Expressing Discrete Geometry using the Conformal Model

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Abstract Primitives and transformations in discrete geometry, such as lines, circles, hyperspheres, hyperplanes, have been defined with classical linear algebra in dimension 2 and 3, leading to different expressions and algorithms. This paper explores the use of the conformal algebra to express these discrete primitives in arbitrary dimensions with a minimum of expressions and then algorithms. Starting with hyperspheres and hyperplanes, a generalization to *k*-sphere is then proposed. This gives one simple and compact formula, valid for all geometric conformal elements in $\mathbb{R}^n$, from the circle to the hypersphere, and the line to the hyperplane.

1 Introduction

1.1 Motivation

Discrete geometry is about handling objects and transformations in $\mathbb{Z}^n$ [6]. In computer graphics, a discrete point will be represented by pixels or voxels (pixels in dimensions greater than two) in images. A major problem or difficulty concerns the description of classical geometric objects in such a world, like lines, circles, planes or spheres. Since J.-P. Réveillès introduced the analytical description of two dimensional lines in 1988 [6], many different type of primitives have been described analytically. They are typically defined as all the discrete points verifying a set of inequalities in a classical linear algebra framework.

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For instance, a Reveilles discrete straight line in $\mathbb{Z}^2$ is analytically defined by:

$$L_2 = \left\{X = (x,y) \in \mathbb{Z}^2 | 0 \leq ax + by + c < \omega\right\} \tag{1}$$

where ω is called the arithmetical width, a parameter controlling the thickness of the discrete line. Here, the coefficients a, b and c define the line in $\mathbb{R}^2$, and are real numbers. An extension to hyperplanes is pretty straightforward [2]. This way of defining a 2D discrete line can be easily adapted to 2D discrete circles and extended to discrete hyperplanes and hyperspheres [1, 2]. A 3D discrete circle can be defined as the intersection of a discrete hyperplane and a discrete sphere but intersections do not necessarily define objects with good properties such as, for instance, connectivity properties. Another representation has been introduced in [7] based on structuring elements. For instance, considering a unit sphere as structuring element (the ball $B_2\left(\frac{1}{2}\right)$ of radius $1/2$ for the Euclidean distance d_2), then the discretisation $D(F)$ of a Euclidean object F is defined by:

$$D(F) = \left\{X \in \mathbb{Z}^n | d_2(X,F) \leq \frac{1}{2}\right\} \tag{2}$$

$$= \left(B_2\left(\frac{1}{2}\right) \oplus F\right) \cap \mathbb{Z}^n \tag{3}$$

The conformal algebra allows the representation of k-flats and k-spheres [3, 4, 5] (pair points, circles, lines, hyperspheres and hyperplanes, ...). This article investigates its use to formalize and unify the definition of discrete objects. A major advantage is the common definition of lines and circles, and of hyperspheres and hyperplanes; another advantage comes from the definition of discrete objects in arbitrary dimension in a compact form.

In this article, we cannot present all the possible discrete set definitions using the conformal model. So, we limit ourselves to a single structuring element, using the Euclidean distance d_2. We propose two different discrete set definitions. The first one uses a signed distance determination for a set of inequalities, similar to classical linear algebra; the second definition involves only one inequality using an intersection test.

1.2 Outlines

The rest of this article is organized as follows: the section 2 recall the conformal model basics, and our notations. Section 3 introduces the discrete sphere, starting in dimension 3, and then extended to any dimension $n \geq 2$. It presents also the discrete hyperplane, a particular case of hypersphere with a point at infinity. Next, section 4 proposes a general definition for discrete rounds, or discrete k-spheres, *i.e.* subspaces of dimension k embedded in a Euclidean space of dimension n. Some examples are the point pair and the circle. By extension, this is also valid for discrete

k-flats. So, section 5 presents example for 1-sphere, firstly with the discrete circle, which are nicely defined in any dimension $n \geq 2$, and secondly the discrete line which is a particular case of the former. The section 6 concludes this paper giving some future work plans.

2 Conformal Model

This section briefly recalls the basics about the conformal model. As many authors use their own notations, we also briefly present the notations used in this paper (based mostly on those of Dorst and al. [3]).

The conformal model is based on a Euclidean space of dimension n, denoted by $\mathbb{E}^n$ with an orthonormal basis $(e_1, e_2, \ldots, e_n)$ which is completed by adding two vectors e_+ and e_- with respective positive and negative signature. The particular vectors representing the origin and the infinity are denoted respectively by $n_o = \frac{1}{2}(e_+ + e_-)$ and $n_\infty = e_- - e_+$. These two vectors are null, since $n_o^2 = n_\infty^2 = 0$. Any Euclidean point (in bold) can be embedded into the conformal model using the following transformation:

$$p = F(\mathbf{p}) = n_o + \mathbf{p} + \frac{1}{2}\mathbf{p}^2 n_\infty \tag{4}$$

In this paper, the following normalisation condition is respected:

$$n_\infty \cdot p = -1 \tag{5}$$

Using the relation $n_o \cdot n_\infty = -1$, this normalisation condition allows to compute the Euclidean distance d_2 between two given Euclidean points $\mathbf{a}$ and $\mathbf{b}$ using the scalar product:

$$\begin{aligned}
a \cdot b &= \left(n_o + \mathbf{a} + \mathbf{a}^2 n_\infty/2\right) \cdot \left(n_o + \mathbf{b} + \mathbf{b}^2 n_\infty/2\right) \\
&= n_o \cdot n_o + n_o \cdot \mathbf{b} + \tfrac{\mathbf{b}^2}{2} n_o \cdot n_\infty + \mathbf{a} \cdot n_o + \mathbf{a} \cdot \mathbf{b} + \tfrac{\mathbf{b}^2}{2} \mathbf{a} \cdot n_\infty \\
&\quad + \tfrac{\mathbf{a}^2}{2} n_\infty \cdot n_o + \tfrac{\mathbf{a}^2}{2} n_\infty \cdot \mathbf{b} + \tfrac{\mathbf{a}^2 \mathbf{b}^2}{4} n_\infty \cdot n_\infty \\
&= \mathbf{a} \cdot \mathbf{b} - \frac{1}{2}\mathbf{a}^2 - \frac{1}{2}\mathbf{b}^2 \\
&= -\frac{1}{2}(\mathbf{a} - \mathbf{b})^2
\end{aligned}$$

The Euclidean and conformal pseudoscalars are respectively denoted by $I_n = e_1 \wedge e_2 \wedge \ldots \wedge e_n$ and $I_{n+1,1} = n_o \wedge I_n \wedge n_\infty$.

3 Discrete Spheres and Planes

3.1 Discrete Spheres centered at the origin

In discrete geometry [1], using a Euclidean distance, a n-dimensional sphere centered at **c** with radius r can be defined as the set of discrete point close to the Euclidean sphere, as:

$$\left\{\mathbf{p} \in \mathbb{Z}^n |\, (r-d)^2 \leq |\mathbf{c}-\mathbf{p}|^2 < (r+d)^2\right\} \tag{6}$$

where the width $d \in \mathbb{R}^{+,*}$ is a positive real number, smaller than r.

Hence, a point lies in the discrete sphere if it is inside a bigger one with radius $r+d$, and outside a smaller one with radius $r-d$.

We propose to define a discrete sphere using the conformal model in a similar spirit. For instance, using a sphere S_1 centered at the origin $n_o$and with radius 1, the intersection with a point **p** is:

$$S_1{}^* \cdot F(\mathbf{p}) = \frac{1}{2}\left(1-\mathbf{p}^2\right) \tag{7}$$

Since we are interested in the sign of this expression to separate the interior and exterior of the sphere, it can be reduced to $1-\mathbf{p}^2$. When this last expression is positive, then $\mathbf{p}^2 < 1$, and so **p** is inside the sphere; conversely, when it is negative we have $\mathbf{p}^2 > 1$, and **p** is outside.

In the general case, we define the following two spheres:

$$S_{\mathbf{c},r+d}{}^* = c - \tfrac{1}{2}(r+d)^2 n_\infty \qquad \text{and} \qquad S_{\mathbf{c},r-d}{}^* = c - \tfrac{1}{2}(r-d)^2 n_\infty$$

Now, we can check the distances, using two inner products:

$$S_{\mathbf{c},r+d}{}^* \cdot p = \frac{1}{2}\left((r+d)^2 - |\mathbf{p}-\mathbf{c}|^2\right) \tag{8}$$

$$S_{\mathbf{c},r-d}{}^* \cdot p = \frac{1}{2}\left((r-d)^2 - |\mathbf{p}-\mathbf{c}|^2\right) \tag{9}$$

As with (7), when (8) is negative, it is obvious that the Euclidean point **p** is outside the sphere defined by the center **c** and the radius $r+d$. So, to find the discrete points corresponding to the discrete sphere, this intersection must be positive.

For the second sphere of radius $r-d$, and assuming $d << r$, the same reasoning applied to (9) means that searching for Euclidean points outside the corresponding sphere implies a negative intersection value. To summarise, in $\mathbb{R}^n$, a discrete sphere centered at **c** with radius r is:

$$\left\{\mathbf{p} \in \mathbb{Z}^n \left| S_{\mathbf{c},(r-d)}{}^* \cdot p < 0 \text{ and } S_{\mathbf{c},(r+d)}{}^* \cdot p \geq 0\right.\right\}$$

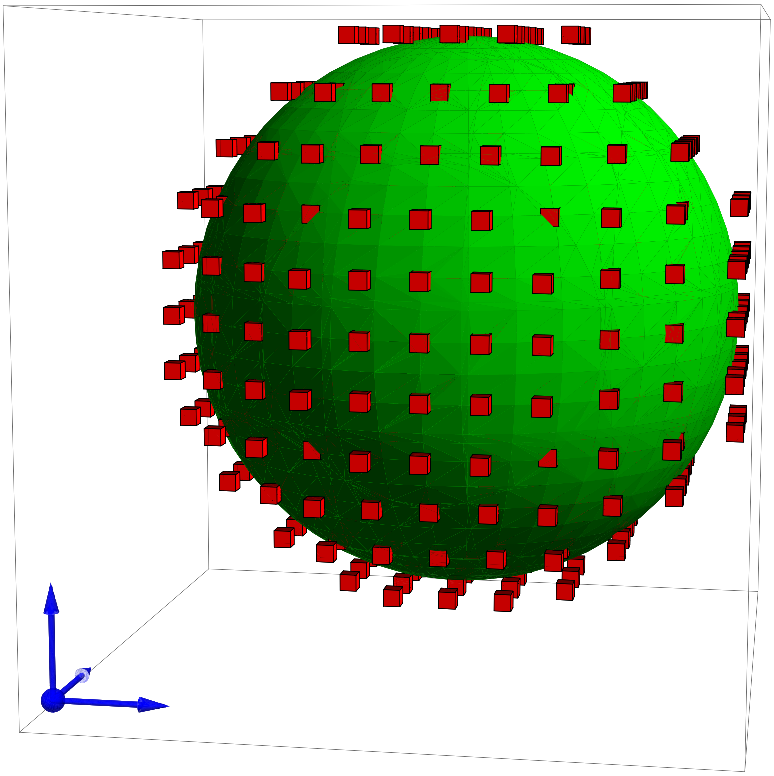

Fig. 1 A discrete sphere in $\mathbb{R}^3$ with center $\mathbf{c} = (6,6,6)$ and radius $r = 5$

As a result example, the Fig. 1 shows the discretization of a sphere defined in the conformal model.

3.2 Discrete Hyperplanes in $\mathbb{R}^n$

A plane is a particular sphere, passing through the infinite point [3, 5]. Therefore, it is the outer product between n_∞ and n conformal points. By duality, it can be expressed in any dimension as the 1-vector:

$$\pi = \mathbf{n} + \delta n_\infty$$

where $\mathbf{n}$ is the Euclidean normal vector, and δ the distance to the origin along $\mathbf{n}$.

For a given width d, using two orthogonal translations of $d\mathbf{n}$ and $-d\mathbf{n}$, we can enclose all the points of its discretization. Then the discrete hyperplane is:

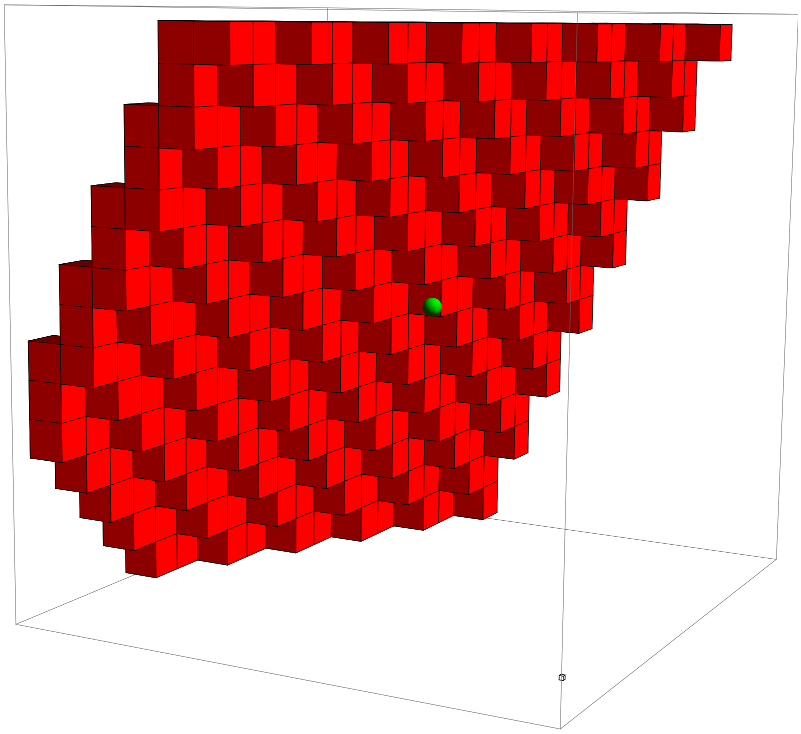

Fig. 2 A discrete plane in $\mathbb{R}^3$, with normal $\mathbf{n} = (2,1,1)$ and distance to origin -5. The discrete width is $\sqrt{2}/2$.

$$\{\mathbf{p} \in \mathbb{Z}^n | (\mathbf{n} + (\delta - d)n_\infty) \cdot p < 0 \text{ and } (\mathbf{n} + (\delta + d)n_\infty) \cdot p \geq 0\}$$

4 Discrete rounds

4.1 General definition

A k-sphere, or round according to Dorst's denomination [3], can be defined as the outer product between $k+2$ linearly independent conformal points representing Euclidean points:

$$\mathscr{R}_k = p_1 \wedge \ldots \wedge p_{k+2}$$

A hypersphere is then a $(n-1)$-sphere or round, while a point pair is a 0-sphere. A k-sphere can be also defined as the intersection between a hypersphere and a Euclidean flat $\mathbf{A}_k$ of dimension k:

$$\mathscr{R}_k = \left(c + \rho^2 n_\infty/2\right) \wedge \left(-c \rfloor \left(\widehat{\mathbf{A}_k} n_\infty\right)\right)$$

where $\widehat{\mathbf{A}_k}$ denotes the grade involution of the flat $\mathbf{A}_k$, so that $\widehat{\mathbf{A}_k} = (-1)^k \mathbf{A}_k$. A first example is the point pair, obtained by intersecting a line (a one dimensional flat space) and a hypersphere, or simply as the wedge between two conformal points; a second example is the circle, using a two dimensional flat space (a plane in dimension $n = 3$), or equivalently wedging three conformal points.

Let us notice that k-flats are particular rounds, and then can also be constructed using these two definitions. Using the former, a k-flat is expressed wedging $k+1$ points with n_∞; equivalently, with the later definition, it is defined using an infinite radius hypersphere.

The most important thing to notice here, is that according to Dorst [3], the intersection of a k-sphere and a hypersphere is a $(k-1)$-sphere.

4.2 Discrete rounds

A discrete round results from the Minkowski sum using the round as Euclidean object F. Using a hypersphere as structuring element, we can easily check if a given point $\mathbf{p}$ lies into the discrete round by verifying that the sphere centered on p intersects the k-sphere in a real $(k-1)$-sphere. The intersection calculation is well-known, and consists in the inner product of one element with the dual of the other. Here, we use the dual of the sphere, encoded as its center minus the infinity multiplied by the square radius over two:

$$\left\{ \mathbf{p} \in \mathbb{Z}^n \,\middle|\, (p - d^2 n_\infty/2) \cdot \mathscr{R}_k \text{ is a real round} \right\} \tag{10}$$

where d is the radius of the structuring element (*i.e.* a hypersphere). Let us notice that this approach is also usable for an hypersphere F, giving us a second method to obtain its discrete representation.

So the last question to answer is what is a real round, with respect to an imaginary one? Again, the solution is well-known [3]: a real round has a positive square radius, while an imaginary has a negative one. The square radius of a round $\mathscr{R}_k$ is computed by:

$$\rho^2 = \frac{\mathscr{R}_k \widehat{\mathscr{R}_k}}{(n_\infty \rfloor \mathscr{R}_k)^2} \tag{11}$$

Hence, a discrete round $\mathscr{R}_k$ can be rewritten as:

$$\left\{ \mathbf{p} \in \mathbb{Z}^n \,\middle|\, (-1)^{k-1} \left[(p - d^2 n_\infty/2) \cdot \mathscr{R}_k\right]^2 \geq 0 \right\} \tag{12}$$

The sign $(-1)^{k-1}$ appears due to the calculation of a $(k-1)$-round radius with a grade involution, using (11). Indeed, the intersection between a hypersphere and a k-sphere is a $(k-1)$-round.

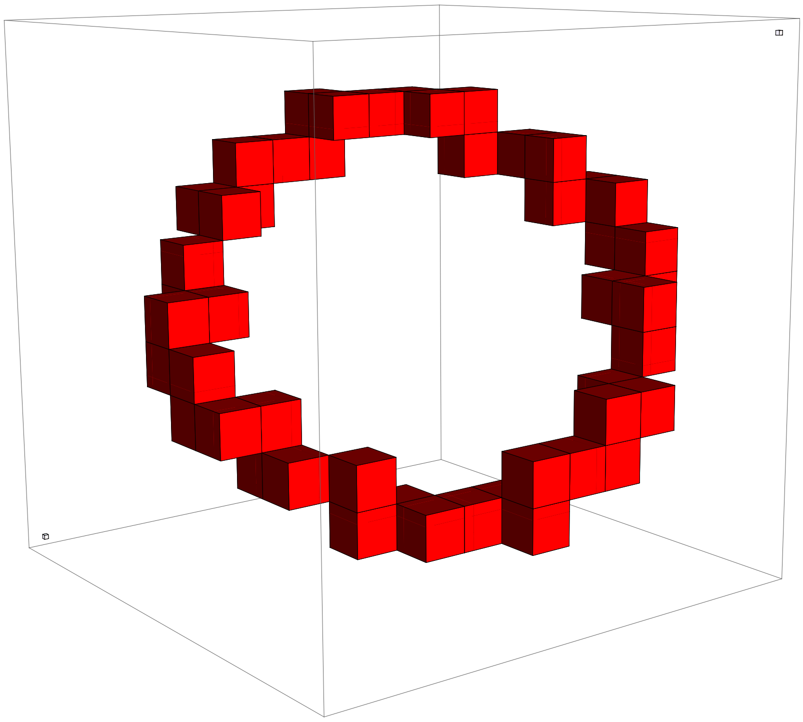

Fig. 3 Example of circle $\mathbb{R}^3$, centered at $\mathbf{c} = (1,1,2)$, in a plane $(e_1 + e_2) \wedge (2e_1 + e_3)$, and with radius $r = \sqrt{6}/2$. The discrete point radius is $d = 5/7$.

5 Discrete Circles and Lines

A circle can be defined as the intersection of a sphere and a plane containing the center of the former. If one attempts to compute the discrete circle using the intersection of the discrete sphere and plane, the result will be erroneous: obviously, it will not correspond to the Minkowski sum by a sphere of radius d. Hence, except for the two dimensional case, it is not equal to a discrete circle.

We propose here a robust and correct definition, using the conformal definition of circle, and by extension of line, and the general discrete set given by (12).

Contrary to the hypersphere, a conformal circle is always defined with 3 conformal points, and not $n+1$. So, in any dimension, a conformal circle is still a 3-vector.

Then, we can generalize the 2 dimensional case to $\mathbb{R}^n$, either using some rotation before the translation to the center of the circle, or directly by defining the circle as the 3-vector through three given Euclidean points.

Hence, the discrete circle passing through the Euclidean points $\mathbf{p_1}$, $\mathbf{p_2}$ and $\mathbf{p_3}$ is:

$$\left\{\mathbf{p} \in \mathbb{Z}^n \,\middle|\, \left[(p_1 \wedge p_2 \wedge p_3) \cdot \left(p - d^2 n_\infty/2\right)\right]^2 \geq 0\right\} \tag{13}$$

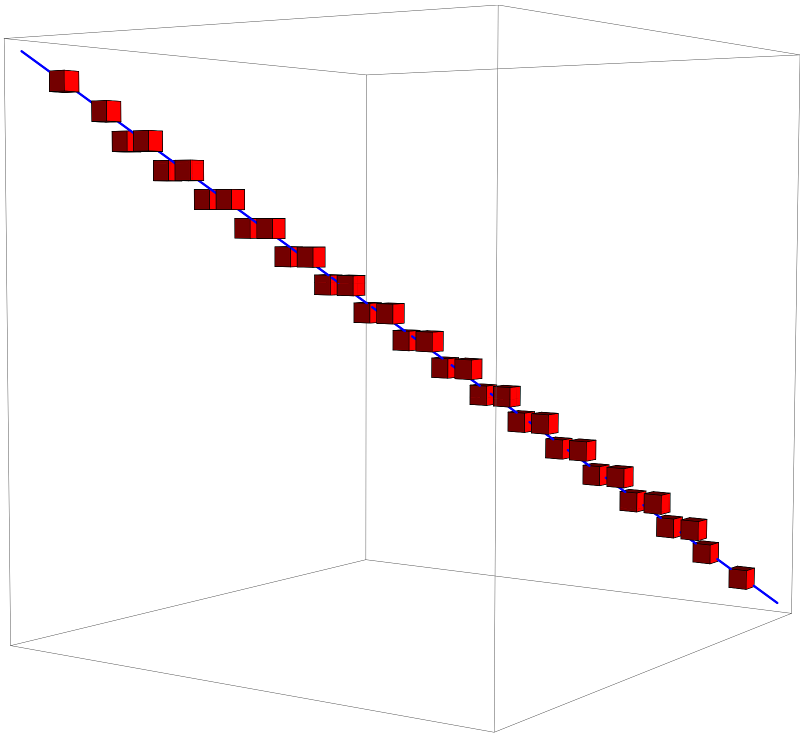

Fig. 4 Example of discrete line $\mathbb{R}^3$, passing through $(-10,-10,-9)$ and $(10,10,10)$, calculated with the width $\sqrt{2}/2$.

Lines are a particular case of circles passing through the infinity point n_∞. So the equation (13) is valid, albeit one of the three point is equal to n_∞. It follows that the discrete line in $\mathbb{R}^n$, passing through **a** and then **b** is:

$$\left\{\mathbf{p} \in \mathbb{Z}^n \,\middle|\, \left[(a \wedge b \wedge n_\infty) \cdot \left(p - d^2 n_\infty/2\right)\right]^2 \geq 0\right\} \tag{14}$$

As examples of result in $\mathbb{R}^3$, the Fig. 3 shows a discrete circle, while Fig. 4 depicts a discrete line, made using conformal model calculation only.

6 Conclusions

This article proposes definitions of discrete objects, including hypersphere and hyperplane, and *k*-sphere. Even if they do not come with efficient algorithms for constructing the sets, they are valid in any dimension $n \geq 2$, and allow unification between circles and lines, and between spheres and planes.

Later, as perspectives, we plan to work on some extensions. First, we aim to work with other structuring element in the Minkowski sum, for instance one representing a d_1 distance; this should be done using translation and hyperplane with each $\mathbb{R}^n$ basis element as normal vector. This might allow us to study new types of discretization, using the conformal formalization.

Another perspective is the k-spheres recognition in $\mathbb{R}^n$; actually, there exist some performant algorithms to recognize lines in 2 or 3 dimensions. Using classical geometry, they do not seem to be naturally extended in dimension *n*, or applied to circles in 2 dimensions, or be used for rounds. Our goal is then to extend them using our discretization, and the fact that lines and circles are similar, and that circles are round, leading to a single algorithm working with any kind of circles and in any dimension.

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