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THE PROBLEM OF DETECTING CORROSION BY AN ELECTRIC MEASUREMENT REVISITED

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ABSTRACT. We establish a logarithmic stability estimate for the problem of detecting corrosion by a single electric measurement. We give a proof based on an adaptation of the method initiated in [3] for solving the inverse problem of recovering the surface impedance of an obstacle from the scattering amplitude. The key idea consists in estimating accurately a lower bound of the local L^2 -norm at the boundary, of the solution of the boundary value problem used in modeling the problem of detection corrosion by an electric measurement.

1. Introduction. Let Ω be a C^3 -smooth bounded domain of \mathbb{R}^n , $n = 2, 3$, so that the following assumption fulfilled: any two points of Ω can be joined by a broken line consisting of at most ℓ segments, where ℓ is a given non negative integer. A domain satisfying this property with $\ell = 1$ is nothing but a starshaped domain.

We denote the boundary of Ω by Γ and we consider the following boundary value problem (abbreviated to BVP in the sequel)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + q(x)u = g & \text{on } \Gamma. \end{cases} \quad (1)$$

In the sequel, $g \in H^{3/2}(\Gamma)$ and g is non identically equal to zero.

For $s \in \mathbb{R}$ and $1 \leq r \leq \infty$, we introduce the vector space

$$B_{s,r}(\mathbb{R}^{n-1}) := \{w \in \mathcal{S}'(\mathbb{R}^{n-1}); (1 + |\xi|^2)^{s/2} \widehat{w} \in L^r(\mathbb{R}^{n-1})\},$$

where $\mathcal{S}'(\mathbb{R}^{n-1})$ is the space of tempered distributions on \mathbb{R}^{n-1} and \widehat{w} is the Fourier transform of w . Equipped with its natural norm

$$\|w\|_{B_{s,r}(\mathbb{R}^{n-1})} := \|(1 + |\xi|^2)^{s/2} \widehat{w}\|_{L^r(\mathbb{R}^{n-1})},$$

$B_{s,r}(\mathbb{R}^{n-1})$ is a Banach space (it is noted that $B_{s,2}(\mathbb{R}^{n-1})$ is merely the usual Sobolev space $H^s(\mathbb{R}^{n-1})$). By using local charts and a partition of unity, we construct $B_{s,r}(\Gamma)$ from $B_{s,r}(\mathbb{R}^{n-1})$ similarly as $H^s(\Gamma)$ is built from $H^s(\mathbb{R}^{n-1})$.

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To carry out our analysis, we need solutions of the BVP (1) with some smoothness. In order to give sufficient conditions on the coefficient q guaranteeing this smoothness, we set

$$\mathcal{Q} = \{q \in B_{3/2,1}(\Gamma); q \geq 0 \text{ and } q \neq 0\}$$

and

$$\mathcal{Q}_M = \{q \in \mathcal{Q}; \|q\|_{B_{3/2,1}(\Gamma)} \leq M\},$$

where $M > 0$ is a given constant.

By [10, Theorem 2.3], for any $q \in \mathcal{Q}$, the BVP (1) has a unique solution $u_q \in H^3(\Omega)$. Moreover,

$$\|u_q\|_{H^3(\Omega)} \leq C_0 \text{ for all } q \in \mathcal{Q}_M. \quad (2)$$

The constant C_0 above can depend only on Ω , g and M .

It is worthwhile to mention that, from the classical embedding theorems, $H^3(\Omega)$ is continuously embedded in $C^2(\overline{\Omega})$ if $n = 2$ and in $C^{1,1/2}(\overline{\Omega})$ if $n = 3$. Due to the regularity of Ω , $C^2(\overline{\Omega})$ is continuously embedded in $C^{1,\beta}(\overline{\Omega})$ for any $0 \leq \beta \leq 1$. Therefore $H^3(\Omega)$ is continuously embedded in $C^{1,1/2}(\overline{\Omega})$ when $n = 2$ or 3 .

A typical example corresponding to the mathematical analysis we develop in the present work is the problem of detecting the corrosion inside a pipe by electric measurements. This is one of the most important topics in engineering, for instance for administering safely a nuclear power station.

Usually, in a BVP modeling the problem of detecting corrosion damage by electric measurements the boundary Γ consists in two parts: $\Gamma = \overline{\Gamma_a \cup \Gamma_i}$, Γ_a and Γ_i being two disjoint relatively open subsets of Γ . Here, Γ_a corresponds to the part of the boundary accessible to measurements and Γ_i is the inaccessible part of the boundary where the corrosion damage may occur.

Henceforth, we assume that the current flux g satisfies $\text{supp}(g) \subset \Gamma_a$. The function q in (1) is known as the corrosion coefficient and it is naturally supported on Γ_i . This motivates the introduction of the following subset of \mathcal{Q}_M :

$$\mathcal{Q}_M^0 = \{q \in \mathcal{Q}_M; \text{supp}(q) \subset \Gamma_i\}.$$

We are interested in the stability issue for the problem consisting in the determination of the boundary coefficient q from the boundary measurement $u_q|_\gamma$, where γ is an open subset of the accessible sub-boundary Γ_a . In the sequel, we assume that γ does not meet $\text{supp}(g)$:

$$\gamma \subset \Gamma_a \setminus \text{supp}(g).$$

We aim to prove the following theorem. We only sketch the main steps of the proof since the most intermediate results consist in an adaptation of the ones already proved in [3] (see also [14]).

Theorem 1.1. *There exist four constants $C_i > 0$ and $\sigma_i > 0$, $i = 0, 1$ so that, for any $q, \tilde{q} \in \mathcal{Q}_M^0 \cap C^{0,\alpha}(\Gamma)$,*

$$\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq C_0 \Psi(C_1 \Phi(\|u - \tilde{u}\|_{L^2(\gamma)})),$$

where $u = u_q$, $\tilde{u} = u_{\tilde{q}}$ and

$$\Psi(\rho) = |\ln \ln \rho|^{-\sigma_1} + \rho, \quad \rho > 0,$$

$$\Phi(\rho) = |\ln \rho|^{-\sigma_2} + \rho, \quad \rho > 0.$$

Theorem 1.1 can be seen as a completion of the results established in [8] in dimension two and in [4] in dimensions two and three. We note that in the above mentioned works the difference of $q - \tilde{q}$ is only estimated in a compact subset of $\{x \in \Gamma_i; u_q(x) \neq 0\}$. However, there is a counterpart in estimating $q - \tilde{q}$ in the whole Γ_i . The stability estimates in [8] and [4] are of single logarithmic type, while the estimate in Theorem 1.1 is of triple logarithmic type. This means that the stability deteriorates near the points where the solution of the BVP (1) vanishes.

There is a wide literature treating the problem of detecting corrosion by electric measurements. We refer to [1, 6, 7, 9, 11, 12, 15, 16, 18, 19] where various type of stability estimates are given. We just quote these few references, but of course there are many others. A neighbor problem is the one consisting in the determination of the surface impedance of an obstacle from the scattering amplitude (e.g. [2, 3, 17] and the reference therein).

To our knowledge, the existing results on stability including the vicinity of the zeroes of the solution, of the BVP under consideration, do not give self contained proofs. They always refer to several previous works. Therefore it is very hard to recover completely the proofs. Although our result seems to be weaker than some of these existing results, our method is direct and it is based only on an elementary Carleman inequality. In fact our result is not really comparable with those of the literature because in the existing results q is only estimated on a fixed subset of Γ_i .

The rest of this text consists in two sections. In section 2 we estimate accurately a lower bound of the local L^2 -norm at the boundary, of the solution of the BVP (1). We show, step by step, how we adapt the method in [3] to the present problem. Section 3 is devoted the proof of Theorem 1.1.

Unless otherwise specified, all the constants we use in the sequel depend only on data.

2. Lower bound for L^2 -norm at the boundary. We first note that as Ω is C^3 -smooth, it has the uniform interior cone property, abbreviated to the **UICP** in the sequel. That is there are $R > 0$ and $\theta \in]0, \pi[$ satisfying, for all $\tilde{x} \in \Gamma$, we find $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$ and

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R \text{ and } (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

The domain Ω satisfies also the uniform exterior sphere property, abbreviated to **UESP** in the sequel: there exists $\rho > 0$ so that, to any $\tilde{x} \in \Gamma$ corresponds $x_0 = x_0(\tilde{x}) \in \mathbb{R}^n \setminus \bar{\Omega}$ for which

$$B(x_0, \rho) \cap \Omega = \emptyset \text{ and } B(x_0, \rho) \cap \bar{\Omega} = \{\tilde{x}\}.$$

Additionally, ξ is definition of **UICP** can be chosen as follows $\xi = (\tilde{x} - x_0)/|\tilde{x} - x_0|$.

For $\delta > 0$, we set

$$\Omega^\delta = \{x \in \Omega; \text{dist}(x, \Gamma) > \delta\}$$

and we recall the following useful three sphere inequality for the H^1 -norm.

Lemma 2.1. *There exist $C > 0$ and $0 < s < 1$ so that: for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω , $y \in \Omega$ and $0 < r < \frac{1}{3} \text{dist}(y, \Gamma)$,*

$$r \|u\|_{H^1(B(y, 2r))} \leq C \|u\|_{H^1(B(y, r))}^s \|u\|_{H^1(B(y, 3r))}^{1-s}.$$

We refer to [3] for a proof. The case of a general divergence form operator is detailed in [13] and [14].

Proposition 1. *Let s be as in the previous lemma and fix $M > 0$. There exists a constant $C > 0$ so that for any $\delta > 0$, $u \in H^1(\Omega)$ satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega)} \leq M$ and, $x, y \in \Omega^\delta$,*

$$(Cr)^{1-s^N} \|u\|_{H^1(B(x,r))} \leq \|u\|_{H^1(B(y,2r))}^{s^N}, \quad 0 < r < \delta/6,$$

where N is the smallest integer satisfying $|x - y| - 2Nr \leq r$.

Proof. The proof is given for $\ell = 1$. For an arbitrary ℓ , the proof is quite similar with slight adaptation. Set

$$d = |y - x|, \quad \eta = \frac{y - x}{|y - x|}$$

and consider the sequence, where $0 < 2r < d$,

$$x_k = y - k(2r)\eta, \quad k \geq 1.$$

Clearly,

$$|x_k - x| = d - k(2r).$$

Let N be the smallest integer such that $d - N(2r) \leq r$, or equivalently

$$\frac{d}{2r} - \frac{1}{2} \leq N < \frac{d}{2r} + \frac{1}{2}.$$

By Lemma 2.1, there exist $C > 0$ and $0 < s < 1$ so that

$$Cr \|u\|_{H^1(B(x_1,r))} \leq \|u\|_{H^1(B(y,r))}^s,$$

where we take into account that $B(x_1, r) \subset B(y, 2r)$.

Similarly, we have

$$Cr \|u\|_{H^1(B(x_{k+1},r))} \leq \|u\|_{H^1(B(x_k,r))}^s, \quad k \geq 1.$$

Hence, an induction argument yields

$$(Cr)^{1-s^N} \|u\|_{H^1(B(x_N,2r))} \leq \|u\|_{H^1(B(y,2r))}^{s^N}. \quad (3)$$

Since $|x_N - x| = d - N(2r) \leq r$, $B(x, r) \subset B(x_N, 2r)$. Whence (3) entails

$$(Cr)^{1-s^N} \|u\|_{H^1(B(x,r))} \leq \|u\|_{H^1(B(y,2r))}^{s^N}. \quad (4)$$

□

We recall that according to Caccioppoli's inequality, for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω and $x \in \Omega$,

$$\|\nabla u\|_{L^2(B(x,2r))^n} \leq Cr^{-1} \|u\|_{L^2(B(x,3r))},$$

for $0 < 3r < \text{dist}(x, \Gamma)$.

Therefore the following corollary is immediate from Proposition 1.

Corollary 1. *Let s be as in the previous lemma and fix $M > 0$. There exists a constant $C > 0$ so that for any $\delta > 0$, $u \in H^1(\Omega)$ satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega)} \leq M$ and, $x, y \in \Omega^\delta$,*

$$(Cr)^{1-s^N} \|u\|_{H^1(B(x,r))} \leq \|u\|_{L^2(B(y,3r))}^{s^N}, \quad 0 < r < \delta/6,$$

where N is the smallest integer satisfying $|x - y| - 2Nr \leq r$.

Corollary 2. Fix $0 < \alpha < 1$ and $0 < \eta \leq M$, and set $\delta_0 = \min((\eta/(2M))^{1/\alpha}, R/2)$. There exists a constant $c > 0$ so that, for any $u \in H^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega})} \leq M$ and $|u(\tilde{x})| \geq \eta$, for some $\tilde{x} \in \Gamma$, and $0 < \delta \leq \delta_0$,

$$e^{-ce^{\frac{c}{\delta}}} \leq \|u\|_{L^2(B(y,r))}, \quad y \in \Omega^{(\delta/2) \sin \theta}, \quad 0 < r < (\delta/3) \sin \theta.$$

Here R and θ are as in the definition of the **UICP**.

Proof. Let $u \in H^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega})} \leq M$ and $|u(\tilde{x})| \geq \eta$, for some $\tilde{x} \in \Gamma$. If $z \in B(\tilde{x}, \delta_0)$ then

$$\begin{aligned} |u(z)| &\geq |u(\tilde{x})| - |u(\tilde{x}) - u(z)| \\ &\geq |u(\tilde{x})| - |\tilde{x} - z|^\alpha [u]_\alpha \\ &\geq \eta - \delta^\alpha M = \eta/2. \end{aligned}$$

Let $\xi = \xi(\tilde{x})$ be as in the definition of **UICP** and set $x = \tilde{x} + (\delta_0/2)\xi$. Since $B(x, (\delta_0/2) \sin \theta) \subset \mathcal{C}(x) \subset \Omega$, $x \in \Omega^{(\delta_0/2) \sin \theta}$. If $\delta \leq \delta_0$, we get by applying Corollary 1, for all $y \in \Omega^{(\delta/2) \sin \theta}$,

$$(Cr)^{1-s^N} \|u\|_{H^1(B(x,r/3))} \leq \|u\|_{L^2(B(y,r/3))}^{s^N}, \quad 0 < r < (\delta/3) \sin \theta, \quad (5)$$

where N is the smallest integer so that $|x - y| - 2Nr \leq r$.

But

$$\|u\|_{H^1(B(x,r/3))} \geq \|u\|_{L^2(B(x,r/3))} \geq (\eta/2) |\mathbb{S}^{n-1}| (r/3)^{n/2}, \quad (6)$$

due to the fact that $B(x, r/3) \subset B(\tilde{x}, \delta_0)$. Hence, after some computations, (5) and (6) yield

$$(Cr)^{\varrho(N)} \leq \|u\|_{L^2(B(y,r))}, \quad 0 < r < (\delta/3) \sin \theta, \quad (7)$$

with

$$\varrho(N) = \frac{n/2 + 1}{s^N} - 1.$$

Shortening C if necessary, we can assume that $Cr < 1$. Hence

$$(Cr)^{\varrho(N)} = e^{(-(n+1/2)e^{N|\ln s|+1})|\ln(Cr)|} \geq e^{\left(-e^{\left((n+1/2)\left(\frac{D}{2r} + \frac{1}{2}\right)\right)|\ln s|+1}\right)|\ln(Cr)|}. \quad (8)$$

Here $D = \text{diam}(\Omega)$.

The expected inequality is derived in a straightforward manner from (7) and (8). \square

If $\tilde{x} \in \Gamma$, and $x_0 = x_0(\tilde{x})$ and ρ are as in the definition of **UESP**, we set

$$\mathcal{B}(\tilde{x}, r) = B(x_0, r + \rho), \quad r > 0.$$

Theorem 2.2. Fix $0 < \alpha < 1$ and $0 < \eta \leq M$. Let δ_0 be as in Corollary 2. There exists a constant $c > 0$ so that, for any $u \in H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ satisfying $\Delta u = 0$, $\|u\|_{H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})} \leq M$ and $|u(\tilde{x})| \geq \eta$, for some $\tilde{x} \in \Gamma$,

$$e^{-ce^{\frac{c}{\delta}}} \leq \|u\|_{L^2(\mathcal{B}(\hat{x},r) \cap \Gamma)} + \|\nabla v\|_{L^2(\mathcal{B}(\hat{x},r) \cap \Gamma)^n}, \quad 0 < r < r_0 = \min(R, 8\delta_0/\sin \theta),$$

for any $\hat{x} \in \Gamma$, where R and θ are as in the definition of **UICP**

Proof. From [3, Corollary 3.1], there exist two constants $C_0 > 0$ and $0 < \beta < 1/2$ so that, for any $v \in H^2(\Omega)$ satisfying $\Delta v = 0$, $\hat{x} \in \Gamma$ and $r > 0$,

$$\begin{aligned} C_0 r^2 \|v\|_{H^1(\mathcal{B}(\hat{x}, \frac{r}{4}) \cap \Omega)} & \\ &\leq \|v\|_{H^2(\Omega)}^{1-\beta} \left(\|u\|_{L^2(\mathcal{B}(\hat{x},r) \cap \Gamma)} + \|\nabla v\|_{L^2(\mathcal{B}(\hat{x},r) \cap \Gamma)^n} \right)^\beta. \end{aligned} \quad (9)$$

Let $u \in H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, satisfying $\Delta u = 0$, $\|u\|_{H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})} \leq M$ and $|u(\tilde{x})| \geq \eta$, for some $\tilde{x} \in \Gamma$.

For $0 < r < R$, let $y = \hat{x} + (r/4)\xi$ with $\xi = \xi(\hat{x})$. Then it is straightforward to check that $B(y, (r/4)\sin\theta) \subset \Omega^{(r/4)\sin\theta} \cap \mathcal{B}(\hat{x}, r)$. Therefore by (9)

$$Cr^2 \|u\|_{L^2(B(y, (r/4)\sin\theta))} \leq (\|u\|_{L^2(\mathcal{B}(\hat{x}, r) \cap \Gamma)} + \|\nabla v\|_{L^2(\mathcal{B}(\hat{x}, r) \cap \Gamma)^n})^\beta. \quad (10)$$

On the other hand from Corollary 2

$$e^{-ce^{\frac{c}{\rho}}} \leq \|u\|_{L^2(B(y, \rho))}, \quad 0 < \rho < (r/6)\sin\theta, \quad (11)$$

provided that $\rho \leq \delta_0$, where δ_0 is defined in Corollary 2.

Combine (10) and (11) with $\rho = (r/8)\sin\theta$ in order to get

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(\mathcal{B}(\hat{x}, r) \cap \Gamma)} + \|\nabla v\|_{L^2(\mathcal{B}(\hat{x}, r) \cap \Gamma)^n}.$$

□

One can proceed as in the proof of [3, Theorem 4.1] to derive from the previous theorem the following corollary.

Corollary 3. *Let Γ_0 be an open non empty subset of Γ . Fix $0 < \alpha < 1$, $\Lambda_0 > 0$ and $0 < \eta \leq M$, and let r_0 be as in Theorem 2.2. There exists a constant $c > 0$ so that, for any $u \in H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ satisfying*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ |\partial_\nu u| \leq \Lambda_0 |u| \text{ on } \Gamma_0, \\ |u(\tilde{x})| \geq \eta \text{ for some } \tilde{x} \in \Gamma, \\ \|u\|_{H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})} \leq M, \end{cases}$$

and $\hat{x} \in \Gamma_0$,

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(\mathcal{B}(\hat{x}, r) \cap \Gamma)}, \quad 0 < r < r_0/2.$$

Therefore, we can mimic the proof of [3, Proposition 4.1] to get the following result.

Proposition 2. *Let Γ_0 be an open non empty subset of Γ . Fix $0 < \alpha, \beta < 1$, $\Lambda_0 > 0$ and $0 < \eta \leq M$. There exist constants $C > 0$ and $\sigma > 0$ so that, for any $u \in H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ satisfying*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ |\partial_\nu u| \leq \Lambda_0 |u| \text{ on } \Gamma_0, \\ |u(\tilde{x})| \geq \eta \text{ for some } \tilde{x} \in \Gamma, \\ \|u\|_{H^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})} \leq M, \end{cases}$$

and for any $f \in C^{0,\beta}(\overline{\Gamma_0})$ with $\|f\|_{C^{0,\beta}(\overline{\Gamma_0})} \leq M$,

$$\|f\|_{L^\infty(\Gamma_0)} \leq C \left(|\ln \ln (\|fu\|_{L^\infty(\Gamma_0)})|^{-\sigma} + \|fu\|_{L^\infty(\Gamma_0)} \right).$$

3. Proof of the stability estimate.

Lemma 3.1. *There exist a constant $\eta > 0$, that can depend on M and g , with the property that, for any $q \in \mathcal{Q}_M^0$, one finds $\tilde{x} \in \overline{\Gamma}$ so that $|u_q(\tilde{x})| \geq \eta$.*

Proof. Let $\gamma_0 \Subset \gamma$ and C_0 be the constant in (2). By [5, Corollary 1], there exist three constants $C > 0$, $\beta > 0$ and $\delta > 0$ so that for any $\lambda > 0$

$$C\|\lambda g\|_{L^\infty(\Gamma)} \leq C_0 \left| \ln \left(C_0^{-1} (\|u_q(\lambda g)\|_{L^2(\gamma_0)} + \|\nabla u_q(\lambda g)\|_{L^2(\gamma_0)}) \right) \right|^{-\beta},$$

whenever $\|u_q(\lambda g)\|_{L^2(\gamma_0)} + \|\nabla u_q(\lambda g)\|_{L^2(\gamma_0)} \leq \delta$.

Using a cutoff function and an interpolation inequality we obtain

$$\|u_q(\lambda g)\|_{L^2(\gamma_0)} + \|\nabla u_q(\lambda g)\|_{L^2(\gamma_0)} = \|u_q(\lambda g)\|_{H^1(\gamma_0)} \leq C_1 \|u_q(\lambda g)\|_{L^2(\gamma)}.$$

Then the choice of $\lambda > 0$ so that $C_1 C_0 \lambda \leq \delta$ gives

$$C\lambda \|g\|_{L^\infty(\Gamma)} \leq C_0 \left| \ln \left(C_0^{-1} \lambda \|u_q(g)\|_{L^2(\gamma)} \right) \right|^{-\beta}$$

Hence

$$C_0 e^{-C_1 \|g\|_{L^\infty(\Gamma)}^{-1/\beta}} \leq \|u_q(g)\|_{L^2(\gamma)} \leq |\gamma|^{1/2} \|u_q\|_{L^\infty(\gamma)}.$$

This estimate implies that there exists $\tilde{x} \in \bar{\gamma}$ so that

$$\eta = |\gamma|^{-1/2} C_0 e^{-C_1 \|g\|_{L^\infty(\Gamma)}^{-1/\beta}} \leq |u_q(\tilde{x})|.$$

□

This lemma in combination with Proposition 2 yields

Proposition 3. *Let $0 < \beta < 1$. There exist two constants $C > 0$ and $\sigma > 0$ so that, for any $q \in \mathcal{D}_M^0$ and $f \in C^{0,\beta}(\Gamma)$ with $\|f\|_{C^{0,\beta}(\Gamma)} \leq M$,*

$$\|f\|_{L^\infty(\gamma)} \leq C \left(\left| \ln \ln (\|f u_q\|_{L^\infty(\gamma)}) \right|^{-\sigma} + \|f u_q\|_{L^\infty(\gamma)} \right). \quad (12)$$

Proof of Theorem 1.1. Let $v = \tilde{u} - u$. Since $\Delta v = 0$. The stability estimate for the Cauchy problem in [5] (see also [14]) yields

$$\|v\|_{W^{1,\infty}(\Gamma)} \leq C \left(\left| \ln (\|v\|_{L^2(\gamma)}) \right|^{-\beta} + \|v\|_{L^2(\gamma)} \right), \quad (13)$$

for some constants $C > 0$ and $\beta > 0$.

But

$$(q - \tilde{q})u = \partial_\nu v + \tilde{q}v. \quad (14)$$

Hence (13) yields

$$\|(q - \tilde{q})u\|_{L^\infty(\Gamma)} \leq C \left(\left| \ln (\|v\|_{L^2(\gamma)}) \right|^{-\beta} + \|v\|_{L^2(\gamma)} \right). \quad (15)$$

In light of (13), we end up getting

$$\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq C_0 \Psi \left(C_1 \Phi (\|u - \tilde{u}\|_{L^2(\gamma)}) \right).$$

The proof of Theorem 1.1 is then complete. □

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