\begin{aligned} \left(\frac{w^{h,n} - w^{h,n-1}}{\delta t}, \phi^h\right)_\Omega &= (H(v^{h,n}, w^{h,n}), \phi^h)_\Omega, \\ \left(\frac{\gamma^{h,n} - \gamma^{h,n-1}}{\delta t}, \phi^h\right)_\Omega &= (G(v^{h,n}, \gamma^{h,n}), \phi^h)_\Omega, \end{aligned}$$

for all $\psi^h \in \mathcal{V}^h$, $\phi^h \in V^h$ and for all $n \in \{1, \dots, N\}$; the initial condition takes the form

$$(v^{h,0}, w^{h,0}, \gamma^{h,0}) = (\mathbb{P}_{V^h}(v_0), \mathbb{P}_{V^h}(w_0), \mathbb{P}_{V^h}(\gamma_0)).$$

Here $I_{\text{app}}^{i,h}(\cdot)$, $I_{\text{app}}^{e,h}(\cdot)$ are time averages over $[(n-1)\delta t, n\delta t]$ of I_{app}^i , I_{app}^e , respectively.

5.2. A glimpse into the convergence proof. Consider solutions to the fully discrete system, and let us indicate some milestones of its convergence analysis. As in the continuous case, using the same kind of test functions, using the convexity inequality $(a^n - a^{n-1})a^n \geq \frac{(a^n)^2}{2} - \frac{(a^{n-1})^2}{2}$ to treat the time-discretized terms, we obtain uniform in h and δt estimates on the discrete solutions:

$$\|\mathbf{u}^h\|_{L^2(0,T;H^1(\Omega)^3)} + \|p^h\|_{L^2(\Omega_T)} + \|u_{i,e}^h\|_{L^2(0,T;H^1(\Omega))} + \|(v^h, w^h, \gamma^h)\|_{L^2(\Omega_T)} \leq \mathcal{C} \quad (5.2)$$

and also the following uniform in h estimate related to linearization of the nonlinearity I_{ion}

$$\sum_{n=1}^N \delta t \int_{\Omega} |v^{h,n}|^2 \mathcal{B}(v^{h,n-1}) \leq \mathcal{C} \quad (5.3)$$

(the latter estimate replaces, in a weaker sense, the $L^4(\Omega_T)$ estimate of v^h). The details of the estimates are very close to those given in [2] for the linearized implicit scheme. Estimate (5.2) implies in particular the invertibility of the matrix of the discrete system that is solved on each time step.

Now, similarly to §3.2 we get space and time translation estimates on $(w^h)_h$ and on $(\gamma^h)_h$. For the time estimates, we introduce, e.g., \bar{w}^h the piecewise affine in t function in $W^{1,\infty}([0,T];V^h)$ interpolating the states $(w^{h,n})_{n=0..N} \subset V^h$ at the points $(n\delta t)_{n=0..N}$. Then we have $\bar{w}_t^h = H(v^h, w^h)$, and due to the linearity of H , with the notation introduced in §3.2 we also have

$$(T^\tau \bar{w}^h)_t = H(T^\tau v^h, T^\tau w^h).$$

We can use test function $(T^\tau w^h)(t, \cdot)$ in this equation to obtain time translation estimates on $(\bar{w}^h)_h$ using Fubini theorem and the bounds (5.2). Further, the estimates of space translates $(J^r w^h)_h$ are obtained in the same way as (3.5), hence analogous estimate on $(J^r \bar{w}^h)_h$ follows because for $t \in ((n-1)\delta t, n\delta t]$, $\bar{w}^h(t, \cdot)$ is a convex combination of $w^{h,n-1}(\cdot)$ and $w^{h,n}(\cdot)$. By the Fréchet-Kolmogorov theorem we deduce strong $L^2(\Omega_T)$ compactness of $(\bar{w}^h)_h$. Further, it is easily seen from the definition of \bar{w}^h , from the equation

$$\frac{w^{h,n} - w^{h,n-1}}{\delta t} = H(v^{h,n}, w^{h,n})$$

and estimates (5.2) that

$$\|\bar{w}^h - w^h\|_{L^2(\Omega_T)}^2 \leq \sum_{n=1}^N \delta t \|w^{h,n-1} - w^{h,n}\|_{L^2(\Omega)}^2 \leq \mathcal{C} \delta t \rightarrow 0 \text{ as } \delta t \rightarrow 0.$$

Finally, we conclude that $(w^h)_h$ is strongly compact in $L^2(\Omega_T)$. The same argument applies to $(\gamma^h)_h$. Unfortunately, the lack of $L^4(\Omega_T)$ estimate on $(v^h)_h$ precludes us from getting the analogous time translation estimate and compactness property on $(v^h)_h$, but we circumvent this difficulty using the Young measures' representation of weakly convergent sequences, in the spirit of [16].

Indeed, extracting convergent subsequences corresponding to the above arguments, firstly we pass to the limit in the linear ODEs and in the elliptic equation on \mathbf{u}^h ; as in § 3.3, we get strong convergence of $\mathbf{D}\mathbf{u}^h$ to $\mathbf{D}\mathbf{u}$ from the Minty-Browder argument. Now, we pass to the limit in the equations on $u_{i,e}^h$; the delicate point is the passage to the limit in the nonlinearity $I_{\text{ion}}^{\text{lin}}(v^h, \tilde{v}^h, w^h)$, where $\tilde{v}^h(t, x) = v^h(t - \delta t, x)$ with the convention that $v^h(t, x) = v^{h,0}(x)$ for $t \leq 0$. More precisely, we have to pass to the limit in the term $v^h \mathcal{B}(\tilde{v}^h)$. Here we exploit estimate (5.3) to get an equi-integrability estimate on this term. As in [2], using the Sobolev embedding of $L^2(0, T; H^1(\Omega))$ into $L^2(0, T; L^6(\Omega))$ and the space interpolation with $L^\infty(0, T; L^2(\Omega))$, we find a uniform $L^{10/3}(\Omega_T)$ bound on v_h . Such a bound yields equi-integrability of $|\tilde{v}^h|^2$ on Ω_T , thus from the weighted Young inequality

$$|v\mathcal{B}(\tilde{v})| \leq \delta |v|^2 \mathcal{B}(\tilde{v}) + \frac{1}{\delta} \mathcal{B}(\tilde{v}),$$

using the growth assumption on $j(\cdot)$ that implies that $\mathcal{B}(\tilde{v}) \leq \mathcal{C}(1 + |\tilde{v}|^2)$, we derive a uniform $L^1(\Omega_T)$ bound and, moreover, the equi-integrability on Ω_T for the family $(v^h \mathcal{B}(\tilde{v}^h))_h$. Therefore up to extraction of a further subsequence, $v^h \mathcal{B}(\tilde{v}^h)$ converges weakly in $L^1(\Omega_T)$ to a limit that we denote by Ψ . Further, multiplying the discrete equation for $v^{h,n}$ by $\frac{1}{2}(v^{h,n} + v^{h,n-1})$ and summing in n from the previously obtained estimates one deduces the uniform in h estimate

$$\left| \sum_{n=1}^N \delta t \int_{\Omega} v^{h,n} v^{h,n-1} \mathcal{B}(v^{h,n-1}) \right| \leq \mathcal{C}. \quad (5.4)$$

Finally, multiplying the same equation for $v^{h,n}$ by $(v^{h,n} - v^{h,n-1})$ and summing in n , using in particular (5.4), we get

$$\sum_{n=1}^N \delta t \int_{\Omega} |v^{h,n} - v^{h,n-1}|^2 \rightarrow 0 \quad \text{as } \delta t \rightarrow 0.$$

Therefore, $\tilde{v}^h - v^h$ converges to zero strongly in $L^2(\Omega_T)$, in particular, the weak $L^2(\Omega_T)$ limit v of v^h is also the weak $L^2(\Omega_T)$ limit of \tilde{v}^h . Now, in order to prove strong convergence of $(v^h)_h$ and to pass to the limit in the nonlinear ionic current term $v^h \mathcal{B}(\tilde{v}^h)$ (we have to prove in particular that $\Psi = v\mathcal{B}(v)$), we introduce the Young measure $(\nu_{(t,x)})_{(t,x) \in \Omega_T}$ which is the limit of the selected subsequence of $(v^h)_h$ (not relabeled):

$$\begin{aligned} v(t, x) &= \int_{\mathbb{R}} \lambda d\nu_{(t,x)}(\lambda) \quad \text{with } \int_{\mathbb{R}} d\nu_{(t,x)}(\lambda) = 1 \text{ for a.e. } (t, x) \in \Omega_T, \text{ moreover,} \\ \text{for all } F \in \mathcal{C}^0(\mathbb{R}; \mathbb{R}) \text{ such that } (F(v^h))_h \text{ is weakly convergent in } L^1(\Omega_T), \quad (5.5) \\ F(v^h) &\rightharpoonup \int_{\mathbb{R}} F(\lambda) d\nu_{(\cdot, \cdot)}(\lambda) \text{ in } L^1(\Omega_T). \end{aligned}$$

Then the convergence proof relies on the following observations.

LEMMA 5.1. *The Young measure $(\nu_{(t,x)})_{(t,x) \in \Omega_T}$ has the following properties:*

- (i) $v(t, x) = \int_{\mathbb{R}} \lambda d\nu_{(t,x)}(\lambda)$ and $\Psi(t, x) = \int_{\mathbb{R}} \lambda \mathcal{B}(\lambda) d\nu_{(t,x)}(\lambda)$;
- (ii) $\int \int_{\Omega_T} \int_{\mathbb{R}} \lambda^2 \mathcal{B}(\lambda) d\nu_{(t,x)}(\lambda) dx dt < \infty$ and $v \in L^4(\Omega_T)$, $\Psi \in L^{4/3}(\Omega_T)$;
- (iii) for a.e. $(t, x) \in \Omega_T$, $\int_{\mathbb{R}} \int_{\mathbb{R}} (\lambda - \mu)(\lambda \mathcal{B}(\lambda) - \mu \mathcal{B}(\mu)) d\nu_{(t,x)}(\lambda) d\nu_{(t,x)}(\mu) \leq 0$;
- (iv) for a.e. $(t, x) \in \Omega_T$, $\nu_{(t,x)}(\lambda) = \delta(\lambda - v(t, x))$;
in particular, $v^h \rightarrow v$ and $\tilde{v}^h \rightarrow v$ a.e. on Ω_T , and we have $\Psi = v\mathcal{B}(v)$ a.e. on Ω_T .

Proof. For (i) and (ii) we use the estimates established above, while for (iii), we use the equations satisfied by v^h and by v in a way similar to the Minty trick. The last point follows by classical properties of Young measures.

(i) The first claim of (i) is a part of the definition of the Young measure. To prove the second point, we first observe that $\Delta_1^h := v^h \mathcal{B}(\tilde{v}^h) - v^h \mathcal{B}(v^h)$ tends to zero in $L^1(\Omega_T)$ as $h \rightarrow 0$. Indeed, recall that v^h, \tilde{v}^h are bounded in $L^{10/3}(\Omega_T)$ due to interpolation and embedding results; therefore Δ_1^h is bounded in $L^{10/9}(\Omega_T)$, and thus it is equi-integrable. Moreover, extracting a further subsequence we can assume that $v^h - \tilde{v}^h \rightarrow 0$ a.e. on Ω_T , so that $\Delta_1^h \rightarrow 0$ a.e. on Ω_T . Then Δ_1^h vanishes in $L^1(\Omega_T)$ due to the Vitali theorem. Now, we have $v^h \mathcal{B}(\tilde{v}^h) \rightharpoonup \Psi$ in $L^1(\Omega_T)$. Consequently, we have as well $v^h \mathcal{B}(v^h) \rightharpoonup \Psi$ in $L^1(\Omega_T)$ as $h \rightarrow 0$, and we can apply the representation formula of (5.5) to Ψ .

(ii) Consider a family of bounded continuous functions $(\mathcal{B}_n)_n$ on \mathbb{R} that increase towards the limit \mathcal{B} pointwise on \mathbb{R} , as $n \rightarrow \infty$. Then $F_n : \lambda \mapsto \lambda^2 \mathcal{B}_n(\lambda)$ can be used in (5.5) because we know that $(v^h)_h$ is an equi-integrable sequence on Ω_T and b_n is bounded for every fixed n . In addition, as in the point (i) we readily see that $\Delta_2^h := (v^h)^2 \mathcal{B}_n(v^h) - (v^h)^2 \mathcal{B}_n(\tilde{v}^h)$ vanishes in $L^1(\Omega_T)$ as $h \rightarrow 0$, up to a subsequence. Therefore we find, with the limits taken in the weak $L^1(\Omega_T)$ sense, the chain of equalities

$$\int_{\mathbb{R}} F_n(\lambda) d\nu_{(\cdot, \cdot)}(\lambda) = \lim_{h \rightarrow 0} F_n(v^h) = \lim_{h \rightarrow 0} (v^h)^2 \mathcal{B}_n(\tilde{v}^h).$$

In particular,

$$\iint_{\Omega_T} \int_{\mathbb{R}} F_n(\lambda) d\nu_{(t,x)}(\lambda) dt dx \leq \lim_{h \rightarrow 0} \iint_{\Omega_T} (v^h)^2 \mathcal{B}_n(\tilde{v}^h) \leq \iint_{\Omega_T} (v^h)^2 \mathcal{B}(\tilde{v}^h) \leq \mathcal{C}$$

due to estimate (5.3). Notice that \mathcal{C} does not depend on n , therefore using the monotonicity of $(\mathcal{B}_n)_n$ we deduce the first claim of (ii) from the monotone convergence theorem. Then, the lower growth bound on \mathcal{B} yields $\int_{\mathbb{R}} \lambda^4 d\nu_{(\cdot, \cdot)}(\lambda) \in L^1(\Omega_T)$, whence by the Jensen inequality for the convex function $\lambda \mapsto \lambda^4$ and the probability measures $\nu_{(t,x)}$ we deduce that $v = \int_{\mathbb{R}} \lambda d\nu_{(\cdot, \cdot)}(\lambda) \in L^4(\Omega_T)$. Using the upper growth bound on \mathcal{B} we find that $\Psi \in L^{3/4}(\Omega_T)$.

(iii) Firstly, we prove the inequality

$$\limsup_{h \rightarrow 0} \iint_{\Omega_T} (T-t)(v^h)^2 \mathcal{B}(\tilde{v}^h) \leq \iint_{\Omega_T} (T-t)v\Psi, \quad (5.6)$$

for sufficiently small T (then, after having proved (iv), we can bootstrap the argument and achieve arbitrarily large time horizon T).

On the one hand, we proceed in a way analogous to the one followed to obtain (3.4) (recall that now we have $\chi = c_m = 1$). Namely, we combine the equations on $u_{i,e}^{h,n}$ (with test function $u_{i,e}^{h,n}$) and on $w^{n,h}$ (with test function $(C/A)w^{h,n}$), use the convexity inequality in the place of chain rule in time, and we end up with

$$\begin{aligned} \int_0^s \int_{\Omega} \left((v^h)^2 \mathcal{B}(\tilde{v}^h) - A_1 (v^h)^2 + \frac{BC}{A} (w^h)^2 + (\mathbf{M}_e(x, \mathbf{D}u^h) \nabla u_e^h) \cdot \nabla u_e^h + (\mathbf{M}_i(x, \mathbf{D}u^h) \nabla u_i^h) \cdot \nabla u_i^h \right) \\ + \int_{\Omega} \frac{1}{2} (v^h)^2 (s, \cdot) \leq \int_{\Omega} \frac{1}{2} (v_0^h)^2 + \int_0^s \int_{\Omega} I_{\text{app}}^{i,h} u_i^h - I_{\text{app}}^{e,h} u_e^h, \end{aligned}$$

for every $s < T$. Then we integrate in $s \in [0, T]$ and we assume $2A_1 T \leq 1$; we use the Fubini theorem to simplify the double time integral, which brings the factor $(T-t)$ under the integrals. Using the strong convergence of $v^{h,0}$, w^h and $\mathbf{D}u^h$ in $L^2(\Omega)$, $L^2(\Omega_T)$ and $L^2(\Omega_T)^3$, respectively, using the weak $L^2(\Omega_T)$ convergence of v^h , of $\sqrt{T-t}v^h$, of $\sqrt{T-t}(\mathbf{M}_{i,e}(x, \mathbf{D}u^h))^{1/2} \nabla u_{i,e}^h$ and the lower semi-continuity of the L^2 norm with respect to weak convergence, at the limit $h \rightarrow 0$ (for the selected subsequence) we find

$$\limsup_{h \rightarrow 0} \int_0^T \int_{\Omega} (T-t) \left((v^h)^2 \mathcal{B}(\tilde{v}^h) + \int_{\Omega} \left(\frac{1}{2} - (T-t)A_1 \right) v^2 + \frac{BC}{A} w^2 \right)$$

$$\begin{aligned}
& + (T-t) \left(\mathbf{M}_e(x, \mathbf{D}\mathbf{u}) \nabla u_e \cdot \nabla u_e + (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}) \nabla u_i) \cdot \nabla u_i \right) \\
& \leq \int_{\Omega} \frac{T}{2} (v_0)^2 + \int_0^T \int_{\Omega} (T-t) (I_{\text{app}}^i u_i - I_{\text{app}}^e u_e). \quad (5.7)
\end{aligned}$$

On the other hand, the following limit equations, as $h \rightarrow 0$, are easily obtained as $h \rightarrow 0$:

$$\partial_t v - \nabla \cdot (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}) \nabla u_i) + \Psi - A_1 v + Cw = I_{\text{app}}^i \quad \text{in } \Omega_T, \quad (5.8)$$

$$\partial_t v + \nabla \cdot (\mathbf{M}_e(x, \mathbf{D}\mathbf{u}) \nabla u_e) + \Psi - A_1 v + Cw = I_{\text{app}}^e \quad \text{in } \Omega_T, \quad (5.9)$$

$$v = u_i - u_e \quad \text{in } \Omega_T, \quad (5.10)$$

$$\partial_t w - H(v, w) = 0 \quad \text{in } \Omega_T. \quad (5.11)$$

In order to combine these equations in the same way as what we have done at the discrete level, we recall that $v \in L^4(\Omega_T)$ and $\Psi \in (L^4(\Omega_T))'$, so that [2, Lemma 2.3] (cf. Remark 2.3 after Definition 2.2) can be used in order to give sense to multiplication of (5.8), (5.9) by u_i , u_e , respectively. We obtain, integrating in $t \in [0, s]$ then in $s \in [0, T]$ the following relation:

$$\begin{aligned}
& \int_0^T \int_{\Omega} (T-t) v \Psi + \int_0^T \int_{\Omega} \left(\left(\frac{1}{2} - (T-t) A_1 \right) v^2 + \frac{BC}{A} w^2 \right) \\
& \quad + (T-t) \left(\mathbf{M}_e(x, \mathbf{D}\mathbf{u}) \nabla u_e \cdot \nabla u_e + (\mathbf{M}_i(x, \mathbf{D}\mathbf{u}) \nabla u_i) \cdot \nabla u_i \right) \\
& = \int_{\Omega} \frac{T}{2} (v_0)^2 + \int_0^T \int_{\Omega} (T-t) (I_{\text{app}}^i u_i - I_{\text{app}}^e u_e). \quad (5.12)
\end{aligned}$$

By comparison of (5.12) and (5.7), we deduce the claim (5.6).

Now, using as in the proof of (ii) an increasing family of functions $(\mathcal{B}_n)_n$ approximating \mathcal{B} , we find for all n ,

$$\begin{aligned}
& \int \int_{\Omega_T} (T-t) \int_{\mathbb{R}} \lambda^2 \mathcal{B}_n(\lambda) d\nu_{(t,x)}(\lambda) dt dx = \lim_{h \rightarrow 0} \int \int_{\Omega_T} (T-t) (v^h)^2 \mathcal{B}_n(\tilde{v}^h) \\
& \leq \limsup_{h \rightarrow 0} \int \int_{\Omega_T} (T-t) (v^h)^2 \mathcal{B}(\tilde{v}^h) \leq \int \int_{\Omega_T} (T-t) v \Psi.
\end{aligned}$$

Then with the monotone convergence theorem and representation of Ψ and v proved in (i), we find

$$\begin{aligned}
& \int \int_{\Omega_T} (T-t) \int_{\mathbb{R}} \lambda^2 \mathcal{B}_n(\lambda) d\nu_{(t,x)}(\lambda) dt dx \\
& \leq \int \int_{\Omega_T} (T-t) \left(\int_{\mathbb{R}} \lambda d\nu_{(t,x)}(\lambda) \right) \left(\int_{\mathbb{R}} \lambda \mathcal{B}(\lambda) d\nu_{(t,x)}(\lambda) \right) dt dx.
\end{aligned}$$

Then, rearranging the terms as in [16], one finds

$$\int_0^T \int_{\Omega} (T-t) \int_{\mathbb{R}} \int_{\mathbb{R}} (\lambda - \mu) (\lambda \mathcal{B}(\lambda) - \mu \mathcal{B}(\mu)) d\nu_{(t,x)}(\lambda) d\nu_{(t,x)}(\mu) dt dx \leq 0.$$

Now the claim of (iii) follows by monotonicity of the map $\lambda \mapsto \lambda \mathcal{B}(\lambda)$.

(iv) Starting from the “div-curl” relation proved in (iii), one deduces the claim (iv) from the general properties of Young measures (see [16]). \square

Property (iv) of the previous lemma, along with equations (5.8), (5.9) concludes the passage to the limit in the scheme. It remains to observe that, if the data of the problem fall within the regularity framework of § 4.2, the extraction of a subsequence is bypassed using the classical argument (the unique solution is the unique accumulation point), and in this case the finite element method converges to the unique solution of the system. *A priori* error estimates can also be obtained in this case, with arguments similar to those of § 4.

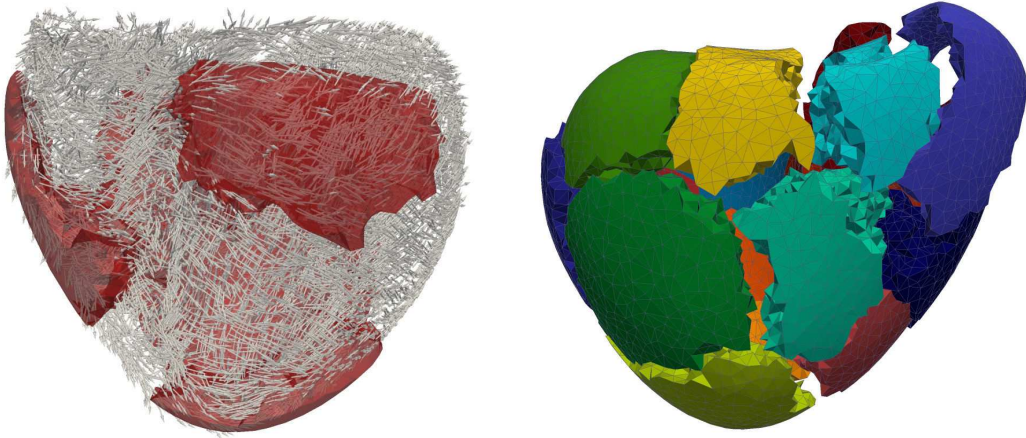


FIG. 5.1. *Fibers distribution with segmented biventricular geometry (left) and partition of the mesh into sixteen non-overlapping subdomains, each color representing a different subdomain (right).*

5.3. Numerical tests. We now illustrate the performance of the coupled finite element method. The linear systems arising from the full discretization of the problem are solved with GMRES (with a tolerance of $\hat{\epsilon}_{\text{tol}} = 10^{-7}$). The average overall CPU time spent per time step is 3.5 seconds. All the simulations in this section have been performed on four cluster nodes with two Intel Xeon processors (quad core, 8MB cache, 2.66Ghz CPU) each. The transmembrane potential is initially at rest $v = -84$ mV and the excitation propagation is initiated with a stimulus of magnitude 100 mV applied on the septum at time $t = 10$. The human heart geometry and fiber directions obtained from CT scan data [34] have been smoothed, rescaled and meshed using the VMTK library (<http://www.vmtk.org>) (see Figure 5.1). The resulting idealized biventricular mesh consists of 729496 four-node elements and 130574 vertices. A time step $\Delta t = 0.01$ is employed. We simulate one full heartbeat and report in Figure 5.3 snapshots of the spatio-temporal evolution of action potential and tissue deformation. We have applied several model simplifications for sake of the analysis, including phenomenological FitzHugh-Nagumo membrane kinetics, linearized mechanical response, and stretch-independent activation. Some features of the cardiac function are difficult to recover in this framework, such as accurate representations of stresses at high strains. We also observe that the following residuals

$$\| \text{Cof}(\mathbf{D}\mathbf{u}) - \mathbf{I} \|_{L^2(\Omega)}, \| \mathbf{D}\mathbf{u} \|_{L^\infty(\Omega)}, \| \det(\mathbf{D}\mathbf{u}) - 1 \|_{L^2(\Omega)},$$

are non-negligible (see the dynamics depicted in Figure 5.2, where the spike in the middle and right plots is due to the electrical stimulus applied at $t = 10$). In the fully nonlinear case, performing Newton or Picard iterations up to a fixed tolerance rapidly decreases the first residual, associated with the nonlinear stress. Nevertheless, the order of magnitude of the second and third residuals remain unchanged, suggesting that total strains and incompressibility are well resolved even in the linear case. Moreover, from Figure 5.3 we readily observe some other key features such as the desired delay of the activation γ with respect to the front of the transmembrane potential, and the subsequent contraction of the muscle.

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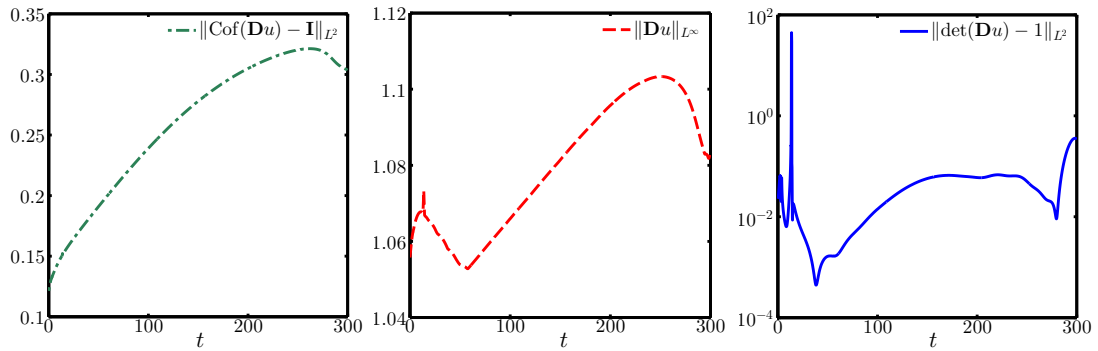


FIG. 5.2. Time evolution over half a heartbeat of the numerical residuals induced by the linearization of the elasticity problem.

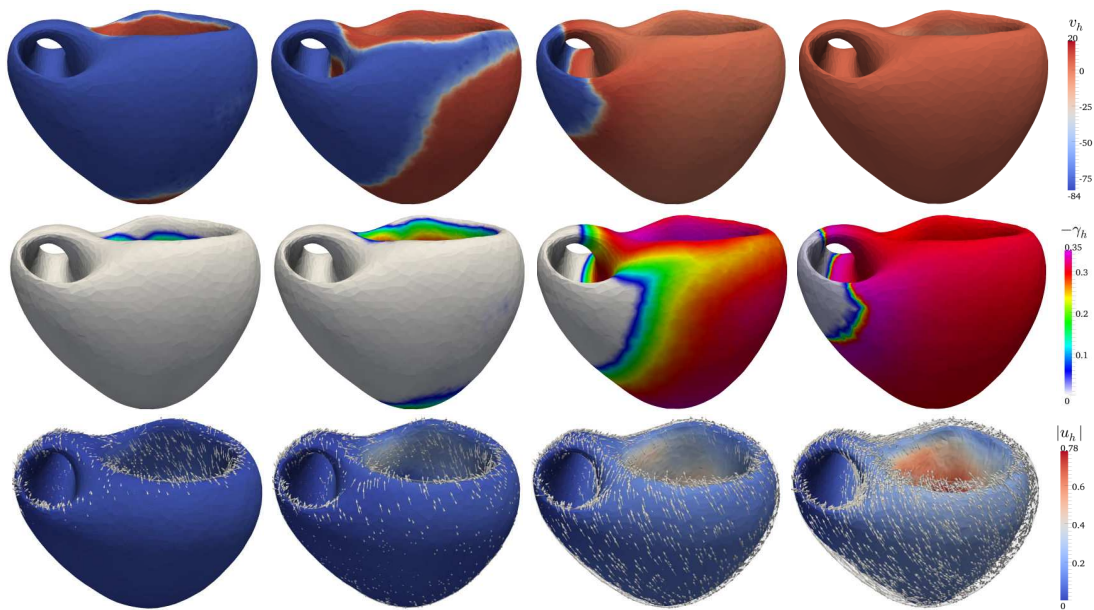


FIG. 5.3. Transmembrane potential (top), activation function (middle) and tissue deformation (bottom) at time instants $t = 30, 60, 120, 180$ (from left to right).

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