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# Bounds on the Cover Time of Parallel Rotor Walks

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## Abstract

The *rotor-router mechanism* was introduced as a deterministic alternative to the random walk in undirected graphs. In this model, a set of  $k$  identical walkers is deployed in parallel, starting from a chosen subset of nodes, and moving around the graph in synchronous steps. During the process, each node maintains a cyclic ordering of its outgoing arcs, and successively propagates walkers which visit it along its outgoing arcs in round-robin fashion, according to the fixed ordering.

We consider the *cover time* of such a system, i.e., the number of steps after which each node has been visited by at least one walk, regardless of the starting locations of the walks. In the case of  $k = 1$ , Yanovski et al. (2003) and Bampas et al. (2009) showed that a single walk achieves a cover time of exactly  $\Theta(mD)$  for any  $n$ -node graph with  $m$  edges and diameter  $D$ , and that the walker eventually stabilizes to a traversal of an Eulerian circuit on the set of all directed edges of the graph. For  $k > 1$  parallel walks, no similar structural behaviour can be observed.

In this work we provide tight bounds on the cover time of  $k$  parallel rotor walks in a graph. We show that this cover time is at most  $\Theta(mD/\log k)$  and at least  $\Theta(mD/k)$  for any graph, which corresponds to a speedup of between  $\Theta(\log k)$  and  $\Theta(k)$  with respect to the cover time of a single walk. Both of these extremal values of speedup are achieved for some graph classes. Our results hold for up to a polynomially large number of walks,  $k = O(\text{poly}(n))$ .

## 1 Introduction

In graph exploration problems, a walker or group of walkers (agents) is placed on a node of a graph and moves between adjacent nodes, with the goal of visiting all the nodes of the graph. The study of graph exploration is closely linked to central problems of theoretical computer science, such as the question of deciding if two nodes of the graph belong to the same connected component (*st-connectivity*). For example, fast approaches to connectivity testing in little memory rely on the deployment of multiple random walks [6, 11]. In these algorithms, the initial locations of the walkers are chosen according to a specific probability distribution.

More recently, multiple walks have been studied in a worst-case scenario where the  $k$  agents are placed on some set of starting nodes and deployed in parallel, in synchronous steps. The considered parameter is the *cover time* of the process, i.e., the number of steps until each node of the graph has been visited by at least one walker. Alon *et al.* [2], Efremenko and Reingold [9], and Elsässer and Sauerwald [10] have studied the notion of the *speedup* of the random walk for an undirected graph  $G$ , defined as the ratio between the cover time of a  $k$ -agent walk in  $G$  for worst-case initial positions of agents and that of a single-agent walk in  $G$  starting from

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a worst-case initial position, as a function of  $k$ . A characterization of the speedup has been achieved for many graph classes with special properties, such as small mixing time compared to cover time. However, a central question posed in [2] still remains open: what are the minimum and maximum values of speed-up of the random walk in arbitrary graphs? The smallest known value of speedup is  $\Theta(\log k)$ , attained e.g. for the cycle, while the largest known value is  $\Theta(k)$ , attained for many graph classes, such as expanders, cliques, and stars.

In this work, we consider a deterministic model of walks on graphs, known as the *rotor-router*. The rotor-router model, introduced by Priezzhev *et al.* [14] and further popularised by James Propp, provides a mechanism for the environment to control the movement of the agent deterministically, mimicking the properties of exploration as the random walk. In the rotor-router, the agent has no operational memory and the whole routing mechanism is provided within the environment. The edges outgoing from each node  $v$  are arranged in a fixed cyclic order known as a *port ordering*, which does not change during the exploration. Each node  $v$  maintains a *pointer* which indicates the edge to be traversed by the agent during its next visit to  $v$ . If the agent has not visited node  $v$  yet, then the pointer points to an arbitrary edge adjacent to  $v$ . The next time when the agent enters node  $v$ , it is directed along the edge indicated by the pointer, which is then advanced to the next edge in the cyclic order of the edges adjacent to  $v$ .

For a single agent, the (deterministic) cover time of the rotor-router and the (expected) cover time of the random walk prove to be surprisingly convergent for many graph classes. In general, it is known that for any  $n$ -node graph of  $m$  edges and diameter  $D$ , the cover time of the rotor-router in a worst-case initialization is precisely  $\Theta(mD)$  [16, 3]. By comparison, the random walk satisfies an upper bound of  $O(mD \log n)$  on the cover time.

The behavior of the rotor-router model with multiple agents appears to be much more complicated. Since the parallel walkers interact with the pointers of a single rotor-router system, they cannot be considered independent (in contrast to the case of parallel random walks). In the first work on the topic, Yanovski *et al.* [16] showed that adding a new agent to a rotor-router system with  $k$  agents cannot increase the cover time, and showed experimental evidence suggesting that a speedup does indeed occur. Klasing *et al.* [13] have provided the first evidence of speedup, showing that for the special case when  $G$  is a cycle, a  $k$ -agent system explores an  $n$ -node cycle  $\Theta(\log k)$  times more quickly than a single agent system.

In this work we completely resolve the question of the possible range of speedups of the parallel rotor-router model in a graph, showing that its value is between  $\Theta(\log k)$  and  $\Theta(k)$ , for any graph. Both of these bounds are tight. Thus, the proven range of speedup for the rotor-router corresponds precisely to the conjectured range of speedup for the random walk.

## 1.1 Related work

**The rotor-router model.** Studies of the rotor-router started with works of Wagner *et al.* [15] who showed that in this model, starting from an arbitrary configuration (arbitrary cyclic orders of edges, arbitrary initial values of the port pointers and an arbitrary starting node) the agent covers all  $m$  edges of an  $n$ -node graph within  $O(nm)$  steps. Bhatt *et al.* [5] showed later that within  $O(nm)$  steps the agent not only covers all edges but enters (establishes) an Eulerian cycle. More precisely, after the initial stabilization period of  $O(nm)$  steps, the agent keeps repeating the same Eulerian cycle of the directed symmetric version  $\vec{G}$  of graph  $G$  (see Section 3 for a definition). Subsequently, Yanovski *et al.* [16] and Bampas *et al.* [3] showed that the Eulerian cycle is in the worst case entered within  $\Theta(mD)$  steps in a graph of diameter  $D$ . Considerations of specific graph classes were performed in [12]. Robustness properties of the rotor-router were further studied in [4], who considered the time required for the rotor-router to stabilize to a (new) Eulerian cycle after an edge is added or removed from the graph. Regarding the terminology, we

note that the rotor-router model has also been referred to as the *Propp machine* [3] or *Edge Ant Walk algorithm* [15, 16], and has also been described in [5] in terms of traversing a maze and marking edges with pebbles. Studies of the multi-agent rotor-router was performed by Yanovski *et al.* [16] and Klasing *et al.* [13], and its speedup was considered for both worst-case and best-case scenarios.

A variant of the multi-agent rotor-router mechanism has been extensively studied in a different setting, in the context of balancing the workload in a network. The single agent is replaced with a number of agents, referred to as *tokens*. Cooper and Spencer [7] study  $d$ -dimensional grid graphs and show a constant bound on the discrepancy, defined as the difference between the number of tokens at a given node  $v$  in the rotor-router model and the expected number of tokens at  $v$  in the random-walk model. Subsequently, Doerr and Friedrich [8] analyze in more detail the distribution of tokens in the rotor-router mechanism on the 2-dimensional grid. Akbari and Berenbrink [1] showed an upper bound of  $O(\log^{3/2} n)$  on the discrepancy for hypercubes and a bound of  $O(1)$  for a constant-dimensional torus.

**Parallel random walks.** Alon *et al.* [2] introduced the notion of the speed-up of  $k$  independent random walks as the ratio of the cover time of a single walk to the cover time of  $k$  random walks. They conjectured that the speed-up is between  $\log k$  and  $k$  for any graph. The speedup was shown to be  $k$  for many graph classes, such as complete graphs [2],  $d$ -dimensional grids [2, 10], hypercubes [2, 10], expanders [2, 10], and different models of random graphs [2, 10]. For the cycle, the speed-up is equal to  $\log k$  [2]. For general graphs, an upper bound  $\min\{k \log n, k^2\}$  on the speed-up was obtained by Efremenko *et al.* [9]. Independently, Elsässer *et al.* [10] showed the  $k \log n$  upper bound. Another measure studied by Efremenko *et al.* [9] concerns the speedup with respect to a different exploration parameter — the maximizing hitting time, i.e., the maximum over all pairs of nodes of the graph of the expected time required by the walk to move from one node to the other. For this parameter, they show a bound on speedup of  $O(k)$ , mentioning that it is tight in many graph classes.

## 1.2 Our results and overview of the paper

In this work we establish bounds on the minimum and maximum possible cover time for a worst-case initialization of a  $k$ -rotor-router system in a graph  $G$  with  $m$  edges and diameter  $D$ .

We start by providing a formal definition of the rotor-router model and recalling its basic properties in Section 2. In Section 3, we first prove that the cover time  $t_C$  satisfies  $t_C \in O(mD/\log k)$ , when  $k < 2^{16D}$ . We then extend this result to the case of  $k \in O(\text{poly}(n))$ , i.e.,  $k < n^c$  for some absolute constant  $c$ . The main part of our proofs relies on a global analysis of the number of visits to edges in successive time steps, depending on the number of times that these edges have been traversed in the past. We first prove a stronger version of local structural lemmas proposed by Yanovski *et al.* [16], and apply them within a global amortization argument over all time steps and all edges in the graph. The extension to the case of  $k \in O(\text{poly}(n))$  relies on a variant of a similar amortized analysis, and also makes use of a technique known as *delayed deployments* introduced by Klasing *et al.* [13], which we briefly recall in Section 2. We remark that by [13], a cover time of  $\Theta(mD/\log k)$  is achieved when  $G$  is a cycle with all agents starting from one node, when  $k < n^{1/11}$ .

In Section 4, we show a complementary lower bound on the cover time of the  $k$ -agent rotor-router in worst case initialization, namely,  $t_C \in \Omega(mD/k)$ . As a starting point, the proof uses a decomposition of the edge set of a graph, introduced by Bampas *et al.* [3], into a “heavy part” containing a constant proportion of the edges and a “deep part”, having diameter linear in  $D$ . The main part of the analysis is to show that an appropriate initialization of  $k$  agents in the

Table 1: A comparison of values of speed-up for  $k$ -agent exploration with the rotor-router and parallel random walks. All results hold at least for  $k \leq n$ , except for those cited from [13] which hold for  $k \leq n^{1/11}$ .

Graph class	Speedup of Rotor-Router		Speedup of Random Walk		
	<i>for cover time</i>		<i>for cover time</i>	<i>for max hitting time</i>	
General case:	$\Omega(\log k), O(k)$	(Thm. 3.7, 4.1)	$O(k^2), O(k \log n)$	[9, 10]	$O(k)$ [10]
Cycle:	$\Theta(\log k)$	[13]	$\Theta(\log k)$	[2]	$\Theta(\log k)$ [2]
Star:	$\Theta(k)$	(Prop. 4.2)	$\Theta(k)$	[2]	$\Theta(k)$ [2]

heavy part takes a long time to reach the most distant nodes of the deep part. The argument also takes advantage of the delayed deployment technique. We close the section by remarking that a cover time of  $\Theta(mD/k)$  is, in fact, achieved for some graphs, such as stars.

Table 1 contains a summary of our results on the speed-up of the  $k$ -agent rotor-router, compared to corresponding results from the literature for parallel random walks. Note that for a deterministic process such as the rotor-router, the notions of cover time and maximum hitting are equivalent, and hence we only refer to cover times.

## 2 Model and preliminaries

Let  $G = (V, E)$  be an undirected connected graph with  $n$  nodes,  $m$  edges and diameter  $D$ . We denote the neighborhood of a node  $v \in V$  by  $\Gamma(v)$ . The directed graph  $\vec{G} = (V, \vec{E})$  is the directed symmetric version of  $G$ , where the set of arcs  $\vec{E} = \{(v, u) : \{v, u\} \in E\}$ . We will denote arc  $(v, u)$  by  $v \rightarrow u$ .

**Model definition.** We consider the rotor-router model (on graph  $G$ ) with  $k \geq 1$  indistinguishable agents, which run in steps, synchronized by a global clock. In each step, each agent moves in discrete steps from node to node along the arcs of graph  $\vec{G}$ . A *configuration* at the current step is defined as a triple  $((\rho_v)_{v \in V}, (\pi_v)_{v \in V}, \{r_1, \dots, r_k\})$ , where  $\rho_v$  is a cyclic order of the arcs (in graph  $\vec{G}$ ) outgoing from node  $v$ ,  $\pi_v$  is an arc outgoing from node  $v$ , which is referred to as *the (current) port pointer at node  $v$* , and  $\{r_1, \dots, r_k\}$  is the (multi-)set of nodes currently containing an agent. For each node  $v \in V$ , the cyclic order  $\rho_v$  of the arcs outgoing from  $v$  is fixed at the beginning of exploration and does not change in any way from step to step.

For an arc  $v \rightarrow u$ , let  $next(v \rightarrow u)$  denote the arc next after arc  $(v \rightarrow u)$  in the cyclic order  $\rho_v$ . The exploration starts from some initial configuration and then keeps running in all future rounds, without ever terminating. During the current step, first each agent  $i$  is moved from node  $r_i$  traversing the arc  $\pi_{r_i}$ , and then the port pointer  $\pi_{r_i}$  at node  $r_i$  is advanced to the next arc outgoing from  $r_i$  (that is,  $\pi_{r_i}$  becomes  $next(\pi_{r_i})$ ). This is performed sequentially for all  $k$  agents. Note that the order in which agents are released within the same step is irrelevant from the perspective of the system, since agents are indistinguishable. For example, if a node  $v$  contained two agents at the start of a step, then it will send one of the agents along the arc  $\pi_v$ , and the other along the arc  $(v, next(\pi_v))$ .

**Notation.** We will denote by  $a^{(t)}(e)$  the number of agents traversing directed arc  $e \in \vec{E}$  during step  $t + 1$ . We recall that multiple agents traversing one arc  $e \in \vec{E}$  in the same time step  $t$  are considered to move simultaneously. By  $d^{(t)}(e)$  we denote the number of traversals of directed arc  $e \in \vec{E}$  till the end of step  $t$ ,  $d^{(t)}(e) = \sum_{t' \in [0, t)} a^{(t')}(e)$ . For a node  $v \in V$ , let

$d^{(t)}(v) = \min_{w \in \Gamma(v)} \{d^{(t)}(v \rightarrow w)\}$  be the number of fully completed rotations of the rotor at node  $v$  at the end of step  $t$ . We note that for any arc  $u \rightarrow v \in \vec{E}$ ,  $0 \leq d^{(t)}(u \rightarrow v) - d^{(t)}(u) \leq 1$  [16].

We also denote  $V_i^{(t)} = \{v \in V : d^{(t)}(v) \leq i\}$  and  $E_i^{(t)} = \{e \in \vec{E} : d^{(t)}(e) \leq i\}$ .  $\mathbb{N}_+$  denotes the set of positive integers, and  $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$ . We also introduce compact notation for discrete intervals of integers:  $[a, b] \equiv \{a, a + 1, \dots, b\}$ , and  $[a, b) \equiv [a, b - 1]$ , for  $a, b \in \mathbb{N}$ . Given a graph  $G = (V, E)$  and a subset  $X \subseteq V$ ,  $G[X]$  denotes the subgraph of  $G$  induced by  $X$ ,  $G[X] = (X, \{\{u, v\} \in E \mid u, v \in X\})$ .

**Delayed deployment technique.** In some of the proofs, we will make use of modified executions of the  $k$ -agent rotor-router system called *delayed deployments* [13], in which some agents may be stopped at a node, skipping their move for some number of rounds. Formally, a delayed deployment  $D$  of  $k$  agents is defined as a function  $D : V \times \mathbb{N} \rightarrow \mathbb{N}$ , where  $D(v, t) \geq 0$  represents the number of agents which are stopped in node  $v$  in step  $t$  of the execution of the system. (The unmodified rotor-router system, denoted  $R$ , corresponds to the deployment  $R(v, t) = 0$ , for all  $v$  and  $t$ ). Delayed deployments may be conveniently viewed as algorithmic procedures for delaying agents, and are introduced for purposes of analysis, only. The following lemma relates the cover time of the rotor-router system to that of its delayed deployment.

**Lemma 2.1.** [13] *Let  $R$  be a  $k$ -rotor router system with an arbitrarily chosen initialization, and let  $D$  be any delayed deployment of  $R$ . Suppose that deployment  $D$  covers all the nodes of the graph after  $T$  rounds, and in at least  $\tau$  of these rounds, all  $k$  agents were active in  $D$ . Then, the cover time  $t_C$  of the rotor-router system  $R$  can be bounded by:  $\tau \leq t_C \leq T$ .*

### 3 Upper bound on cover time

In this section, we will show that a  $k$ -agent parallel rotor-router system explores a graph in  $O(mD/\log k)$  steps, regardless of initialization. We start by providing an informal intuition of the main idea of the proof. After some initialization phase of duration  $t_0$ , but before exploration is completed at time  $t_C$ , we consider a shortest path connecting the arc of the graph which has already been visited many times at time  $t_0$ , with an arc which will remain unvisited at time  $t_C$ . We look at the number of visits to consecutive arcs on this path. It turns out that the rotor-router admits a property which can be informally stated as follows: if, up to some step  $t$  of exploration, an arc  $e_{t+1}$  of the considered path has been traversed more times than the next arc  $e_t$  on the path by some difference of  $\delta$ , then in the next step  $t + 1$  of exploration, at least  $\delta - O(1)$  agents will traverse arcs which have, so far, been visited not more often (up to a constant additive factor) than  $e_t$ . In this way, the larger the discrepancy between the number of visits to adjacent arcs, the more activity will the rotor-router perform to even out this discrepancy, by traversing under-visited arcs. This load-balancing behavior of the system will be shown to account for the  $(\log k)$ -speedup in cover time with respect to the case of a single agent.

We start by proving two structural lemmas which generalize the results of Yanovski *et al.* [16, Theorem 2]. The first lemma establishes a connection between the existence of an arc entering a subset of nodes  $S \subseteq V$  that has been traversed more times than all arcs outgoing from  $S$ , and the number of agents currently located within set  $S$ .

**Lemma 3.1.** *For any time  $t \in \mathbb{N}$  and  $d \in \mathbb{N}$ , consider the partition of the set of nodes  $V = S \cup T$  such that each node in set  $S$  (set  $T$ ) has completed at most  $d$  (more than  $d$ ) full cycles of if its rotor,  $S = V_d^{(t)}$  and  $T = V \setminus S$ . Suppose that for some nodes  $v \in S$ ,  $u \in T$ , and some  $\delta \in \mathbb{N}$ , there exists an arc  $u \rightarrow v$ , such that  $d^{(t)}(u \rightarrow v) \geq d + \delta$ . Then, the set of arcs having their tail at a node of  $S$  will be traversed by at least  $\delta - 1$  agents in total in step  $t + 1$ .*

*Proof.* Denote by  $S \rightarrow T$  (resp.,  $T \rightarrow S$ ) the set of arcs connecting nodes from  $S$  with nodes from  $T$  (resp., nodes from  $T$  with nodes from  $S$ ), and let  $l = |S \rightarrow T| = |T \rightarrow S|$ . By the basic property of the rotor-router process, all arcs outgoing from some node  $w$  have been traversed either  $d^{(t)}(w)$  or  $d^{(t)}(w) + 1$  times by the end of step  $t$ . It follows the definition of sets  $S$  and  $T$  that any arc outgoing from  $S$  was traversed at most  $d + 1$  times and any arc outgoing from  $T$  was traversed at least  $d + 1$  times. The arc  $u \rightarrow v \in T \rightarrow S$  was traversed  $d + \delta$  times. Hence:

$$\begin{aligned} \sum_{e \in S \rightarrow T} d^{(t)}(e) &\leq l \cdot (d + 1), \\ \sum_{e \in T \rightarrow S} d^{(t)}(e) &\geq (l - 1) \cdot (d + 1) + d + \delta \geq \sum_{e \in S \rightarrow T} d^{(t)}(e) + \delta - 1. \end{aligned}$$

Thus, at least  $\delta - 1$  more agents moved from  $T$  to  $S$  than in the opposite direction until the end of step  $t$ . So, at the end of time step  $t$ , we have at least  $\delta - 1$  agents located at nodes from set  $S$ . It follows that during step  $t + 1$ , at least  $\delta - 1$  agents traverse arcs outgoing from nodes from the set  $S$ .  $\square$

By an application of the above lemma, we obtain the key property of a pair of consecutive arcs which have a different number of traversals at time  $t$ .

**Lemma 3.2.** *Let  $G = (V, E)$  be any undirected graph and let  $e_2 = u \rightarrow v$ ,  $e_1 = v \rightarrow w$  be two consecutive arcs of  $\vec{G}$ . Fix a time step  $t \in \mathbb{N}_+$ . Then, for any  $x \geq d^{(t)}(e_1) + 1$ , the number of agents that traverse arcs from set  $E_x^{(t)}$  in time step  $t + 1$  satisfies:*

$$\sum_{e \in E_x^{(t)}} a^{(t)}(e) \geq d^{(t)}(e_2) - d^{(t)}(e_1) - 1.$$

*Proof.* We can assume that  $d^{(t)}(e_2) - d^{(t)}(e_1) \geq 2$ , otherwise the claim is trivial. By the definition of the rotor-router, we know that  $0 \leq d^{(t)}(e_1) - d^{(t)}(v) \leq 1$  and  $d^{(t)}(u) \geq d^{(t)}(e_2) - 1 \geq d^{(t)}(e_1) + 1 \geq d^{(t)}(v)$ . We now apply Lemma 3.1 for  $d = d^{(t)}(v)$ , putting  $S = V_{d^{(t)}(v)}^{(t)}$  and  $T = V \setminus S$ . Note that  $v \in S$ ,  $u \in T$ , and  $d^{(t)}(u \rightarrow v) = d + \delta$  for  $\delta = d^{(t)}(e_2) - d^{(t)}(v) \geq d^{(t)}(e_2) - d^{(t)}(e_1)$ . It follows from the Lemma that during step  $t + 1$ , at least  $d^{(t)}(e_2) - d^{(t)}(e_1) - 1$  agents traverse arcs outgoing from nodes from the set  $S$ . Since  $S = V_{d^{(t)}(v)}^{(t)}$ , all arcs  $e^*$  outgoing from nodes from set  $S$  have a number of traversals which satisfies  $d^{(t)}(e^*) \leq d^{(t)}(v) + 1 \leq d^{(t)}(e_1) + 1$ , so  $e^* \in E_{d^{(t)}(e_1)+1}^{(t)}$ . Thus,  $d^{(t)}(e_2) - d^{(t)}(e_1) - 1$  agents in step  $t + 1$  traverse edges in  $E_{d^{(t)}(e_1)+1}^{(t)}$ , and moreover  $E_{d^{(t)}(e_1)+1}^{(t)} \subseteq E_x^{(t)}$  for all  $x \geq d^{(t)}(e_1) + 1$ .  $\square$

The property of the rotor-router captured by the above lemma is, in fact, sufficient to prove the main results of the section, following the general approach outlined at the beginning of the section. To show a bound of  $t_C \in O(mD/\log k)$ , we will apply two separate arguments, first one for the range of relative small  $k$  ( $k \in 2^{O(D)}$ , which corresponds to  $t_C \in \Omega(m)$ ), and then one for values of  $k$  which are larger, but polynomially bounded with respect to  $n$ .

**Theorem 3.3.** *Let  $G = (V, E)$  be any undirected graph with arbitrary initialization of pointers and let  $D$  be the diameter of  $G$ . If  $k \leq 2^{16D}$ , then a team of  $k$  agents performing in parallel the rotor-router movement explores  $G$  in less than  $500mD/\log k$  steps, regardless of the initial positions of agents.*

*Proof.* First, assume that  $k > 2^{160}$  and fix  $b = \lfloor (\log k)/2 \rfloor$ . Consider the first  $t_0$  steps, where  $t_0 = \lceil 2^{b+1}mD/k \rceil$ . Since in every step exactly  $k$  arcs are traversed by agents, the total number of arc traversals during the first  $t_0$  steps is at least  $2^{b+1}mD$ . We have  $2m$  arcs in total. Thus,

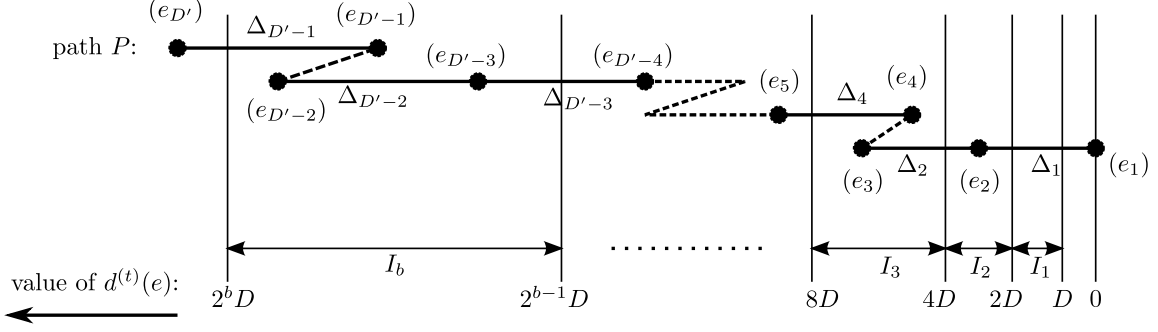


Figure 1: An illustration of sets  $I_i$  and  $\Delta_i$  in the proof of Theorem 3.3.

there exists an arc  $e'$  such that  $d^{(t_0)}(e') \geq 2^b D$ . These first  $t_0$  steps we will call as a form of setup stage, after which we begin to analyze the behavior of the rotor-router process.

Denote by  $t_C$  the cover time of  $G$  with  $k$  agents for a given initialization. We will assume that  $t_C > t_0$ , i.e., at least one arc of the graph has not been explored at time  $t_0$ ; otherwise,  $t_C \leq t_0 = \lceil 2^{b+1} mD/k \rceil \leq \lceil 2mD/\sqrt{k} \rceil$ , since  $b = \lfloor (\log k)/2 \rfloor$ , and the claim of the theorem holds for all  $k$ .

Take  $e'' \in \vec{E}$  to be an arc which is explored for the first time in step  $t_C$ , i.e., such that  $d^{(t_C-1)}(e'') = 0$ . Since the diameter of  $G$  is  $D$ , there exists a path  $\mathcal{P} = \langle e'' = e_1, e_2, \dots, e_{D'} = e' \rangle$  such that  $D' \leq D + 2$ , and for each  $l \in [1, D']$ ,  $e_l = v_{l+1} \rightarrow v_l$  where  $v_l, v_{l+1} \in V$ .

Fix a time step  $t \in [t_0, t_C)$ . We will place some of the arcs of path  $\mathcal{P}$  in groups (buckets)  $I_1, I_2, \dots, I_b$ , such that all arcs in bucket  $I_i$  have been traversed between  $2^{i-1}D$  and  $2^i D$  times until step  $t$ . Formally, denote:

$$I_i = \left\{ l : d^{(t)}(e_l) \in [2^{i-1}D, 2^i D) \right\} \subseteq [1, D'], \quad \text{for } i \in [1, b].$$

We now analyze which buckets successive arcs of the path  $\mathcal{P}$  fall into. For  $l \in [1, D']$ , define

$$\Delta_l = \begin{cases} [d^{(t)}(e_l), d^{(t)}(e_{l+1})], & \text{if } d^{(t)}(e_l) < d^{(t)}(e_{l+1}), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that the union of all  $\Delta_l$  covers the interval  $[0, 2^b D)$ , since for any  $x \in [0, 2^b D)$  there exists  $l^* \in [1, D']$  such that  $x \in \Delta_{l^*}$  because  $d^{(t)}(e_1) = 0$  and  $d^{(t)}(e_{D'}) \geq 2^b D$  (see Fig. 1 for an illustration). The intuition of the proof is now as follows: Since there are at most  $D'$  non-empty intervals  $\Delta_l$  spanning the total range  $[0, 2^b D)$  of all buckets  $I_1, I_2, \dots, I_b$ , in a constant proportion of all buckets  $I_i$ , the average length of an intervals  $\Delta_l$  starting in bucket  $I_i$  will be at least  $|I_i|b/D = 2^{i-1}b$ , up to a constant factor. The existence of such long intervals  $\Delta_l$  beginning in  $I_i$  will allow us to exploit Lemma 3.2 to show that arcs  $e_l, e_{l+1}$  differ in the number of traversals by a constant times  $2^{i-1}b$ . This implies that for the considered bucket indices  $i$ , the number of agents active at time  $t$  on edges from buckets  $I_1, \dots, I_i$  will be at least  $2^{i-1}b$ , up to constant factors and minor shifts at bucket boundaries. We now proceed to formalize the above arguments.

For  $i \in [1, b]$ , denote by  $\mathcal{X}_i$  the set of intervals  $\Delta_l$  beginning in bucket  $I_i$ :  $\mathcal{X}_i = \bigcup_{l \in I_i} \Delta_l$ . Consider any  $x \in [0, 2^b D)$ , and let  $l^*$  be such that  $x \in \Delta_{l^*}$ . We have  $d^{(t)}(e_{l^*}) \leq x < 2^b D$ , hence  $l^* \in I_{i^*}$ , for some  $i^* \in [1, b]$ , and  $x \in \mathcal{X}_{i^*}$ . It follows that:

$$[0, 2^b D) \subseteq \bigcup_{i \in [1, b]} \mathcal{X}_i. \quad (1)$$



For  $i \in \mathbb{N}$ , denote by  $a_i^{(t)}$  the number of agents that traverse arcs from set  $E_{2^i D}^{(t)}$  in step  $t + 1$ ,  $a_i^{(t)} \equiv \sum_{e \in E_{2^i D}^{(t)}} a^{(t)}(e)$ , and let  $a_{-1}^{(t)} = 0$ . (We remark that  $E_{2^i D}^{(t)} \supseteq I_1 \cup \dots \cup I_i$ .) First, note that for all  $i \in [1, b]$  and for  $l \in I_i$ , we have  $d^{(t)}(e_l) < 2^i D$ . So, by Lemma 3.2:

$$a_i^{(t)} \geq d^{(t)}(e_{l+1}) - d^{(t)}(e_l) - 1 = |\Delta_l| - 1 \implies |\Delta_l| \leq a_i^{(t)} + 1. \quad (2)$$

Now, observe that for any  $i \in [1, b]$ :

$$\max \mathcal{X}_i = \max_{l \in I_i} (\max \Delta_l) \leq \max_{l \in I_i} (d^{(t)}(e_l) + |\Delta_l| - 1) < 2^i D + a_i^{(t)}, \quad (3)$$

where we took into account inequality (2) and that  $d^{(t)}(e_l) < 2^i D$  for  $l \in I_i$ .

Next, we will show that for all  $i \in [1, b]$ :

$$2^{i-1} D - a_{i-1}^{(t)} \leq |\mathcal{X}_i| \leq |I_i|(a_i^{(t)} + 1). \quad (4)$$

The right inequality in (4) is proved as follows:  $|\mathcal{X}_i| \leq \sum_{l \in I_i} |\Delta_l| \leq |I_i|(a_i^{(t)} + 1)$ , where the latter inequality is a consequence of (2).

We now prove the left inequality in (4). If  $a_{i-1}^{(t)} \geq 2^{i-1} D$ , then the bound is trivial. In the case when  $a_{i-1}^{(t)} < 2^{i-1} D$ , we will first prove that:

$$[2^{i-1} D + a_{i-1}, 2^i D] \subseteq \mathcal{X}_i. \quad (5)$$

To this end, take any  $x \in [2^{i-1} D + a_{i-1}, 2^i D]$  and observe that by (1), there exists some  $j \in [1, b]$  such that  $x \in \mathcal{X}_j$ . Moreover, note that:

1. For any  $j < i$ ,  $x \notin \mathcal{X}_j$ , because, by (3),  $\max \mathcal{X}_j < 2^j D + a_j^{(t)} \leq 2^{i-1} D + a_{i-1}^{(t)} \leq x$ .
2. For any  $j > i$ ,  $x \notin \mathcal{X}_j$ , because:  $\min \mathcal{X}_j = \min_{l \in I_j, \Delta_l \neq \emptyset} \min \Delta_l = \min_{l \in I_j, \Delta_l \neq \emptyset} d^{(t)}(e_l) \geq 2^{j-1} D \geq 2^j D > x$ .

Thus,  $x \in \mathcal{X}_i$ , and (5) follows. Equation (5) implies that  $|\mathcal{X}_i| \geq 2^{i-1} D - a_{i-1}^{(t)}$ , which completes the proof of (4). Next, by (4),

$$|I_i| \geq \frac{2^{i-1} D - a_{i-1}^{(t)}}{a_i^{(t)} + 1} \quad \text{for all } i \in [1, b].$$

The buckets  $I_1, I_2, \dots, I_b$  are pairwise disjoint by definition and contain at most  $D'$  elements altogether, which gives:

$$D + 2 \geq D' \geq \sum_{i=1}^b |I_i| \geq \sum_{i=1}^b \frac{2^{i-1} D - a_{i-1}^{(t)}}{a_i^{(t)} + 1} \geq \sum_{i=1}^b \frac{2^{i-1} D}{a_i^{(t)} + 1} - b,$$

where in the last inequality we used the fact that  $a_i^{(t)} \geq a_{i-1}^{(t)}$  for  $i \in [2, b]$ . Dividing the sum in the last inequality by  $bD$ , we get the following expression for the arithmetic average:

$$\frac{1}{b} \sum_{i=1}^b \frac{2^{i-1}}{a_i^{(t)} + 1} \leq \frac{D + b + 2}{bD} = \frac{1}{b} + \frac{1 + 2/b}{D} < \frac{9.2}{b},$$

where in the last inequality we took into account that  $k \leq 2^{16D}$  and  $b \leq (\log k)/2$  by assumption, hence  $D \geq (\log k)/16 \geq b/8$ , and that  $b = \lfloor (\log k)/2 \rfloor \geq 80$ . All the elements of the considered

sum are positive, hence by Markov's inequality, there exists a subset of indices  $S^{(t)} \subseteq [1, b]$ , with  $|S| \geq b/2$ , such that for all  $j \in S^{(t)}$  we have:

$$\frac{2^{j-1}}{a_j^{(t)} + 1} \leq 2 \cdot \frac{1}{b} \sum_{i=1}^b \frac{2^{i-1}}{a_i^{(t)} + 1} \leq \frac{18.4}{b}.$$

This implies that for all  $j \in S^{(t)}$ :

$$a_j^{(t)} \geq \frac{b}{18.4} \cdot 2^{j-1} - 1 > \frac{b}{25} \cdot 2^{j-1}, \quad (6)$$

where we again took into account that  $b \geq 80$ .

Fix  $t_1 = \lceil 100mD/b \rceil$ . We now prove that

$$t_C \leq t_0 + 2t_1 + 4m. \quad (7)$$

Suppose, by contradiction, that  $t_C > t_0 + 2t_1 + 4m$ . We will say that an index  $j \in [1, b]$  is *good* after time  $t$  if  $j \in S^{(t)}$ . Since for all  $t \in [t_0, t_C)$  we have  $|S^{(t)}| \geq b/2$  and  $S^{(t)} \subseteq [1, b]$ , by the pigeon-hole principle there must exist an index  $j^*$  that is good in at least  $(t_C - t_0)/2 = t_1 + 2m$  steps in  $[t_0, t_C)$ ; we will call these steps *good steps*.

For an arc  $e$  of the graph, we denote by  $t_e$  the so called *exit time step* for arc  $e$ , after which the total number of visits to arc  $e$  of the graph for the first time exceeds  $2^{j^*}D$ :  $d^{(t_e)}(e) \leq 2^{j^*}D < d^{(t_e+1)}(e)$ . The set of all exit time steps, taken over all arcs of the graph, is denoted  $\hat{T} = \{t_e : e \in \vec{E}\}$ . Note that  $e \in E_{2^{j^*}D}^{(t)}$  if and only if  $t \leq t_e$ , and therefore we may write:

$$\sum_{t \in [0, t_C) \setminus \hat{T}} a_{j^*}^{(t)} = \sum_{t \in [0, t_C) \setminus \hat{T}} \sum_{e \in E_{2^{j^*}D}^{(t)}} a^{(t)}(e) \leq \sum_{e \in \vec{E}} \sum_{t=0}^{t_e-1} a^{(t)}(e) = \sum_{e \in \vec{E}} d^{(t_e)}(e) \leq 2m \cdot 2^{j^*}D. \quad (8)$$

Now, recall that there are at least  $t_1 + 2m$  good time steps  $t \in [t_0, t_C)$  for which index  $j^*$  satisfies (6), and that  $|\hat{T}| \leq 2m$ . It follows that:

$$\sum_{t \in [0, t_C) \setminus \hat{T}} a_{j^*}^{(t)} > t_1 \cdot \frac{b}{25} \cdot 2^{j^*-1} = \left\lceil \frac{100mD}{b} \right\rceil \frac{b}{25} \cdot 2^{j^*-1} \geq 2m \cdot 2^{j^*}D,$$

a contradiction with (8). Thus, we have proved (7).

By (7), we obtain

$$\begin{aligned} t_C &\leq t_0 + 2t_1 + 4m = \left\lceil \frac{2^{b+1}mD}{k} \right\rceil + 2 \left\lceil \frac{100mD}{b} \right\rceil + 4m \leq \\ &\leq \frac{mD}{\log k} \left( \frac{2^{b+1} \log k}{k} + \frac{200 \log k}{b} + \frac{4 \log k}{D} + \frac{3 \log k}{mD} \right) \end{aligned} \quad (9)$$

Taking into account that  $b = \lfloor (\log k)/2 \rfloor$ ,  $k \leq 2^{16D}$ , and  $k > 2^{160}$ , we obtain that the expression in the above bracket can be bounded by a constant, giving:  $t_C < 500 \frac{mD}{\log k}$ . This completes the proof for the case  $k > 2^{160}$ .

Suppose now that  $k \leq 2^{160}$ . Yanovski *et al.* [16] showed that a single agent explores the graph in at most  $2mD$  steps regardless of the initialization, and moreover, that adding agents cannot decrease the number of traversals on any edge. We thus trivially obtain the claim:  $t_C \leq 2mD < 500 \frac{mD}{\log k}$ .  $\square$

We now consider the case when  $k \geq 2^{16D}$ . Here, we first make the additional assumption that each agent starts from a distinct node. We show that additional assumption implies that no arc is traversed by more than one agent in a single step. The proof then proceeds along similar lines as that of Theorem 3.3, and we show that in many time steps  $t$ , there exists a pair of arcs  $e_{l+1}, e_l$  in  $\mathcal{P}$  with a large difference in the number of traversals up to time  $t$ . However, instead of counting the number of long arcs on path  $\mathcal{P}$  belonging to a bucket  $I_i$ , in this proof we take advantage of the fact that the length of the path  $D' \leq D + 2$  is small compared to  $\log k$ , which can be used to infer the existence of the sought arc pairs.

**Lemma 3.4.** *Let  $G = (V, E)$  be any undirected graph with arbitrary initialization of pointers and let  $D$  be the diameter of  $G$ . If  $k \geq 2^{16D}$ , then a team of  $k$  agents performing parallel rotor-router movement, with each agent starting from a distinct node of the graph, explores  $G$  in time  $16mD/\log k$ .*

*Proof.* We first prove that in every step  $t \in \mathbb{N}$  of the exploration, every arc is traversed by at most one agent. Assume, to the contrary, that  $t^* \in \mathbb{N}$  is the first step when two agents traverse the same arc, and let this arc be  $e = u \rightarrow v$ . Then, by virtue of the rotor-router principle, the number of agents located at  $u$  at the end of step  $t^* - 1$  must have been at least  $\deg(u) + 1$ . This in particular implies that  $t^* > 1$ . Since there are exactly  $\deg(u)$  incoming arcs to  $u$ , one of them was traversed by more than one agent in step  $t^* - 1$ . This contradicts the minimality of  $t^*$ . Thus, we have  $a^{(t)}(e) \leq 1$ , for all  $e \in \vec{E}$  and  $t \in \mathbb{N}$ .

Denote by  $t_C$  the cover time of graph  $G$ . For  $i \in \mathbb{N}_+$ , let  $X = \lfloor k^{1/(2D+6)} \rfloor$  and let  $Y_i = \sum_{j=0}^{i-1} X^j = \frac{X^i - 1}{X - 1}$ . Note that since  $k \geq 2^{16D}$ , we have:

$$X \geq 2 \quad \text{and} \quad Y_i < X^i \quad \text{for all } i \in \mathbb{N}. \quad (10)$$

Similarly as in proof of Theorem 3.3, we first consider a setup phase, consisting of steps  $[1, t_0)$  of exploration, this time defining  $t_0$  as:

$$t_0 = 2 \lceil mX^{2D+5}/k \rceil \leq 2 \lceil m/X \rceil. \quad (11)$$

During the setup stage, the total number of edge traversals is at least  $2mX^{2D+5}$ . Thus, there exists an arc  $e'$  such that  $d^{(t_0)}(e') \geq X^{2D+5}$ . There also exists an arc  $e''$  such that  $d^{(t_C-1)}(e'') = 0$ . Thus, for each  $t \in [t_0, t_C)$ ,

$$d^{(t)}(e'') = 0 \quad \text{and} \quad d^{(t)}(e') \geq X^{2D+5} > Y_{2D+5}. \quad (12)$$

Since  $D$  is the diameter of  $G$ , there exists a path  $\mathcal{P} = \langle e'' = e_1, e_2, \dots, e_{D'} = e' \rangle$ , such that  $D' \leq D + 2$  and for all  $i \in [1, D')$ ,  $e_i = v_i \rightarrow v_{i+1}$  where  $v_i, v_{i+1} \in V$ .

For each time step  $t$  and  $i \geq 2$ , let  $a_i^{(t)}$  be the number of agents that during step  $t + 1$  traverse those arcs which were traversed at most  $Y_i$  times until the end of step  $t$ ,  $a_i^{(t)} \equiv \sum_{e \in E_{Y_i}^{(t)}} a^{(t)}(e)$ .

We have for any  $i \geq 2$ :

$$\sum_{t=t_0}^{t_C-1} a_i^{(t)} \leq 2m(Y_i + 1) < 3mY_i, \quad (13)$$

because otherwise we would have an arc  $e$  that contributes at least  $Y_i + 2$  to the above sum. Then, since in each time step  $t \in N$  each arc is traversed at most once, there exist steps  $t_0 < t_1 < t_2 < \dots < t_{Y_i+2} \leq t_C$  in which  $e$  is traversed, and moreover  $e \in E_{Y_i}^{(t_{Y_i+2}-1)}$ . However, till the end of step  $t_{Y_i+2} - 1 \geq t_{Y_i+1}$  the arc  $e$  has been traversed  $Y_i + 1$  times, so,  $e \notin E_{Y_i}^{(t_{Y_i+2}-1)}$ , and we obtain a contradiction, proving (13).

We now prove that

$$t_C \leq t_0 + 6 \left\lceil \frac{m}{X-1} \right\rceil. \quad (14)$$

Suppose, by contradiction, that  $t_C > t_0 + 6\lceil m/(X-1) \rceil$ . For each time step  $t$ , we will call the set of arcs  $E_{Y_i}^{(t)} \setminus E_{Y_{i-1}}^{(t)}$  the  $i$ -th zone at time  $t$ , for  $i \geq 2$ .

Each zone that does not contain any arc of path  $\mathcal{P}$  in a given time step is called *free*. The path  $\mathcal{P}$  has at most  $D'$  arcs and hence at least  $D'$  zones with indices in the interval  $[2, 2D'+1]$  are free in each time step. Thus, by the pigeonhole principle, during the time period  $[t_0, t_C)$  there must exist an index  $i^* \in [2, 2D'+1]$  such that the  $i^*$ -th zone is free during a set of time steps  $T \subseteq [t_0, t_C)$ , with:

$$|T| \geq (t_C - t_0)/2 > 3\lceil m/(X-1) \rceil.$$

By (12), the arc  $e'$  belongs to a zone with index at least  $2D+6 \geq 2D'+2$  in each time step  $t \in T$ , while arc  $e''$  belongs to zone 1. Since the  $i^*$ -th zone is free at time  $t$ , by following path  $\mathcal{P}$  from arc  $e'$  to  $e''$ , we will necessarily encounter an index  $j \in [1, D')$ , such that  $d^{(t)}(e_{j+1}) \geq Y_{i^*+1} + 1$  and  $d^{(t)}(e_j) \leq Y_{i^*}$ , which gives:

$$d^{(t)}(e_{j+1}) - d^{(t)}(e_j) \geq Y_{i^*+1} + 1 - Y_{i^*} = X^{i^*} + 1.$$

By Lemma 3.2, for each  $t \in T$ , at least  $X^{i^*}$  agents traverse arcs from set  $E_{Y_{i^*}}^{(t)}$  in step  $t+1$ , i.e.,  $a_{i^*}^{(t)} \geq X^{i^*}$ . Thus,

$$\sum_{t \in T} a_{i^*}^{(t)} \geq |T|X^{i^*} \geq 3 \left\lceil \frac{m}{X-1} \right\rceil X^{i^*} > 3mY_{i^*}.$$

This contradicts (13), completing the proof of (14). Note that:

$$X = \left\lfloor k^{1/(2D+6)} \right\rfloor$$

By (14), (11), and the definition of  $X$ , we have:

$$t_C \leq 2 \left\lceil \frac{m}{X} \right\rceil + 6 \left\lceil \frac{m}{X-1} \right\rceil \leq 8 \frac{m}{X-1} + 8 \leq \frac{mD}{\log k} \left( \frac{\log k}{D(k^{1/(8D)} - 2)} + 8 \right).$$

Observe that for fixed  $D$ , the expression in the above bracket is strictly decreasing with  $k$  for  $k > 2^{8D}$ , and for  $k = 2^{16D}$  takes a value of 16. Knowing that  $k \geq 2^{16D}$ , we therefore obtain  $t_C < 16 \frac{mD}{\log k}$ .  $\square$

It remains to consider the case not covered by the above lemma, when not all agents start from distinct positions. In fact, we will reduce such a case to the one already considered by making use of the concept of delayed deployments discussed in Section 2.

**Lemma 3.5.** *Let  $R$  and  $R'$  be two starting configurations of the  $k$ -agent rotor-router system with cover times  $t_C$  and  $t'_C$ , respectively. Suppose that there exists a delayed deployment  $D$  of  $R$  whose execution transforms the starting configuration of  $R$  into the starting configuration of  $R'$  in  $\hat{t}$  time steps. Then,  $t_C \leq \hat{t} + t'_C$ .*

*Proof.* Observe that the concatenation of the execution of deployment  $D$  for  $\hat{t}$  steps and  $R'$  for  $t'_C$  steps is a delayed deployment of  $R$  which explores the graph in  $t_C \leq \hat{t} + t'_C$  steps. The claim follows by Lemma 2.1.  $\square$

The next lemma provides an upper bound on the time of transforming a rotor-router configuration with at most  $n$  agents into one in which agents occupy distinct starting nodes.

**Lemma 3.6.** *For any initialization  $R$  of the rotor-router system with  $k$  agents,  $k \leq n$ , there exists a delayed deployment  $D$  of  $R$  which terminates in a configuration in which all agents occupy distinct positions after  $\hat{t} \leq k^4$  steps.*

*Proof.* In deployment  $D$ , we release agents sequentially from their starting positions in  $R$ , moving one agent only at a time until it is located at a node unoccupied by another agent. Consider the phase in which we move a fixed agent  $a$  in this deployment. In the worst case,  $a$  has to explore the graph induced by all nodes occupied to date. The agent acts a single-agent rotor router system with respect to this graph. Recall that the cover time of a graph with  $m$  edges and diameter  $D$  by a single agent is at most  $2mD$ , regardless of the initial configuration [16]. Since in the considered system there are at most  $k$  occupied nodes with at most  $k^2/2$  edges between them, and the graph of occupied nodes has diameter at most  $k$ ,  $a$  finds an unoccupied node within  $2 \cdot k^2/2 \cdot k = k^3$  steps. This has to be done by each of  $k$  agents, thus total time of all phases of the delayed deployment is  $\hat{t} \leq k^4$ .  $\square$

When  $1 < k \leq \lceil n^{1/5} \rceil$ , we can bound the time  $\hat{t}$  in the above lemma as:  $\hat{t} \leq k^4 \leq 32n/k \leq 64m/k \leq 128 \frac{mD}{\log k}$ .

Combining the above result with Lemmas 3.4 and 3.5, we obtain that for any rotor router initialization with  $k$  agents,  $k \leq \lceil n^{1/5} \rceil$  and  $k \geq 2^{16D}$ , exploration is completed within time  $t_C = \hat{t} + t'_C \leq 128 \frac{mD}{\log k} + 16 \frac{mD}{\log k} = 144 \frac{mD}{\log k}$ . On the other hand, when  $k < 2^{16D}$ , by Theorem 3.3, the cover time is  $t_C \leq 500 \frac{mD}{\log k}$ . It follows that the bound  $t_C \leq 500 \frac{mD}{\log k}$  holds for all starting configurations with  $k \leq \lceil n^{1/5} \rceil$ .

When  $k > \lceil n^{1/5} \rceil$ , we can make use of a result of Yanovski *et al.* [16], stating that the worst-case initialization of a rotor-router system with  $k$  agents cannot have greater cover time than the worst-case initialization of a system with  $k' < k$  agents. Putting  $k' = \lceil n^{1/5} \rceil$ , for any  $k > \lceil n^{1/5} \rceil$  we obtain:  $t_C \leq 500 \frac{mD}{\log k'} \leq 2500 \frac{mD}{\log n}$ . Finally, combining the results for  $k \leq \lceil n^{1/5} \rceil$  and  $k > \lceil n^{1/5} \rceil$  gives the following theorem.

**Theorem 3.7.** *Let  $G = (V, E)$  be any undirected graph with arbitrary initialization of pointers and let  $D$  be the diameter of  $G$ . A team of  $k$  agents performing in parallel the rotor-router movement explores  $G$  in time  $\max\{500mD/\log k, 2500mD/\log n\}$ , regardless of the initial positions of agents. In particular, if  $k \leq n^c$  for some  $c > 0$ , then the cover time is at most  $2500c \cdot mD/\log k$ .  $\square$*

Theorems 3.3 and 3.7 imply that the cover time of the rotor-router is  $O(mD/\log k)$  for all graphs, whenever  $k \in 2^{O(D)}$  or  $k \in O(\text{poly}(n))$ . On the other hand, the cover time of the rotor-router is trivially lower-bounded as  $\Omega(D)$  for a team of agents starting from a single node, regardless of the number of agents. It follows that it is not possible to extend the bound of  $O(mD/\log k)$  on cover time beyond the range  $k \in 2^{O(n)}$ . We leave as open the question of whether the considered bound can be achieved for the (rather special) range of values of  $k$  not covered by Theorems 3.3 and 3.7.

## 4 Lower bound on cover time

**Theorem 4.1.** *Let  $G = (V, E)$  be any undirected graph of diameter  $D$ . There exists a port labeling of the edges of  $G$ , an initialization of pointers and an assignment of starting positions to a team of  $k$  agents, such that the exploration performed in parallel with the rotor-router movement has cover time  $t_C \geq \frac{1}{5}mD/k$ .*

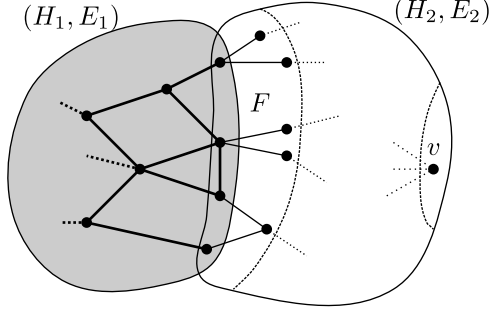


Figure 2: Graph decomposition used in the proof of Theorem 4.1.

*Proof.* If  $k > m$ , we make all agents start from an arbitrarily chosen single node, and choose an arbitrary pointer initialization. In such scenario, the exploration will be completed after time at least  $D > \frac{mD}{k}$ . Thus, we can safely assume that  $k \leq m$ .

Without the loss of generality, we can assume that  $D \geq 10$ . Otherwise, we can bound the exploration time by  $2m/k \geq \frac{1}{5}mD/k$ .

For any graph  $G = (V, E)$ , as shown in [3, Theorem 2], there exists a partition of the edge set  $E = E_1 \cup E_2$ , such that (see Fig. 2 for an illustration):

- (i)  $|E_1| \geq \frac{m}{2}$ ,
- (ii) there exist  $V_1 \subseteq V$  and  $V_2 \subseteq V$  such that the subgraphs  $H_1 = G[V_1]$  and  $H_2 = G[V_2]$  are connected and their edge sets are  $E_1$  and  $E_2$ , respectively,
- (iii) there exists a node  $v \in V_2$  being at distance at least  $\frac{D}{2}$  from each node of  $H_1$ .

Denote by  $F \subset E_2$  the set of edges incident to some node from  $H_1$ .

Now, let  $C = \{e_1, e_2, \dots, e_{2|E_1|}\}$  be a directed Eulerian cycle in  $\vec{H}_1$  (the bidirected subgraph corresponding to  $H_1$ ) traversing every edge in  $E_1$  exactly once in each direction. To simplify notation, let  $\Delta = \left\lfloor \frac{2|E_1|}{k} \right\rfloor$ .

We choose an arbitrary set of indexes  $1 = j_1 < j_2 < \dots < j_k \leq 2|E_1|$  such that they are spread (almost-)equidistantly in  $\{1, \dots, 2|E_1|\}$ , that is:

$$\forall_{1 \leq i < k} \quad j_{i+1} - j_i \in \{\Delta, \Delta + 1\} \quad \text{and} \quad j_1 - j_k + 2|E_1| \in \{\Delta, \Delta + 1\}.$$

This is possible because, due to (i),  $2|E_1| \geq k$ .

We partition  $E_1$  into  $\Delta$  sets  $S_1, \dots, S_\Delta$  of size  $k$ :

$$S_{i+1} = \{e_{j_1+i}, e_{j_2+i}, \dots, e_{j_k+i}\}, \quad \text{for } 0 \leq i < \Delta,$$

and one set for all remaining edges:  $R = E_1 \setminus \bigcup_{t=1}^{\Delta} S_t$ .

We choose the starting positions of  $k$  agents, the port assignment, and the initialization of pointers for the edges in  $E_1$  such that in their first  $\Delta + 1$  steps, the  $k$  agents traverse all edges in  $E_1$  in the following delayed deployment: for each  $t \in \{1, \dots, \Delta\}$ , in the  $t$ -th step, exactly the edges in  $S_t$  are traversed, whereas in the  $(\Delta + 1)$ -th step we delay some agents so that exactly the edges in  $R$  are traversed. We achieve this by setting outgoing ports so that, for every node  $u$  in  $H_1$ , we order the edges in  $E_1$  incident to  $u$  by assigning smaller ports to edges in  $S_t$  than to the edges in  $S_{t+1}$ , for each  $t \in \{1, \dots, \Delta\}$ , where  $S_{\Delta+1} = R$ . Such a port ordering is enough to explore the graph  $H_1$ , with delayed deployment, with the property that every edge is visited once every  $\Delta + 1$  steps.

Now we assign ports to the edges in  $F$ . To this end, we consider the subgraph of  $G$ , denoted by  $\tilde{G}$ , consisting of the edges in  $E_1 \cup F$ . In other words, we take  $H_1$  (together with the port assignment obtained above) and we add the edges in  $F$ , obtaining  $\tilde{G}$ . Note that, by (ii), each edge in  $F$  has one endpoint in  $V_1$  and the other endpoint in  $V \setminus V_1$ . The ports in  $F$  are determined by analyzing the behavior of agents in the graph  $\tilde{G}$  in the delayed deployment described above. Whenever any set of agents are about to leave  $H_1$  and traverse any edge from  $F$ , we select a single agent in a deterministic way (for example, by choosing the agent located on a node with the smallest index, having indexes assigned to nodes). We stop all other agents and perform traversals only with the selected agent, until it returns to  $H_1$ . We set the ports of the edges in  $F$  so that whenever an agent leaves  $H_1$  through an edge  $(v \rightarrow u) \in F$  ( $v \in V_1, u \notin V_1$ ), it returns to  $H_1$  through the edge  $(u \rightarrow v)$  (we call this property *the property of return*). Having the property of return, we achieve that the agents patrol  $E_1$ , and whenever an agent is about to leave  $H_1$ , the other agents are delayed until the agent returns to the same node. Since the selection of agents is done deterministically, the edges in  $F$  are always traversed in separated periods of time (when one agent is traversing edges from  $F$ , all other agents are stopped) in a cyclic fashion, i.e., the sequence of traversal of the edges in  $F$  is  $(f_1, f'_1, f_2, f'_2, \dots, f_{|F|}, f'_{|F|})^*$ , where  $f'$  means the reversed edge to an edge  $f$ , i.e., if  $f = (u \rightarrow v)$ , then  $f' = (v \rightarrow u)$ . Denote  $f_i = (u_i \rightarrow v_i)$  for each  $i \in \{1, \dots, |F|\}$ .

It remains to assign port labels to the edges in  $E_2 \setminus F$ , and to initialize the pointers for the nodes in  $V \setminus V(\tilde{G})$ . This is done by first constructing a multigraph  $G'$  and then by analyzing a single agent movement in  $G'$ . The node set of  $G'$  is  $\{h\} \cup (V \setminus V_1)$ . For each  $(u \rightarrow v) \in E_2 \setminus F$ , let  $(u \rightarrow v)$  be an edge of  $G'$ , and for each  $i \in \{1, \dots, |F|\}$ , let  $(h, v_i)$  and  $(v_i, h)$  be the edges of  $G'$ . In other words, we construct  $G'$  by taking  $G$ , leaving the edges in  $E \setminus E_1$  untouched, and contracting (identifying) the nodes of  $H_1$  into the single node  $h$ . (The loops at  $h$  formed by the edges in  $E_1$  are discarded.) For each  $i \in \{1, \dots, |F|\}$ , the ports of  $(h \rightarrow v_i)$  and  $(v_i \rightarrow h)$  equal the ports of  $(u_i, v_i)$  and  $(v_i, u_i)$ , respectively.

We set the remaining ports in  $G'$  and pointer initialization so that a single agent that starts at  $h$  explores  $G'$  in the following way:

- (a) The edges in  $F$  are traversed according to the order

$$((h \rightarrow v_1), (v_1 \rightarrow h), (h \rightarrow v_2), (v_2 \rightarrow h), \dots, (h \rightarrow v_{|F|}), (v_{|F|} \rightarrow h)).$$

Later on, we use the port labeling of  $G'$  to assign port labels to the edges in  $E_2$  in  $G$ , and the above allows us to maintain the return property in  $G$ .

- (b) The agent requires at least  $(D/2 - 1)$  traversals through each of the edges in  $F$ . This follows from the fact that, due to (iii), there exists a node in  $G'$  being at distance at least  $D/2$  from  $h$ .

The above process assigns port labels to the edges in  $E_2$  and sets initial values of all pointers in  $G'$ , which completes the construction of  $G$  and the initial setup of the rotor-router.

Now we analyze the delayed deployment performed by the  $k$  agents in  $G$ . We divide the exploration of  $G$  into phases. The  $i$ -th phase starts in the step in which each edge in  $S_1$  is traversed for the  $i$ -th time, and ends in the step preceding the beginning of the  $(i + 1)$ -th stage. Note that each stage contains at least  $\Delta$  steps in which all agent move simultaneously. By (a), the property of return holds in  $G$ , and therefore each edge in  $F$  is traversed exactly once in each phase. Thus, by (b), at least  $(D/2 - 1)$  phases are required in the delayed deployment to explore  $G$ . This means that we need  $\tau$  steps in which all agents move simultaneously to fully explore

the graph  $G$ , where (since  $D \geq 10$ ):

$$\tau \geq \Delta \cdot (D/2 - 1) = \left\lfloor \frac{2|E_1|}{k} \right\rfloor \cdot (D/2 - 1) \geq \left\lfloor \frac{m}{k} \right\rfloor \cdot (D/2 - 1) \geq \frac{1}{2} \cdot \frac{m}{k} \cdot (D/2 - 1) \geq \frac{1}{5}mD/k$$

We can now apply Lemma 2.1 for the considered deployment, obtaining that the cover time of  $G$  is  $t_C \geq \tau \geq \frac{1}{5}mD/k$ .  $\square$

The bound in Theorem 4.1 is asymptotically tight for some graph classes, for example for stars. We leave the following simple observation without proof.

**Proposition 4.2.** *Let  $G$  be a star on  $n$  nodes. A team of  $k \leq n$  agents covers  $G$  in time  $t_C \leq 2\lceil n/k \rceil$ , for any initialization of the rotor-router and any initial positions of agents.*  $\square$

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