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THE PROBLEM OF DETECTING CORROSION BY AN ELECTRIC MEASUREMENT REVISITED

†MOURAD CHOULLI AND ‡AYMEN JBALIA

ABSTRACT. We establish a logarithmic stability estimate for the problem of detecting corrosion by a single electric measurement. We give a proof based on an adaptation of the method initiated in [BCJ] for solving the inverse problem of recovering the surface impedance of an obstacle from the scattering amplitude. The key idea consists in estimating accurately a lower bound of the L^2 -norm, locally at the boundary, of the solution of the boundary value problem used in modeling the problem of detection corrosion by an electric measurement.

Key words: Logarithmic stability estimate, detecting corrosion, boundary measurement.

MSC: 35R30.

1. INTRODUCTION

Let Ω be a C^n -smooth bounded domain of \mathbb{R}^n , $n = 2, 3$. We denote its boundary by Γ and we consider the following boundary value problem (abbreviated to BVP in the sequel)

$$(1.1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + q(x)u = g & \text{on } \Gamma. \end{cases}$$

We assume in the sequel that $g \in H^{n-3/2}(\Gamma)$ and g is non identically equal to zero.

For $s \in \mathbb{R}$ and $1 \leq r \leq \infty$, we introduce the vector space

$$B_{s,r}(\mathbb{R}^{n-1}) := \{w \in \mathcal{S}'(\mathbb{R}^{n-1}); (1 + |\xi|^2)^{s/2} \widehat{w} \in L^r(\mathbb{R}^{n-1})\},$$

where $\mathcal{S}'(\mathbb{R}^{n-1})$ is the space of temperate distributions on \mathbb{R}^{n-1} and \widehat{w} is the Fourier transform of w . Equipped with its natural norm

$$\|w\|_{B_{s,r}(\mathbb{R}^{n-1})} := \|(1 + |\xi|^2)^{s/2} \widehat{w}\|_{L^r(\mathbb{R}^{n-1})},$$

$B_{s,r}(\mathbb{R}^{n-1})$ is a Banach space (it is noted that $B_{s,2}(\mathbb{R}^{n-1})$ is merely the usual Sobolev space $H^s(\mathbb{R}^{n-1})$). By using local charts and a partition of unity, we construct $B_{s,r}(\Gamma)$ from $B_{s,r}(\mathbb{R}^{n-1})$ similarly as $H^s(\Gamma)$ is built from $H^s(\mathbb{R}^{n-1})$.

To carry out our analysis, we need solutions of the BVP (1.1) with some smoothness. In order to give sufficient conditions on the coefficient q guaranteeing this smoothness, we set

$$\mathcal{Q} = \{q \in B_{n-1/2,1}(\Gamma); q \geq 0 \text{ and } q \neq 0\}$$

and

$$\mathcal{Q}_M = \{q \in \mathcal{Q}; \|q\|_{B_{n-1/2,1}(\Gamma)} \leq M\},$$

where $M > 0$ is a given constant.

By Theorem 2.3 in [Ch1], observing that $B_{n-1/2,1}(\Gamma)$ is continuously embedded in $B_{n-3/2,1}(\Gamma)$, we obtain that, for any $q \in \mathcal{Q}$, the BVP (1.1) has a unique solution $u_q \in H^n(\Omega)$. Moreover,

$$(1.2) \quad \|u_q\|_{H^n(\Omega)} \leq C \text{ for all } q \in \mathcal{Q}_M.$$

The constant C above can depend only on Ω , g and M .

Usually, in a BVP modeling the problem of detecting corrosion damage by electric measurements the boundary Γ consists in two parts: $\Gamma = \overline{\Gamma_a} \cup \overline{\Gamma_i}$, Γ_a and Γ_i being two disjoint open subsets of Γ . Γ_a corresponds to the part of the boundary accessible to measurements and Γ_i is the inaccessible part of the boundary where the corrosion damage can occur.

Henceforth, we assume that the current flux g satisfies $\text{supp}(g) \subset \Gamma_a$. The function q in (1.1) is known as the corrosion coefficient and it is naturally supported on Γ_i . This motivates the introduction of the following subset of \mathcal{Q}_M :

$$\mathcal{Q}_M^0 = \{q \in \mathcal{Q}_M; \text{supp}(q) \subset \Gamma_i\}.$$

We are interested in the stability issue for the problem consisting in the determination of the boundary coefficient q from the boundary measurement $u_q|_\gamma$, where γ is an open subset of the accessible sub-boundary Γ_a . In the sequel, we assume that γ does not meet $\text{supp}(g)$:

$$\gamma \subset \Gamma_a \setminus \text{supp}(g).$$

We need to extend the usual notion of a starshaped domain. We say that D is multiply-starshaped if there exists a finite number of points in D , say x_1, \dots, x_k , such that

- (i) $\cup_{i=1}^{k-1} [x_i, x_{i+1}] \subset D$,
- (ii) any point in D can be connected by a line segment to at least one of the points x_i .

In this case, any two points in D can be connected by a broken line consisting of at most $k + 1$ line segments. Obviously, the case $k = 1$ corresponds to the usual definition of a starshaped domain.

Even if it is not always necessary, we assume in all of this text that Ω is multiply-starshaped and, for each $\tilde{x} \in \Gamma$, Γ is locally located at one side of the tangent plane $T_{\tilde{x}}$ to Γ at \tilde{x} . Precisely, to each $\tilde{x} \in \Gamma$, there is a neighborhood V of \tilde{x} such that

$$\Gamma \cap V \subset \{x \in \mathbb{R}^n; (x - \tilde{x}) \cdot \nu(\tilde{x}) \leq 0\}.$$

The later definition means that Γ is convex at each of its points (corresponding to the definition in chapter 13, page 95 of [Th]). We notice that the convexity of Γ implies the convexity at each of its points, but the converse is not true in general.

We aim in the present work to prove the following theorem.

Theorem 1.1. *There are three positive constants A , B and σ satisfying: for any $q \in \mathcal{Q}_M^0 \cap C^\alpha(\Gamma)$, we find $\epsilon = \epsilon(q)$ so that for all $\tilde{q} \in \mathcal{Q}_M^0 \cap C^\alpha(\Gamma)$ such that $\|q - \tilde{q}\|_{L^\infty(\Gamma_i)} \leq \epsilon$,*

$$\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq \frac{A}{|\ln |\ln (B\|u - \tilde{u}\|_{L^2(\gamma)})||^\sigma},$$

with $u = u_q$ and $\tilde{u} = u_{\tilde{q}}$.

Theorem 1.1 can be seen as a completion of the results established in [CCL] in dimension two and in [BCC] in dimensions two and three. We note that in the above mentioned works the difference of $q - \tilde{q}$ is only estimated in a compact subset of $\{x \in \Gamma_i; u_q(x) \neq 0\}$. However, there is a counterpart in estimating $q - \tilde{q}$ in the whole Γ . The stability estimates in [CCL] and [BCC] are of single logarithmic type, while the estimate in Theorem 1.1 is of double logarithmic type.

There is a wide literature treating the problem of detecting corrosion by electric measurements. We refer to [ADR, CFJL, CJ, CCY, Ch2, Ch3, FI, In, Si2] where various type of stability estimate are given. We just quote these few references, but of course there are many others. A neighbor problem is the one consisting in the determination of the surface impedance of an obstacle from the scattering amplitude (e.g [ASV, BCJ, Si1] and the reference therein).

The rest of this text consists in two sections. In section 2 we estimate accurately a lower bound of the L^2 -norm, locally at the boundary, of the solution of the BVP (1.1). We show, step by step, how we adapt the method in [BCJ] to the present problem. Section 3 is devoted the proof of Theorem 1.1.

Unless otherwise specified, all the constants we use in the sequel depend only on data.

2. LOWER BOUND FOR L^2 -NORM AT THE BOUNDARY

We prove the following theorem.

Theorem 2.1. *Let $M > 0$, there is $c > 0$ so that: for all $q \in \mathcal{Q}_M^0$ and $\tilde{x} \in \Gamma$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u_q\|_{L^2(B(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*,$$

where r^* is a constant that can depend on q .

We need some preliminary results before proving Theorem 2.1.

For sake of simplicity, we assume in the sequel that Ω is in addition starshaped. From the proof of Proposition 2.1 below, one can see that the extension to the case where Ω is multiply-starshaped is obvious.

For $\delta > 0$, we set

$$\Omega^\delta = \{x \in \Omega; \text{dist}(x, \Gamma) > \delta\}$$

and we recall the following useful three sphere inequality.

Lemma 2.1. *There exist $C > 0$ and $0 < s < 1$ so that: for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω , $y \in \Omega$ and $0 < r < \frac{1}{3} \text{dist}(y, \Gamma)$,*

$$r \|u\|_{H^1(B(y, 2r))} \leq C \|u\|_{H^1(B(y, r))}^s \|u\|_{H^1(B(y, 3r))}^{1-s}.$$

Proposition 2.1. *We assume that Ω is starshaped with respect to $x^* \in \Omega$ and we choose $\delta > 0$ such that $x^* \in \Omega^\delta$. Let $M > 0$, there are two constants $c > 0$ and $r_\delta > 0$ so that: for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega)} \leq M$ and, $x, y \in \Omega^\delta$, we have*

$$e^{-ce^{\frac{c}{r}}} \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(y, 4r))}, \quad 0 < r < r_\delta.$$

¹We refer to [BCJ] for a proof. The case of a general divergence form operator is detailed in [CT].

Proof. We set

$$d_1 = |x - x^*|, \quad \eta = \frac{x^* - x}{|x^* - x|}$$

and we consider the sequence, where $0 < 2r < d_1$,

$$x_k = x^* - k(2r)\eta, \quad k \geq 1.$$

Clearly,

$$|x_k - x| = d_1 - k(2r).$$

Let N_1 be the smallest integer such that $d_1 - N_1(2r) \leq r$, or equivalently

$$\frac{d_1}{2r} - \frac{1}{2} \leq N_1 < \frac{d_1}{2r} + \frac{1}{2}.$$

By Lemma 2.1, it follows that

$$(2.1) \quad \tilde{C}r^t \|u\|_{H^1(B(x_{N_1}, 2r))} \leq \|u\|_{H^1(B(y^*, 2r))}^{s^{N_1}} \quad \text{with } t = \frac{1}{1-s}.$$

Since $|y_{N_1} - x| = d_1 - N_1(2r) \leq r$, $B(x_0, r) \subset B(y_{N_1}, 2r)$. Whence (2.1) entails

$$(2.2) \quad Cr^t \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(x^*, 2r))}^{s^{N_1}}.$$

The same argument between x^* and y gives

$$(2.3) \quad Cr^t \|u\|_{H^1(B(x^*, r))} \leq \|u\|_{H^1(B(y, 2r))}^{s^{N_2}}.$$

Here N_2 is defined by the relation

$$\frac{d_2}{2r} - \frac{1}{2} \leq N_2 < \frac{d_2}{2r} + \frac{1}{2} \quad \text{with } d_2 = |y - x^*|.$$

A combination of (2.2) and (2.3) implies

$$(Cr)^{1+s^{N_1}} \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(y, 4r))}^{s^{N_1+N_2}}.$$

Or equivalently

$$(2.4) \quad (Cr)^\kappa \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(y, 4r))},$$

with

$$\kappa = \frac{1 + s^{N_1}}{s^{N_1+N_2}}.$$

Henceforth, we assume that r is sufficiently small in such a way that $Cr < 1$ in (2.4). Letting $D = \text{diam}(\Omega)$, we obtain by a direct computation

$$\kappa \leq (1 + s^{-1/2}) e^{\frac{2D|\ln s|}{r}}.$$

This estimate in (2.4) yields

$$e^{-ce^{\frac{c}{r}}} \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(y, 4r))},$$

as it is the expected. \square

We recall that according to Caccioppoli's inequality, for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω and $x \in \Omega$,

$$\|\nabla u\|_{L^2(B(x, r))^n} \leq Cr^{-1} \|u\|_{L^2(B(x, 2r))},$$

for a sufficiently small r .

Therefore the following corollary is immediate from Proposition 2.1.

Corollary 2.1. *Under the assumptions of Proposition 2.1 and for $M > 0$, there are two constants $c > 0$ and $r_\delta > 0$ so that: for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega)} \leq M$ and $x, y \in \Omega^\delta$,*

$$e^{-ce^{\frac{c}{r}}} \|u\|_{H^1(B(x,r))} \leq \|u\|_{L^2(B(y,8r))}, \quad 0 < r < r_\delta.$$

By an elementary continuity argument, we get from this corollary

Corollary 2.2. *We fix $\eta > 0$ and $M > 0$. There is $c > 0$ with the property that, to any $u \in H^1(\Omega)$, satisfying*

$\Delta u = 0$, $\|u\|_{H^1(\Omega)} \leq M$ and there is $\hat{x} \in \Gamma$ such that $u \in C(B(\hat{x}, \hat{r}) \cap \Omega)$, for some $\hat{r} > 0$, and $|u(\hat{x})| \geq \eta$,

corresponds $\delta > 0$ and $r_\delta > 0$ for which, for all $y \in \Omega^\delta$,

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(B(y,r))}, \quad 0 < r < r_\delta.$$

Note here that δ and r_δ may depend also on u .

Now, because Γ is convex at each of its points and bearing in mind that Ω is located at one side of Γ , Ω has the uniform exterior ball property. That is, there is $\rho > 0$ so that, for all $\tilde{x} \in \Gamma$, we find $x' \in \mathbb{R}^n \setminus \overline{\Omega}$ such that

$$B(x', \rho) \cap \Omega = \emptyset \quad \text{and} \quad \overline{B(x', \rho)} \cap \overline{\Omega} = \{\tilde{x}\}.$$

Let

$$(2.5) \quad \mathcal{B}(\tilde{x}, r) = B(x', \rho + r), \quad \tilde{x} \in \Gamma.$$

Then, as a peculiar case of Corollary 3.1 in [BCJ], we have

Proposition 2.2. *There exist two constants $C > 0$ and $0 < \gamma < 1/2$ so that, for any $0 < r \leq D$ and $u \in H^2(\Omega)$ satisfying $\Delta u = 0$, the following estimate holds true*

$$(2.6) \quad Cr^2 \|u\|_{H^1(\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega)} \leq \|u\|_{H^2(\Omega)}^{1-\gamma} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma^n)})^\gamma.$$

As Ω is C^n -smooth, it has also the uniform interior cone property: there are $R > 0$ and $\theta \in]0, 2\pi[$ satisfying, for all $\tilde{x} \in \Gamma$, we find $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$ and

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R \text{ and } (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

We note that ξ can be chosen in the following form

$$\xi = \frac{\tilde{x} - x'}{|\tilde{x} - x'|},$$

where x' is the same as in the definition of uniform exterior ball property.

In other words, following the definition in [BCJ], Ω possesses the uniform exterior ball-interior cone property. Then a slight modification of the first part of the proof of Theorem 4.1 in [BCJ] yields

Proposition 2.3. *We pick $\tilde{x} \in \Gamma$. Then, for sufficiently small r , we can choose $x_0 \in \Omega$, $y_0 \in \Omega$ two points in the line segment passing through \tilde{x} and directed by ξ such that $B(x_0, r/2) \subset \mathcal{B}(\tilde{x}, r) \cap \Omega$ and $B(y_0, \kappa r) \subset \Omega^{R/2}$, where κ is constant depending only on θ . Let $M > 0$, there are $C > 0$, $\eta > 1$, and $r^* > 0$, not depending on x_0 and y_0 , such that for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω and $\|u\|_{H^1(\Omega)} \leq M$,*

$$e^{-\frac{C}{r^\eta}} \|u\|_{H^1(B(y_0, \kappa r))} \leq \|u\|_{H^1(B(x_0, r))}, \quad 0 < r \leq r^*.$$

A combination of Corollary 2.2, Proposition 2.2 and Proposition 2.3 gives

Theorem 2.2. *Let $\eta > 0$, $M > 0$ and $\hat{x} \in \Gamma$. There is a constant $c > 0$ such that for all $u \in H^2(\Omega)$ satisfying $\Delta u = 0$ in Ω , $|u(\hat{x})| \geq \eta$, $\|u\|_{H^2(\Omega)} \leq M$ and $\tilde{x} \in \Gamma$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\partial_\nu u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*,$$

where r^* can depend on u .

For $\tilde{x} \in \Gamma$, pick $\psi \in C_0^\infty(\mathcal{B}(\tilde{x}, 2r))$ satisfying $\psi = 1$ in a neighborhood of $\mathcal{B}(\tilde{x}, r)$ and $|\partial^\beta \psi| \leq Cr^{-|\beta|}$ for any $\beta \in \mathbb{R}^n$, $|\beta| \leq 2$. Let $u \in H^2(\Omega)$. Using the interpolation inequality

$$\|\psi u\|_{H^1(\Gamma)} \leq C \|\psi u\|_{H^{3/2}(\Gamma)}^{2/3} \|\psi u\|_{L^2(\Gamma)}^{1/3}$$

together with the properties of ψ and the continuity of the trace operator $v \in H^2(\Omega) \rightarrow v|_\Gamma \in H^{3/2}(\Gamma)$, we get in a straightforward manner

$$(2.7) \quad \|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} \leq Cr^{-4/3} \|u\|_{H^{3/2}(\Omega)}^{2/3} \|u\|_{L^2(\mathcal{B}(\tilde{x}, 2r) \cap \Gamma)}^{1/3}.$$

On the other hand, we claim that

$$(2.8) \quad \mathcal{B}(\tilde{x}, r) \cap \Gamma \subset B(\tilde{x}, 2\rho r) \cap \Gamma.$$

Here $\mathcal{B}(\tilde{x}, r)$ is defined by (2.5).

Indeed, under a rigid transformation, it is enough to prove this inclusion when $\tilde{x} = 0$, $x' = (0, -\rho) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $T_{\tilde{x}} = \{x_n = 0\}$ and Γ is located locally in $\{x_n \geq 0\}$. In that case, by an elementary calculation, we get

$$B(x', \rho + r) \cap \{x_n \geq 0\} \subset B(0, 2\rho r) \cap \{x_n \geq 0\}.$$

Clearly, this inclusion entails the following one

$$B(x', \rho + r) \cap \Gamma \subset B(0, 2\rho r) \cap \Gamma.$$

Therefore, in light of (2.7) and (2.8), a direct consequence of Theorem 2.2 is the following corollary.

Corollary 2.3. *Let $\eta > 0$, $M > 0$ and $\hat{x} \in \Gamma$. There is a constant $c > 0$ so that for all $u \in H^2(\Omega)$ satisfying $\Delta u = 0$ in Ω , $|u(\hat{x})| \geq \eta$, $\|u\|_{H^2(\Omega)} \leq M$ and $\tilde{x} \in \Gamma$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\partial_\nu u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*,$$

where r^* can depend on u .

Moreover, if $|\partial_\nu u| \leq N|u|$ on Γ , for some constant N , then

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*.$$

We are now able to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We need to prove that there are $\hat{x} \in \Gamma$ and $\eta > 0$ for which $|u_q(\hat{x})| \geq \eta$ for any $q \in \mathcal{Q}_M^0$.

We fix Γ_0 an arbitrary nonempty open subset of $\Gamma \setminus \text{supp}(g)$. By Corollary 1 in [Bo], there is a constant $A > 0$ such that, for all $u \in H^2(\Omega)$ satisfying $\Delta u = 0$ and $\|u\|_{H^2(\Omega)} \leq M$, we have

$$(2.9) \quad \|u\|_{H^1(\Omega)} \leq \frac{A}{|\ln(M^{-1}\delta)|^{1/2}},$$

where $\delta = \|u\|_{H^1(\Gamma_0)} + \|\partial_\nu u\|_{L^2(\Gamma_0)}$.²

²Note that the smallness condition on δ in Corollary 1 in [Bo] can be easily removed.

Let Γ_1 be an open subset of Γ satisfying $\text{supp}(g) \subset \Gamma_1 \Subset \Gamma$. Proceeding as before, we deduce from an usual interpolation inequality

$$\begin{aligned} \|g\|_{L^2(\Gamma_a)} &= \|\partial_\nu u_q\|_{L^2(\Gamma_1)} \leq C \|\partial_\nu u_q\|_{H^{-1/2}(\Gamma)}^{1/2} \|\partial_\nu u_q\|_{H^{1/2}(\Gamma)}^{1/2} \\ &\leq C \|u_q\|_{H^1(\Omega)}^{1/2} \|u_q\|_{H^2(\Omega)}^{1/2} \\ &\leq C \|u_q\|_{H^1(\Omega)}^{1/2}. \end{aligned}$$

This and (2.9) imply

$$\|g\|_{L^2(\Gamma_a)} \leq \frac{C}{|\ln(M^{-1}\delta)|^{1/4}}, \text{ with } \delta = \|u_q\|_{H^1(\Gamma_0)} + \|\partial_\nu u_q\|_{L^2(\Gamma_0)},$$

or equivalently

$$\tilde{\eta} = M e^{-C^4 \|g\|_{L^2(\Gamma_a)}^{-4}} \leq \|u_q\|_{H^1(\Gamma_0)} + \|\partial_\nu u_q\|_{L^2(\Gamma_0)}.$$

Replacing Γ_0 by a smaller subset and proceeding as in the proof of Corollary 2.3, we get

$$(2.10) \quad \tilde{\eta} = M e^{-c^4 \|g\|_{L^2(\Gamma_1)}^{-4}} \leq \|u_q\|_{L^2(\Gamma_0)}.$$

Now since $H^n(\Omega)$ is continuously embedded in $C(\overline{\Omega})$, we derive from (2.10)

$$\eta = \tilde{\eta} |\Gamma_0|^{-1/2} \leq |u_q(\hat{x})| = \max_{\Gamma_0} |u_q|.$$

□

3. PROOF OF THE STABILITY ESTIMATE

First, we mimick the proof of Proposition 4.1 in [BCJ] to get that there are $B > 0$ and $\sigma > 0$ satisfying: for any $q \in \mathcal{Q}_M^0$, we find $\epsilon(q) > 0$ so that, for any $f \in C^\alpha(\Gamma)$ with

$$[f]_\alpha = \sup\{|f(x) - f(y)| |x - y|^{-\alpha}; x, y \in \Gamma, x \neq y\} \leq M$$

and $\|f\|_{L^\infty(\Gamma)} \leq \epsilon(q)$,

$$(3.1) \quad \|f\|_{L^\infty(\Gamma)} \leq \frac{B}{|\ln \|f u_q\|_{L^\infty(\Gamma)}|^\sigma}.$$

Proof of Theorem 1.1. Let $v = \tilde{u} - u$. Since $\Delta v = 0$, the same argument as in the proof of Theorem 2.1 yields

$$\|\partial_\nu v\|_{L^2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}^{1/2}.$$

Hence,

$$(3.2) \quad \|v\|_{L^2(\Gamma)} + \|\partial_\nu v\|_{L^2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}^{1/2}.$$

Let $\gamma_0 \Subset \gamma$. Again, by Corollary 1 in [Bo], there is a constant $A > 0$ for which

$$(3.3) \quad \|v\|_{H^1(\Omega)} \leq \frac{A}{|\ln((2M)^{-1}\delta)|^{1/2}},$$

with $\delta = \|v\|_{H^1(\gamma_0)} + \|\partial_\nu v\|_{L^2(\gamma_0)}$.

As previously, we get by applying an interpolation inequality

$$\|v\|_{H^1(\gamma_0)} \leq C \|v\|_{L^2(\gamma)}^{1/3},$$

and since $\partial_\nu v = 0$ on γ , (3.3) implies

$$(3.4) \quad \|v\|_{H^1(\Omega)} \leq \frac{A}{|\ln(B\|v\|_{L^2(\gamma)})|^{1/2}}.$$

In light of (3.2), (3.4) leads

$$(3.5) \quad \|v\|_{L^2(\Gamma)} + \|\partial_\nu v\|_{L^2(\Gamma)} \leq \frac{A}{|\ln(B\|v\|_{L^2(\gamma)})|^{1/4}},$$

Let $f = (q - \tilde{q})u$. We fix θ satisfying $2/3 < \theta < 1$ if $n = 2$ and $3/5 < \theta < 1$ if $n = 3$ and set $s = 3\theta/2$ for $n = 2$ and $s = 5\theta/2$ for $n = 3$. By this peculiar choice of s , $H^s(\Gamma)$ is continuously embedded in $L^\infty(\Gamma)$. Therefore, using the interpolation inequalities

$$\begin{aligned} \|f\|_{H^s(\Gamma)} &\leq C\|f\|_{H^{3/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 2, \\ \|f\|_{H^s(\Gamma)} &\leq C\|f\|_{H^{5/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 3, \end{aligned}$$

we obtain

$$\begin{aligned} \|f\|_{L^\infty(\Gamma)} &\leq C\|f\|_{H^{3/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 2, \\ \|f\|_{L^\infty(\Gamma)} &\leq C\|f\|_{H^{5/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 3, \end{aligned}$$

Or

$$\|f\|_{H^{n-1/2}(\Gamma)} = \|(q - \tilde{q})u\|_{H^{n-1/2}(\Gamma)} \leq C\|q - \tilde{q}\|_{B_{n-1/2,1}(\Gamma)}\|u\|_{H^{n-1/2}(\Gamma)}.$$

Consequently,

$$(3.6) \quad \|(q - \tilde{q})u\|_{L^\infty(\Gamma)} \leq C\|(q - \tilde{q})u\|_{L^2(\Gamma)}^{1-\theta}.$$

But

$$(3.7) \quad (q - \tilde{q})u = \partial_\nu v + \tilde{q}v.$$

Then a combination of (3.5), (3.6) and (3.7) yields

$$(3.8) \quad \|(q - \tilde{q})u\|_{L^\infty(\Gamma)} \leq \frac{A}{|\ln(B\|v\|_{L^2(\gamma)})|^{(1-\theta)/4}},$$

In light of (3.1), we end up getting

$$\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq \frac{A}{|\ln |\ln(B\|u - \tilde{u}\|_{L^2(\gamma)})||^\sigma}.$$

□

REFERENCES

- [ADR] G. ALESSANDRINI, L. DEL PIERO AND L. RONDI, *Stable determination of corrosion by a single electrostatic boundary measurement*, Inverse problems, 19 (4) (2003), 973-984.
- [ASV] G. ALESSANDRINI, E. SINCICH AND S. VESSELLA *Stable determination of surface impedance on a rough obstacle by far field data*, Inverse Problems and Imaging, 7 (2) (2013), 341-351.
- [BCJ] M. BELLASSOUED, M. CHOULLI AND A. JBALIA, *Stability of the determination of the surface impedance of an obstacle from the scattering amplitude*, Math. Methods Appl. Sci. 36 (18) (2013) 2429-2448.
- [BCC] M. BELLASSOUED, J. CHENG AND M. CHOULLI, *Stability estimate for an inverse boundary coefficient problem in thermal imaging*, J. Math. Anal. Appl. 343 (2008) 328-336.
- [Bo] L. BOURGEOIS, *About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$ domains*, Math. Model. Numer. Anal. 44 (4) (2010), 715-735.

- [CFJL] S. CHAABANE, I. FELLAH, M. JAOUA and J. LEBLOND, *Logarithmic stability estimates for a robin coefficient in two-dimensional Laplace inverse problems*, Inverse Problems 20, (2004), 47-59.
- [CJ] S. CHAABANE and M. JAOUA, *Identification of Robin coefficients by means of boundary measurements*, Inverse Problems 15, (1999), 1425-1438.
- [CCL] J. CHENG, M. CHOULLI and J. LIN, *Stable determination of a boundary coefficient in an elliptic equation*, M3AS 18 (1) (2008) 107-123.
- [CCY] J. CHENG, M. CHOULLI and X. YANG, *An iterative BEM for the inverse problem of detecting corrosion in a pipe*, Numer. Math. J. Chinese Univ. 14 (3) (2005) 252-266.
- [Ch1] M. CHOULLI, *Stability estimates for an inverse elliptic problem*, J. Inverse Ill-Posed Probl. 10 (6) (2002), 601-610.
- [Ch2] M. CHOULLI, *An inverse problem in corrosion detection: stability estimates*, J. Inverse Ill-Posed Probl. 12 (4) (2004), 349-367.
- [Ch3] M. CHOULLI, *Une introduction aux problèmes inverses elliptiques et paraboliques*, SMAI-Springer, Berlin, 2009.
- [CT] M. CHOULLI and F. TRIKI, *On an inverse medium problem with internal data*, preprint
- [FI] D. FASINO AND G. INGLESE, *An inverse Robin problem for Laplace's equation: theoretical and numerical methods*, Inverse Problems 15 (1999), 41-48.
- [In] G. INGLESE, *An inverse problem in corrosion detection*, Inverse Problems 13 (1997), 977-994.
- [Si1] SINCICH, *Stable determination of the surface impedance of an obstacle by far field measurements*, SIAM J. Math. Anal., 38 (2) (2006), 434-451.
- [Si2] E. SINCICH, *Lipschitz stability for the inverse Robin problem*, Inverse Problems 23 (2007), 1311-1326.
- [Th] J. A. THORPE, *Elementary topics in differential geometry*, Springer-Verlag, Berlin, 1979.

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