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THE PROBLEM OF DETECTING CORROSION BY ELECTRIC MEASUREMENTS REVISITED

†MOURAD CHOULLI AND ‡AYMEN JBALIA

ABSTRACT. We establish a logarithmic stability estimate for the problem of detecting corrosion by electric measurements. We propose a proof based on an adaptation of the method initiated in [BCJ]. Roughly speaking, it consists in estimating a lower bound of the local L^2 -norm at the boundary of the solution of the boundary value problem used in modeling the problem of detection corrosion by electric measurements.

Key words: Stability estimate, detecting corrosion, boundary measurement.

AMS subject classifications: 35R30.

1. INTRODUCTION

Let Ω be a C^n -smooth bounded domain of \mathbb{R}^n , $n = 2, 3$. We denote its boundary by Γ and we consider the following boundary value problem (abbreviated to BVP)

$$(1.1) \quad \begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_\nu u + q(x)u = g, & \text{on } \Gamma. \end{cases}$$

In all of this paper we assume that $g \in H^{n-3/2}(\Gamma)$ and g is non identically equal to zero. For $s \in \mathbb{R}$ and $1 \leq r \leq \infty$, we introduce the vector space

$$B_{s,r}(\mathbb{R}^{n-1}) := \{w \in \mathcal{S}'(\mathbb{R}^{n-1}); (1 + |\xi|^2)^{s/2} \widehat{w} \in L^r(\mathbb{R}^{n-1})\},$$

where $\mathcal{S}'(\mathbb{R}^{n-1})$ is the space of temperate distributions on \mathbb{R}^{n-1} and \widehat{w} is the Fourier transform of w . Equipped with its natural norm

$$\|w\|_{B_{s,r}(\mathbb{R}^{n-1})} := \|(1 + |\xi|^2)^{s/2} \widehat{w}\|_{L^r(\mathbb{R}^{n-1})},$$

$B_{s,r}(\mathbb{R}^{n-1})$ is a Banach space (it is noted that $B_{s,2}(\mathbb{R}^{n-1})$ is merely the usual Sobolev space $H^s(\mathbb{R}^{n-1})$). By local charts and partition of unity, we construct $B_{s,r}(\Gamma)$ from $B_{s,r}(\mathbb{R}^{n-1})$ in the same manner as $H^s(\Gamma)$ is built from $H^s(\mathbb{R}^{n-1})$.

We shall need that the solution of the BVP (1.1) has some smoothness. In order to give sufficient conditions on data guaranteeing this smoothness, we first define the set of boundary coefficients. Let

$$\mathcal{Q} = \{q \in B_{n-1/2,1}(\Gamma); q \geq 0 \text{ and } q \not\equiv 0\}$$

and

$$\mathcal{Q}_M = \{q \in \mathcal{Q}; \|q\|_{B_{n-1/2,1}(\Gamma)} \leq M\},$$

where $M > 0$ is a given constant.

By Theorem 2.3 in [Ch1], observing that $B_{n-1/2,1}(\Gamma)$ is continuously embedded in $B_{n-3/2,1}(\Gamma)$, we have that, for any $q \in \mathcal{Q}$, the BVP (1.1) has a unique solution $u_q \in H^n(\Omega)$. In addition

$$(1.2) \quad \|u_q\|_{H^n(\Omega)} \leq C \text{ for all } q \in \mathcal{Q}_M.$$

The constant C above can depend only on Ω , g and M .

Usually in a BVP modeling the problem of detecting corrosion damage by electric measurements the boundary Γ consists in two parts: $\Gamma = \overline{\Gamma_a} \cup \overline{\Gamma_i}$, Γ_a and Γ_i being two disjoint open subsets of Γ . Γ_a corresponds to the part of the boundary accessible to measurements and Γ_i is the inaccessible part of the boundary where the corrosion damage occurs.

Henceforth, we assume that the current flux g satisfies $\text{supp}(g) \subset \Gamma_a$. The function q in (1.1) is known as the corrosion coefficient and it is supported on Γ_i . This motivates the introduction of the following set

$$\mathcal{Q}_M^0 = \{q \in \mathcal{Q}_M; \text{supp}(q) \subset \Gamma_i\}.$$

We are interested in the stability issue for the problem consisting in the determination of the boundary coefficient q from the boundary measurement $u_{q|\gamma}$, where γ is a subset of the accessible sub-boundary Γ_a for which we assume that the following condition holds true:

$$\gamma \subset \Gamma_a \setminus \text{supp}(g).$$

Next, we introduce the notion of multiply-starshaped domain. We say that D is multiply-starshaped if there exists a finite number of points in D , say x_1, \dots, x_k , such that any point in D can be connected by a line segment to one of x_i . In this case, any two points in D can be connected by a broken line consisting of at most $k + 1$ line segments. Obviously, the case $k = 1$ corresponds to the usual notion of starshapedness.

The main result in the present note is the following theorem.

Theorem 1.1. *We fix $0 < \alpha < 1$ and we assume that $\bar{\Omega}$ is locally convex¹ and Ω is multiply-starshaped. There are three positive constants A and B and σ satisfying for any $q \in \mathcal{Q}_M^0 \cap C^\alpha(\Gamma)$, we find $\epsilon = \epsilon(q)$ with the property that for all $\tilde{q} \in \mathcal{Q}_M^0 \cap C^\alpha(\Gamma)$ such that $\|q - \tilde{q}\|_{L^\infty(\Gamma_i)} \leq \epsilon$, we have*

$$\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq \frac{A}{|\ln |\ln(B\|u - \tilde{u}\|_{L^2(\gamma)})||^\sigma},$$

with $u = u_q$ and $\tilde{u} = u_{\tilde{q}}$.

The result in Theorem 1.1 can be seen as an improvement of those already established in [CCL] in dimension two and in [BCC] in dimensions two and three. We note that in these above mentioned works the difference of $q - \tilde{q}$ is only estimated in a compact subset of $\{x \in \Gamma_i; u_q(x) \neq 0\}$. However there is a counterpart in estimating $q - \tilde{q}$ in the whole Γ . The stability estimates in [CCL] and [BCC] are of logarithmic type, while the estimate in Theorem 1.1 is of double logarithmic type.

There is a wide literature treating the problem of detecting corrosion by electric measurements. We refer to [CFJL, CJ, CCY, Ch2, Ch3, FI, In, Si] where various type of stability estimate are given. We just quote these few references, but of course there are many others.

Unless otherwise specified, all the constants we use in the sequel depend only on data.

2. LOWER BOUND FOR THE LOCAL L^2 -NORM AT THE BOUNDARY

We aim to prove the following theorem.

Theorem 2.1. *We assume that $\bar{\Omega}$ is locally convex and Ω is multiply-starshaped. Let $M > 0$, there is $c > 0$ such that, for all $q \in \mathcal{Q}_M^0$ and all $\tilde{x} \in \Gamma$, we have*

$$e^{-ce^{\frac{c}{r}}} \leq \|u_q\|_{L^2(B(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*,$$

where r^* is a constant that can depend on q .

We need several preliminary results before proving Theorem 2.1. We start by introducing some definitions. As usual, we say that Ω has the uniform exterior ball property (abbreviated to UEBP) if there is $\rho > 0$ for which, for all $\tilde{x} \in \Gamma$, we find $x' \in \mathbb{R}^n \setminus \bar{\Omega}$ such that $B(x', \rho) \cap \Omega = \emptyset$ and $\bar{B}(x', \rho) \cap \bar{\Omega} = \{\tilde{x}\}$.

Next, we recall that Ω has the uniform interior cone property (abbreviated to UICP) if there are $R > 0$ and $\theta \in]0, 2\pi[$ satisfying, for all $\tilde{x} \in \Gamma$, we find $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$ and

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R \text{ and } (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

¹This means that any point of the topological vector space $\bar{\Omega}$ has a convex neighborhood

Also, we say that Ω has the uniform interior cone-exterior ball property (abbreviated to UICEBP) if UEBP and UICP are both satisfied at any point $\tilde{x} \in \Gamma$ and in addition

$$\xi = \frac{\tilde{x} - x'}{|\tilde{x} - x'|},$$

where x' and ξ are the same as in the definitions of UEBP and UICP respectively.

Now let (\mathcal{G}) be the following assumption: There exist $C > 0$ and $0 < r_0$ such that for all $\tilde{x} \in \Gamma$ and $0 < r \leq r_0$,

$$\mathcal{B}(\tilde{x}, r) \cap \Gamma \subset B(\tilde{x}, Cr) \cap \Gamma,$$

with

$$\mathcal{B}(\tilde{x}, r) = B(x', \rho + r), \quad \tilde{x} \in \Gamma,$$

where x' and ρ are the same as in the definition of UEBP.

One can easily check that if $\overline{\Omega}$ is locally convex, then Ω possesses both UICESP and (\mathcal{G}) .

For sake of simplicity, we replace in the sequel the assumption that Ω is multiply-starshaped by a stronger one. Precisely, we assume that Ω is starshaped. From the proof of Proposition 2.1 below, one can see that the extension to the case where Ω is multiply-starshaped is obvious.

For $\delta > 0$, we set

$$\Omega^\delta = \{x \in \Omega; \text{dist}(x, \Gamma) > \delta\}$$

and we recall the following useful three sphere inequality.

Lemma 2.1. *There exist $C > 0$ and $0 < s < 1$ such that, for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω , $y \in \Omega$ and $0 < r < \frac{1}{3}\text{dist}(y, \Gamma)$,*

$$r\|u\|_{H^1(B(y, 2r))} \leq C\|u\|_{H^1(B(y, r))}^s \|u\|_{H^1(B(y, 3r))}^{1-s}.$$

Proposition 2.1. *We assume that Ω is starshaped with respect to $x^* \in \Omega$ and we choose $\delta > 0$ such that $x^* \in \Omega^\delta$. Let $M > 0$, there are two constants $c > 0$ and $r_\delta > 0$ such that, for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ and $\|u\|_{H^1(\Omega)} \leq M$ and for all $x, y \in \Omega^\delta$, we have*

$$e^{-ce^{\frac{c}{r}}} \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(y, 4r))}, \quad 0 < r < r_\delta.$$

Proof. We set

$$d_1 = |x - x^*|, \quad \eta = \frac{x^* - x}{|x^* - x|}$$

and we consider the sequence, where $0 < 2r < d$,

$$x_k = x^* - k(2r)\eta, \quad k \geq 1.$$

We have

$$|x_k - x| = d_1 - k(2r).$$

Let N_1 be the smallest integer such that $d_1 - N_1(2r) \leq r$, or equivalently

$$\frac{d_1}{2r} - \frac{1}{2} \leq N_1 < \frac{d_1}{2r} + \frac{1}{2}.$$

By Lemma 2.1, it follows that

$$(2.1) \quad \tilde{C}r^t \|u\|_{H^1(B(x_{N_1}, 2r))} \leq \|u\|_{H^1(B(y^*, 2r))}^{s^{N_1}} \quad \text{with } t = \frac{1}{1-s}.$$

Since $|y_{N_1} - x| = d_1 - N_1(2r) \leq r$, $B(x_0, r) \subset B(y_{N_1}, 2r)$. Whence (2.1) entail

$$(2.2) \quad Cr^t \|u\|_{H^1(B(x, r))} \leq \|u\|_{H^1(B(x^*, 2r))}^{s^{N_1}}.$$

The same argument between x^* and y gives

$$(2.3) \quad Cr^t \|u\|_{H^1(B(x^*, r))} \leq \|u\|_{H^1(B(y, 2r))}^{s^{N_2}}.$$

²We refer to [BCJ] for a proof. The case of a general divergence form operator is detailed in [CT].

Here N_2 is defined by the relation

$$\frac{d_2}{2r} - \frac{1}{2} \leq N_2 < \frac{d_2}{2r} + \frac{1}{2} \text{ with } d_2 = |x - x^*|.$$

A combination of (2.2) and (2.3) imply

$$(Cr)^{1+s^{N_1}} \|u\|_{H^1(B(x,r))} \leq \|u\|_{H^1(B(y,4r))}^{s^{N_1+N_2}}.$$

Or equivalently

$$(2.4) \quad (Cr)^\kappa \|u\|_{H^1(B(x,r))} \leq \|u\|_{H^1(B(y,4r))}$$

with

$$\kappa = \frac{1 + s^{N_1}}{s^{N_1+N_2}}.$$

Henceforth, we assume that r is sufficiently small in such a way that $Cr < 1$ in (2.4). Letting $D = \text{diam}(\Omega)$, we obtain by a direct computation

$$\kappa \leq (1 + s^{-1/2}) e^{\frac{2D|\ln s|}{r}}.$$

This estimate in (2.4) yields

$$e^{-ce^{\frac{c}{r}}} \|u\|_{H^1(B(x,r))} \leq \|u\|_{H^1(B(y,4r))},$$

which is the expected estimate. \square

We recall that according to Caccioppoli's inequality, for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω and all $x \in \Omega$, we have, for a sufficiently small r ,

$$\|\nabla u\|_{L^2(B(x,r))^n} \leq Cr^{-1} \|u\|_{L^2(B(x,2r))}.$$

Therefore the following corollary is immediate from Proposition 2.1.

Corollary 2.1. *Under the assumptions of Proposition 2.1 and for $M > 0$, there are two constants $c > 0$ and $r_\delta > 0$ such that, for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ and $\|u\|_{H^1(\Omega)} \leq M$ and for all $x, y \in \Omega^\delta$, we have*

$$e^{-ce^{\frac{c}{r}}} \|u\|_{H^1(B(x,r))} \leq \|u\|_{L^2(B(y,8r))}, \quad 0 < r < r_\delta.$$

By an elementary continuity argument, we get from this corollary

Corollary 2.2. *We fix $\eta > 0$ and $M > 0$. There is $c > 0$ with the property that for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$, $\|u\|_{H^1(\Omega)} \leq M$ and there exists $\hat{x} \in \Gamma$ such that $u \in C(B(\hat{x}, \hat{r}) \cap \Omega)$, for some $\hat{r} > 0$, and $|u(\hat{x})| \geq \eta$. Then we find $\delta > 0$ and $r_\delta > 0$ for which, for all $y \in \Omega^\delta$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(B(y,r))}, \quad 0 < r < r_\delta.$$

Note here that δ and r_δ may depend also on u .

We recall that $\mathcal{B}(\tilde{x}, r) = B(x', \rho + r)$, $\tilde{x} \in \Gamma$, where x' and ρ are the same as in the definition of UEBP. As a peculiar case of Corollary 3.1 in [BCJ], we have

Proposition 2.2. *There exist two constants $C > 0$ and $0 < \gamma < 1/2$ with the property that, for any $0 < r \leq D$ and any $u \in H^2(\Omega)$ satisfying $\Delta u = 0$, the following estimate holds true*

$$(2.5) \quad Cr^2 \|u\|_{H^1(\mathcal{B}(\tilde{x}, \frac{r}{4}) \cap \Omega)} \leq \|u\|_{H^2(\Omega)}^{1-\gamma} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)})^\gamma.$$

Also, a slight modification of the first part of the proof of Theorem 4.1 in [BCJ] yields

Proposition 2.3. *We assume that Ω has UICEBP and we pick $\tilde{x} \in \Gamma$. For sufficiently small r , we can choose $x_0 \in \Omega$, $y_0 \in \Omega$ two points in the line segment passing through \tilde{x} and directed by ξ such that $B(x_0, r/2) \subset \mathcal{B}(\tilde{x}, r) \cap \Omega$ and $B(y_0, \kappa r) \subset \Omega^{R/2}$, where κ is constant depending only on θ . Let $M > 0$, there are $C > 0$, $\eta > 1$, and $r^* > 0$, not depending on x_0 and y_0 , such that for all $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω and $\|u\|_{H^1(\Omega)} \leq M$,*

$$e^{-\frac{C}{r^\eta}} \|u\|_{H^1(B(y_0, \kappa r))} \leq \|u\|_{H^1(B(x_0, r))}, \quad 0 < r \leq r^*.$$

A combination of Corollary 2.2, Proposition 2.2 and Proposition 2.3 gives

Theorem 2.2. *Let $\eta > 0$, $M > 0$, $\hat{x} \in \Gamma$ and assume that Ω has UICEBP and it is starshaped. There is a constant $c > 0$ such that for all $u \in H^2(\Omega)$ satisfying $\Delta u = 0$ in Ω , $|u(\hat{x})| \geq \eta$, $\|u\|_{H^2(\Omega)} \leq M$, and for all $\tilde{x} \in \Gamma$, we have*

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\partial_\nu u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*,$$

where r^* can depend on u .

For $\tilde{x} \in \Gamma$, pick $\psi \in C_0^\infty(\mathcal{B}(\tilde{x}, 2r))$ satisfying $\psi = 1$ in a neighborhood of $\mathcal{B}(\tilde{x}, r)$ and $|\partial^\beta \psi| \leq Cr^{-|\beta|}$ for any $\beta \in \mathbb{R}^n$, $|\beta| \leq 2$. Let $u \in H^2(\Omega)$. Using the interpolation inequality

$$\|\psi u\|_{H^1(\Gamma)} \leq C \|\psi u\|_{H^{3/2}(\Gamma)}^{2/3} \|\psi u\|_{L^2(\Gamma)}^{1/3}$$

together with the properties of ψ and the continuity of the trace operator $v \in H^2(\Omega) \rightarrow v|_\Gamma \in H^{3/2}(\Gamma)$, we obtain in an easy manner

$$\|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} \leq Cr^{-4/3} \|u\|_{H^2(\Omega)}^{2/3} \|u\|_{L^2(\mathcal{B}(\tilde{x}, 2r) \cap \Gamma)}^{1/3}.$$

Therefore, we get from Theorem 2.2

Corollary 2.3. *Let $\eta > 0$, $M > 0$, $\hat{x} \in \Gamma$ and assume that Ω has both UICEBP and (\mathcal{G}) and it is starshaped. There is a constant $c > 0$ such that for all $u \in H^2(\Omega)$ satisfying $\Delta u = 0$ in Ω , $|u(\hat{x})| \geq \eta$, $\|u\|_{H^2(\Omega)} \leq M$, and for all $\tilde{x} \in \Gamma$, we have*

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\partial_\nu u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*,$$

where r^* can depend on u .

If in addition $|\partial_\nu u| \leq N|u|$ on Γ for some constant N , then

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)}, \quad 0 < r \leq r^*.$$

We are now able to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We need to prove that there are $\hat{x} \in \Gamma$ and $\eta > 0$ for which $|u_q(\hat{x})| \geq \eta$ for any $q \in \mathcal{Q}_M^0$.

We fix Γ_0 an arbitrary nonempty open subset of $\Gamma \setminus \text{supp}(g)$. By Corollary 1 in [Bo], there is a constant $A > 0$ such that, for all $u \in H^2(\Omega)$ satisfying $\Delta u = 0$ and $\|u\|_{H^2(\Omega)} \leq M$, we have

$$(2.6) \quad \|u\|_{H^1(\Omega)} \leq \frac{A}{|\ln(M^{-1}\delta)|^{1/2}},$$

where $\delta = \|u\|_{H^1(\Gamma_0)} + \|\partial_\nu u\|_{L^2(\Gamma_0)}$.³

Let Γ_1 be an open subset of Γ satisfying $\text{supp}(g) \subset \Gamma_1 \Subset \Gamma$. Proceeding as previously, we deduce from an usual interpolation inequality

$$\begin{aligned} \|g\|_{L^2(\Gamma_a)} = \|\partial_\nu u_q\|_{L^2(\Gamma_1)} &\leq C \|\partial_\nu u_q\|_{H^{-1/2}(\Gamma)}^{1/2} \|\partial_\nu u_q\|_{H^{1/2}(\Gamma)}^{1/2} \\ &\leq C \|u_q\|_{H^1(\Omega)}^{1/2} \|u_q\|_{H^2(\Omega)}^{1/2} \\ &\leq C \|u_q\|_{H^1(\Omega)}^{1/2}. \end{aligned}$$

This and (2.6) imply

$$\|g\|_{L^2(\Gamma_a)} \leq \frac{C}{|\ln(M^{-1}\delta)|^{1/4}}, \quad \text{with } \delta = \|u_q\|_{H^1(\Gamma_0)} + \|\partial_\nu u_q\|_{L^2(\Gamma_0)},$$

or equivalently

$$\tilde{\eta} = Me^{-C^4 \|g\|_{L^2(\Gamma_a)}^{-4}} \leq \|u_q\|_{H^1(\Gamma_0)} + \|\partial_\nu u_q\|_{L^2(\Gamma_0)}.$$

³Note that the smallness condition on δ in Corollary 1 in [Bo] can be easily removed.

Replacing Γ_0 by a smaller subset and proceeding as in the proof of Corollary 2.3, we get

$$(2.7) \quad \tilde{\eta} = Me^{-c^4 \|g\|_{L^2(\Gamma_1)}^{-4}} \leq \|u_q\|_{L^2(\Gamma_0)}.$$

Now since $H^n(\Omega)$ is continuously embedded in $C(\overline{\Omega})$, we derive from (2.7)

$$\eta = \tilde{\eta} |\Gamma_0|^{-1/2} \leq |u_q(\hat{x})| = \max_{\overline{\Gamma_0}} |u_q|.$$

□

3. PROOF OF THE STABILITY ESTIMATE

First, we paraphrase the proof of Proposition 4.1 in [BCJ] to get that there are $B > 0$ and $\sigma > 0$ such that for any $q \in \mathcal{Q}_M^0$, we find $\epsilon(q) > 0$ with the property that for any $f \in C^\alpha(\Gamma)$ satisfying

$$[f]_\alpha = \sup\{|f(x) - f(y)| |x - y|^{-\alpha}; x, y \in \Gamma, x \neq y\} \leq M$$

and $\|f\|_{L^\infty(\Gamma)} \leq \epsilon(q)$, we have

$$(3.1) \quad \|f\|_{L^\infty(\Gamma)} \leq \frac{B}{|\ln \|f u_q\|_{L^\infty(\Gamma)}|^\sigma}.$$

Proof of Theorem 1.1. Let $v = \tilde{u} - u$. Since $\Delta v = 0$, the same argument as in the proof of Theorem 2.1 leads to

$$\|\partial_\nu v\|_{L^2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}^{1/2}.$$

Hence,

$$(3.2) \quad \|v\|_{L^2(\Gamma)} + \|\partial_\nu v\|_{L^2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}^{1/2}.$$

Let $\gamma_0 \Subset \gamma$. Again, by Corollary 1 in [Bo], there is a constant $A > 0$ for which

$$(3.3) \quad \|v\|_{H^1(\Omega)} \leq \frac{A}{|\ln((2M)^{-1}\delta)|^{1/2}},$$

with $\delta = \|v\|_{H^1(\gamma_0)} + \|\partial_\nu v\|_{L^2(\gamma_0)}$.

As we have done previously, we obtain by an interpolation inequality that

$$\|v\|_{H^1(\gamma_0)} \leq C \|v\|_{L^2(\gamma)}^{1/3},$$

and since $\partial_\nu v = 0$ on γ , (3.3) implies

$$(3.4) \quad \|v\|_{H^1(\Omega)} \leq \frac{A}{|\ln(B\|v\|_{L^2(\gamma)})|^{1/2}}.$$

In light of (3.2), we get from (3.4)

$$(3.5) \quad \|v\|_{L^2(\Gamma)} + \|\partial_\nu v\|_{L^2(\Gamma)} \leq \frac{A}{|\ln(B\|v\|_{L^2(\gamma)})|^{1/4}},$$

Let $f = (q - \tilde{q})u$. We fix θ satisfying $2/3 < \theta < 1$ if $n = 2$ and $3/5 < \theta < 1$ if $n = 3$ and set $s = 3\theta/2$ for $n = 2$ and $s = 5\theta/2$ for $n = 3$. By this choice of s , $H^s(\Gamma)$ is continuously embedded in $L^\infty(\Gamma)$. Therefore, using the interpolation inequalities

$$\begin{aligned} \|f\|_{H^s(\Gamma)} &\leq C \|f\|_{H^{3/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 2, \\ \|f\|_{H^s(\Gamma)} &\leq C \|f\|_{H^{5/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 3, \end{aligned}$$

we obtain

$$\begin{aligned} \|f\|_{L^\infty(\Gamma)} &\leq C \|f\|_{H^{3/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 2, \\ \|f\|_{L^\infty(\Gamma)} &\leq C \|f\|_{H^{5/2}(\Gamma)}^\theta \|f\|_{L^2(\Gamma)}^{1-\theta} \quad \text{if } n = 3, \end{aligned}$$

Or

$$\|f\|_{H^{n-1/2}(\Gamma)} = \|(q - \tilde{q})u\|_{H^{n-1/2}(\Gamma)} \leq C \|q - \tilde{q}\|_{B_{n-1/2,1}(\Gamma)} \|u\|_{H^{n-1/2}(\Gamma)}.$$

Consequently

$$(3.6) \quad \|(q - \tilde{q})u\|_{L^\infty(\Gamma)} \leq C \|(q - \tilde{q})u\|_{L^2(\Gamma)}^{1-\theta}.$$

Returning to the definition of v , we get

$$(3.7) \quad (q - \tilde{q})u = \partial_\nu v + \tilde{q}v.$$

A combination of (3.5), (3.6) and (3.7) yields

$$(3.8) \quad \|(q - \tilde{q})u\|_{L^\infty(\Gamma)} \leq \frac{A}{|\ln(B\|v\|_{L^2(\gamma)})|^{(1-\theta)/4}},$$

In light of (3.1), we end up getting

$$\|q - \tilde{q}\|_{L^\infty(\Gamma)} \leq \frac{A}{|\ln |\ln(B\|u - \tilde{u}\|_{L^2(\gamma)})|^\sigma|}.$$

□

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