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# ON DWORK'S $p$ -ADIC FORMAL CONGRUENCES THEOREM AND HYPERGEOMETRIC MIRROR MAPS

E. DELAYGUE, T. RIVOAL AND J. ROQUES

ABSTRACT. Using Dwork's theory, we prove a broad generalisation of his famous  $p$ -adic formal congruences theorem. This enables us to prove certain  $p$ -adic congruences for the generalized hypergeometric series with rational parameters; in particular, they hold for *any* prime number  $p$  and not only for almost all primes. Along the way, using Christol's functions, we provide an explicit formula for the "Eisenstein constant" of any globally bounded hypergeometric series with rational parameters. As an application of these results, we obtain an arithmetic statement of a new type concerning the integrality of Taylor coefficients of the associated mirror maps. It essentially contains all the similar univariate integrality results in the litterature.

## 1. INTRODUCTION

Mirror maps are power series which occur in Mirror Symmetry as the inverse for composition of power series  $q(z) = \exp(\omega_2(z)/\omega_1(z))$ , called local  $q$ -coordinates, where  $\omega_1$  and  $\omega_2$  are particular solutions of the Picard-Fuchs equation associated with certain one-parameter families of Calabi-Yau varieties. They can be viewed as higher dimensional generalisations of the classical modular forms, and in several cases, it has been observed that such mirror maps and  $q$ -coordinates have integral Taylor coefficients at the origin, like the  $q$ -expansion of Eisenstein series for instance. The arithmetical study of mirror maps began with the famous example of a family of mirror for quintic threefolds in  $\mathbb{P}^4$  given by Candelas et al. [6] and associated with the Picard-Fuchs equation

$$\theta^4\omega - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4)\omega = 0, \quad \theta = z\frac{d}{dz}.$$

This equation is (a rescaling of) a generalized hypergeometric differential equation with two linearly independent local solutions at  $z = 0$  given by

$$\omega_1(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n \quad \text{and} \quad \omega_2(z) = G(z) + \log(z)\omega_1(z),$$

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where

$$G(z) = \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} (5H_{5n} - 5H_n) z^n \quad \text{and} \quad H_n := \sum_{k=1}^n \frac{1}{k}.$$

The  $q$ -coordinate  $\exp(\omega_2(z)/\omega_1(z))$  occurs in enumerative geometry and in the Mirror Conjecture associated with quintics threefolds in  $\mathbb{P}^4$  (see [25]). The integrality of its Taylor coefficients at the origin has been proved by Lian and Yau in [26].

In a more general context, Batyrev and van Straten conjectured the integrality of the Taylor coefficients at the origin of a large class of  $q$ -coordinates [2, Conjecture 6.3.4] built on  $A$ -hypergeometric series. (See [33] for an introduction to these series, which generalize the classical hypergeometric series to the multivariate case). Furthermore they provided a lot of examples of univariate  $q$ -coordinates whose Taylor coefficients were subsequently proved to be integers in many cases by Zudilin [34] and Krattenthaler-Rivoal [18].

Motivated by the search for differential operators  $\mathcal{L}$  associated with particular families of Calabi-Yau varieties, Almkvist et al. [1] and Bogner and Reiter [5] introduced the notion of “Calabi-Yau operators”. Even if both notions slightly differ, both

require that an irreducible differential operator  $\mathcal{L} \in \mathbb{Q}(z)[d/dz]$  of Calabi-Yau type satisfies

- ( $P_1$ )  $\mathcal{L}$  has a solution  $\omega_1(z) \in 1 + z\mathbb{C}[[z]]$  at  $z = 0$  which is  $N$ -integral.
- ( $P_2$ )  $\mathcal{L}$  has a linearly independent solution  $\omega_2 = G(z) + \log(z)\omega_1(z)$  at  $z = 0$  with  $G(z) \in z\mathbb{C}[[z]]$  and  $\exp(\omega_2(z)/\omega_1(z))$  is  $N$ -integral.

We say that a power series  $f(z) \in \mathbb{C}[[z]]$  is  $N$ -integral if there exists  $c \in \mathbb{Q}$  such that  $f(cz) \in \mathbb{Z}[[z]]$ . The constant  $c$  might be called the Eisenstein constant of  $f$ , in reference to Eisenstein’s theorem that such a constant  $c$  exists when  $f$  is a holomorphic algebraic function over  $\mathbb{Q}(z)$ .

One of the main results of this article is an effective criterion for an irreducible generalized hypergeometric differential operator  $\mathcal{L} \in \mathbb{Q}(z)[d/dz]$  to satisfy properties ( $P_1$ ) and ( $P_2$ ).

For all tuples  $\alpha := (\alpha_1, \dots, \alpha_r)$  and  $\beta := (\beta_1, \dots, \beta_s)$  of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , we write  $\mathcal{L}_{\alpha, \beta}$  for the generalized hypergeometric differential operator associated with  $(\alpha, \beta)$  and defined by

$$\mathcal{L}_{\alpha, \beta} := \prod_{i=1}^s (\theta + \beta_i - 1) - z \prod_{i=1}^r (\theta + \alpha_i), \quad \theta = z \frac{d}{dz}.$$

We recall that  $\mathcal{L}_{\alpha, \beta}$  is irreducible if and only if, for all  $i \in \{1, \dots, r\}$  and all  $j \in \{1, \dots, s\}$ , we have  $\alpha_i \not\equiv \beta_j \pmod{\mathbb{Z}}$ . The equation  $\mathcal{L}_{\alpha, \beta} \cdot \omega = 0$  admits a formal solution  $F_{\alpha, \beta}(z) \in 1 + z\mathbb{C}[[z]]$  if and only if there exists  $i \in \{1, \dots, s\}$  such that  $\beta_i = 1$ . In this case,  $F_{\alpha, \beta}$  is uniquely determined by  $\alpha$  and  $\beta$  and is

$$F_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} z^n, \quad (1.1)$$

where  $(x)_n$  denotes Pochhammer symbol  $(x)_n = x(x+1)\cdots(x+n-1)$  if  $n \geq 1$  and  $(x)_0 = 1$  otherwise.  $F_{\alpha, \beta}$  is a generalized hypergeometric series and if one assumes that

$\beta_s = 1$ , then our definition (1.1) agrees with the classical notation

$$F_{\alpha, \beta}(z) = {}_rF_{s-1} \left[ \begin{matrix} \alpha_1, \dots, \alpha_r \\ \beta_1, \dots, \beta_{s-1} \end{matrix} ; z \right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_{s-1})_n} \frac{z^n}{n!}.$$

An elementary computation of the  $p$ -adic valuation of Pochhammer symbols leads to the following result.

**Proposition 1.** *Let  $\alpha$  and  $\beta$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . Then,  $F_{\alpha, \beta}$  is  $N$ -integral if and only if, for almost all primes  $p$ , we have  $F_{\alpha, \beta}(z) \in \mathbb{Z}_p[[z]]$ .*

*Remark.* We say that an assertion  $\mathcal{A}_p$  is true for almost all primes  $p$  if there is  $C \in \mathbb{N}$  such that  $\mathcal{A}_p$  is true for all primes  $p \geq C$ .

Using Proposition 1 in combination with a result of Christol (Proposition 1 in [7]), one obtains an effective criterion for  $F_{\alpha, \beta}$  to be  $N$ -integral, *i.e.* for  $\mathcal{L}_{\alpha, \beta}$  to satisfy  $(P_1)$ . To state this criterion, we introduce some notations.

For all  $x \in \mathbb{Q}$ , we write  $\langle x \rangle$  for the unique element in  $(0, 1]$  such that  $x - \langle x \rangle \in \mathbb{Z}$ , *i.e.*  $\langle x \rangle = 1 - \{1 - x\}$ , where  $\{\cdot\}$  is the fractional part function. In particular, we have  $x - \langle x \rangle = -\lfloor 1 - x \rfloor$ , where  $\lfloor \cdot \rfloor$  denotes the floor function. We write  $\preceq$  for the total order on  $\mathbb{R}$  defined by

$$x \preceq y \iff \left( \langle x \rangle < \langle y \rangle \quad \text{or} \quad (\langle x \rangle = \langle y \rangle \quad \text{and} \quad x \geq y) \right).$$

Given tuples  $\alpha = (\alpha_1, \dots, \alpha_r)$  and  $\beta = (\beta_1, \dots, \beta_s)$  of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , we write  $d_{\alpha, \beta}$  for the least common multiple of the exact denominators of elements of  $\alpha$  and  $\beta$ . For all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and all  $x \in \mathbb{R}$ , we set

$$\xi_{\alpha, \beta}(a, x) := \#\{1 \leq i \leq r : a\alpha_i \preceq x\} - \#\{1 \leq j \leq s : a\beta_j \preceq x\}.$$

We now state Christol's criterion for the  $N$ -integrality of  $F_{\alpha, \beta}$ .

**Theorem A** (Christol, [7]). *Let  $\alpha := (\alpha_1, \dots, \alpha_r)$  and  $\beta := (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . Then, the following assertions are equivalent:*

- (i)  $F_{\alpha, \beta}$  is  $N$ -integral;
- (ii) For all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and all  $x \in \mathbb{R}$ , we have  $\xi_{\alpha, \beta}(a, x) \geq 0$ .

*Remark.* Formally, Christol proved Theorem A (Proposition 1 in [7]) under the assumptions that  $r = s$ , that there is  $j \in \{1, \dots, s\}$  such that  $\beta_j \in \mathbb{N}$  and that all elements  $\alpha \in \mathbb{N}$  of  $\alpha$  and  $\beta$  satisfies  $\alpha \geq \beta_j$ . However, his proof does not use these assumptions.

Theorem A provides a criterion for an irreducible operator  $\mathcal{L}_{\alpha, \beta}$  to satisfy  $(P_1)$  but do not give any information on the rational numbers  $C$  satisfying  $F_{\alpha, \beta}(Cz) \in \mathbb{Z}[[z]]$ . If  $F_{\alpha, \beta}$  is  $N$ -integral then it is not hard to see that the set of all  $C \in \mathbb{Q}$  satisfying  $F_{\alpha, \beta}(Cz) \in \mathbb{Z}[[z]]$  is  $C_{\alpha, \beta}\mathbb{Z}$  for some  $C_{\alpha, \beta} \in \mathbb{Q} \setminus \{0\}$ . Our first result, Theorem 1 below, gives some arithmetical properties of  $C_{\alpha, \beta}$  and gives a simple formula for  $C_{\alpha, \beta}$  when  $r = s$  and when all elements of  $\alpha$  and  $\beta$  lie in  $(0, 1]$ . Before stating this result, we introduce some notations.

Let  $\mathbb{Z}_p$  denotes the ring of  $p$ -adic integers. Then, for all primes  $p$ , we define

$$\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \#\{1 \leq i \leq r : \alpha_i \in \mathbb{Z}_p\} - \#\{1 \leq j \leq s : \beta_j \in \mathbb{Z}_p\}.$$

Note that if  $\alpha \in \mathbb{Q} \setminus \{0\}$ , then  $\alpha \in \mathbb{Z}_p$  if and only if  $p$  does not divide the exact denominator of  $\alpha$ . Furthermore, we write  $\mathcal{P}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  for the set of all primes  $p$  such that  $p$  divides  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  or  $p \leq r - s + 1$ . In particular, if  $r = s$ , then  $\mathcal{P}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  is the set of the prime divisors of  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ . Finally, for all rational numbers  $a$ , we write  $d(a)$  for the exact denominator of  $a$ . Our result on  $C_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  is the following.

**Theorem 1.** *Let  $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$  and  $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  is  $N$ -integral. Then, there exists  $C \in \mathbb{N}$  such that*

$$C_{\boldsymbol{\alpha}, \boldsymbol{\beta}} = C \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \prod_{p \in \mathcal{P}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}} p^{-\lfloor \frac{\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p-1} \rfloor}. \quad (1.2)$$

Furthermore, if  $r = s$  and if all elements of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  lie in  $(0, 1]$ , then we have  $C = 1$ .

*Remarks.* The following comments are detailed in Section 2.1.

- If  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are tuples of same length of parameters in  $(0, 1]$  such that  $F_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \in \mathbb{Q}[[z]]$  is algebraic over  $\mathbb{Q}(z)$ , then Theorem 1 gives a simple formula for the Eisenstein constant of  $F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ .

- When all the elements of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  lie in  $(0, 1]$ , Theorem 1 also gives a necessary condition on the numerators of the elements of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  for  $F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  to be  $N$ -integral.

- Given tuples  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$ , one can follow the proofs of Theorem 1 and Lemma 5 to determine an explicit bound for the prime divisors of  $C$ .

Let  $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$  and  $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\beta_s = 1$  and  $F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  is  $N$ -integral, so that  $\mathcal{L}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  satisfies property  $(P_1)$ . Then, a simple computation (see Equation (7) in [2]) shows that  $\mathcal{L}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  has a formal solution  $G_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z) + \log(z)F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z)$  with  $G_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z) \in z\mathbb{C}[[z]]$  if and only if there exists  $i \in \{1, \dots, s-1\}$  such that  $\beta_i = 1$ . In this case  $G_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  is uniquely determined by  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  and it is explicitly given by

$$G_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z) := \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \left( \sum_{i=1}^r H_{\alpha_i}(n) - \sum_{j=1}^s H_{\beta_j}(n) \right),$$

where, for all  $n \in \mathbb{N}$  and all  $x \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ ,  $H_x(n) := \sum_{k=0}^{n-1} \frac{1}{x+k}$ .

For all tuples  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , we define

$$q_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z) := \exp \left( \frac{G_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z) + \log(z)F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z)} \right) = z \exp \left( \frac{G_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z)}{F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(z)} \right), \quad (1.3)$$

so that  $\mathcal{L}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  satisfies  $(P_2)$  if and only if there are at least two elements equal to 1 in  $\boldsymbol{\beta}$  and  $q_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  is  $N$ -integral. The mirror map  $z_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  associated with  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  is the compositional inverse of  $q_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ . For all  $C \in \mathbb{Q}$ , we have  $z_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(Cq) \in \mathbb{Z}[[q]]$  if and only if  $q_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(Cz) \in \mathbb{Z}[[z]]$ , thus our results and comments on  $N$ -integrality of  $q$ -coordinates also hold for the corresponding mirror maps.

For a detailed description of the known results on  $N$ -integrality of  $q_{\alpha,\beta}$  and  $z_{\alpha,\beta}$ , we refer the reader to Section 2.2 below. The tuples  $\alpha$  and  $\beta$  such that  $F_{\alpha,\beta}$  and  $q_{\alpha,\beta}$  are  $N$ -integral are entirely determined:

- when  $\alpha$  and  $\beta$  are  $R$ -partitioned. Throughout this article, we say that  $\alpha$  is  $R$ -partitioned if  $\alpha$  is the concatenation of tuples of the form  $(b/m)_{b \in \{1, \dots, m\}, \gcd(b,m)=1}$  for  $m \in \mathbb{N}$ ,  $m \geq 1$ , up to permutation<sup>(1)</sup>. The characterization for this class of parameters is a consequence of Delaygue's Theorems 1 and 3 in [8];

- when all parameters of  $\alpha$  lie in  $(0, 1]$ ,  $\beta = (1, \dots, 1)$  and  $\alpha$  and  $\beta$  have the same number of parameters  $r \geq 2$ . The characterization for this class of parameters is a consequence of Roques' articles [31] and [32].

The starting point of the proofs of these characterizations is to reduce the problem to a  $p$ -adic statement for any prime  $p$ , according to the following simple principle:

If  $x \in \mathbb{Q}$ , then  $x \in \mathbb{Z}$  if and only if, for all prime  $p$ ,  $x$  belongs to the ring of  $p$ -adic integers  $\mathbb{Z}_p$ .

Then, given a fixed prime  $p$ , one can apply the following lemma of Dieudonné and Dwork [17, Chap. IV, Sec. 2, Lemma 3] and its corollary (see [34, Lemma 5]) to get rid of the exponential map in (1.3).

**Lemma 1** (Dieudonné, Dwork). *Given a prime  $p$  and  $F(z) \in 1 + z\mathbb{Q}_p[[z]]$ , we have  $F(z) \in 1 + z\mathbb{Z}_p[[z]]$  if and only if  $F(z^p)/F(z)^p \in 1 + pz\mathbb{Z}_p[[z]]$ .*

**Corollary 1.** *Given a prime  $p$  and  $f(z) \in z\mathbb{Q}[[z]]$ , we have  $\exp(f(z)) \in 1 + z\mathbb{Z}_p[[z]]$  if and only if  $f(z^p) - pf(z) \in pz\mathbb{Z}_p[[z]]$ .*

By Corollary 1,  $q_{\alpha,\beta}$  is  $N$ -integral if and only if there exists  $C \in \mathbb{Q}$  such that, for all primes  $p$ , we have

$$\frac{G_{\alpha,\beta}(Cz^p)}{F_{\alpha,\beta}(Cz^p)} - p \frac{G_{\alpha,\beta}(Cz)}{F_{\alpha,\beta}(Cz)} \in pz\mathbb{Z}_p[[z]]. \quad (1.4)$$

One of the main results of this article (Theorem 2 below) provides an analogous version of (1.4) for a large class of tuples  $\alpha$  and  $\beta$  and with  $\mathbb{Z}_p$  replaced by certain algebras of  $\mathbb{Z}_p$ -valued functions. This result enables us to prove a complete characterization (Theorem 3 below) of tuples  $\alpha$  and  $\beta$  such that  $F_{\alpha,\beta}$  and  $q_{\alpha,\beta}$  are  $N$ -integral, without any restriction on the shape of  $\alpha$  nor  $\beta$ .

**1.1. Additional notations.** To state Theorem 2, we first define some algebras of  $\mathbb{Z}_p$ -valued functions and a constant associated with  $(\alpha, \beta)$ .

- For all primes  $p$  and all positive integers  $n$ , we write  $\mathfrak{A}_{p,n}$ , respectively  $\mathfrak{A}_{p,n}^*$ , for the  $\mathbb{Z}_p$ -algebra of the functions  $f : (\mathbb{Z}_p^\times)^n \rightarrow \mathbb{Z}_p$  such that, for all positive integers  $m$ , all  $\mathbf{x} \in (\mathbb{Z}_p^\times)^n$  and all  $\mathbf{a} \in \mathbb{Z}_p^n$ , we have

$$f(\mathbf{x} + \mathbf{a}p^m) \equiv f(\mathbf{x}) \pmod{p^m\mathbb{Z}_p},$$

respectively  $f(\mathbf{x} + \mathbf{a}p^m) \equiv f(\mathbf{x}) \pmod{p^{m-1}\mathbb{Z}_p}$ .

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<sup>1</sup>We say that, up to permutation,  $(\alpha_1, \dots, \alpha_n) = (\alpha'_1, \dots, \alpha'_n)$  if there exists a permutation  $\sigma$  of  $\{1, \dots, n\}$  such that, for all  $i \in \{1, \dots, n\}$ , we have  $\alpha_i = \alpha'_{\sigma(i)}$ .

• If  $D$  is a positive integer coprime to  $p$ , then, for all  $\nu \in \mathbb{N}$ , and all  $b \in \{1, \dots, D\}$  coprime to  $D$ , we write  $\Omega_b(p^\nu, D)$  for the set of all  $t \in \{1, \dots, p^\nu D\}$  coprime to  $p^\nu D$  satisfying  $t \equiv b \pmod{D}$ .

• We write  $\mathcal{A}_b(p^\nu, D)$ , respectively  $\mathcal{A}_b(p^\nu, D)^*$ , for the  $\mathbb{Z}_p$ -algebra of the functions  $f : \Omega_b(p^\nu, D) \rightarrow \mathbb{Z}_p$  such that, for all positive integers  $m$  and all  $t_1, t_2 \in \Omega_b(p^\nu, D)$  satisfying  $t_1 \equiv t_2 \pmod{p^m}$ , we have  $f(t_1) \equiv f(t_2) \pmod{p^m \mathbb{Z}_p}$ , respectively  $f(t_1) \equiv f(t_2) \pmod{p^{m-1} \mathbb{Z}_p}$ .

• For all  $t \in \Omega_b(p^\nu, D)$  and all  $r \in \mathbb{N}$ , we write  $t^{(r)}$  for the unique element of  $\{1, \dots, p^\nu D\}$  satisfying  $t^{(r)} \equiv t \pmod{p^\nu}$  and  $p^r t^{(r)} \equiv t \pmod{D}$ .

• Furthermore, if  $\beta \notin \mathbb{Z}^s$ , then we write  $\mathbf{m}_{\alpha, \beta}$  for the number of elements of  $\alpha$  and  $\beta$  with exact denominator divisible by 4. We write  $d_{\alpha, \beta}^*$  for the integer obtained by dividing  $d_{\alpha, \beta}$  by the product of its prime divisors. We set  $C'_{\alpha, \beta} = 2C_{\langle \alpha \rangle, \langle \beta \rangle}$  and  $d'_{\alpha, \beta} = 2d_{\alpha, \beta}^*$  if  $\beta \notin \mathbb{Z}^s$  and if  $\mathbf{m}_{\alpha, \beta}$  is odd, and we set  $C'_{\alpha, \beta} = C_{\langle \alpha \rangle, \langle \beta \rangle}$  and  $d'_{\alpha, \beta} = d_{\alpha, \beta}^*$  otherwise.

**1.2. Statements of the main results.** By Theorem A, the  $N$ -integrality of  $F_{\alpha, \beta}$  depends on the graphs of Christol's functions  $\xi_{\alpha, \beta}(a, \cdot)$ . The  $N$ -integrality of  $q_{\alpha, \beta}$  also strongly depends on the graphs of these functions. More precisely, let  $m_{\alpha, \beta}(a)$  denote the smallest element in the ordered set  $(\{a\alpha_1, \dots, a\alpha_r, a\beta_1, \dots, a\beta_s\}, \leq)$ . Let  $H_{\alpha, \beta}$  denotes the assertion

$H_{\alpha, \beta}$ : “For all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and all  $x \in \mathbb{R}$  satisfying  $m_{\alpha, \beta}(a) \leq x < a$ , we have  $\xi_{\alpha, \beta}(a, x) \geq 1$ .”

One of our main results is the following.

**Theorem 2.** *Let  $\alpha$  and  $\beta$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  with the same number of elements such that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint (this is equivalent to the irreducibility of  $\mathcal{L}_{\alpha, \beta}$ ) and such that  $H_{\alpha, \beta}$  holds.*

*Let  $p$  be a fixed prime and write  $d_{\alpha, \beta} = p^\nu D$  with  $\nu, D \in \mathbb{N}$  and  $D$  coprime to  $p$ . Let  $b \in \{1, \dots, D\}$  be coprime to  $D$ . Then, there exists a sequence  $(R_{k, b})_{k \geq 0}$  of elements in  $\mathcal{A}_b(p^\nu, D)^*$  such that, for all  $t \in \Omega_b(p^\nu, D)$ , we have*

$$\frac{G_{\langle t^{(1)} \alpha \rangle, \langle t^{(1)} \beta \rangle}}{F_{\langle t^{(1)} \alpha \rangle, \langle t^{(1)} \beta \rangle}}(C'_{\alpha, \beta} z^p) - p \frac{G_{\langle t \alpha \rangle, \langle t \beta \rangle}}{F_{\langle t \alpha \rangle, \langle t \beta \rangle}}(C'_{\alpha, \beta} z) = p \sum_{k=0}^{\infty} R_{k, b}(t) z^k.$$

*Furthermore, if  $p$  is a prime divisor of  $d_{\alpha, \beta}$ , then, for all  $k \in \mathbb{N}$ ,*

- *if  $\beta \in \mathbb{Z}^r$ , then we have  $R_{k, b} \in p^{-1 - \lfloor \lambda_p / (p-1) \rfloor} \mathcal{A}_b(p^\nu, D)$ ;*
- *if  $\beta \notin \mathbb{Z}^r$  and  $p-1 \nmid \lambda_p$ , then we have  $R_{k, b} \in \mathcal{A}_b(p^\nu, D)$ ;*
- *if  $\beta \notin \mathbb{Z}^r$ ,  $\mathbf{m}_{\alpha, \beta}$  is odd and  $p=2$ , then we have  $R_{k, b} \in \mathcal{A}_b(p^\nu, D)$ .*

In Theorem 2 and throughout this article, when  $\alpha = (\alpha_1, \dots, \alpha_r)$  and  $f$  is a map defined on  $\{\alpha_1, \dots, \alpha_r\}$ , we write  $f(\alpha)$  for  $(f(\alpha_1), \dots, f(\alpha_r))$ .

*Remarks.* • If  $\alpha$  and  $\beta$  satisfy hypothesis of Theorem 2, then using Theorem A, we obtain that  $F_{\alpha, \beta}$  is  $N$ -integral.

• If  $\beta \in \mathbb{Z}^r$ , then  $\lambda_p \leq -1$  and  $-1 - \lfloor \lambda_p / (p-1) \rfloor \geq 0$  so that  $p^{-1 - \lfloor \lambda_p / (p-1) \rfloor} \mathcal{A}_b(p^\nu, D) \subset \mathcal{A}_b(p^\nu, D) \subset \mathcal{A}_b(p^\nu, D)^*$ .

The  $N$ -integrality of  $q_{\alpha,\beta}$  is closely related to the  $N$ -integrality of a product  $\exp(S_{\alpha,\beta}(z))$  of  $q$ -coordinates associated with  $(\alpha, \beta)$ , that we now define. We set

$$S_{\alpha,\beta}(z) := \sum_{\substack{a=1 \\ \gcd(a,d)=1}}^d \frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)},$$

with  $d = d_{\alpha,\beta}$ , so that

$$\exp(S_{\alpha,\beta}(z)) = \frac{1}{z^{\varphi(d)}} \prod_{\substack{a=1 \\ \gcd(a,d)=1}}^d q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z),$$

where  $\varphi$  denotes Euler's totient function. Our criterion for the  $N$ -integrality of  $q_{\alpha,\beta}(z)$  and  $\exp(S_{\alpha,\beta}(z))$  is the following.

**Theorem 3.** *Let  $\alpha := (\alpha_1, \dots, \alpha_r)$  and  $\beta := (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint (this is equivalent to the irreducibility of  $\mathcal{L}_{\alpha,\beta}$ ) and such that  $F_{\alpha,\beta}$  is  $N$ -integral. Then,*

- (1) *For all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , all Taylor coefficients at the origin of  $q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$  are positive, but its constant term which is 0;*
- (2) *The following assertions are equivalent.*
  - (i)  *$q_{\alpha,\beta}(z)$  is  $N$ -integral;*
  - (ii)  *$q_{\alpha,\beta}(C'_{\alpha,\beta}z) \in \mathbb{Z}[[z]]$ ;*
  - (iii) *Assertion  $H_{\alpha,\beta}$  holds, we have  $r = s$  and, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have  $q_{\alpha,\beta}(z) = q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$ .*

*Furthermore, if (i) holds, then we have either  $\alpha = (1/2)$  and  $\beta = (1)$ , or  $s \geq 2$  and there are at least two 1's in  $\langle \beta \rangle$ .*

- (3) *If  $r = s$  and if  $H_{\alpha,\beta}$  holds, then  $\exp(S_{\alpha,\beta}(z))$  is  $N$ -integral and we have*

$$\exp\left(\frac{S_{\alpha,\beta}(C'_{\alpha,\beta}z)}{\mathbf{n}_{\alpha,\beta}}\right) \in \mathbb{Z}[[z]],$$

where

$$\mathbf{n}_{\alpha,\beta} := d_{\alpha,\beta} \prod_{p|d_{\alpha,\beta}} p^{-2-\lfloor \frac{\lambda_p}{p-1} \rfloor} \text{ if } \beta \in \mathbb{Z}^s, \quad \text{and} \quad \mathbf{n}_{\alpha,\beta} := d'_{\alpha,\beta} \prod_{\substack{p|d'_{\alpha,\beta} \\ p-1|\lambda_p}} p^{-1} \text{ otherwise.}$$

As a consequence, we obtain the following relation between  $N$ -integrality of  $q_{\alpha,\beta}(z)$  and  $\exp(S_{\alpha,\beta}(z))$ .

**Corollary 2.** *Let  $\alpha := (\alpha_1, \dots, \alpha_r)$  and  $\beta := (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint (this is equivalent to the irreducibility of  $\mathcal{L}_{\alpha,\beta}$ ) and such that  $F_{\alpha,\beta}$  is  $N$ -integral. Then the following assertions are equivalent:*

- (1)  *$q_{\alpha,\beta}(z)$  is  $N$ -integral;*



(2)  $\exp(S_{\alpha,\beta}(z))$  is  $N$ -integral and  $\exp(S_{\alpha,\beta}(z)) = (q_{\alpha,\beta}(z)/z)^{\varphi(d_{\alpha,\beta})}$ .

Under the assumptions of Theorem 3 for the tuples  $\alpha$  and  $\beta$  and if  $q_{\alpha,\beta}(z)$  is  $N$ -integral, then Assertion (3) of Theorem 3 leads to

$$\left(\frac{1}{C'_{\alpha,\beta}z}q_{\alpha,\beta}(C'_{\alpha,\beta}z)\right)^{\varphi(d_{\alpha,\beta})/n_{\alpha,\beta}} \in \mathbb{Z}[[z]]. \quad (1.5)$$

It follows that in some cases, we obtain the integrality of the Taylor coefficients of a non-trivial root of the  $q$ -coordinate. But when all the elements of  $\beta$  are integers, Theorem 2 in combination with Corollary 1 and Assertion (2) of Theorem 3 provides a better result than (1.5). Indeed, we obtain the following result.

**Corollary 3.** *Let  $\alpha$ , respectively  $\beta$ , be a tuple of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , respectively of positive integers, such that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint. If  $F_{\alpha,\beta}(z)$  and  $q_{\alpha,\beta}(z)$  are  $N$ -integral, then we have*

$$\left(\frac{1}{C'_{\alpha,\beta}z}q_{\alpha,\beta}(C'_{\alpha,\beta}z)\right)^{1/n'_{\alpha,\beta}} \in \mathbb{Z}[[z]],$$

with

$$n'_{\alpha,\beta} = \prod_{p|d_{\alpha,\beta}} p^{-1-\lfloor \frac{\lambda_p}{p-1} \rfloor}.$$

Corollary 3 is stronger than (1.5) because  $n_{\alpha,\beta}/n'_{\alpha,\beta} = d_{\alpha,\beta}^*$  divides  $\varphi(d_{\alpha,\beta})$ . Let us now make some remarks on Theorems 2 and 3 and their corollaries.

- Note that, by [14, Lemma 5], if  $f(z) \in \mathbb{Z}[[z]]$  and if  $V$  is the greatest positive integer satisfying  $f(z)^{1/V} \in \mathbb{Z}[[z]]$ , then the positive integers  $U$  satisfying  $f(z)^{1/U} \in \mathbb{Z}[[z]]$  are exactly the positive divisors of  $V$ . Furthermore, by [24, Introduction], for all positive integers  $v$  and all  $C \in \mathbb{Q}$ , we have  $((Cq)^{-1}z_{\alpha,\beta}(Cq))^{1/v} \in \mathbb{Z}[[z]]$  if and only if  $((Cz)^{-1}q_{\alpha,\beta}(Cz))^{1/v} \in \mathbb{Z}[[z]]$ . We deduce that Corollary 3 also gives the integrality of the Taylor coefficients at the origin of roots of mirror maps.

- In Section 7, we prove Proposition 6 which generalizes Assertion (1) of Theorem 3. Furthermore, if  $q_{\alpha,\beta}(z)$  is  $N$ -integral, then, according to Assertions (1) and (2) of Theorem 3, all Taylor coefficients at  $z = 0$  of  $q_{\alpha,\beta}(C'_{\alpha,\beta}z)$  are positive integers, but its constant term = 0. This leads to a natural question: do these coefficients count any object related to the geometric origin of  $q_{\alpha,\beta}(z)$ ?

- In all cases,  $n_{\alpha,\beta}$  and  $n'_{\alpha,\beta}$  are positive integers.

- Suppose that  $\mathcal{L}_{\alpha,\beta}$  is an irreducible operator satisfying  $(P_1)$ . We can formally consider  $q_{\alpha,\beta}$  without assuming that there are at least two 1's in  $\beta$ . But if  $q_{\alpha,\beta}$  is  $N$ -integral, then  $q_{\alpha,\beta} = q_{\langle \alpha \rangle, \langle \beta \rangle}$  and if furthermore  $s \geq 2$ , then there are at least two 1's in  $\langle \beta \rangle$  so that  $q_{\alpha,\beta}$  is the exponential of a ratio of power series canceled by  $\mathcal{L}_{\langle \alpha \rangle, \langle \beta \rangle}$ . The operator  $\mathcal{L}_{\alpha,\beta}$  may not satisfy  $(P_2)$ , but (see section 2.1.2)  $\mathcal{L}_{\langle \alpha \rangle, \langle \beta \rangle}$  is an irreducible operator satisfying  $(P_1)$  and  $(P_2)$ . Furthermore, if  $q_{\alpha,\beta}$  is  $N$ -integral, then we have  $r = s$  so that  $\mathcal{L}_{\alpha,\beta}$  and  $\mathcal{L}_{\langle \alpha \rangle, \langle \beta \rangle}$  are Fuchsian.

- As explained in more details in Section 2.2, Theorem 3 generalizes previous results on the integrality of the Taylor coefficients at the origin of  $q$ -coordinates associated with generalized hypergeometric functions.

- Let us explain the reason why Theorem 3 provides an effective criterion for the  $N$ -integrality of  $q$ -coordinates  $q_{\alpha,\beta}$ . Given tuples  $\alpha$  and  $\beta$ , Assertion  $H_{\alpha,\beta}$  can easily be checked and the identity  $q_{\alpha,\beta}(z) = q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$  is equivalent to  $F_{\alpha,\beta}(z)G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z) = F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)G_{\alpha,\beta}(z)$ . Let us assume that there are at least two 1's in  $\beta$  so that  $G_{\alpha,\beta}(z) + \log(z)F_{\alpha,\beta}(z)$  is canceled by  $\mathcal{L}_{\alpha,\beta}$ .

On the one hand, if  $r \neq s$ , then  $q_{\alpha,\beta}$  is not  $N$ -integral. On the other hand, if  $r = s$ , then  $F_{\alpha,\beta}(z)G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$  and  $F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)G_{\alpha,\beta}(z)$  are analytic functions at  $z = 0$  canceled by the tensor product  $\mathcal{L}'$  of the differential operators  $\mathcal{L}_{\alpha,\beta}$  and  $\mathcal{L}_{\langle a\alpha \rangle, \langle a\beta \rangle}$ . Since the order of  $\mathcal{L}'$  is less than or equal to  $r^2$  then we have  $q_{\alpha,\beta} = q_{\langle a\alpha \rangle, \langle a\beta \rangle}$  if and only if the first  $r^2$  Taylor coefficients at the origin of  $F_{\alpha,\beta}(z)G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$  and  $F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)G_{\alpha,\beta}(z)$  are equal, which can be checked in a finite number of elementary algebraic operations.

- If  $q_{\alpha,\beta}(z)$  is  $N$ -integral, then the power series  $q_{\alpha,\beta}(C'_{\alpha,\beta}z)/(C'_{\alpha,\beta}z)$  to the power  $\varphi(d_{\alpha,\beta})/\mathfrak{n}_{\alpha,\beta}$  lies in  $\mathbb{Z}[[z]]$  so that, in some cases, a non-trivial root of  $q_{\alpha,\beta}(z)$  is  $N$ -integral. This suggests that one might be able to improve Assertion (3) of Theorem 3 by replacing  $\mathfrak{n}_{\alpha,\beta}$  by  $\varphi(d_{\alpha,\beta})$  or  $d_{\alpha,\beta}$  <sup>(2)</sup>. But this statement is not always true. Indeed, a counterexample is given by  $\alpha = (1/7, 1/4, 3/7, 6/7)$  and  $\beta = (1, 1, 1, 1)$ , where we have  $d_{\alpha,\beta} = 28$ ,  $C'_{\alpha,\beta} = C_{\alpha,\beta} = 2^3 7^2$ ,  $\varphi(28) = 12$ ,  $\mathfrak{n}_{\alpha,\beta} = 2$ ,

$$\exp\left(\frac{S_{\alpha,\beta}(2^3 7^2 z)}{12}\right) \in 1 + 4802z + \frac{81541341}{2}z^2 + \frac{1328534273395}{3}z^3 + z^4\mathbb{Q}[[z]]$$

and

$$\exp\left(\frac{S_{\alpha,\beta}(2^3 7^2 z)}{28}\right) \in 1 + 2058z + \frac{29299137}{2}z^2 + z^3\mathbb{Q}[[z]].$$

Before ending this introduction, we would like to mention that this article also contains two useful results, that play a central role in the proof of Theorem 3, but we need too many definitions to state them here. The first one is Proposition 5 stated in Section 3.3 which gives an useful formula for the  $p$ -adic valuation of the Taylor coefficients at the origin of  $F_{\alpha,\beta}(C_{\alpha,\beta}z)$  when  $p$  is a prime divisor of  $d_{\alpha,\beta}$ . The second one is Theorem 4 stated in Section 5 which generalizes Dwork's theorem on formal congruences [12, Theorem 1.1] also used by Dwork in [13] to obtain the analytic continuation of certain  $p$ -adic functions.

While working on this article, we found an error in a lemma in Lang's book [23, Lemma 1.1, Section 1, Chapter 14] about the arithmetic properties of Mojita's  $p$ -adic Gamma function. This lemma has been used in several articles on the integrality of the Taylor coefficients of mirror maps including papers of the authors. Even if we do not use this lemma in this article, we give in Section 2.4 a corrected version and we explain why the initial error does not change the validity of our previous results.

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<sup>2</sup>Note that  $\exp(S_{\alpha,\beta}(z)/\varphi(d_{\alpha,\beta}))$  is the geometric mean of the  $\frac{1}{z}q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$ .

**1.3. Structure of the paper.** In Section 2, we make comments on results stated in introduction and we compare this results with previous ones on  $N$ -integrality of mirror maps associated with generalized hypergeometric functions. Furthermore, we give a corrected version of a lemma of Lang on Mojita's  $p$ -adic Gamma function at the end of this section.

Section 3 is devoted to a detailed study of the  $p$ -adic valuation of Pochhammer symbol. In particular, we prove Proposition 1 and we define and study step functions  $\Delta_{\alpha,\beta}$  associated with tuples  $\alpha$  and  $\beta$  which play a central role in proofs of Theorems 1-3.

We prove Theorem 1 in Section 4.

Section 5 is devoted to the statement and the proof of Theorem 4 on formal congruences between formal power series. We also compare Theorem 4 with previous generalizations of Dwork's theorem on formal congruences [12, Theorem 1.1]. Theorem 4 is the most important tool in the proof of Theorem 2.

We prove Theorem 2 in Section 6, which is by far the longest and the most technical part of this article.

Sections 7, 8 and 9 are respectively dedicated to the proofs of Assertions (1), (3) and (2) of Theorem 3.

## 2. COMMENTS ON THE MAIN RESULTS AND COMPARISON WITH PREVIOUS ONES

This section is devoted to a detailed study of certain consequences of Theorems 1-3. In particular, we compare these theorems with previous results on the integrality of the Taylor coefficients at the origin of generalized hypergeometric series and their associated (roots of) mirror maps. This section also contains some results that we use throughout this article.

**2.1. Comments on the main results.** We provide precisions on Theorems A, 1 and 3.

**2.1.1. An example of application of Theorem 1.** We illustrate Theorem A and Theorem 1 with an example. Let  $\alpha := (1/6, 1/2, 2/3)$  and  $\beta := (1/3, 1, 1)$  so that we have  $d_{\alpha,\beta} = 6$ . According to Theorem A,  $F_{\alpha,\beta}$  is  $N$ -integral if and only if, for all  $a \in \{1, 5\}$  and all  $x \in \mathbb{R}$ , we have  $\xi_{\alpha,\beta}(a, x) \geq 0$ .

We have  $1/6 \prec 1/3 \prec 1/2 \prec 2/3 \prec 1$  thus, for all  $x \in \mathbb{R}$ , we get  $\xi_{\alpha,\beta}(1, x) \geq 0$ . Furthermore, we have  $1/3 + 3 = 10/3 \prec 5/2 \prec 5/3 \prec 5/6 \prec 5$  and thus, for all  $x \in \mathbb{R}$ , we get  $\xi_{\alpha,\beta}(5, x) \geq 0$ . This shows that  $F_{\alpha,\beta}$  is  $N$ -integral.

Moreover, we have  $r = s$ , all elements of  $\alpha$  and  $\beta$  lie in  $(0, 1]$ ,  $\lambda_2(\alpha, \beta) = 1 - 3 = -2$  and  $\lambda_3(\alpha, \beta) = 1 - 2 = -1$  thus, according to Theorem 1, we get

$$C_{\alpha,\beta} = \frac{6 \cdot 2 \cdot 3}{3} 2^{-\lfloor -2 \rfloor} 3^{-\lfloor -1/2 \rfloor} = 2^4 3^2.$$

**2.1.2.  $N$ -integrality of  $F_{\langle\alpha\rangle, \langle\beta\rangle}$ .** We show that if  $F_{\alpha,\beta}$  is  $N$ -integral then  $F_{\langle\alpha\rangle, \langle\beta\rangle}$  is also  $N$ -integral. The converse is false in general, a counterexample being given by  $\alpha = (1/2, 1/2)$  and  $\beta = (3/2, 1)$  since we have  $3/2 \prec 1/2 \prec 1$  and  $\langle\alpha\rangle = (1/2, 1/2)$ ,  $\langle\beta\rangle = (1/2, 1)$ . But, if  $\langle\alpha\rangle$  and  $\langle\beta\rangle$  are disjoint, then, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ ,  $\langle a\alpha\rangle$  and  $\langle a\beta\rangle$  are disjoint. Hence, applying Theorem A, we obtain that  $(F_{\langle\alpha\rangle, \langle\beta\rangle} \text{ is } N\text{-integral}) \Rightarrow (F_{\alpha,\beta} \text{ is } N\text{-integral})$ .

$N$ -integral). More precisely, we shall prove the following proposition that we use several times in this article.

**Proposition 2.** *Let  $\alpha$  and  $\beta$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  and  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$ . Then we have  $d_{\langle a\alpha \rangle, \langle a\beta \rangle} = d_{\alpha, \beta}$ . Let  $c \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and  $x \in \mathbb{R}$  be fixed and let  $b \in \{1, \dots, d_{\alpha, \beta}\}$  be such that  $b \equiv ca \pmod{d_{\alpha, \beta}}$ . Then we have*

$$\xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) = \begin{cases} \xi_{\alpha, \beta}(b, \langle x \rangle -) & \text{if } x > c; \\ r - s & \text{if } x \leq c \text{ and } x \in \mathbb{Z}; \\ \xi_{\alpha, \beta}(b, \langle x \rangle -) \text{ or } \xi_{\alpha, \beta}(b, \langle x \rangle +) & \text{otherwise,} \end{cases} .$$

where  $r$ , respectively  $s$ , is the number of elements of  $\alpha$ , respectively of  $\beta$ .

*Remark.* For all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$ ,  $r - s$  is the limit of  $\xi_{\alpha, \beta}(a, n)$  when  $n \in \mathbb{Z}$  tends to  $-\infty$ .

In Proposition 2 and throughout this article, if  $f$  is a function defined on  $\mathcal{D} \subset \mathbb{R}$  and  $x \in \mathcal{D}$ , then we adopt the notations

$$f(x+) := \lim_{\substack{y \rightarrow x \\ y \in \mathcal{D}, y > x}} f(y) \quad \text{and} \quad f(x-) := \lim_{\substack{y \rightarrow x \\ y \in \mathcal{D}, y < x}} f(y).$$

*Proof.* For all elements  $\alpha$  and  $\beta$  of  $\alpha$  or  $\beta$ , we have  $\langle c\langle a\alpha \rangle \rangle = \langle ca\alpha \rangle = \langle b\alpha \rangle$  and  $\langle b\alpha \rangle = \langle b\beta \rangle$  if and only if  $\langle \alpha \rangle = \langle \beta \rangle$ . If  $\langle b\alpha \rangle = \langle x \rangle$ , then we have  $c\langle a\alpha \rangle \preceq x \Leftrightarrow c\langle a\alpha \rangle \geq x$ . It follows that if  $x > c$ , then we have

$$\begin{aligned} \xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) &= \#\{1 \leq i \leq r : \langle b\alpha_i \rangle < \langle x \rangle\} - \#\{1 \leq j \leq s : \langle b\beta_j \rangle < \langle x \rangle\} \\ &= \xi_{\alpha, \beta}(b, \langle x \rangle -). \end{aligned}$$

If  $x \in \mathbb{Z}$  and  $x \leq c$ , then we have  $\langle x \rangle = 1$  and  $\xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) = r - s$ . Now we assume that  $x \leq c$  and  $x \notin \mathbb{Z}$ . If  $\alpha$  and  $\beta$  are elements of  $\alpha$  or  $\beta$  satisfying  $\langle x \rangle = \langle b\alpha \rangle = \langle b\beta \rangle$ , then  $\langle \alpha \rangle = \langle \beta \rangle$  so  $\langle a\alpha \rangle = \langle a\beta \rangle$  and we obtain that  $c\langle a\alpha \rangle \preceq x \Leftrightarrow c\langle a\beta \rangle \preceq x$ . Thus we have

$$\begin{aligned} \xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) &= \begin{cases} \#\{1 \leq i \leq r : \langle b\alpha_i \rangle < \langle x \rangle\} - \#\{1 \leq j \leq s : \langle b\beta_j \rangle < \langle x \rangle\} \\ \text{or} \\ \#\{1 \leq i \leq r : \langle b\alpha_i \rangle \leq \langle x \rangle\} - \#\{1 \leq j \leq s : \langle b\beta_j \rangle \leq \langle x \rangle\} \end{cases} \\ &= \begin{cases} \xi_{\alpha, \beta}(b, \langle x \rangle -) \\ \text{or} \\ \xi_{\alpha, \beta}(b, \langle x \rangle +) \end{cases} \end{aligned}$$

because  $\langle x \rangle < 1$ . □

By Proposition 2 with  $a = 1$  together with Theorem A, we obtain that, if  $F_{\alpha, \beta}$  is  $N$ -integral, then  $F_{\langle \alpha \rangle, \langle \beta \rangle}$  is also  $N$ -integral. Similarly, if  $H_{\alpha, \beta}$  holds then  $H_{\langle \alpha \rangle, \langle \beta \rangle}$  also holds. More precisely, we have the following result, used several times in the proof of Theorem 3.

**Lemma 2.** *Let  $\alpha$  and  $\beta$  be two disjoint tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  with the same number of elements and such that  $H_{\alpha,\beta}$  holds. Then, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , Assertion  $H_{\langle a\alpha \rangle, \langle a\beta \rangle}$  holds.*

*Proof.* Let  $c \in \{1, \dots, d_{\alpha,\beta}\}$  be coprime to  $d_{\alpha,\beta}$  and  $x \in \mathbb{R}$  be such that  $m_{\langle a\alpha \rangle, \langle a\beta \rangle}(c) \preceq x \prec c$ . We shall prove that  $\xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) \geq 1$  by applying Proposition 2.

Let  $b \in \{1, \dots, d_{\alpha,\beta}\}$  be such that  $b \equiv ac \pmod{d_{\alpha,\beta}}$ . First, note that there exists an element  $\alpha$  of  $\alpha$  or  $\beta$  such that  $c\langle a\alpha \rangle \preceq x$ , that is  $\langle x \rangle > \langle b\alpha \rangle$  or  $(\langle x \rangle = \langle b\alpha \rangle$  and  $c\langle a\alpha \rangle \geq x)$ . We distinguish three cases.

- If  $x > c$  then we have  $\langle x \rangle > \langle b\alpha \rangle$  and  $\xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) = \xi_{\alpha,\beta}(b, \langle x \rangle -)$ . Thus there exists  $y \in \mathbb{R}$ ,  $m_{\alpha,\beta}(b) \preceq y \prec b$  such that  $\xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) = \xi_{\alpha,\beta}(b, y) \geq 1$ .

- If  $x \leq c$  and  $x \notin \mathbb{Z}$ , then we have  $\langle x \rangle < 1$  and  $\xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) = \xi_{\alpha,\beta}(b, \langle x \rangle -)$  or  $\xi_{\alpha,\beta}(b, \langle x \rangle +)$ . Since  $\langle x \rangle \geq \langle b\alpha \rangle$ , there exists  $y \in \mathbb{R}$ ,  $m_{\alpha,\beta}(b) \preceq y \prec b$  such that  $\xi_{\alpha,\beta}(b, \langle x \rangle +) = \xi_{\alpha,\beta}(b, y) \geq 1$ . Furthermore, if  $\langle x \rangle > \langle b\alpha \rangle$  then we have  $\xi_{\alpha,\beta}(b, \langle x \rangle -) \geq 1$  as in the case  $x > c$ . Now we assume that, for all elements  $\beta$  of  $\alpha$  or  $\beta$ , we have  $\langle x \rangle \leq \langle b\beta \rangle$ . Hence we have  $\langle x \rangle = \langle b\alpha \rangle$  and, as explained in the proof of Proposition 2, we have

$$\begin{aligned} \xi_{\langle a\alpha \rangle, \langle a\beta \rangle}(c, x) &= \#\{1 \leq i \leq r : \langle b\alpha_i \rangle \leq \langle x \rangle\} - \#\{1 \leq j \leq s : \langle b\beta_j \rangle \leq \langle x \rangle\} \\ &= \xi_{\alpha,\beta}(b, \langle x \rangle +) \geq 1. \end{aligned}$$

- It remains to consider the case  $x \leq c$  and  $x \in \mathbb{Z}$ . But in this case we do not have  $x \prec c$  thus  $H_{\langle a\alpha \rangle, \langle a\beta \rangle}$  is proved.  $\square$

**2.1.3. Numerators of the elements of  $\alpha$  and  $\beta$ .** Let  $\alpha = (\alpha_1, \dots, \alpha_r)$  and  $\beta = (\beta_1, \dots, \beta_r)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . Then Theorem 1 gives a necessary condition on the numerators of elements of  $\alpha$  and  $\beta$  for  $F_{\alpha,\beta}$  to be  $N$ -integral. Indeed, let us assume that  $F_{\alpha,\beta}$  is  $N$ -integral. Then, according to Section 2.1.2,  $F_{\langle \alpha \rangle, \langle \beta \rangle}$  is also  $N$ -integral. We write  $n_i$ , respectively  $n'_j$ , for the exact numerator of  $\langle \alpha_i \rangle$ , respectively of  $\langle \beta_j \rangle$ . Then by Theorem 1, the first-order Taylor coefficient at the origin of  $F_{\langle \alpha \rangle, \langle \beta \rangle}(C_{\langle \alpha \rangle, \langle \beta \rangle} z)$  is <sup>(3)</sup>

$$\frac{\prod_{i=1}^r n_i}{\prod_{j=1}^r n'_j} \prod_{p|d_{\alpha,\beta}} p^{-\left\lfloor \frac{\lambda_p(\alpha,\beta)}{p-1} \right\rfloor} \in \mathbb{Z},$$

so that, for all primes  $p$ , we have

$$v_p \left( \frac{\prod_{i=1}^r n_i}{\prod_{j=1}^r n'_j} \right) \geq \left\lfloor \frac{\lambda_p(\alpha, \beta)}{p-1} \right\rfloor.$$

For instance, the last inequality is not satisfied with  $p = 2$ ,  $\alpha = (1/5, 1/3, 3/5)$  and  $\beta = (1/2, 1, 1)$ , or with  $p = 3$ ,  $\alpha = (1/7, 2/7, 4/7, 5/7)$  and  $\beta = (3/4, 1, 1, 1)$ . Thus in both cases the associated generalized hypergeometric series  $F_{\alpha,\beta}$  is not  $N$ -integral.

<sup>3</sup>Note that, for all primes  $p$ , we have  $\lambda_p(\alpha, \beta) = \lambda_p(\langle \alpha \rangle, \langle \beta \rangle)$ .

2.1.4. *The Eisenstein constant of algebraic generalized hypergeometric series.* Let  $\alpha = (\alpha_1, \dots, \alpha_r)$  and  $\beta = (\beta_1, \dots, \beta_r)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . If  $F_{\alpha, \beta}(z)$  is algebraic over  $\mathbb{Q}(z)$  then  $F_{\alpha, \beta}$  is  $N$ -integral (Eisenstein's theorem) and one can apply Theorem 1 to get arithmetical properties of the Eisenstein constant of  $F_{\alpha, \beta}$ . For the sake of completeness, let us remind the reader of a result of Beukers and Heckman [3, Theorem 1.5] proved in [4] on algebraic hypergeometric functions:

“Assume that  $\beta_r = 1$  and that  $\mathcal{L}_{\alpha, \beta}$  is irreducible. Then the set of solutions of the hypergeometric equation associated with  $\mathcal{L}_{\alpha, \beta}$  consists of algebraic functions (over  $\mathbb{C}(z)$ ) if and only if the sets  $\{a\alpha_i : 1 \leq i \leq r\}$  and  $\{a\beta_i : 1 \leq i \leq r\}$  interlace modulo 1 for every integer  $a$  with  $1 \leq a \leq d_{\alpha, \beta}$  and  $\gcd(a, d_{\alpha, \beta}) = 1$ .”

The sets  $\{\alpha_i : 1 \leq i \leq r\}$  and  $\{\beta_i : 1 \leq i \leq r\}$  interlace modulo 1 if the points of the sets  $\{e^{2\pi i \alpha_j} : 1 \leq j \leq r\}$  and  $\{e^{2\pi i \beta_j} : 1 \leq j \leq r\}$  occur alternatively when running along the unit circle.

Beukers-Heckman criterion can be reformulated in terms of Christol's functions as follows.

“Assume that  $\beta_r = 1$  and that  $\mathcal{L}_{\alpha, \beta}$  is irreducible. Then the solution set of the hypergeometric equation associated with  $\mathcal{L}_{\alpha, \beta}$  consists of algebraic functions (over  $\mathbb{C}(z)$ ) if and only if, for every integer  $a$  with  $1 \leq a \leq d_{\alpha, \beta}$  and  $\gcd(a, d_{\alpha, \beta}) = 1$ , we have  $\xi_{\alpha, \beta}(a, \mathbb{R}) = \{0, 1\}$ .”

## 2.2. Comparison with previous results.

2.2.1. *Theorem 2 and previous results.* The first result on  $p$ -adic integrality of  $q_{\alpha, \beta}$  is due to Dwork [12, Theorem 4.1]. This result enables us to prove that, for particular tuples  $\alpha$  and  $\beta$ , we have  $q_{\alpha, \beta}(z) \in \mathbb{Z}_p[[z]]$  for almost all primes  $p$ . It follows without much trouble that  $q_{\alpha, \beta}$  is  $N$ -integral. Thus we know that there exists  $C \in \mathbb{N}$ ,  $C \geq 1$ , such that  $q_{\alpha, \beta}(Cz) \in \mathbb{Z}[[z]]$  but the only information on  $C$  given by Dwork's result is that we can choose  $C$  with prime divisors in an explicit finite set associated with  $(\alpha, \beta)$ . Hence, improvements of Dwork's method consist in finding explicit formulas for  $C$  and we discuss such previous improvements in the next section. But Theorem 2 is more general and, in order to compare this theorem with Dwork's result [12, Theorem 4.1], we introduce some notations that we use throughout this article. Until the end of this section, we restrict ourself to the case where  $\alpha$  and  $\beta$  have the same numbers of elements.

- For all primes  $p$  and all  $p$ -adic integers  $\alpha$  in  $\mathbb{Q}$ , we write  $\mathfrak{D}_p(\alpha)$  for the unique  $p$ -adic integer in  $\mathbb{Q}$  satisfying  $p\mathfrak{D}_p(\alpha) - \alpha \in \{0, \dots, p-1\}$ . The operator  $\alpha \mapsto \mathfrak{D}_p(\alpha)$  has been used by Dwork in [12] and denoted by  $\alpha \mapsto \alpha'$  <sup>(4)</sup>.

- For all primes  $p$ , all  $x \in \mathbb{Q} \cap \mathbb{Z}_p$  and all  $a \in [0, p)$  we define

$$\rho_p(a, x) := \begin{cases} 0 & \text{if } a \leq p\mathfrak{D}_p(x) - x; \\ 1 & \text{if } a > p\mathfrak{D}_p(x) - x. \end{cases}$$

---

<sup>4</sup>See Section 3 for a detailed study of Dwork's map  $\mathfrak{D}_p$ .

• We write  $\alpha = (\alpha_1, \dots, \alpha_r)$  and  $\beta = (\beta_1, \dots, \beta_r)$ . Let  $r'$  be the number of elements  $\beta_i$  of  $\beta$  such that  $\beta_i \neq 1$ . We rearrange the subscripts so that  $\beta_i \neq 1$  for  $i \leq r'$ . For all  $a \in [0, p)$  and all  $k \in \mathbb{N}$ , we set

$$N_{p,\alpha}^k(a) = \sum_{i=1}^r \rho_p(a, \mathfrak{D}_p^k(\alpha_i)) \quad \text{and} \quad N_{p,\beta}^k(a) = \sum_{i=1}^{r'} \rho_p(a, \mathfrak{D}_p^k(\beta_i)).$$

• For a given prime  $p$  not dividing  $d_{\alpha,\beta}$ , we define two assertions:

- (v)<sub>p</sub> for all  $i \in \{1, \dots, r'\}$  and all  $k \in \mathbb{N}$ , we have  $\mathfrak{D}_p^k(\beta_i) \in \mathbb{Z}_p^\times$ ;
- (vi)<sub>p</sub> for all  $a \in [0, p)$  and all  $k \in \mathbb{N}$ , we have either  $N_{p,\alpha}^k(a) = N_{p,\beta}^k(a+) = 0$  or  $N_{p,\alpha}^k(a) - N_{p,\beta}^k(a+) \geq 1$ .

Dwork's result [12, Theorem 4.1] restricted to the case where  $\alpha$  and  $\beta$  have the same number of elements is the following.

**Theorem B** (Dwork). *Let  $\alpha$  and  $\beta$  be two tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  with the same number of elements. Let  $p$  be a prime not dividing  $d_{\alpha,\beta}$  such that  $\alpha$  and  $\beta$  satisfy (v)<sub>p</sub> and (vi)<sub>p</sub>. Then we have*

$$\frac{G_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}}{F_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}}(z^p) - p \frac{G_{\alpha, \beta}(z)}{F_{\alpha, \beta}}(z) \in pz\mathbb{Z}_p[[z]].$$

Now let us assume that  $\alpha$  and  $\beta$  are disjoint with elements in  $(0, 1]$  and that  $H_{\alpha,\beta}$  holds. For all primes  $p$  not dividing  $d_{\alpha,\beta}$ , we have  $\mathfrak{D}_p(\alpha) = \langle \omega\alpha \rangle$  and  $\mathfrak{D}_p(\beta) = \langle \omega\beta \rangle$  where  $\omega \in \mathbb{Z}$  satisfies  $\omega p \equiv 1 \pmod{d_{\alpha,\beta}}$  (see Section 3.2 below). Then, by Theorem 2 for a fixed prime  $p$  and  $b = t = 1$ , we obtain that

$$\frac{G_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}}{F_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}}(C'_{\alpha,\beta} z^p) - p \frac{G_{\alpha,\beta}}{F_{\alpha,\beta}}(C'_{\alpha,\beta} z) \in pz\mathbb{Z}_p[[z]]. \quad (2.1)$$

Thus, contrary to Theorem B, there is no restriction on the primes  $p$  because of the constant  $C'_{\alpha,\beta}$ . Furthermore, in the proof of Lemma 21 in Section 9.1.3, we show that if  $H_{\alpha,\beta}$  holds then  $\alpha$  and  $\beta$  satisfy Assertions (v)<sub>p</sub> and (vi)<sub>p</sub> for almost all primes  $p$ . By Theorems 3 and B, the converse holds when, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have

$$\frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}}(z) = \frac{G_{\alpha,\beta}}{F_{\alpha,\beta}}(z).$$

Indeed, in this case, Theorem B in combination with Corollary 1 implies that, for almost all primes  $p$ , we have  $q_{\alpha,\beta}(z) \in \mathbb{Z}_p[[z]]$ . Then it is a simple exercise to show that  $q_{\alpha,\beta}$  is  $N$ -integral and, by Theorem 3, we obtain that  $H_{\alpha,\beta}$  holds.

The main improvement in Theorem 2 is the use of algebras of  $\mathbb{Z}_p$ -valued functions instead of  $\mathbb{Z}_p$ . This is precisely this generalization which enables us to prove the integrality of the Taylor coefficients of certain roots of  $S_{\alpha,\beta}(C'_{\alpha,\beta} z)$ .

2.2.2. *Theorem 3 and previous results.* The constants  $C \in \mathbb{Q}$  such that an  $N$ -integral canonical coordinate  $q_{\alpha,\beta}$  satisfies  $q_{\alpha,\beta}(Cz) \in \mathbb{Z}[[z]]$  was first studied when there exist some disjoint tuples of positive integers  $\mathbf{e} = (e_1, \dots, e_u)$ ,  $\mathbf{f} = (f_1, \dots, f_v)$  and a constant  $C_0 \in \mathbb{Q}$  such that

$$F_{\alpha,\beta}(C_0z) = \sum_{n=0}^{\infty} \frac{(e_1n)! \cdots (e_un)!}{(f_1n)! \cdots (f_vn)!} z^n \in \mathbb{Z}[[z]]. \quad (2.2)$$

The results obtained by Lian and Yau [26], Zudilin [34], Krattenthaler-Rivoal [18] and Delaygue [8] led to an effective criterion [8, Theorem 1] based on simple analytical properties of Landau's function

$$\Delta_{\mathbf{e},\mathbf{f}}(x) := \sum_{i=1}^u \lfloor e_i x \rfloor - \sum_{j=1}^v \lfloor f_j x \rfloor.$$

By combining and reformulating this criterion and [8, Theorem 3], we obtain the following result.

**Theorem C** (Delaygue). *If (2.2) holds, then the following assertions are equivalent:*

- (1)  $q_{\alpha,\beta}(z)$  is  $N$ -integral;
- (2)  $q_{\alpha,\beta}(C_0z) \in \mathbb{Z}[[z]]$ ;
- (3) we have  $\sum_{i=1}^u e_i = \sum_{j=1}^v f_j$  and, for all  $x \in [1/M_{\mathbf{e},\mathbf{f}}, 1[$ , we have  $\Delta_{\mathbf{e},\mathbf{f}}(x) \geq 1$ , where  $M_{\mathbf{e},\mathbf{f}}$  is the largest element of  $\mathbf{e}$  and  $\mathbf{f}$ .

According to [8, Proposition 2], one can write (a rescaling of)  $F_{\alpha,\beta}$  as the generating function of a sequence of factorial ratios if and only if  $\alpha$  and  $\beta$  are  $R$ -partitioned. In this case, Landau's criterion [22] asserts that the additional condition of integrality in (2.2) is equivalent to the nonnegativity of  $\Delta_{\mathbf{e},\mathbf{f}}$  on  $[0, 1]$ , which can be checked easily because, by [8, Proposition 3], for all  $x \in [0, 1]$ , we have

$$\Delta_{\mathbf{e},\mathbf{f}}(x) = \#\{i : x \geq \alpha_i\} - \#\{j : x \geq \beta_j\}. \quad (2.3)$$

Furthermore, by [8, Proposition 2], if  $\alpha = (\alpha_1, \dots, \alpha_r)$ , respectively  $\beta = (\beta_1, \dots, \beta_s)$ , is the concatenation of tuples  $(b/N_i)_{b \in \{1, \dots, N_i\}, \gcd(b, N_i)=1}$ ,  $1 \leq i \leq r'$ , respectively of tuples  $(b/N'_j)_{b \in \{1, \dots, N'_j\}, \gcd(b, N'_j)=1}$ ,  $1 \leq j \leq s'$ , then we have

$$C_0 = \frac{\prod_{i=1}^{r'} N_i^{\varphi(N_i)} \prod_{p|N_i} p^{\frac{\varphi(N_i)}{p-1}}}{\prod_{j=1}^{s'} N'_j{}^{\varphi(N'_j)} \prod_{p|N'_j} p^{\frac{\varphi(N'_j)}{p-1}}} \quad \text{and} \quad \sum_{i=1}^u e_i - \sum_{j=1}^v f_j = r - s. \quad (2.4)$$

Let us show that Theorem 3 implies Theorem C. Let  $\alpha$  and  $\beta$  be disjoint tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that (2.2) holds. Then  $\alpha$  and  $\beta$  are  $R$ -partitioned and their elements lie in  $(0, 1]$  so that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint and  $F_{\alpha,\beta}$  is  $N$ -integral. First we prove that if  $r = s$ , then we have  $C'_{\alpha,\beta} = C_{\alpha,\beta} = C_0$ . We write  $\lambda_p$  for  $\lambda_p(\alpha, \beta)$ . Since  $\alpha$  and  $\beta$  are  $R$ -partitioned, the number of elements of  $\alpha$  and  $\beta$  with exact denominator divisible by 4 is a sum of multiple of integers of the form  $\varphi(2^k)$  with  $k \in \mathbb{N}$ ,  $k \geq 2$ , so this number is



even. Thus, we have  $C'_{\alpha,\beta} = C_{\alpha,\beta}$ . Furthermore, for all primes  $p$ , we have

$$\lambda_p = r - \sum_{\substack{i=1 \\ p|N_i}}^{r'} \varphi(N_i) - s + \sum_{\substack{j=1 \\ p|N'_j}}^{s'} \varphi(N'_j) = - \sum_{\substack{i=1 \\ p|N_i}}^{r'} \varphi(N_i) + \sum_{\substack{j=1 \\ p|N'_j}}^{s'} \varphi(N'_j).$$

If  $p$  divides  $N_i$  then  $p-1$  divides  $\varphi(N_i)$  so that  $-\lfloor \lambda_p/(p-1) \rfloor = -\lambda_p/(p-1)$  and  $C_{\alpha,\beta} = C_0$  as expected. Now we assume that (2.2) and Theorem 3 hold and we prove that Assertions (1), (2) and (3) of Theorem C are equivalent.

• (1)  $\Rightarrow$  (2): If  $q_{\alpha,\beta}(z)$  is  $N$ -integral, then we obtain that  $q_{\alpha,\beta}(C'_{\alpha,\beta}z) \in \mathbb{Z}[[z]]$  and  $r = s$  so that  $C'_{\alpha,\beta} = C_0$  and Assertion (2) of Theorem C holds.

• (2)  $\Rightarrow$  (3): If  $q_{\alpha,\beta}(C_0z) \in \mathbb{Z}[[z]]$  then  $q_{\alpha,\beta}(z)$  is  $N$ -integral and, according to Theorem 3, we have  $r = s$  and  $H_{\alpha,\beta}$  is true. We deduce that we have  $\sum_{i=1}^u e_i = \sum_{j=1}^v f_j$ . Now, since  $\alpha$  and  $\beta$  are disjoint tuples with elements in  $(0, 1]$ , equation (2.3) ensures that the assertions “for all  $x \in [1/M_{\mathbf{e},\mathbf{f}}, 1[$ , we have  $\Delta_{\mathbf{e},\mathbf{f}}(x) \geq 1$ ” and “for all  $x \in \mathbb{R}$ ,  $m_{\alpha,\beta}(1) \preceq x \prec 1$ , we have  $\xi_{\alpha,\beta}(1, x) \geq 1$ ” are equivalent. Thus Assertion (3) of Theorem C holds.

• (3)  $\Rightarrow$  (1): We assume that  $\sum_{i=1}^u e_i = \sum_{j=1}^v f_j$ , that is  $r = s$ , and that, for all  $x \in [1/M_{\mathbf{e},\mathbf{f}}, 1[$ , we have  $\Delta_{\mathbf{e},\mathbf{f}}(x) \geq 1$ . Since  $\alpha$  and  $\beta$  are  $R$ -partitioned, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$  we have  $\langle a\alpha \rangle = \alpha$  and  $\langle a\beta \rangle = \beta$ , and these tuples are disjoint. We deduce that for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$  and all  $x \in \mathbb{R}$ ,  $m_{\alpha,\beta}(a) \preceq x \prec a$ , Equation (2.3) gives us that  $\xi_{\alpha,\beta}(a, x) \geq 1$ , so that  $H_{\alpha,\beta}$  holds. Thus Assertion (iii) of Theorem 3 holds and  $q_{\alpha,\beta}(z)$  is  $N$ -integral as expected. This finishes the proof that Theorem 3 implies Theorem C.

Furthermore, when (2.2) holds, Delaygue [11, Theorem 8] generalized some of the results of Krattenthaler-Rivoal [21] and proved that all Taylor coefficients at the origin of  $q_{\alpha,\beta}(C_0z)$  are positive, but its constant term which is 0. Assertion (1) of Theorem 3 generalizes this result since it does not use the assumption that  $\alpha$  and  $\beta$  are  $R$ -partitioned.

Later, Roques studied (see [31] and [32]) the integrality of the Taylor coefficients of canonical coordinates  $q_{\alpha,\beta}$  without assuming that (2.2) holds, in the case  $\alpha$  and  $\beta$  have the same number of elements  $r \geq 2$ , all the elements of  $\beta$  are equal to 1 and all the elements of  $\alpha$  lie in  $(0, 1] \cap \mathbb{Q}$ . In this case, we have  $r = s$  and it is easy to prove that  $H_{\alpha,\beta}$  holds but  $\alpha$  is not necessarily  $R$ -partitioned. Roques proved that  $q_{\alpha,\beta}(z)$  is  $N$ -integral if and only if, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have  $q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z) = q_{\alpha,\beta}(z)$  in accordance with Theorem 3. Furthermore, when  $r = 2$ , he found the exact finite set <sup>(5)</sup> of tuples  $\alpha$  such that  $q_{\alpha,\beta}(z)$  is  $N$ -integral (see [31, Theorem 3]) and, when  $r \geq 3$ , he proved (see [32]) that  $q_{\alpha,\beta}(z)$  is  $N$ -integral if and only if  $\alpha$  is  $R$ -partitioned (the “if part” is proved by Krattenthaler-Rivoal in [18]). Note that if  $\beta = (1, \dots, 1)$ , then it is easy to prove that  $F_{\alpha,\beta}(z)$  is  $N$ -integral.

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<sup>5</sup>This sets contains 28 elements amongst which 4 are  $R$ -partitioned.

In the case  $r = 2$ , Roques gave constants  $C_1$  such that  $q_{\alpha,\beta}(C_1 z) \in \mathbb{Z}[[z]]$  which equal  $C'_{\alpha,\beta}$  unless for  $\alpha = (1/2, 1/4)$  or  $(1/2, 3/4)$  where  $C_1 = 256$  and Theorem 3 improves this constant since  $C'_{\alpha,\beta} = 32$ .

The integrality of Taylor coefficients of roots of  $z^{-1}q_{\alpha,\beta}(z)$  has been studied in case (2.2) holds by Lian-Yau [24], Krattenthaler-Rivoal [19], and by Delaygue [9]. For a detailed survey of these results, we refer the reader to [9, Section 1.2].

- In [24], Lian-Yau studied the case  $\mathbf{e} = (p)$  and  $\mathbf{f} = (1, \dots, 1)$  with  $p$  1's in  $\mathbf{f}$  and where  $p$  is a prime. In this case, we have  $\beta = (1, \dots, 1)$  and  $\mathbf{n}'_{\alpha,\beta} = 1$ , thus we do not obtain a root with Corollary 3.

- In [19], Krattenthaler-Rivoal studied the case  $\mathbf{e} = (N, \dots, N)$  with  $k$   $N$ 's in  $\mathbf{e}$  and  $\mathbf{f} = (1, \dots, 1)$  with  $kN$  1's in  $\mathbf{f}$ . In this case, we also have  $\beta = (1, \dots, 1)$ . For all prime divisors  $p$  of  $N$ , we write  $N = p^{\eta_p} N_p$  with  $\eta_p, N_p \in \mathbb{N}$  and  $N_p$  not divisible by  $p$ . A simple computation of the associated tuples  $\alpha$  and  $\beta$  shows that  $d_{\alpha,\beta} = N$  and  $\lambda_p = k(N_p - N)$ . Thus, for all prime divisors  $p$  of  $N$ ,  $p - 1$  divides  $\lambda_p$  and we have

$$\mathbf{n}'_{\alpha,\beta} = \prod_{p|N} p^{-1+k\frac{N-N_p}{p-1}},$$

but it seems that the roots found by Krattenthaler-Rivoal are always better than  $\mathbf{n}'_{\alpha,\beta}$  in these cases.

- However, in a lot of cases, our root  $\mathbf{n}'_{\alpha,\beta}$  improves the one found by Delaygue in [9]. For example, if  $\mathbf{e} = (4, 2)$  and  $\mathbf{f} = (1, 1, 1, 1, 1, 1)$ , then [9, Corollary 1.1] gives us the root 4 while  $\beta = (1, \dots, 1)$  and  $\mathbf{n}'_{\alpha,\beta} = 32$ .

**2.3. Open questions.** We formulate some open questions about  $N$ -integrality of mirror maps and  $q$ -coordinates.

- Does the equivalence in Assertion (2) of Theorem 3 still hold if we do not assume that  $F_{\alpha,\beta}(z)$  is  $N$ -integral?

- One of the conditions for  $q_{\alpha,\beta}(z)$  to be  $N$ -integral is that, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have  $q_{\alpha,\beta}(z) = q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$ . According to [31] and [32], we know that, when  $\beta = (1, \dots, 1)$  and all elements of  $\alpha$  belong to  $(0, 1]$ , this condition implies a stronger characterization related to the exact forms of  $\alpha$  and  $\beta$ . Is it possible to deduce a similar characterization in the general case?

**2.4. A corrected version of a lemma of Lang.** As mentioned in the Introduction, while working on this article, we noticed an error in a lemma stated by Lang [23, Lemma 1.1, Section 1, Chapter 14] about arithmetic properties of Mojita's  $p$ -adic Gamma function. This lemma has been used in several articles on the integrality of the Taylor coefficients of mirror maps including papers of the authors. First we give a corrected version of Lang's lemma, then we explain why this error does not change the validity of our previous results.

Let  $p$  be a fixed prime. For all  $n \in \mathbb{N}$ , we define the  $p$ -adic Gamma function  $\Gamma_p$  by

$$\Gamma_p(n) := (-1)^n \prod_{\substack{k=1 \\ \gcd(k,p)=1}}^{n-1} k.$$

In particular,  $\Gamma_p(0) = 1$ ,  $\Gamma_p(1) = -1$  and  $\Gamma_p$  can be extended to  $\mathbb{Z}_p$ .

**Proposition 3.** *For all  $k, m, s \in \mathbb{N}$ , we have*

$$\Gamma_p(k + mp^s) \equiv \begin{cases} \Gamma_p(k) \pmod{p^s} & \text{if } p^s \neq 4; \\ (-1)^m \Gamma_p(k) \pmod{p^s} & \text{if } p^s = 4. \end{cases}.$$

The case  $p^s \neq 4$  in Proposition 3 is proved by Morita in [28]. We provide a complete proof of the proposition.

*Proof.* If  $s = 0$  or if  $m = 0$  this is trivial. We assume in the sequel that  $s \geq 1$  and  $m \geq 1$ . Then

$$\begin{aligned} \frac{\Gamma_p(k + mp^s)}{\Gamma_p(k)} &= (-1)^{mp^s} \prod_{\substack{i=k \\ \gcd(i,p)=1}}^{k+mp^s-1} i = (-1)^{mp^s} \prod_{\substack{i=0 \\ \gcd(k+i,p)=1}}^{p^s-1} \prod_{j=0}^{m-1} (k + i + jp^s) \\ &\equiv (-1)^{mp^s} \prod_{\substack{i=0 \\ \gcd(k+i,p)=1}}^{p^s-1} (k + i)^m \pmod{p^s} \\ &\equiv (-1)^{mp^s} \prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j^m \pmod{p^s}, \end{aligned} \tag{2.5}$$

because, for all  $j \in \{0, \dots, p^s - 1\}$ , there exists a unique  $i \in \{0, \dots, p^s - 1\}$  such that  $k + i \equiv j \pmod{p^s}$ .

We first assume that  $p \geq 3$ . In this case, the group  $(\mathbb{Z}/p^s\mathbb{Z})^\times$  is cyclic and contains just one element of order 2. Collecting each element of  $(\mathbb{Z}/p^s\mathbb{Z})^\times$  of order  $\geq 3$  with its inverse, we obtain

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j \equiv -1 \pmod{p^s}.$$

Together, with (2.5), we get

$$\frac{\Gamma_p(k + mp^s)}{\Gamma_p(k)} \equiv 1 \pmod{p^s},$$

because  $p$  is odd.

Let us now assume that  $p = 2$ . If  $s = 1$ , then

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j = 1$$

and by (2.5) this yields  $\Gamma_p(k + mp^s) \equiv \Gamma_p(k) \pmod{p^s}$ . If  $s = 2$ , then

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j = 3 \equiv -1 \pmod{p^s},$$

and by (2.5), this yields  $\Gamma_p(k + mp^s) \equiv (-1)^m \Gamma_p(k) \pmod{p^s}$ . It remains to deal with the case  $s \geq 3$ . The group  $(\mathbb{Z}/2^s\mathbb{Z})^\times$  is isomorphic to  $\mathbb{Z}/2^{s-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Moreover,

$$\sum_{k=0}^{2^{s-2}-1} \sum_{j=0}^1 (k, j) = \left( 2 \sum_{k=0}^{2^{s-2}-1} k, 2^{s-2} \right) = (2^{s-2}(2^{s-2} - 1), 2^{s-2}) \in 2^{s-2}\mathbb{Z} \times 2\mathbb{Z},$$

because  $s \geq 3$ . Hence,

$$\prod_{\substack{j=0 \\ \gcd(j,p)=1}}^{p^s-1} j \equiv 1 \pmod{p^s}$$

and by (2.5), this yields

$$\frac{\Gamma_p(k + mp^s)}{\Gamma_p(k)} \equiv 1 \pmod{p^s},$$

which completes the proof of the proposition.  $\square$

The error in Lang's version is that he wrote  $\Gamma_p(k + mp^s) \equiv \Gamma_p(k) \pmod{p^s}$  if  $p^s = 4$ , forgetting the factor  $(-1)^m$ . He gives the proof only for  $p \geq 3$  and he claims that the proof goes through similarly when  $p = 2$ , overlooking the subtlety. Delaygue and Krattenthaler-Rivoal used Lang's version in [8, Lemma 11], [10, Lemma 8] and [18]. Fortunately, the resulting mistakes in these papers are purely local and can be fixed. Indeed, the factor  $(-1)^\ell$  (that should have been added when  $p^s = 4$ ) would have occurred for an *even* value of  $\ell$  and thus would have immediately disappeared without changing the rest of the proof.

### 3. THE $p$ -ADIC VALUATION OF POCHHAMMER SYMBOLS

We introduce certain step functions, defined over  $\mathbb{R}$ , that enable us to compute the  $p$ -adic valuation of Pochhammer symbols. We will then provide a connection between the values of these functions and the functions  $\xi_{\alpha,\beta}(a, \cdot)$ . This construction is inspired by various works of Christol [7], Dwork [12] and Katz [16]. We first prove Proposition 1.

**3.1. Proof of Proposition 1.** Let  $\alpha$  and  $\beta$  be two sequences taking their values in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . If there exists  $C \in \mathbb{Q}^*$  such that  $F_{\alpha, \beta}(Cz) \in \mathbb{Z}[[z]]$ , then for all primes  $p$  such that  $v_p(C) \leq 0$ , we have  $F_{\alpha, \beta}(z) \in \mathbb{Z}_p[[z]]$ . Hence, there exists only a finite number of primes  $p$  such that  $F_{\alpha, \beta}(z) \notin \mathbb{Z}_p[[z]]$ .

Conversely, let us assume there exists only a finite number of primes  $p$  such that  $F_{\alpha, \beta}(z) \notin \mathbb{Z}_p[[z]]$ . To prove Proposition 1, it is enough to prove that, for all prime  $p$ , there exists  $m \in \mathbb{Z}_{\leq 0}$  such that for all  $n \in \mathbb{N}$ , we have

$$v_p \left( \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) \geq mn. \quad (3.1)$$

Let  $x \in \mathbb{Q}$ ,  $x = a/b$  with  $a, b \in \mathbb{Z}$ ,  $b \geq 1$ , and  $a$  and  $b$  coprime. If  $b$  is not divisible by  $p$ , then for all  $n \in \mathbb{N}$ , we have  $v_p((x)_n) \geq 0$ . On the other hand, if  $p$  divides  $b$ , then  $v_p((x)_n) = v_p(x)n$ .

Let us now assume that  $x \notin \mathbb{Z}_{\leq 0}$ . Then, for all  $n \in \mathbb{N}$ ,  $n \geq 1$ ,

$$\begin{aligned} v_p \left( \frac{1}{(x)_n} \right) &= v_p \left( \frac{b^n}{a(a+b) \cdots (a+b(n-1))} \right) \geq v_p \left( \frac{b^n}{|a|!(|a|+bn)!} \right) \\ &\geq \left( v_p(b) - \frac{b}{p-1} \right) n - 2 \frac{|a|}{p-1}, \end{aligned}$$

because

$$v_p((|a|+bn)!) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{|a|+bn}{p^\ell} \right\rfloor < \sum_{\ell=1}^{\infty} \frac{|a|+bn}{p^\ell} = \frac{|a|}{p-1} + \frac{b}{p-1}n.$$

Hence, (3.1) holds and Proposition 1 is proved.  $\square$

**3.2. Dwork's map  $\mathfrak{D}_p$ .** Given a prime  $p$  and some  $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$ , we recall that  $\mathfrak{D}_p(\alpha)$  denotes the unique element in  $\mathbb{Z}_p \cap \mathbb{Q}$  such that

$$p\mathfrak{D}_p(\alpha) - \alpha \in \{0, \dots, p-1\}.$$

The map  $\alpha \mapsto \mathfrak{D}_p(\alpha)$  was used by Dwork in [12] (denoted there as  $\alpha \mapsto \alpha'$ ). We observe that the unique element  $k \in \{0, \dots, p-1\}$  such that  $k + \alpha \in p\mathbb{Z}_p$  is  $k = p\mathfrak{D}_p(\alpha) - \alpha$ . More precisely, the  $p$ -adic expansion of  $-\alpha$  in  $\mathbb{Z}_p$  is

$$-\alpha = \sum_{\ell=0}^{\infty} (p\mathfrak{D}_p^{\ell+1}(\alpha) - \mathfrak{D}_p^\ell(\alpha))p^\ell,$$

where  $\mathfrak{D}_p^\ell$  is the  $\ell$ -th iteration of  $\mathfrak{D}_p$ . In particular, for all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ ,  $\mathfrak{D}_p^\ell(\alpha)$  is the unique element in  $\mathbb{Z}_p \cap \mathbb{Q}$  such that  $p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha \in \{0, \dots, p^\ell - 1\}$ .

For all primes  $p$ , we have  $\mathfrak{D}_p(1) = 1$ . Let us now assume that  $\alpha$  is in  $\mathbb{Z}_p \cap \mathbb{Q} \cap (0, 1)$ . Set  $N \in \mathbb{N}$ ,  $N \geq 2$  and  $r \in \{1, \dots, N-1\}$ ,  $\gcd(r, N) = 1$ , such that  $\alpha = r/N$ . Let  $s_N$

be the unique right inverse of the canonical morphism  $\pi_N : \mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z}$  with values in  $\{0, \dots, N-1\}$ . Then (see [31] for details)

$$\mathfrak{D}_p(\alpha) = \frac{s_N(\pi_N(p)^{-1}\pi_N(r))}{N}.$$

Hence, for all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we obtain

$$\mathfrak{D}_p^\ell(\alpha) = \frac{s_N(\pi_N(p)^{-\ell}\pi_N(r))}{N}. \quad (3.2)$$

In particular, if  $\alpha \in (0, 1)$ , then  $\mathfrak{D}_p(\alpha)$  depends only on the congruence class of  $p$  modulo  $N$ . If  $a \in \mathbb{Z}$  satisfies  $ap \equiv 1 \pmod{N}$ , then  $\mathfrak{D}_p^\ell(\alpha) = \{a^\ell \alpha\} = \langle a^\ell \alpha \rangle$  because  $a$  is coprime to  $N$ , hence  $a^\ell \alpha \notin \mathbb{Z}$ . This formula is still valid when  $\alpha = 1$  and  $a$  is any integer.

**Lemma 3.** *Let  $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$ . Then for any prime  $p$  such that  $\alpha \in \mathbb{Z}_p$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , such that  $p^\ell \geq d(\alpha)(\lfloor |1 - \alpha| \rfloor + \langle \alpha \rangle)$ , we have  $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle \alpha \rangle) = \langle \omega \alpha \rangle$ , where  $\omega \in \mathbb{Z}$  satisfies  $\omega p^\ell \equiv 1 \pmod{d(\alpha)}$ .*

*Proof.* Let  $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  and  $p$  be such that  $\alpha \in \mathbb{Z}_p$  and  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$  be such that  $p^\ell \geq d(\alpha)(\lfloor |1 - \alpha| \rfloor + \langle \alpha \rangle)$ . By definition,  $\mathfrak{D}_p^\ell(\alpha)$  is the unique rational number in  $\mathbb{Z}_p$  such that  $p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha \in \{0, \dots, p^\ell - 1\}$ . We set  $\alpha = \langle \alpha \rangle + k$ ,  $k \in \mathbb{Z}$  and  $r := \mathfrak{D}_p^\ell(\langle \alpha \rangle) + \lfloor k/p^\ell \rfloor + a$ , with  $a = 0$  if  $k - p^\ell \lfloor k/p^\ell \rfloor \leq p^\ell \mathfrak{D}_p^\ell(\langle \alpha \rangle) - \langle \alpha \rangle$  and  $a = 1$  otherwise. We obtain

$$p^\ell r - \alpha = p^\ell \mathfrak{D}_p^\ell(\langle \alpha \rangle) - \langle \alpha \rangle + p^\ell \left\lfloor \frac{k}{p^\ell} \right\rfloor - k + p^\ell a \in \{0, \dots, p^\ell - 1\},$$

because  $p^\ell \mathfrak{D}_p^\ell(\langle \alpha \rangle) - \langle \alpha \rangle$  and  $k - p^\ell \lfloor k/p^\ell \rfloor$  are in  $\{0, \dots, p^\ell - 1\}$ . Since  $r \in \mathbb{Z}_p$ , we get  $\mathfrak{D}_p^\ell(\alpha) = r$ . We have  $d(\alpha)(\lfloor |k| + \langle \alpha \rangle \rfloor) > |k|$  thus  $\lfloor k/p^\ell \rfloor \in \{-1, 0\}$ .

If  $\lfloor k/p^\ell \rfloor = 0$ , then since  $\mathfrak{D}_p^\ell(\langle \alpha \rangle) \geq 1/d(\alpha)$ , we get  $p^\ell \mathfrak{D}_p^\ell(\langle \alpha \rangle) - \langle \alpha \rangle \geq |k|$  and thus  $a = 0$ . In this case, we have  $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle \alpha \rangle)$ .

Let us now assume that  $\lfloor k/p^\ell \rfloor = -1$ , i.e.  $k \leq -1$ . We have  $\langle \alpha \rangle < 1$  because  $\alpha \notin \mathbb{Z}_{\leq 0}$ , hence  $d(\alpha) \geq 2$ . We have

$$\begin{aligned} p^\ell \mathfrak{D}_p^\ell(\langle \alpha \rangle) - \langle \alpha \rangle - (k + p^\ell) &\leq p^\ell \left( \frac{d(\alpha) - 1}{d(\alpha)} - 1 \right) - \langle \alpha \rangle - k \leq -\frac{p^\ell}{d(\alpha)} - \langle \alpha \rangle - k \\ &\leq -|k| - 2\langle \alpha \rangle - k \leq -2\langle \alpha \rangle < 0, \end{aligned}$$

thus  $a = 1$  and  $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle \alpha \rangle)$ . □

**3.3. Analogues of Landau functions.** We now define the step functions that will enable us to compute the  $p$ -adic valuation of the Taylor coefficients at  $z = 0$  of  $F_{\alpha, \beta}(z)$ . For all primes  $p$ , all  $\alpha \in \mathbb{Q} \cap \mathbb{Z}_p$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we denote by  $\delta_{p, \ell}(\alpha, \cdot)$  the step function defined, for all  $x \in \mathbb{R}$ , by

$$\left( \delta_{p, \ell}(\alpha, x) = k \iff x - \mathfrak{D}_p^\ell(\alpha) - \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} \in [k - 1, k) \right), \quad k \in \mathbb{Z}.$$

In particular, if  $\alpha \in (0, 1]$ , then for all  $k \in \mathbb{Z}$ , we have

$$\delta_{p,\ell}(\alpha, x) = k \iff x - \mathfrak{D}_p^\ell(\alpha) \in [k-1, k).$$

Let  $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$  and  $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$  be two sequences taking their values in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . For any  $p$  that does not divide  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ , and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we denote by  $\Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p,\ell}$  the step function defined, for all  $x \in \mathbb{R}$ , by

$$\Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p,\ell}(x) := \sum_{i=1}^r \delta_{p,\ell}(\alpha_i, x) - \sum_{j=1}^s \delta_{p,\ell}(\beta_j, x).$$

The motivation behind the functions  $\Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p,\ell}$  is given by the following result.

**Proposition 4.** *Let  $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_r)$  and  $\boldsymbol{\beta} := (\beta_1, \dots, \beta_s)$  be two sequences taking their values in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . Let  $p$  be such that  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are in  $\mathbb{Z}_p$ . Then, for all  $n \in \mathbb{N}$ , we have*

$$v_p \left( \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) = \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p,\ell} \left( \frac{n}{p^\ell} \right) = \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + (r-s)v_p(n!).$$

*Remark.* This proposition is a reformulation of results in Section III of [7], proved by Christol in order to compute the  $p$ -adic valuation of the Pochhammer symbol  $(x)_n$  for  $x \in \mathbb{Z}_p$ .

*Proof.* For any  $p$ , any  $n := \sum_{k=0}^{\infty} n_k p^k \in \mathbb{Z}_p$  with  $n_k \in \{0, \dots, p-1\}$ , and any  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we set  $T_p(n, \ell) := \sum_{k=0}^{\ell-1} n_k p^k$ . For all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have

$$T_p(-\alpha, \ell) = p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha.$$

We fix a  $p$ -adic integer  $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . For all  $k \in \mathbb{Z}$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have

$$\begin{aligned} \delta_{p,\ell} \left( \alpha, \frac{n}{p^\ell} \right) = k &\iff \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} + k - 1 \leq \frac{n}{p^\ell} < \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} + k \\ &\iff p^\ell \mathfrak{D}_p^\ell(\alpha) + \lfloor 1 - \alpha \rfloor + (k-1)p^\ell \leq n < p^\ell \mathfrak{D}_p^\ell(\alpha) + \lfloor 1 - \alpha \rfloor + kp^\ell \\ &\iff p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha + (k-1)p^\ell < n \leq p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha + kp^\ell \\ &\iff \left\lceil \frac{n - T_p(-\alpha, \ell)}{p^\ell} \right\rceil = k, \end{aligned} \tag{3.3}$$

where, for all  $x \in \mathbb{R}$ ,  $\lceil x \rceil$  is the smallest integer larger than  $x$ . We have used in (3.3) the fact that  $-\alpha = -\langle \alpha \rangle + \lfloor 1 - \alpha \rfloor$ ,  $-1 \leq -\langle \alpha \rangle < 0$  and  $p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha \in \mathbb{N}$ . We then obtain

$$\delta_{p,\ell} \left( \alpha, \frac{n}{p^\ell} \right) = \left\lceil \frac{n - T_p(-\alpha, \ell)}{p^\ell} \right\rceil. \tag{3.4}$$

Christol proved in [7] that for all  $\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$  and all  $n \in \mathbb{N}$ , we have

$$v_p((\alpha)_n) = \sum_{\ell=1}^{\infty} \left\lceil \frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} \right\rceil. \tag{3.5}$$

For all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have

$$\frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} \in \frac{1}{p^\ell} \mathbb{Z},$$

so that if  $k \in \mathbb{Z}$  is such that

$$k \leq \frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} < k + 1,$$

then

$$k - 1 < \frac{n - T_p(-\alpha, \ell)}{p^\ell} \leq k.$$

Hence, we get

$$\left\lfloor \frac{n + p^\ell - 1 - T_p(-\alpha, \ell)}{p^\ell} \right\rfloor = \left\lceil \frac{n - T_p(-\alpha, \ell)}{p^\ell} \right\rceil.$$

By (3.4) and (3.5), it follows that

$$v_p((\alpha)_n) = \sum_{\ell=1}^{\infty} \delta_{p,\ell} \left( \alpha, \frac{n}{p^\ell} \right) = \sum_{\ell=1}^{\infty} \delta_{p,\ell} \left( \alpha, \left\{ \frac{n}{p^\ell} \right\} \right) + v_p(n!),$$

because  $\delta_{p,\ell}(\alpha, n/p^\ell) = \delta_{p,\ell}(\alpha, \{n/p^\ell\}) + \lfloor n/p^\ell \rfloor$  and  $v_p(n!) = \sum_{\ell=1}^{\infty} \lfloor n/p^\ell \rfloor$ .  $\square$

The following lemma provides an upper for the abscissae of the jumps of the functions  $\Delta_{\alpha,\beta}^{p,\ell}$ .

**Lemma 4.** *Let  $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . There exists a constant  $M(\alpha) > 0$  such that, for all  $p$  such that  $\alpha \in \mathbb{Z}_p$ , and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have*

$$\frac{1}{M(\alpha)} \leq \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} \leq 1.$$

*Remark.* In particular, if  $\alpha$  and  $\beta$  are two sequences taking their values in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ , there exists a constant  $M(\alpha, \beta) > 0$  such that for all  $p$  that does not divide  $d_{\alpha,\beta}$ , all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , and all  $x \in [0, 1/M(\alpha, \beta))$ , we have  $\Delta_{\alpha,\beta}^{p,\ell}(x) = 0$ .

*Proof.* Set  $a := p^\ell \mathfrak{D}_p^\ell(\alpha) - \alpha \in \{0, \dots, p^\ell - 1\}$ . We have

$$\mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} = \frac{a}{p^\ell} + \frac{\langle \alpha \rangle}{p^\ell} \in (0, 1],$$

because  $0 < \langle \alpha \rangle \leq 1$ . By Lemma 3, if  $p^\ell \geq d(\alpha)(\lfloor 1 - \alpha \rfloor + \langle \alpha \rangle)$ , then  $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle \alpha \rangle) \geq 1/d(\langle \alpha \rangle)$  and hence

$$\mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} \geq \frac{1}{d(\alpha)} \left( \frac{\langle \alpha \rangle}{\lfloor 1 - \alpha \rfloor + \langle \alpha \rangle} \right).$$

This completes the proof of Lemma 4 because there exists only a finite number of couples  $(p, \ell)$  such that  $p^\ell < d(\alpha)(\lfloor 1 - \alpha \rfloor + \langle \alpha \rangle)$ .  $\square$



Finally, our next lemma enables us to connect the functions  $\Delta_{\alpha,\beta}^{p,\ell}$  to the values of the functions  $\xi_{\alpha,\beta}(a, \cdot)$ . This is useful to decide if  $F_{\alpha,\beta}$  is  $N$ -integral.

**Lemma 5.** *Let  $\alpha$  and  $\beta$  be two sequences taking their values in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . There exists a constant  $\mathcal{N}_{\alpha,\beta}$  such that for all elements  $\alpha$  and  $\beta$  of the sequence  $\alpha$  or  $\beta$ , for all  $p$  that does not divide  $d_{\alpha,\beta}$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$  such that  $p^\ell \geq \mathcal{N}_{\alpha,\beta}$ , we have*

$$a\alpha \preceq a\beta \iff \mathfrak{D}_p^\ell(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p^\ell} \leq \mathfrak{D}_p^\ell(\beta) + \frac{\lfloor 1 - \beta \rfloor}{p^\ell},$$

where  $a \in \{1, \dots, d_{\alpha,\beta}\}$  satisfies  $p^\ell a \equiv 1 \pmod{d_{\alpha,\beta}}$ . Moreover, if the sequence  $\alpha$  and  $\beta$  take their values in  $(0, 1]$ , then we can take  $\mathcal{N}_{\alpha,\beta} = 1$ .

*Proof.* Let  $p$  be such that the sequences  $\alpha$  and  $\beta$  take their values in  $\mathbb{Z}_p$ . By Lemma 3, there exists a constant  $\mathcal{N}_1$  such that, for all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$  such that  $p^\ell \geq \mathcal{N}_1$ , and all elements  $\alpha$  of  $\alpha$  or  $\beta$ , we have  $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_p^\ell(\langle \alpha \rangle)$ . Moreover, if  $\alpha$  and  $\beta$  take their values in  $(0, 1]$ , we can take  $\mathcal{N}_1 = 1$  because  $\alpha = \langle \alpha \rangle$ . We set

$$\mathcal{N}_2 := \max \{d_{\alpha,\beta} | [1 - \alpha_i] - [1 - \beta_j] | : 1 \leq i \leq r, 1 \leq j \leq s\} + 1$$

and  $\mathcal{N}_{\alpha,\beta} := \max(\mathcal{N}_1, \mathcal{N}_2)$ . In particular, if  $\alpha$  and  $\beta$  take their values in  $(0, 1]$ , then  $\mathcal{N}_{\alpha,\beta} = 1$ . Let  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$  be such that  $p^\ell \geq \mathcal{N}_{\alpha,\beta}$  and  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$  such that  $p^\ell a \equiv 1 \pmod{d_{\alpha,\beta}}$ .

Let  $\alpha$  and  $\beta$  be elements of  $\alpha$  or  $\beta$ . We set  $k_1 := \lfloor 1 - \alpha \rfloor$  and  $k_2 := \lfloor 1 - \beta \rfloor$ . By (3.2), we have  $a\langle \alpha \rangle - \mathfrak{D}_p^\ell(\langle \alpha \rangle) \in \mathbb{Z}$ . Hence,

$$a\alpha = a\langle \alpha \rangle - ak_1 = \mathfrak{D}_p^\ell(\langle \alpha \rangle) + a\langle \alpha \rangle - \mathfrak{D}_p^\ell(\langle \alpha \rangle) - ak_1,$$

with  $\mathfrak{D}_p^\ell(\langle \alpha \rangle) \in (0, 1]$  and  $a\langle \alpha \rangle - \mathfrak{D}_p^\ell(\langle \alpha \rangle) - ak_1 \in \mathbb{Z}$ . Moreover, if  $\mathfrak{D}_p^\ell(\langle \alpha \rangle) = \mathfrak{D}_p^\ell(\langle \beta \rangle)$ , then still by (3.2), we have  $\langle \alpha \rangle = \langle \beta \rangle$ . By definition of the total order  $\prec$ , we obtain

$$\begin{aligned} a\alpha \preceq a\beta &\iff \mathfrak{D}_p^\ell(\langle \alpha \rangle) < \mathfrak{D}_p^\ell(\langle \beta \rangle) \quad \text{or} \quad (\mathfrak{D}_p^\ell(\langle \alpha \rangle) = \mathfrak{D}_p^\ell(\langle \beta \rangle) \quad \text{and} \quad a\alpha \geq a\beta) \\ &\iff \mathfrak{D}_p^\ell(\langle \alpha \rangle) < \mathfrak{D}_p^\ell(\langle \beta \rangle) \quad \text{or} \quad (\mathfrak{D}_p^\ell(\langle \alpha \rangle) = \mathfrak{D}_p^\ell(\langle \beta \rangle) \quad \text{and} \quad k_2 \geq k_1) \\ &\iff \mathfrak{D}_p^\ell(\langle \alpha \rangle) - \mathfrak{D}_p^\ell(\langle \beta \rangle) \leq \frac{k_2 - k_1}{p^\ell} \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\iff \mathfrak{D}_p^\ell(\langle \alpha \rangle) + \frac{k_1}{p^\ell} \leq \mathfrak{D}_p^\ell(\langle \beta \rangle) + \frac{k_2}{p^\ell} \\ &\iff \mathfrak{D}_p^\ell(\alpha) + \frac{k_1}{p^\ell} \leq \mathfrak{D}_p^\ell(\beta) + \frac{k_2}{p^\ell}, \end{aligned} \tag{3.7}$$

where in (3.6) we have used the fact that if  $\mathfrak{D}_p^\ell(\langle \alpha \rangle) \neq \mathfrak{D}_p^\ell(\langle \beta \rangle)$ , then  $|\mathfrak{D}_p^\ell(\langle \alpha \rangle) - \mathfrak{D}_p^\ell(\langle \beta \rangle)| \geq 1/d_{\alpha,\beta}$ . The equivalence (3.7) finishes the proof of Lemma 5.  $\square$

Proposition 4 shows that the functions  $\Delta_{\alpha,\beta}^{p,\ell}$  allow to compute the  $p$ -adic valuation of  $(\alpha)_n/(\beta)_n$  when  $p$  does not divide  $d_{\alpha,\beta}$ . If  $\alpha$  and  $\beta$  have the same number of parameters and if these parameters are in  $(0, 1]$ , the constant  $C_{\alpha,\beta}$  enables us to get a very convenient formula for the computation of the  $p$ -adic valuation of  $C_{\alpha,\beta}^n(\alpha)_n/(\beta)_n$  when  $p$  divides  $d_{\alpha,\beta}$ .

This formula, stated in the next proposition, is key to the proof of Theorem 1 and is also used many times in the proof of Theorem 2.

**Proposition 5.** *Let  $\alpha$  and  $\beta$  be two tuples of  $r$  parameters in  $\mathbb{Q} \cap (0, 1]$  such that  $F_{\alpha, \beta}$  is  $N$ -integral. Let  $p$  be a prime divisor of  $d_{\alpha, \beta}$ . We set  $d_{\alpha, \beta} = p^f D$ ,  $f \geq 1$ , with  $D \in \mathbb{N}$ ,  $D$  not divisible by  $p$ . For all  $a \in \{1, \dots, p^f\}$  not divisible by  $p$ , and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we choose a prime  $p_{a, \ell}$  such that*

$$p_{a, \ell} \equiv p^\ell \pmod{D} \quad \text{and} \quad p_{a, \ell} \equiv a \pmod{p^f}. \quad (3.8)$$

Then, for all  $n \in \mathbb{N}$ , we have

$$v_p \left( C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) = \frac{1}{\varphi(p^f)} \sum_{\substack{a=1 \\ \gcd(a, p)=1}}^{p^f} \sum_{\ell=1}^{\infty} \Delta_{\alpha, \beta}^{p_{a, \ell}, 1} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + n \left\{ \frac{\lambda_p(\alpha, \beta)}{p-1} \right\}, \quad (3.9)$$

where

$$C_0 = \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^r d(\beta_j)} \prod_{p|d_{\alpha, \beta}} p^{-\lfloor \frac{\lambda_p(\alpha, \beta)}{p-1} \rfloor}.$$

*Proof.* We denote by  $\tilde{\alpha}$ , respectively  $\tilde{\beta}$ , the (possibly empty) sequence of elements of  $\alpha$ , respectively of  $\beta$ , whose denominator is not divisible by  $p$ . We also set  $\lambda_p := \lambda_p(\alpha, \beta)$ . For all  $n \in \mathbb{N}$ , we have

$$v_p \left( C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) = \sum_{\ell=1}^{\infty} \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + \lambda_p v_p(n!) - n \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor. \quad (3.10)$$

Let  $\alpha$  be an element of  $\alpha$  or  $\beta$ . Let  $N$  be the denominator of  $\alpha$ . If  $p$  does not divide  $N$ , then  $N$  divides  $D$  and, for all  $a \in \{1, \dots, p^f\}$ ,  $\gcd(a, p) = 1$ , and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have  $p_{a, \ell} \equiv p^\ell \pmod{N}$ . Hence,  $\mathfrak{D}_p^\ell(\alpha) = \mathfrak{D}_{p_{a, \ell}}^\ell(\alpha)$  because  $\alpha \in (0, 1]$ .

On the other hand, if  $p$  divides  $N$ , then for all  $n, \ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we define  $\omega_\ell(\alpha, n)$  as the number of elements  $a \in \{1, \dots, p^f\}$ ,  $\gcd(a, p) = 1$ , such that  $\{n/p^\ell\} \geq \mathfrak{D}_{p_{a, \ell}}(\alpha)$ . Thus for all  $n, \ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we get

$$\sum_{\substack{a=1 \\ \gcd(a, p)=1}}^{p^f} \Delta_{\alpha, \beta}^{p_{a, \ell}, 1} \left( \left\{ \frac{n}{p^\ell} \right\} \right) = \varphi(p^f) \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + \sum_{\substack{i=1 \\ \alpha_i \notin \mathbb{Z}_p}}^r \omega_\ell(\alpha_i, n) - \sum_{\substack{j=1 \\ \beta_j \notin \mathbb{Z}_p}}^r \omega_\ell(\beta_j, n). \quad (3.11)$$

Let  $\alpha$  be an element of  $\alpha$  or  $\beta$  such that  $p$  divides  $d(\alpha)$ . We now compute  $\sum_{\ell=1}^{\infty} \omega_\ell(\alpha, n)$ . Let  $\alpha = r/(p^e N)$  where  $1 \leq e \leq f$ ,  $N$  divides  $D$ ,  $1 \leq r \leq p^e N$  and  $r$  is coprime to  $p^e N$ . Given  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , there exists  $r_{a, \ell} \in \{1, \dots, p^e N\}$  coprime to  $p^e N$  such that  $\mathfrak{D}_{p_{a, \ell}}(\alpha) = r_{a, \ell}/(p^e N)$  and  $p_{a, \ell} r_{a, \ell} - r \equiv 0 \pmod{p^e N}$ . In particular, by (3.8), we have

$$p^\ell r_{a, \ell} - r \equiv 0 \pmod{N} \quad \text{and} \quad a r_{a, \ell} - r \equiv 0 \pmod{p^e},$$

*i. e.*

$$r_{a,\ell} \equiv s_N \left( \frac{\pi_N(r)}{\pi_N(p^{\ell+e})} \right) p^e + s_{p^e} \left( \frac{\pi_{p^e}(r)}{\pi_{p^e}(aN)} \right) N \pmod{p^e N}.$$

In the rest of the proof, if  $a/b$  is a rational number written in irreducible form and the integer  $c \geq 1$  is coprime to  $b$ , we set

$$\varpi_c \left( \frac{a}{b} \right) := s_c \left( \frac{\pi_c(a)}{\pi_c(b)} \right).$$

Then,

$$\frac{r_{a,\ell}}{p^e N} \equiv \frac{\varpi_N(r/p^{\ell+e})}{N} + \frac{\varpi_{p^e}(r/(aN))}{p^e} \pmod{1}. \quad (3.12)$$

For all  $\ell \in \mathbb{N}$ , we have  $p^{\ell+1}\varpi_N(r/p^{\ell+1}) - p^\ell\varpi_N(r/p^\ell) \equiv 0 \pmod{N}$ , hence, since  $p$  and  $N$  are coprime, we obtain  $p\varpi_N(r/p^{\ell+1}) - \varpi_N(r/p^\ell) \equiv 0 \pmod{N}$ , *i. e.*

$$\mathfrak{D}_p \left( \frac{\varpi_N(r/p^\ell)}{N} \right) = \frac{\varpi_N(r/p^{\ell+1})}{N},$$

yielding

$$\frac{\varpi_N(r/p^{\ell+1})}{N} = \mathfrak{D}_p^{\ell+1} \left( \frac{r}{N} \right).$$

Let  $-r/N = \sum_{k=0}^{\infty} a_k p^k$  be the  $p$ -adic expansion of  $-r/N$ . For all  $\ell \in \mathbb{N}$ , we have

$$p^{\ell+1}\mathfrak{D}_p^{\ell+1} \left( \frac{r}{N} \right) - \frac{r}{N} = \sum_{k=0}^{\ell} a_k p^k$$

and thus

$$\frac{\varpi_N(r/p^{\ell+e})}{N} = \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell+e-1} a_k p^k}{p^{\ell+e}} = \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{\sum_{k=0}^{e-1} a_{\ell+k} p^k}{p^e}. \quad (3.13)$$

Moreover,  $p\varpi_N(r/p) \equiv r \pmod{N}$  but  $p\varpi_N(r/p) \neq r$  because  $r$  is not divisible by  $p$ . Hence,  $p\varpi_N(r/p) - r \geq N$  and  $a_0 \geq 1$ .

The elements of the multiset (*i. e.*, a set where repetition of elements is permitted)

$$\left\{ \left\{ \varpi_{p^e} \left( \frac{r}{aN} \right) : 1 \leq a \leq p^f, \gcd(a, p) = 1 \right\} \right\}$$

are those  $b \in \{1, \dots, p^e\}$  not divisible by  $p$ , where each  $b$  is repeated exactly  $p^{f-e}$  times. We fix  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ . We have

$$0 < \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} \leq \frac{1}{p^{\ell+e}} + \frac{p^\ell - 1}{p^{\ell+e}} \leq \frac{1}{p^e} \quad \text{and} \quad \frac{r_{a,\ell}}{p^e N} \in (0, 1].$$

By (3.12) et (3.13), the multiset

$$\Phi_\ell(\alpha) := \left\{ \left\{ \frac{r_{a,\ell}}{p^e N} : 1 \leq a \leq p^f, \gcd(a, p) = 1 \right\} \right\}$$

has the elements

$$\eta_{\ell,b} := \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{b}{p^e},$$

where  $b = \sum_{k=0}^{e-1} b_k p^k$ ,  $b_k \in \{0, \dots, p-1\}$ ,  $b_0 \neq a_\ell$  and each  $\eta_{\ell,b}$  is repeated exactly  $p^{f-e}$  times. In the sequel, we fix  $n = \sum_{k=0}^{\infty} n_k p^k$  with  $n_k \in \{0, \dots, p-1\}$  and, for all  $k \geq K$ ,  $n_k = 0$ , where  $K \in \mathbb{N}$ . For all  $\ell \in \mathbb{N}$ , we let  $\Lambda_\ell(\alpha, n) = 1$  if

$$\sum_{k=0}^{\ell-e-1} n_k p^k > \sum_{k=e}^{\ell-1} a_k p^{k-e},$$

and  $\Lambda_\ell(\alpha, n) = 0$  otherwise. Let us compute the number  $\omega_\ell(\alpha, n)$  of elements in  $\Phi_\ell(\alpha)$  which are  $\leq \{n/p^\ell\}$ .

If  $\ell \leq e-1$ , then

$$\begin{aligned} \left\{ \frac{n}{p^\ell} \right\} \geq \eta_{\ell,b} &\iff \frac{\sum_{k=0}^{\ell-1} n_k p^k}{p^\ell} \geq \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{\sum_{k=0}^{e-1} b_k p^k}{p^e} \\ &\iff \sum_{k=0}^{\ell-1} n_k p^k \geq \frac{r}{p^e N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^e} + \frac{\sum_{k=0}^{e-1} b_k p^{k+\ell}}{p^e} \\ &\iff \sum_{k=0}^{\ell-1} n_k p^k > \sum_{k=0}^{\ell-1} b_{e-\ell+k} p^k, \end{aligned}$$

because

$$0 < \frac{r}{p^e N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^e} + \frac{\sum_{k=0}^{e-\ell-1} b_k p^{k+\ell}}{p^e} \leq \frac{1}{p^e} + \frac{p^\ell - 1}{p^e} + \frac{p^\ell(p^{e-\ell} - 1)}{p^e} \leq 1.$$

Thus

$$\omega_\ell(\alpha, n) = \left( (p-1)p^{e-\ell-1} \sum_{k=0}^{\ell-1} n_k p^k \right) p^{f-e}.$$

If  $\ell \geq e$ , then

$$\begin{aligned} \left\{ \frac{n}{p^\ell} \right\} \geq \eta_{\ell,b} &\iff \frac{\sum_{k=0}^{\ell-1} n_k p^k}{p^\ell} \geq \frac{r}{p^{\ell+e}N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^{\ell+e}} + \frac{\sum_{k=0}^{e-1} b_k p^k}{p^e} \\ &\iff \sum_{k=0}^{\ell-1} n_k p^k \geq \frac{r}{p^e N} + \frac{\sum_{k=0}^{\ell-1} a_k p^k}{p^e} + \sum_{k=0}^{e-1} b_k p^{k+\ell-e} \\ &\iff \sum_{k=0}^{\ell-1} n_k p^k > \sum_{k=e}^{\ell-1} a_k p^{k-e} + \sum_{k=0}^{e-1} b_k p^{k+\ell-e}, \end{aligned} \tag{3.14}$$

because

$$0 < \frac{r}{p^e N} + \frac{\sum_{k=0}^{e-1} a_k p^k}{p^e} \leq \frac{1}{p^e} + \frac{p^e - 1}{p^e} \leq 1.$$

If we have

$$\sum_{k=\ell-e+1}^{\ell-1} n_k p^k > \sum_{k=1}^{e-1} b_k p^{k+\ell-e},$$

then (3.14) holds and we obtain

$$(p-1) \frac{\sum_{k=\ell-e+1}^{\ell-1} n_k p^k}{p^{\ell-e+1}}$$

numbers  $b$  satisfying the above inequality. Let us now assume that

$$\sum_{k=\ell-e+1}^{\ell-1} n_k p^k = \sum_{k=1}^{e-1} b_k p^{k+\ell-e}.$$

Then (3.14) is the same thing as

$$\sum_{k=0}^{\ell-e} n_k p^k > \sum_{k=e}^{\ell-1} a_k p^{k-e} + b_0 p^{\ell-e}. \quad (3.15)$$

If  $n_{\ell-e} \geq a_\ell + 1$ , then there are  $n_{\ell-e} - 1$  elements  $b_0 \in \{0, \dots, p-1\} \setminus \{a_\ell\}$  such that  $n_{\ell-e} > b_0$ , and, for  $b_0 = n_{\ell-e}$ , we have (3.15) if and only if  $\Lambda_\ell(\alpha, n) = 1$ . Moreover, when  $n_{\ell-e} \geq a_\ell + 1$ , we have  $\Lambda_{\ell+1}(\alpha, n) = 1$ . Hence, if  $n_{\ell-e} \geq a_\ell + 1$ , we have  $n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)$  numbers  $b_0$  such that (3.15) holds.

If  $n_{\ell-e} = a_\ell$ , then there are  $n_{\ell-e}$  numbers  $b_0$  such that (3.15) holds. Furthermore, we have  $\Lambda_\ell(\alpha, n) = \Lambda_{\ell+1}(\alpha, n)$  and in this case we also have  $n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)$  numbers  $b_0$  such that (3.15) holds.

If  $n_{\ell-e} \leq a_\ell - 1$ , then there are  $n_{\ell-e}$  numbers  $b_0$  such that  $b_0 < n_{\ell-e}$ , and for  $b_0 = n_{\ell-e}$ , we have (3.15) if and only if  $\Lambda_\ell(\alpha, n) = 1$ . Moreover, if  $n_{\ell-e} \leq a_\ell - 1$ , then  $\Lambda_{\ell+1}(\alpha, n) = 0$  and again there are  $n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)$  numbers  $b_0$  satisfying (3.15).

It follows that if  $\ell \geq e$ , then,

$$\omega_\ell(\alpha, n) = \left( n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n) + (p-1) \sum_{k=\ell-e+1}^{\ell-1} n_k p^{k-\ell+e-1} \right) p^{f-e}.$$

Hence, for all  $m \in \mathbb{N}$ ,  $m \geq K + e$ , we get

$$\begin{aligned} p^{e-f} \sum_{\ell=1}^m \omega_\ell(\alpha, n) &= (p-1) \sum_{\ell=1}^{e-1} p^{e-\ell-1} \sum_{k=0}^{\ell-1} n_k p^k \\ &+ \sum_{\ell=e}^m (n_{\ell-e} + \Lambda_\ell(\alpha, n) - \Lambda_{\ell+1}(\alpha, n)) + (p-1) \sum_{\ell=e}^m \sum_{k=\ell-e+1}^{\ell-1} n_k p^{k-\ell+e-1}. \end{aligned} \quad (3.16)$$

Let us compute the coefficients  $h_k$  of  $n_k$ ,  $0 \leq k \leq K$ , on the right hand side of (3.16), so that

$$p^{e-f} \sum_{\ell=1}^m \omega_\ell(\alpha, n) = \Lambda_e(\alpha, n) - \Lambda_{m+1}(\alpha, n) + \sum_{k=0}^K h_k n_k. \quad (3.17)$$

If  $e = 1$ , then for all  $k \in \{0, \dots, K\}$ , we have  $h_k = 1 = p^{e-1}$ . Let us assume that  $e \geq 2$ . We have

$$h_0 = (p-1) \sum_{\ell=1}^{e-1} p^{e-\ell-1} + 1 = p^{e-1}.$$

If  $1 \leq k \leq e-2$ , then

$$h_k = (p-1) \sum_{\ell=k+1}^{e-1} p^{k-\ell+e-1} + 1 + (p-1) \sum_{\ell=e}^{k+e-1} p^{k-\ell+e-1} = p^{e-1} - p^k + 1 + p^k - 1 = p^{e-1}.$$

Finally, if  $k \geq e-1$ , then

$$h_k = 1 + (p-1) \sum_{\ell=k+1}^{k+e-1} p^{k-\ell+e-1} = 1 + p^{e-1} - 1 = p^{e-1}.$$

Hence, we obtain

$$p^{e-f} \sum_{\ell=1}^m \omega_\ell(\alpha, n) = \Lambda_e(\alpha, n) - \Lambda_{m+1}(\alpha, n) + p^{e-1} \mathfrak{s}_p(n),$$

where  $\mathfrak{s}_p(n) := \sum_{k=0}^{\infty} n_k = \sum_{k=0}^K n_k$ .

Moreover, we have  $\Lambda_e(\alpha, n) = 0$  and there exists  $K' \geq K+e$  such that, for all  $m \geq K'$ , we have  $\Lambda_{m+1}(\alpha, n) = 0$ . Indeed,  $\sum_{k=0}^{\infty} a_k p^k$  is the  $p$ -adic expansion of  $-r/N \notin \mathbb{N}$ . Thus, there exists  $K' \geq K+e$  such that  $a_{K'} \neq 0$  and hence, for all  $m \geq K'$ , we have  $\Lambda_{m+1}(\alpha, n) = 0$ . Consequently, for all large enough  $\ell$ , we have  $\omega_\ell(\alpha, n) = 0$  and

$$\sum_{\ell=1}^{\infty} \omega_\ell(\alpha, n) = \varphi(p^f) \frac{\mathfrak{s}_p(n)}{p-1}. \quad (3.18)$$

By (3.11) and (3.18), we obtain, for all  $n \in \mathbb{N}$ ,

$$\sum_{\ell=1}^{\infty} \sum_{\substack{a=1 \\ \gcd(a,p)=1}}^{p^f} \Delta_{\alpha, \beta}^{p_{a, \ell}, 1} \left( \left\{ \frac{n}{p^\ell} \right\} \right) = \varphi(p^f) \sum_{\ell=1}^{\infty} \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + (r-s-\lambda_p) \varphi(p^f) \frac{\mathfrak{s}_p(n)}{p-1}.$$

Together with (3.10), this implies that

$$v_p \left( C_0^m \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) = \frac{1}{\varphi(p^f)} \sum_{\ell=1}^{\infty} \sum_{\substack{a=1 \\ \gcd(a,p)=1}}^{p^f} \Delta_{\alpha, \beta}^{p_{a, \ell}, 1} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + \lambda_p \left( \frac{\mathfrak{s}_p(n)}{p-1} + v_p(n!) \right) - n \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor. \quad (3.19)$$

But for all  $n \in \mathbb{N}$ , we have

$$v_p(n!) = \frac{n - \mathfrak{s}_p(n)}{p-1},$$

so that for all  $n \in \mathbb{N}$ ,

$$\lambda_p \left( \frac{\mathfrak{s}_p(n)}{p-1} + v_p(n!) \right) - n \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor = n \left\{ \frac{\lambda_p}{p-1} \right\}. \quad (3.20)$$

Hence, using (3.20) in (3.19), we get equation (3.9), which completes the proof of Proposition 5.  $\square$

#### 4. PROOF OF THEOREM 1

Let  $\alpha$  and  $\beta$  be two sequences taking their values in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . Let us assume that  $F_{\alpha, \beta}$  is  $N$ -integral. We first prove (1.2).

We fix a prime  $p$ . We denote by  $\tilde{\alpha}$ , respectively  $\tilde{\beta}$ , the (possibly empty) sequence  $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_u)$ , respectively  $(\tilde{\beta}_1, \dots, \tilde{\beta}_v)$ , made from the elements of  $\alpha$ , respectively of  $\beta$ , and whose denominator is not divisible by  $p$ . In particular, we have  $\lambda_p(\alpha, \beta) = u - v$ . By Proposition 4, for all  $n \in \mathbb{N}$ , we thus have

$$\begin{aligned} v_p \left( \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) &= -nv_p \left( \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \right) + v_p \left( \frac{(\tilde{\alpha}_1)_n \cdots (\tilde{\alpha}_u)_n}{(\tilde{\beta}_1)_n \cdots (\tilde{\beta}_v)_n} \right) \\ &= -nv_p \left( \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \right) + \sum_{\ell=1}^{\infty} \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + \lambda_p(\alpha, \beta) v_p(n!). \end{aligned} \quad (4.1)$$

By Lemma 4, there exists a constant  $M > 0$  such that, for any prime  $p$  that does not divide  $d_{\tilde{\alpha}, \tilde{\beta}}$ , for any  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , and any  $x \in [0, 1/M)$ , we have  $\Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell}(x) = 0$ . Hence, for all  $n \in \mathbb{N}$ , we have

$$-v \lfloor \log_p(nM) \rfloor \leq \sum_{\ell=1}^{\infty} \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) \leq u \lfloor \log_p(nM) \rfloor,$$

so that

$$\frac{1}{n} \sum_{\ell=1}^{\infty} \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (4.2)$$

Moreover, for all  $n \in \mathbb{N}$ , we have  $v_p(n!) = \sum_{\ell=1}^{\infty} \lfloor n/p^\ell \rfloor$ , hence

$$\sum_{\ell=1}^{\lfloor \log_p(n) \rfloor} \frac{n}{p^\ell} - \lfloor \log_p(n) \rfloor \leq v_p(n!) \leq \sum_{\ell=1}^{\lfloor \log_p(n) \rfloor} \frac{n}{p^\ell}$$

and

$$\frac{1}{n} v_p(n!) \xrightarrow{n \rightarrow +\infty} \frac{1}{p-1}. \quad (4.3)$$

We now use (4.2) and (4.3) in (4.1), and we obtain

$$\frac{1}{n} v_p \left( \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) \xrightarrow{n \rightarrow +\infty} -v_p \left( \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \right) + \frac{\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p-1}.$$

But for all  $n \in \mathbb{N}$ ,

$$C_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \in \mathbb{Z}_p.$$

It follows that for all  $n \in \mathbb{N}$ ,  $n \geq 1$ ,

$$v_p(C_{\boldsymbol{\alpha}, \boldsymbol{\beta}}) \geq -\frac{1}{n} v_p \left( \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) \xrightarrow{n \rightarrow +\infty} v_p \left( \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \right) - \frac{\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p-1}$$

and thus

$$v_p(C_{\boldsymbol{\alpha}, \boldsymbol{\beta}}) \geq v_p \left( \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \right) - \left\lfloor \frac{\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p-1} \right\rfloor,$$

because  $v_p(C_{\boldsymbol{\alpha}, \boldsymbol{\beta}}) \in \mathbb{Z}$ . Furthermore, if  $p$  does not divide  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  and if  $p \geq r - s + 2$ , then  $\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) = r - s$  and  $\lfloor \lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) / (p-1) \rfloor = 0$ . This proves the existence of  $C \in \mathbb{N}^*$  such that

$$C_{\boldsymbol{\alpha}, \boldsymbol{\beta}} = C \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \prod_{p \in \mathcal{P}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}} p^{-\lfloor \frac{\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p-1} \rfloor}. \quad (4.4)$$

We now define

$$C_0 := \frac{\prod_{i=1}^r d(\alpha_i)}{\prod_{j=1}^s d(\beta_j)} \prod_{p | d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}} p^{-\lfloor \frac{\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p-1} \rfloor}.$$

In the sequel, we assume that both sequences  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  take their values in  $(0, 1]$  and that  $r = s$ . We show that in this case  $C = 1$  and for this it is enough to prove that  $F_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(C_0 z) \in \mathbb{Z}[[z]]$ .

Consider a prime  $p$  that does not divide  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ , so that  $\lambda_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) = r - s = 0$ . Together with (4.1), this yields

$$v_p \left( C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) = \sum_{\ell=1}^{\infty} \Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p, \ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right).$$

By Lemma 5 and Theorem A, for all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have

$$\Delta_{\boldsymbol{\alpha}, \boldsymbol{\beta}}^{p, \ell}([0, 1]) = \xi_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(a, \mathbb{R}) \subset \mathbb{N},$$



where  $a \in \{1, \dots, d_{\alpha, \beta}\}$  satisfies  $p^\ell a \equiv 1 \pmod{d_{\alpha, \beta}}$ . Hence, we obtain that  $F_{\alpha, \beta}(C_0 z) \in \mathbb{Z}_p[[z]]$ . It remains to show that for any prime  $p$  that divides  $d_{\alpha, \beta}$ , we also have that  $F_{\alpha, \beta}(C_0 z) \in \mathbb{Z}_p[[z]]$ .

Consider a prime  $p$  that divides  $d_{\alpha, \beta}$ . With the notations of Proposition 5, for all  $n \in \mathbb{N}$ , we have

$$v_p \left( C_0^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right) = \frac{1}{\varphi(p^f)} \sum_{a=1}^{p^f} \sum_{\ell=1}^{\infty} \Delta_{\alpha, \beta}^{p_{a, \ell}, 1} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + n \left\{ \frac{\lambda_p(\alpha, \beta)}{p-1} \right\}.$$

Since none of the primes  $p_{a, \ell}$  divides  $d_{\alpha, \beta}$ , we have  $\Delta_{\alpha, \beta}^{p_{a, \ell}, 1}([0, 1]) \subset \mathbb{N}$  so that  $F_{\alpha, \beta}(C_0 z) \in \mathbb{Z}_p[[z]]$ . This completes the proof of Theorem 1.  $\square$

## 5. FORMAL CONGRUENCES

To prove Theorem 2, we need a ‘‘formal congruences’’ result, stated in Theorem 4 below that we prove in this section.

We fix a prime  $p$  and denote by  $\Omega$  the completion of the algebraic closure of  $\mathbb{Q}_p$ , and by  $\mathcal{O}$  the ring of integers of  $\Omega$ .

To state the main result of this section, we introduce some notations. If  $\mathcal{N} := (\mathcal{N}_r)_{r \geq 0}$  is a sequence of subsets of  $\bigcup_{t \geq 1} (\{0, \dots, p^t - 1\} \times \{t\})$ , then for all  $r \in \mathbb{Z}$ ,  $r \geq -1$  and all  $s \in \mathbb{N}$ , we denote by  $\Psi_{\mathcal{N}}(r, s)$  the set of the  $u \in \{0, \dots, p^s - 1\}$  such that, for all  $(n, t) \in \mathcal{N}_{r+s-t+1}$ , with  $t \leq s$ , and all  $j \in \{0, \dots, p^{s-t} - 1\}$ , we have  $u \neq j + p^{s-t}n$ . In particular, for all  $r \geq -1$ , we have  $\Psi_{\mathcal{N}}(r, 0) = \{0\}$ .

For completeness, let us recall some basic notions. Let  $\mathcal{A}$  be a commutative algebra (with a unit) over a commutative ring (with a unit)  $\mathcal{Z}$ . An element  $a \in \mathcal{A}$  is *regular* if, for all  $b \in \mathcal{A}$ , we have  $(ab = 0 \Rightarrow b = 0)$ . We define  $\mathcal{S}$  as the set of the regular elements of  $\mathcal{A}$ . Hence  $\mathcal{S}$  is a multiplicative set of  $\mathcal{A}$  and the ring  $\mathcal{S}^{-1}\mathcal{A}$  with the map

$$\begin{aligned} \mathcal{Z} \times \mathcal{S}^{-1}\mathcal{A} &\longrightarrow \mathcal{S}^{-1}\mathcal{A} \\ (\lambda, a/s) &\mapsto (\lambda \cdot a)/s \end{aligned}$$

is a  $\mathcal{Z}$ -algebra. Moreover, the morphism of algebra  $a \in \mathcal{A} \mapsto a/1 \in \mathcal{S}^{-1}\mathcal{A}$  is injective and enables us to identify  $\mathcal{A}$  with a sub-algebra of  $\mathcal{S}^{-1}\mathcal{A}$ . This is what we do in the statement of Theorem 4.

**Theorem 4.** *Let  $\mathcal{Z}$  denote a sub-ring of  $\mathcal{O}$  and  $\mathcal{A}$  a  $\mathcal{Z}$ -algebra (commutative with a unit) such that 2 is a regular element of  $\mathcal{A}$ . We consider a sequence of maps  $(\mathbf{A}_r)_{r \geq 0}$  from  $\mathbb{N}$  into  $\mathcal{S}$ , and a sequence of maps  $(\mathbf{g}_r)_{r \geq 0}$  from  $\mathbb{N}$  into  $\mathcal{Z} \setminus \{0\}$ . We assume there exists a sequence  $\mathcal{N} := (\mathcal{N}_r)_{r \geq 0}$  of subsets of  $\bigcup_{t \geq 1} (\{0, \dots, p^t - 1\} \times \{t\})$  such that, for all  $r \geq 0$ , we have the following properties:*

- (i)  $\mathbf{A}_r(0)$  is invertible in  $\mathcal{A}$ ;
- (ii) for all  $m \in \mathbb{N}$ , we have  $\mathbf{A}_r(m) \in \mathbf{g}_r(m)\mathcal{A}$ ;
- (iii) for all  $s, m \in \mathbb{N}$ , we have:

(a) for all  $u \in \Psi_{\mathcal{N}}(r, s)$  and all  $v \in \{0, \dots, p-1\}$ , we have

$$\frac{\mathbf{A}_r(v + up + mp^{s+1})}{\mathbf{A}_r(v + up)} - \frac{\mathbf{A}_{r+1}(u + mp^s)}{\mathbf{A}_{r+1}(u)} \in p^{s+1} \frac{\mathbf{g}_{r+s+1}(m)}{\mathbf{A}_r(v + up)} \mathcal{A};$$

(a<sub>1</sub>) moreover, if  $v + up \in \Psi_{\mathcal{N}}(r-1, s+1)$ , then

$$\mathbf{g}_r(v + up) \left( \frac{\mathbf{A}_r(v + up + mp^{s+1})}{\mathbf{A}_r(v + up)} - \frac{\mathbf{A}_{r+1}(u + mp^s)}{\mathbf{A}_{r+1}(u)} \right) \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A};$$

(a<sub>2</sub>) however, if  $v + up \notin \Psi_{\mathcal{N}}(r-1, s+1)$ , then

$$\mathbf{g}_r(v + up) \frac{\mathbf{A}_{r+1}(u + mp^s)}{\mathbf{A}_{r+1}(u)} \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A};$$

(b) for all  $(n, t) \in \mathcal{N}_r$ , we have  $\mathbf{g}_r(n + mp^t) \in p^t \mathbf{g}_{r+t}(m) \mathcal{Z}$ .

Then, for all  $a \in \{0, \dots, p-1\}$  and all  $m, s, r, K \in \mathbb{N}$ , we have

$$\begin{aligned} S_r(a, K, s, p, m) &:= \\ &\sum_{j=mp^s}^{(m+1)p^s-1} \left( \mathbf{A}_r(a + (K-j)p) \mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(K-j) \mathbf{A}_r(a + jp) \right) \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}, \end{aligned} \tag{5.1}$$

where  $\mathbf{A}_r(n) = 0$  if  $n < 0$ .

Theorem 4 is a generalisation of a result due to Dwork [12, Theorem 1.1], first used (in weaker version [13]) to obtain the analytic continuation of certain  $p$ -adic functions. Dwork then developed in [12] a method to prove the  $p$ -adic integrality of the Taylor coefficients of  $q$ -coordinates. This method is the basis of the proofs of the  $N$ -integrality of  $q_{\alpha, \beta}(z)$ . In the literature, one finds many generalisations of Dwork's formal congruences used to prove the integrality of Taylor coefficients of  $q$ -coordinates with increasing generality (see [20], [8] and [10]).

If we consider only the univariate case, then Theorem 4 encompasses all the analogous results in [20] and [10]. Its interest is due to the two following improvements.

- Theorem 4 can be applied to  $\mathbb{Z}_p$ -algebras more “abstract” than  $\mathcal{O}$ . We use this possibility in this paper, where we consider algebras of functions taking values in  $\mathbb{Z}_p$ . This improvement enables us to consider the integer  $\mathbf{n}_{\alpha, \beta}$  in Assertion (3) of Theorem 3.

- Beside this difference, Theorem 4 is a univariate version of Theorem 4 in [10] that allows to consider a set  $\mathcal{N}$  that depends on  $r$ . This property is crucial when we deal with the case of non  $R$ -partitioned tuples  $\alpha$  and  $\beta$ .

There also exist in the literature other types of generalisations of Dwork's formal congruences, such as the truncated version of Ota [29] and the recent version of Mellit and Vlasenko [27] (applied to constant terms of powers of Laurent polynomials).

**5.1. Proof of Theorem 4.** For all  $s \in \mathbb{N}$ ,  $s \geq 1$ , we denote by  $\alpha_s$  the following assertion: “For all  $a \in \{0, \dots, p-1\}$ , all  $u \in \{0, \dots, s-1\}$ , all  $m, r \in \mathbb{N}$  and all  $K \in \mathbb{Z}$ , we have

$$\mathbf{S}_r(a, K, u, p, m) \in p^{u+1} \mathbf{g}_{r+u+1}(m) \mathcal{A}.”$$

For all  $s \in \mathbb{N}$ ,  $s \geq 1$ , and all  $t \in \{0, \dots, s\}$ , we denote by  $\beta_{t,s}$  the following assertion: “For all  $a \in \{0, \dots, p-1\}$ , all  $m, r \in \mathbb{N}$  and all  $K \in \mathbb{Z}$ , we have

$$\begin{aligned} \mathbf{S}_r(a, K + mp^s, s, p, m) &\equiv \\ &\sum_{j \in \Psi_{\mathcal{N}}(r+t, s-t)} \frac{\mathbf{A}_{r+t+1}(j + mp^{s-t})}{\mathbf{A}_{r+t+1}(j)} \mathbf{S}_r(a, K, t, p, j) \pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}}. \end{aligned}$$

For all  $a \in \{0, \dots, p-1\}$ , all  $K \in \mathbb{Z}$  and all  $r, j \in \mathbb{N}$ , we define

$$\mathbf{U}_r(a, K, p, j) := \mathbf{A}_r(a + (K - j)p) \mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(K - j) \mathbf{A}_r(a + jp).$$

Then, we have

$$\mathbf{S}_r(a, K, s, p, m) = \sum_{j=0}^{p^s-1} \mathbf{U}_r(a, K, p, j + mp^s).$$

We now state four lemmas that will be needed to prove (5.1).

**Lemma 6.** *Assertion  $\alpha_1$  holds.*

**Lemma 7.** *For all  $s, r, m \in \mathbb{N}$ , all  $a \in \{0, \dots, p-1\}$ , all  $j \in \Psi_{\mathcal{N}}(r, s)$  and all  $K \in \mathbb{Z}$ , we have*

$$\mathbf{U}_r(a, K + mp^s, p, j + mp^s) \equiv \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a, K, p, j) \pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}}.$$

**Lemma 8.** *For all  $s \in \mathbb{N}$ ,  $s \geq 1$ , if  $\alpha_s$  holds, then, for all  $a \in \{0, \dots, p-1\}$ , all  $K \in \mathbb{Z}$  and all  $r, m \in \mathbb{N}$ , we have*

$$\mathbf{S}_r(a, K, s, p, m) \equiv \sum_{j \in \Psi_{\mathcal{N}}(r, s)} \mathbf{U}_r(a, K, p, j + mp^s) \pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}};$$

**Lemma 9.** *For all  $s \in \mathbb{N}$ ,  $s \geq 1$ , all  $t \in \{0, \dots, s-1\}$ , Assertions  $\alpha_s$  and  $\beta_{t,s}$  imply Assertion  $\beta_{t+1,s}$ .*

Before we prove these lemmas, let us check that they imply (5.1). We show that  $\alpha_s$  holds for all  $s \geq 1$  by induction on  $s$ , which gives the conclusion of Theorem 4. By Lemma 6,  $\alpha_1$  holds. Let us assume that  $\alpha_s$  holds for some  $s \geq 1$ . We observe that  $\beta_{0,s}$  is the assertion

$$\begin{aligned} \beta_{0,s} : \mathbf{S}_r(a, K + mp^s, s, p, m) &\equiv \\ &\sum_{j \in \Psi_{\mathcal{N}}(r, s)} \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{S}_r(a, K, 0, p, j) \pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}}. \end{aligned}$$

Since  $\mathbf{S}_r(a, K, 0, p, j) = \mathbf{U}_r(a, K, p, j)$ , we have

$$\sum_{j \in \Psi_{\mathcal{N}}(r, s)} \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{S}_r(a, K, 0, p, j) = \sum_{j \in \Psi_{\mathcal{N}}(r, s)} \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a, K, p, j)$$

and, by Lemma 7, we obtain, modulo  $p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ , that

$$\begin{aligned} \sum_{j \in \Psi_{\mathcal{N}}(r, s)} \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a, K, p, j) &\equiv \sum_{j \in \Psi_{\mathcal{N}}(r, s)} \mathbf{U}_r(a, K + mp^s, p, j + mp^s) \\ &\equiv \mathbf{S}_r(a, K + mp^s, s, p, m), \end{aligned} \quad (5.2)$$

where (5.2) is obtained *via* Lemma 8.

Consequently, Assertion  $\beta_{0,s}$  holds. We then obtain the validity of  $\beta_{1,s}$  by means of Lemma 9. Iterating Lemma 9, we finally obtain  $\beta_{s,s}$  which, modulo  $p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ , can be written

$$\begin{aligned} \mathbf{S}_r(a, K + mp^s, s, p, m) &\equiv \sum_{j \in \Psi_{\mathcal{N}}(r+s, 0)} \frac{\mathbf{A}_{r+s+1}(j + m)}{\mathbf{A}_{r+s+1}(j)} \mathbf{S}_r(a, K, s, p, j) \\ &\equiv \frac{\mathbf{A}_{r+s+1}(m)}{\mathbf{A}_{r+s+1}(0)} \mathbf{S}_r(a, K, s, p, 0), \end{aligned} \quad (5.3)$$

where we have used in (5.3) the fact that  $\Psi_{\mathcal{N}}(r + s, 0) = \{0\}$ .

Let us now prove that, for all  $a \in \{0, \dots, p-1\}$ , all  $r \in \mathbb{N}$  and all  $K \in \mathbb{Z}$ , we have  $\mathbf{S}_r(a, K, s, p, 0) \in p^{s+1} \mathcal{A}$ . For all  $N \in \mathbb{Z}$ , we denote by  $P_N$  the assertion: “For all  $a \in \{0, \dots, p-1\}$  and all  $r \in \mathbb{N}$ , we have  $\mathbf{S}_r(a, N, s, p, 0) \in p^{s+1} \mathcal{A}$ ”.

If  $N < 0$ , then for all  $j \in \{0, \dots, p^s-1\}$ , we have  $\mathbf{A}_r(a + (N-j)p) = 0$  and  $\mathbf{A}_{r+1}(N-j) = 0$ , so that  $\mathbf{S}_r(a, N, s, p, 0) = 0 \in p^{s+1} \mathcal{A}$ . To find a contradiction, let us assume the existence of a minimal element  $N \in \mathbb{N}$  such that  $P_N$  does not hold. Consider  $m \in \mathbb{N}$ ,  $m \geq 1$ , and set  $N' := N - mp^s$ . Using (5.3) with  $N'$  instead of  $K$ , we obtain

$$\mathbf{S}_r(a, N, s, p, m) \equiv \frac{\mathbf{A}_{r+s+1}(m)}{\mathbf{A}_{r+s+1}(0)} \mathbf{S}_r(a, N', s, p, 0) \pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}}.$$

Since  $m \geq 1$ , we have  $N' < N$ , which, by definition of  $N$ , yields that  $\mathbf{S}_r(a, N', s, p, 0) \in p^{s+1} \mathcal{A}$ . By Condition (i),  $\mathbf{A}_{r+s+1}(0)$  is an invertible of  $\mathcal{A}$  and thus

$$\mathbf{S}_r(a, N, s, p, m) \in p^{s+1} \mathcal{A}.$$

Hence, for all  $m \in \mathbb{N}$ ,  $m \geq 1$ , we have  $\mathbf{S}_r(a, N, s, p, m) \in p^{s+1}\mathcal{A}$ . Consider  $T \in \mathbb{N}$  such that  $(T+1)p^s > N$ . Then,

$$\begin{aligned} & \sum_{m=0}^T \mathbf{S}_r(a, N, s, p, m) \\ &= \sum_{m=0}^T \sum_{j=mp^s}^{(m+1)p^s-1} \left( \mathbf{A}_r(a + (N-j)p) \mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(N-j) \mathbf{A}_r(a + jp) \right) \\ &= \sum_{j=0}^N \left( \mathbf{A}_r(a + (N-j)p) \mathbf{A}_{r+1}(j) - \mathbf{A}_{r+1}(N-j) \mathbf{A}_r(a + jp) \right) \end{aligned} \quad (5.4)$$

$$= 0, \quad (5.5)$$

where we have used in (5.4) the fact that  $\mathbf{A}_r(n) = 0$  if  $n < 0$ . Equation (5.5) holds because 2 is a regular element of  $\mathcal{A}$  and the term of the sum (5.4) is changed to its opposite when we change the indice  $j$  to  $N-j$ . It follows that we have

$$\mathbf{S}_r(a, N, s, p, 0) = - \sum_{m=1}^T \mathbf{S}_r(a, N, s, p, m) \in p^{s+1}\mathcal{A}.$$

This contradicts the definition of  $N$ . Hence, for all  $N \in \mathbb{Z}$ ,  $P_N$  holds.

Moreover, Conditions (i) and (ii) respectively imply that  $\mathbf{A}_{r+s+1}(0)$  is an invertible element of  $\mathcal{A}$  and that  $\mathbf{A}_{r+s+1}(m) \in \mathfrak{g}_{r+s+1}(m)\mathcal{A}$ . By (5.3), we deduce that

$$\mathbf{S}_r(a, K + mp^s, s, p, m) \in p^{s+1}\mathfrak{g}_{r+s+1}(m)\mathcal{A}.$$

The latter congruence holds for all  $a \in \{0, \dots, p-1\}$ , all  $K \in \mathbb{Z}$  and all  $m, r \in \mathbb{N}$ , which proves that Assertion  $\alpha_{s+1}$  holds, and finishes the induction on  $s$ . It remains to prove Lemmas 6, 7, 8 and 9.

5.1.1. *Proof of Lemma 6.* Let  $a \in \{0, \dots, p-1\}$ ,  $K \in \mathbb{Z}$  and  $m, r \in \mathbb{N}$ . We have

$$\mathbf{S}_r(a, K, 0, p, m) = \mathbf{A}_r(a + (K-m)p) \mathbf{A}_{r+1}(m) - \mathbf{A}_{r+1}(K-m) \mathbf{A}_r(a + mp). \quad (5.6)$$

If  $K-m \notin \mathbb{N}$ , then  $\mathbf{A}_r(a + (K-m)p) = 0$  and  $\mathbf{A}_{r+1}(K-m) = 0$  so that  $\mathbf{S}_r(a, K, 0, p, m) = 0 \in p\mathfrak{g}_{r+1}(m)\mathcal{A}$ , as stated. We can thus assume that  $K-m \in \mathbb{N}$ . We write (5.6) as follows:

$$\begin{aligned} \mathbf{S}_r(a, K, 0, p, m) &= \mathbf{A}_r(a) \left( \mathbf{A}_{r+1}(m) \left( \frac{\mathbf{A}_r(a + (K-m)p)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(K-m)}{\mathbf{A}_{r+1}(0)} \right) \right. \\ &\quad \left. - \mathbf{A}_{r+1}(K-m) \left( \frac{\mathbf{A}_r(a + mp)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(m)}{\mathbf{A}_{r+1}(0)} \right) \right). \end{aligned} \quad (5.7)$$

Since  $\Psi_{\mathcal{N}}(r, 0) = \{0\}$ , we can use Hypothesis (a) of Theorem 4 with 0 instead of  $u$ , and  $a$  instead of  $v$ . We get this way

$$\frac{\mathbf{A}_r(a + (K - m)p)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(K - m)}{\mathbf{A}_{r+1}(0)} \in p \frac{\mathbf{g}_{r+1}(K - m)}{\mathbf{A}_r(a)} \mathcal{A}$$

and

$$\frac{\mathbf{A}_r(a + mp)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(m)}{\mathbf{A}_{r+1}(0)} \in p \frac{\mathbf{g}_{r+1}(m)}{\mathbf{A}_r(a)} \mathcal{A}.$$

Therefore,

$$\begin{aligned} \mathbf{A}_r(a) \mathbf{A}_{r+1}(m) \left( \frac{\mathbf{A}_r(a + (K - m)p)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(K - m)}{\mathbf{A}_{r+1}(0)} \right) &\in p \mathbf{g}_{r+1}(K - m) \mathbf{A}_{r+1}(m) \mathcal{A} \\ &\in p \mathbf{g}_{r+1}(m) \mathcal{A} \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} \mathbf{A}_r(a) \mathbf{A}_{r+1}(K - m) \left( \frac{\mathbf{A}_r(a + mp)}{\mathbf{A}_r(a)} - \frac{\mathbf{A}_{r+1}(m)}{\mathbf{A}_{r+1}(0)} \right) &\in p \mathbf{g}_{r+1}(m) \mathbf{A}_{r+1}(K - m) \mathcal{A} \\ &\in p \mathbf{g}_{r+1}(m) \mathcal{A}, \end{aligned} \quad (5.9)$$

where we have used, in (5.8), Condition (ii) that yields  $\mathbf{A}_{r+1}(m) \in \mathbf{g}_{r+1}(m) \mathcal{A}$ . Using (5.8) and (5.9) in (5.7), we obtain  $\mathbf{S}_r(a, K, 0, p, m) \in p \mathbf{g}_{r+1}(m) \mathcal{A}$ , as expected.

5.1.2. *Proof of Lemma 7.* We have

$$\begin{aligned} \mathbf{U}_r(a, K + mp^s, p, j + mp^s) - \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \mathbf{U}_r(a, K, p, j) \\ = -\mathbf{A}_{r+1}(K - j) \mathbf{A}_r(a + jp) \left( \frac{\mathbf{A}_r(a + jp + mp^{s+1})}{\mathbf{A}_r(a + jp)} - \frac{\mathbf{A}_{r+1}(j + mp^s)}{\mathbf{A}_{r+1}(j)} \right). \end{aligned} \quad (5.10)$$

Since  $j \in \Psi_{\mathcal{N}}(r, s)$ , Hypothesis (a) implies that the right hand side of (5.10) is in

$$\mathbf{A}_{r+1}(K - j) \mathbf{A}_r(a + jp) p^{s+1} \frac{\mathbf{g}_{r+s+1}(m)}{\mathbf{A}_r(a + jp)} \mathcal{A}.$$

These estimates show that the left hand side of (5.10) is in  $p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ , which concludes the proof of the lemma.

5.1.3. *Proof of Lemma 8.* We consider  $s \in \mathbb{N}$ ,  $s \geq 1$ , such that  $\alpha_s$  holds. We fix  $r \in \mathbb{N}$ . If  $\Psi_{\mathcal{N}}(r, s) = \{0, \dots, p^s - 1\}$ , Lemma 8 is trivial. In the sequel, we assume that  $\Psi_{\mathcal{N}}(r, s) \neq \{0, \dots, p^s - 1\}$ .

We have  $u \in \{0, \dots, p^s - 1\} \setminus \Psi_{\mathcal{N}}(r, s)$  if and only if there exist  $(n, t) \in \mathcal{N}_{r+s-t+1}$ ,  $t \leq s$ , and  $j \in \{0, \dots, p^{s-t} - 1\}$  such that  $u = j + p^{s-t}n$ . We denote by  $\mathcal{M}$  the set of the  $(n, t) \in \mathcal{N}_{r+s-t+1}$  with  $t \leq s$ . We thus have

$$\{0, \dots, p^s - 1\} \setminus \Psi_{\mathcal{N}}(r, s) = \bigcup_{(n,t) \in \mathcal{M}} \{j + p^{s-t}n : 0 \leq j \leq p^{s-t} - 1\}.$$

In particular, the set  $\mathcal{M}$  is non-empty.

We will show that there exist  $k \in \mathbb{N}$ ,  $k \geq 1$ , and  $(n_1, t_1), \dots, (n_k, t_k) \in \mathcal{M}$  such that the sets

$$J(n_i, t_i) := \{j + p^{s-t_i}n_i : 0 \leq j \leq p^{s-t_i} - 1\}$$

form a partition of  $\{0, \dots, p^s - 1\} \setminus \Psi_{\mathcal{N}}(r, s)$ . We observe that

$$\mathcal{M} \subset \bigcup_{t=1}^s \left( \{0, \dots, p^t - 1\} \times \{t\} \right)$$

and thus  $\mathcal{M}$  is finite. Hence, it is enough to show that if  $(n, t), (n', t') \in \mathcal{M}$ ,  $j \in \{0, \dots, p^{s-t} - 1\}$  and  $j' \in \{0, \dots, p^{s-t'} - 1\}$  satisfy  $j + p^{s-t}n = j' + p^{s-t'}n'$ , then we have either  $J(n, t) \subset J(n', t')$  or  $J(n', t') \subset J(n, t)$ .

Let us assume, for instance, that  $t \leq t'$ . Then there exists  $j_0 \in \{0, \dots, p^{t'-t} - 1\}$  such that  $j = j' + p^{s-t'}j_0$ , so that  $p^{s-t}n = p^{s-t'}n' + p^{s-t'}j_0$  and thus  $J(n', t') \subset J(n, t)$ . Similarly, if  $t \geq t'$ , then  $J(n, t) \subset J(n', t')$ . Hence, we obtain

$$\begin{aligned} \mathbf{S}_r(a, K, s, p, m) = & \\ & \sum_{j \in \Psi_{\mathcal{N}}(r, s)} \mathbf{U}_r(a, K, p, j + mp^s) + \sum_{j \in \{0, \dots, p^s - 1\} \setminus \Psi_{\mathcal{N}}(r, s)} \mathbf{U}_r(a, K, p, j + mp^s), \end{aligned} \quad (5.11)$$

where

$$\sum_{j \in \{0, \dots, p^s - 1\} \setminus \Psi_{\mathcal{N}}(r, s)} \mathbf{U}_r(a, K, p, j + mp^s) = \sum_{i=1}^k \sum_{j=0}^{p^{s-t_i}-1} \mathbf{U}_r(a, K, p, j + p^{s-t_i}n_i + mp^s). \quad (5.12)$$

We will now prove that for all  $i \in \{1, \dots, k\}$ , we have

$$\sum_{j=0}^{p^{s-t_i}-1} \mathbf{U}_r(a, K, p, j + p^{s-t_i}n_i + mp^s) \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}. \quad (5.13)$$

Let  $i \in \{1, \dots, k\}$ . By definition of  $\mathbf{U}_r$ , we have

$$\sum_{j=0}^{p^{s-t_i}-1} \mathbf{U}_r(a, K, p, j + p^{s-t_i}n_i + mp^s) = \mathbf{S}_r(a, K, s - t_i, p, n_i + mp^{t_i}).$$

Since  $t_i \geq 1$ , we get *via*  $\alpha_s$  that

$$\mathbf{S}_r(a, K, s - t_i, p, n_i + mp^{t_i}) \in p^{s-t_i+1} \mathbf{g}_{r+s-t_i+1}(n_i + mp^{t_i}) \mathcal{A}.$$

We have  $(n_i, t_i) \in \mathcal{N}_{r+s-t_i+1}$  and thus we can apply Hypothesis (b) of Theorem 4 with  $r + s - t_i + 1$  instead of  $r$ :

$$p^{s-t_i+1} \mathbf{g}_{r+s-t_i+1}(n_i + mp^{t_i}) \in p^{s-t_i+1} p^{t_i} \mathbf{g}_{r+s+1}(m) \mathcal{Z} = p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{Z}.$$

It follows that for all  $i \in \{1, \dots, k\}$ , we have (5.13).

Congruence (5.13), together with (5.12) and (5.11), shows that

$$\mathbf{S}_r(a, K, s, p, m) \equiv \sum_{j \in \Psi_{\mathcal{N}}(r, s)} \mathbf{U}_r(a, K, p, j + mp^s) \pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}},$$

which completes the proof of Lemma 8.

5.1.4. *Proof of Lemma 9.* In this proof,  $i$  is an element of  $\{0, \dots, p-1\}$  and  $u$  is an element of  $\{0, \dots, p^{s-t-1} - 1\}$ . For  $t < s$ , we write  $\beta_{t,s}$  as

$$\begin{aligned} \mathbf{S}_r(a, K + mp^s, s, p, m) &\equiv \\ \sum_{i+up \in \Psi_{\mathcal{N}}(r+t, s-t)} \frac{\mathbf{A}_{r+t+1}(i + up + mp^{s-t})}{\mathbf{A}_{r+t+1}(i + up)} \mathbf{S}_r(a, K, t, p, i + up) &\pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}}. \end{aligned} \quad (5.14)$$

We want to prove the congruence  $\beta_{t+1,s}$ , which can be written

$$\begin{aligned} \mathbf{S}_r(a, K + mp^s, s, p, m) &\equiv \\ \sum_{u \in \Psi_{\mathcal{N}}(r+t+1, s-t-1)} \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \mathbf{S}_r(a, K, t+1, p, u) &\pmod{p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}}. \end{aligned}$$

We see that  $\mathbf{S}_r(a, K, t+1, p, u) = \sum_{i=0}^{p-1} \mathbf{S}_r(a, K, t, p, i + up)$ . Hence, with

$$\begin{aligned} X := \mathbf{S}_r(a, K + mp^s, s, p, m) & \\ - \sum_{i=0}^{p-1} \sum_{u \in \Psi_{\mathcal{N}}(r+t+1, s-t-1)} \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \mathbf{S}_r(a, K, t, p, i + up), & \end{aligned}$$

it remains to show that  $X \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ . We have

$$i + up \in \Psi_{\mathcal{N}}(r+t, s-t) \Rightarrow u \in \Psi_{\mathcal{N}}(r+t+1, s-t-1). \quad (5.15)$$

Indeed if  $u \notin \Psi_{\mathcal{N}}(r+t+1, s-t-1)$ , then there exist  $(n, k) \in \mathcal{N}_{r+s-k+1}$ ,  $k \leq s-t-1$ , and  $j \in \{0, \dots, p^{s-t-1-k} - 1\}$  such that  $u = j + p^{s-t-1-k}n$ . Hence,  $i + up = i + jp + p^{s-t-k}n$ , so that  $i + up \notin \Psi_{\mathcal{N}}(r+t, s-t)$ . By  $\beta_{t,s}$  in the form (5.14) and modulo  $p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ , we obtain

$$\begin{aligned} X &\equiv \sum_{i+up \in \Psi_{\mathcal{N}}(r+t, s-t)} \mathbf{S}_r(a, K, t, p, i+up) \left( \frac{\mathbf{A}_{r+t+1}(i + up + mp^{s-t})}{\mathbf{A}_{r+t+1}(i + up)} - \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \right) \\ &\quad - \sum_{\substack{u \in \Psi_{\mathcal{N}}(r+t+1, s-t-1) \\ i+up \notin \Psi_{\mathcal{N}}(r+t, s-t)}} \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \mathbf{S}_r(a, K, t, p, i + up). \end{aligned}$$



But, by Hypothesis  $(a_1)$  of Theorem 4 applied with  $s - t - 1$  for  $s$  and  $r + t + 1$  for  $r$ , we have

$$\mathbf{g}_{r+t+1}(i + up) \left( \frac{\mathbf{A}_{r+t+1}(i + up + mp^{s-t})}{\mathbf{A}_{r+t+1}(i + up)} - \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \right) \in p^{s-t} \mathbf{g}_{r+s+1}(m) \mathcal{A}.$$

Moreover, since  $t < s$  and  $\alpha_s$  holds, we have

$$\mathbf{S}_r(a, K, t, p, i + up) \in p^{t+1} \mathbf{g}_{r+t+1}(i + up) \mathcal{A} \quad (5.16)$$

and, modulo  $p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ , we obtain

$$X \equiv - \sum_{\substack{u \in \Psi_{\mathcal{N}}(r+t+1, s-t-1) \\ i+up \notin \Psi_{\mathcal{N}}(r+t, s-t)}} \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \mathbf{S}_r(a, K, t, p, i + up). \quad (5.17)$$

Finally, when  $i + up \notin \Psi_{\mathcal{N}}(r + t, s - t)$ , we can apply Condition  $(a_2)$  of Theorem 4 with  $s - t - 1$  for  $s$ ,  $i$  for  $v$  and  $r + t + 1$  for  $r$ , so that

$$\mathbf{g}_{r+t+1}(i + up) \frac{\mathbf{A}_{r+t+2}(u + mp^{s-t-1})}{\mathbf{A}_{r+t+2}(u)} \in p^{s-t} \mathbf{g}_{r+s+1}(m) \mathcal{A}. \quad (5.18)$$

Using (5.16) and (5.18) in (5.17), we thus have  $X \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}$ . This completes the proof of Lemma 9 and consequently that of Theorem 4.  $\square$

## 6. PROOF OF THEOREM 2

The aim of this section is to prove Theorem 2. We will first prove some elementary properties of the algebras of functions  $\mathcal{A}_b$  and  $\mathcal{A}_b^*$ .

**6.1. Algebras of functions taking values into  $\mathbb{Z}_p$ .** We gather in the following lemma a few properties of the algebras  $\mathfrak{A}_{p,n}$  and  $\mathfrak{A}_{p,n}^*$ .

**Lemma 10.** *We fix a prime  $p$  and  $n \in \mathbb{N}$ ,  $n \geq 1$ .*

- (1) *An element  $f$  of  $\mathfrak{A}_{p,n}$ , respectively of  $\mathfrak{A}_{p,n}^*$ , is invertible in  $\mathfrak{A}_{p,n}$ , respectively in  $\mathfrak{A}_{p,n}^*$ , if and only if  $f((\mathbb{Z}_p^\times)^n) \subset \mathbb{Z}_p^\times$ ;*
- (2) *the algebra  $\mathfrak{A}_{p,n}$  contains the rational functions*

$$f : \begin{array}{ccc} (\mathbb{Z}_p^\times)^n & \rightarrow & \mathbb{Z}_p \\ (x_1, \dots, x_n) & \mapsto & \frac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)}, \end{array}$$

where  $P, Q \in \mathbb{Z}_p[X_1, \dots, X_n]$  and, for all  $x_1, \dots, x_n \in \mathbb{Z}_p^\times$ , we have  $Q(x_1, \dots, x_n) \in \mathbb{Z}_p^\times$ ;

- (3) *if  $f \in \mathfrak{A}_{p,n}^\times$  and if  $\mathfrak{E}_s$ ,  $s \geq 1$ , is the function Euler quotient defined by*

$$\mathfrak{E}_s : \begin{array}{ccc} \mathbb{Z}_p^\times & \rightarrow & \mathbb{Z}_p \\ x & \mapsto & (x^{\varphi(p^s)} - 1)/p^s, \end{array}$$

then we have  $\mathfrak{E}_s \circ f \in \mathfrak{A}_{p,n}^*$ .

*Proof.* Let  $f \in \mathfrak{A}_{p,n}$ . For  $f$  to be invertible in  $\mathfrak{A}_{p,n}$ , we clearly need that  $f((\mathbb{Z}_p^\times)^n) \subset \mathbb{Z}_p^\times$  and in this case, for all  $\mathbf{x} \in (\mathbb{Z}_p^\times)^n$ , all  $\mathbf{a} \in \mathbb{Z}_p^n$  and all  $m \in \mathbb{N}$ ,  $m \geq 1$ , we have

$$\frac{1}{f(\mathbf{x} + \mathbf{a}p^m)} = \frac{1}{f(\mathbf{x}) + \eta p^m} = \frac{1}{f(\mathbf{x})} \frac{1}{1 + \frac{\eta}{f(\mathbf{x})} p^m} \equiv \frac{1}{f(\mathbf{x})} \pmod{p^m \mathbb{Z}_p},$$

because  $f(\mathbf{x}) \in \mathbb{Z}_p^\times$ ,  $\eta \in \mathbb{Z}_p$ , and  $(1 + p^m \mathbb{Z}_p, \times)$  is a group. The case  $f \in \mathfrak{A}_{p,n}^*$  being similar, Assertion (1) is proved.

To prove Assertion (2), we apply Assertion (1) because any polynomial function  $f : \mathbf{x} \in (\mathbb{Z}_p^\times)^n \mapsto P(\mathbf{x})$ , with  $P \in \mathbb{Z}_p[X_1, \dots, X_n]$  is in  $\mathfrak{A}_{p,n}$ .

Let us now prove Assertion (3). For all  $s \in \mathbb{N}$ ,  $s \geq 1$ , the cardinal of  $(\mathbb{Z}_p/p^s \mathbb{Z}_p)^\times$  is  $\varphi(p^s)$  because  $\mathbb{Z}_p/p^s \mathbb{Z}_p$  is isomorphic to  $\mathbb{Z}/p^s \mathbb{Z}$ . Hence, for all  $x \in \mathbb{Z}_p^\times$ , we have  $x^{\varphi(p^s)} \equiv 1 \pmod{p^s \mathbb{Z}_p}$  and the function  $\mathfrak{E}_s$  is well defined.

We fix  $s \in \mathbb{N}$ ,  $s \geq 1$ . To prove Assertion (3), it is enough to prove that for all  $x \in \mathbb{Z}_p^\times$ , all  $a \in \mathbb{Z}_p$  and all  $m \in \mathbb{N}$ ,  $m \geq 1$ , we have  $\mathfrak{E}_s(x + ap^m) \equiv \mathfrak{E}_s(x) \pmod{p^{m-1} \mathbb{Z}_p}$ . We have

$$\begin{aligned} (x + ap^m)^{\varphi(p^s)} &= \sum_{k=0}^{\varphi(p^s)} \binom{\varphi(p^s)}{k} \frac{a^k}{x^k} p^{km} x^{\varphi(p^s)} \\ &\equiv x^{\varphi(p^s)} + \sum_{k=1}^{\varphi(p^s)} \binom{\varphi(p^s)}{k} \frac{a^k}{x^k} p^{km} \pmod{p^{s+m} \mathbb{Z}_p}, \end{aligned}$$

because  $x^{\varphi(p^s)} \equiv 1 \pmod{p^s \mathbb{Z}_p}$ . By a result of Kummer, the  $p$ -adic valuation of  $\binom{\varphi(p^s)}{k}$  is the number of carries in the addition of  $k$  and  $\varphi(p^s) - k$  in base  $p$ . Let us show that this number is equal to  $s - 1 - v_p(k)$ .

Indeed, if  $v_p(k) = 0$ , then this number is  $s - 1$  because  $\varphi(p^s) = (p - 1)p^{s-1}$ . If  $v_p(k) = \alpha \geq 1$ , then we write  $k = k'p^\alpha$  and  $\varphi(p^s) - k = p^\alpha((p - 1)p^{s-1-\alpha} - k')$  with  $v_p(k') = 0$ , so that the number of carries of the addition of  $k$  and  $\varphi(p^s) - k$  in base  $p$  is the number of carries in the addition of  $k'$  and  $\varphi(p^{s-\alpha}) - k'$ , *i.e.*  $s - 1 - \alpha = s - 1 - v_p(k)$ .

In particular, we obtain that, for all  $k \geq 1$ ,

$$v_p \left( \binom{\varphi(p^s)}{k} \frac{a^k}{x^k} p^{km} \right) \geq s + m + (k - 1)m - v_p(k) - 1 \geq s + m - 1,$$

hence  $(x + ap^m)^{\varphi(p^s)} \equiv x^{\varphi(p^s)} \pmod{p^{s+m-1} \mathbb{Z}_p}$ . Consequently, we have  $\mathfrak{E}_s(x + ap^m) \equiv \mathfrak{E}_s(x) \pmod{p^{m-1} \mathbb{Z}_p}$ , and the proof of Lemma 10 is complete.  $\square$

**Lemma 11.** *Let  $\nu, D \in \mathbb{N}$ ,  $D \geq 1$ , and  $b \in \{1, \dots, D\}$ ,  $\gcd(b, D) = 1$ .*

- (1) *We have  $\mathcal{A}_b(p^\nu, D) \subset \mathcal{A}_b(p^\nu, D)^*$  and  $p\mathcal{A}_b(p^\nu, D)^* \subset \mathcal{A}_b(p^\nu, D)$ ;*
- (2) *An element  $f$  of  $\mathcal{A}_b(p^\nu, D)$ , respectively of  $\mathcal{A}_b(p^\nu, D)^*$ , is invertible in  $\mathcal{A}_b(p^\nu, D)$ , respectively in  $\mathcal{A}_b(p^\nu, D)^*$ , if and only if  $f(\Omega_b(p^\nu, D)) \subset \mathbb{Z}_p^\times$ ;*
- (3) *Any constant function from  $\Omega_b(p^\nu, D)$  into  $\mathbb{Z}_p$  is in  $\mathcal{A}_b(p^\nu, D)$ ;*
- (4) *If  $r \in \mathbb{N}$  and  $\alpha \in \mathbb{Q}$  satisfy  $d(\alpha) = p^\mu D'$ , with  $1 \leq \mu \leq \nu$  and  $D' \mid D$ , then the map  $t \in \Omega_b(p^\nu, D) \mapsto d(\alpha)\langle t^{(r)} \alpha \rangle$  is in  $\mathcal{A}_b(p^\nu, D)^\times$ ;*

- (5) If  $\alpha \in \mathbb{Q} \cap \mathbb{Z}_p$  and  $k \in \mathbb{N}$ , then the map  $t \in \Omega_b(p^\nu, D) \mapsto \varpi_{p^k}(t\alpha)$  is in  $\mathcal{A}_b(p^\nu, D)$ ;  
(6) If  $n \in \mathbb{N}$ ,  $n \geq 1$ ,  $f_1, \dots, f_n \in \mathcal{A}_b(p^\nu, D)^\times$ ,  $g \in \mathfrak{A}_{p,n}$  and  $h \in \mathfrak{A}_{p,n}^*$ , then  $g' := g \circ (f_1, \dots, f_n) \in \mathcal{A}_b(p^\nu, D)$  and  $h' := h \circ (f_1, \dots, f_n) \in \mathcal{A}_b(p^\nu, D)^*$ . Furthermore if  $g$  is invertible in  $\mathfrak{A}_{p,n}$ , respectively  $h$  is invertible in  $\mathfrak{A}_{p,n}^*$ , then  $g'$  is invertible in  $\mathcal{A}_b(p^\nu, D)$ , respectively  $h'$  is invertible in  $\mathcal{A}_b(p^\nu, D)^*$ ;  
(7) If  $f \in \mathcal{A}_b(p^\nu, D)$  and  $g \in \mathcal{A}_b(p^\nu, D)^*$ , then

$$\sum_{t \in \Omega_b(p^\nu, D)} f(t) \in p^{\nu-1}\mathbb{Z}_p \quad \text{and} \quad \sum_{t \in \Omega_b(p^\nu, D)} g(t) \in p^{\nu-2}\mathbb{Z}_p.$$

*Proof.* Assertions (1) and (3) are obvious. The proof of Assertion (2) is similar to that of Assertion (2) of Lemma 10.

Let us prove Assertion (4). For all  $t \in \Omega_b(p^\nu, D)$ , the number  $d(\alpha)\langle t^{(r)}\alpha \rangle$  is the numerator of  $\langle t^{(r)}\alpha \rangle$  and thus it is in  $\mathbb{Z}_p^\times$  because  $p$  divides  $d(\alpha)$ .

Let  $\alpha = \kappa/d(\alpha)$ ,  $t_1, t_2 \in \Omega_b(p^\nu, D)$  and  $m \in \mathbb{N}$ ,  $m \geq 1$  be such that  $t_1 \equiv t_2 \pmod{p^m}$ . Since  $t_1 \equiv t_2 \equiv b \pmod{D}$ , we get  $t_1^{(r)} \equiv t_2^{(r)} \pmod{D}$ .

If  $m \geq \mu$ , then  $t_1^{(r)} \equiv t_2^{(r)} \pmod{p^\mu}$  and the chinese remainder theorem gives  $t_1^{(r)} \equiv t_2^{(r)} \pmod{p^\mu D}$ . Since  $D' \mid D$ , we obtain  $t_1^{(r)}\kappa \equiv t_2^{(r)}\kappa \pmod{d(\alpha)}$  and thus  $d(\alpha)\langle t_1^{(r)}\alpha \rangle = d(\alpha)\langle t_2^{(r)}\alpha \rangle$ , as expected.

On the other hand, if  $m < \mu$ , then  $t_1^{(r)} \equiv t_2^{(r)} \pmod{p^m}$ . Since  $D' \mid D$  and  $d(\alpha)\langle t_i^{(r)}\alpha \rangle \equiv t_i^{(r)}\kappa \pmod{d(\alpha)}$  for  $i \in \{1, 2\}$ , we obtain  $d(\alpha)\langle t_1\alpha \rangle \equiv d(\alpha)\langle t_2\alpha \rangle \pmod{p^m}$ , which proves Assertion (4).

Assertion (5) is obvious and Assertion (6) is a direct consequence of the definitions and of Assertion (2).

Let us prove Assertion (7) by induction on  $\nu$  in the case  $f \in \mathcal{A}_b(p^\nu, D)$ . We denote by  $A_\nu$  the assertion

$$\sum_{t \in \Omega_b(p^\nu, D)} f(t) \in p^{\nu-1}\mathbb{Z}_p.$$

Assertion  $A_1$  trivially holds. Let  $\nu \in \mathbb{N}$ ,  $\nu \geq 1$  be such that  $A_\nu$  holds.

The set  $\Omega_b(p^{\nu+1}, D)$  is the set of the  $t_{\ell, \nu+1} \in \{1, \dots, p^{\nu+1}D\}$  such that  $t_{\ell, \nu+1} \equiv b \pmod{D}$  and  $t_{\ell, \nu+1} \equiv \ell \pmod{p^{\nu+1}}$ , with  $\ell \in \{1, \dots, p^{\nu+1}\}$ ,  $\gcd(\ell, p) = 1$ . Let  $\ell := u + vp^\nu$  with  $u \in \{1, \dots, p^\nu\}$ ,  $\gcd(u, p) = 1$  and  $v \in \{0, \dots, p-1\}$ . Then, we have  $t_{\ell, \nu+1} \equiv u \pmod{p^\nu}$

and by the chinese remainder theorem, we obtain  $t_{\ell,\nu+1} \equiv t_{u,\nu} \pmod{p^\nu D}$ , so that

$$\begin{aligned}
\sum_{t \in \Omega_b(p^{\nu+1}, D)} f(t) &= \sum_{\substack{\ell=1 \\ \gcd(\ell, p)=1}}^{p^{\nu+1}} f(t_{\ell,\nu+1}) = \sum_{\substack{u=1 \\ \gcd(u, p)=1}}^{p^\nu} \sum_{v=0}^{p-1} f(t_{u+vp^\nu, \nu+1}) \\
&\equiv p \sum_{\substack{u=1 \\ \gcd(u, p)=1}}^{p^\nu} f(t_{u,\nu}) \pmod{p^\nu \mathbb{Z}_p} \\
&\equiv p \sum_{t \in \Omega_b(p^\nu, D)} f(t) \pmod{p^\nu \mathbb{Z}_p} \\
&\equiv 0 \pmod{p^\nu \mathbb{Z}_p},
\end{aligned}$$

by Assertion  $A_\nu$ . Hence, Assertion  $A_{\nu+1}$  holds, which completes the proof of Assertion (7) when  $f \in \mathcal{A}_b(p^\nu, D)$ . The case  $f \in \mathcal{A}_b(p^\nu, D)^*$  is similar.  $\square$

**6.2. Proof of Theorem 2.** In this section, we fix two  $r$ -tuples  $\alpha$  and  $\beta$  with parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ . We assume that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint and that  $H_{\alpha, \beta}$  holds.

We set  $C = C_{\langle \alpha \rangle, \langle \beta \rangle}$ ,  $C' = C'_{\alpha, \beta}$ ,  $\mathbf{n} = \mathbf{n}_{\alpha, \beta}$ ,  $\mathbf{m} = \mathbf{m}_{\alpha, \beta}$  and  $\lambda_p = \lambda_p(\alpha, \beta)$ . We write  $d_{\alpha, \beta} = p^\nu D$  with  $\nu \geq 0$  and  $\gcd(D, p) = 1$ . For all  $t \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and all  $r \in \mathbb{N}$ , we recall that  $t^{(r)}$  is the unique element in  $\{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  such that  $t^{(r)} \equiv t \pmod{p^\nu}$  and  $p^r t^{(r)} \equiv t \pmod{D}$ .

We fix  $b \in \{1, \dots, D\}$  coprime to  $D$  and set  $\Omega_b := \Omega_b(p^\nu, D)$ ,  $\mathcal{A}_b := \mathcal{A}_b(p^\nu, D)$ ,  $\mathcal{A}_b^* := \mathcal{A}_b(p^\nu, D)^*$ . We recall that if  $\nu = 0$ , then  $\Omega_b = \{b\}$  and that  $\mathcal{A}_b = \mathcal{A}_b^*$  is the algebra of functions from  $\{b\}$  into  $\mathbb{Z}_p$ .

For all  $t \in \Omega_b$  and all  $r, n \in \mathbb{N}$ , we set

$$\mathcal{Q}_{r,t}(n) := (C')^n \frac{\langle \langle t^{(r)} \alpha \rangle \rangle_n}{\langle \langle t^{(r)} \beta \rangle \rangle_n} \quad \text{and} \quad \mathcal{Q}_{r,\cdot}(n) := (t \in \Omega_b \mapsto \mathcal{Q}_{r,t}(n)).$$

For all  $c \in \{1, \dots, p^\nu\}$  not divisible by  $p$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we fix a prime  $p_{c,\ell}$  such that  $p_{c,\ell} \equiv p^\ell \pmod{D}$  and  $p_{c,\ell} \equiv c \pmod{p^\nu}$ . For all  $t \in \Omega_b$  and all  $r \in \mathbb{N}$ , we set

$$\Delta_{r,t}^{c,\ell} := \Delta_{\langle t^{(r)} \alpha \rangle, \langle t^{(r)} \beta \rangle}^{p_{c,\ell}, 1}.$$

If  $\tilde{\alpha}$ , respectively  $\tilde{\beta}$ , is the sequence of elements of  $\langle t^{(r)} \alpha \rangle$ , respectively of  $\langle t^{(r)} \beta \rangle$ , whose denominator is not divisible by  $p$ , then we set  $\tilde{\Delta}_{r,t}^{p,\ell} := \Delta_{\tilde{\alpha}, \tilde{\beta}}^{p,\ell}$ . We gather in the following lemma a few properties of the sequences  $\mathcal{Q}_{r,\cdot}$ . We set  $\iota = 1$  if  $\mathbf{m}$  is odd and if  $\beta \notin \mathbb{Z}^r$ , and  $\iota = 0$  otherwise.

**Lemma 12.** For all  $n, r \in \mathbb{N}$ , there exists  $\Lambda_{b,r}(n) \in \mathbb{Z}_p$  such that  $\mathcal{Q}_{r,\cdot}(n) \in 2^{rn} \Lambda_{b,r}(n) \mathcal{A}_b^\times$ , where

$$\begin{aligned} v_p(\Lambda_{b,r}(n)) &= \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n \left\{ \frac{\lambda_p}{p-1} \right\} \\ &= \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + n \left\{ \frac{\lambda_p}{p-1} \right\}. \end{aligned}$$

If  $p$  divides  $d_{\alpha,\beta}$ , then for all  $n, r \in \mathbb{N}$ ,  $n \geq 1$ , we have  $v_p(\Lambda_{b,r}(n)) \geq 1$  and if  $\beta \in \mathbb{Z}^r$  then

$$v_p(\Lambda_{b,r}(n)) \geq - \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor.$$

*Proof.* For all  $t \in \Omega_b$ , we have  $\mathcal{Q}_{r,t}(n) = 2^{rn} \Lambda_{b,r}(n) \mathfrak{R}_r(n, t)$  with

$$\Lambda_{b,r}(n) := \left( C \frac{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)} \right)^n \frac{\prod_{\alpha_i \in \mathbb{Z}_p} (\langle t^{(r)} \alpha_i \rangle)_n}{\prod_{\beta_i \in \mathbb{Z}_p} (\langle t^{(r)} \beta_i \rangle)_n}$$

and

$$\mathfrak{R}_r(n, t) := \frac{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)^n (\langle t^{(r)} \alpha_i \rangle)_n}{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)^n (\langle t^{(r)} \beta_i \rangle)_n} = \frac{\prod_{\alpha_i \notin \mathbb{Z}_p} \prod_{k=0}^{n-1} (d(\alpha_i) \langle t^{(r)} \alpha_i \rangle + kd(\alpha_i))}{\prod_{\beta_i \notin \mathbb{Z}_p} \prod_{k=0}^{n-1} (d(\beta_i) \langle t^{(r)} \beta_i \rangle + kd(\beta_i))}.$$

By Assertions (2) and (4) of Lemma 11, we have  $\mathfrak{R}_r(n, \cdot) \in \mathcal{A}_b^\times$ . Moreover if  $\alpha$  is a term of the sequences  $\alpha$  or  $\beta$  whose denominator is not divisible by  $p$ , then  $\langle t^{(r)} \alpha \rangle$  depends only of the class of  $t^{(r)}$  in  $\mathbb{Z}/D\mathbb{Z}$  which is that of  $\varpi_D(p^{-r}b)$  when  $t \in \Omega_b$ . Indeed, if  $\langle \alpha \rangle = 1$ , then  $\langle t^{(r)} \alpha \rangle = 1$  and if  $\langle \alpha \rangle = k/N \neq 1$ , where  $N$  is a divisor of  $D$ , then  $N \langle t^{(r)} \alpha \rangle = N \{t^{(r)} \langle \alpha \rangle\} = \varpi_N(t^{(r)} k)$ . For all  $t \in \Omega_b$  and all  $r \in \mathbb{N}$ , we have  $p^r t^{(r)} \equiv b \pmod{D}$ , so that  $\varpi_N(t^{(r)} k) = \varpi_N(bp^{-r}k)$ . It follows that  $\Lambda_{b,r}(n)$  depends only on  $b, r$  and  $n$ . By Proposition 5, we have

$$\begin{aligned} v_p(\Lambda_{b,r}(n)) &= v_p \left( C^n \frac{(\langle t^{(r)} \alpha \rangle)_n}{(\langle t^{(r)} \beta \rangle)_n} \right) = \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n \left\{ \frac{\lambda_p}{p-1} \right\} \\ &= \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + n \left\{ \frac{\lambda_p}{p-1} \right\}. \end{aligned}$$

In the sequel, we assume that  $p$  divides  $d_{\alpha,\beta}$ . Let us now show that if  $n \geq 1$ , then  $v_p(\Lambda_{b,r}(n)) \geq 1$ . Let  $\alpha$  be a term of the sequences  $\langle t^{(r)} \alpha \rangle$  or  $\langle t^{(r)} \beta \rangle$  whose denominator is divisible by  $p$ . By (3.18), the number of elements  $\mathfrak{D}_{p,c,\ell}(\alpha)$ ,  $\ell \geq 1$ ,  $c \in \{1, \dots, p^\nu\}$ ,  $\gcd(c,p) = 1$ , that satisfy  $\{n/p^\ell\} \geq \mathfrak{D}_{p,c,\ell}(\alpha)$  is equal to  $\varphi(p^\nu) \mathfrak{s}_p(n)/(p-1)$ . In particular, if  $n \geq 1$ , then there exist at least one  $\ell \geq 1$  and one  $c \in \{1, \dots, p^\nu\}$ ,  $\gcd(c,p) = 1$ , such that  $\{n/p^\ell\} \geq \mathfrak{D}_{p,c,\ell}(\alpha)$ .

Thus, there exists one term  $\alpha' \in (0, 1)$  of the sequence  $\langle t^{(r)}\alpha \rangle$  or  $\langle t^{(r)}\beta \rangle$  such that  $\Delta_{r,t}^{c,\ell}(\{n/p^\ell\}) = \Delta_{r,t}^{c,\ell}(\mathfrak{D}_{p_{c,\ell}}(\alpha'))$ . By Lemma 5, we obtain  $\Delta_{r,t}^{c,\ell}(\{n/p^\ell\}) = \xi_{\langle t^{(r)}\alpha \rangle, \langle t^{(r)}\beta \rangle}(a, a\alpha')$ , where  $a \in \{1, \dots, d_{\alpha,\beta}\}$  satisfies  $p_{c,\ell}a \equiv 1 \pmod{d_{\alpha,\beta}}$ . Since  $\alpha' \notin \mathbb{Z}$ , we have  $m_{\alpha,\beta}(a) \preceq a\alpha' \prec a$  and by Lemma 2, Assertion  $H_{\langle t^{(r)}\alpha \rangle, \langle t^{(r)}\beta \rangle}$  holds, so that  $\Delta_{r,t}^{c,\ell}(\{n/p^\ell\}) \geq 1$ . Hence,  $v_p(\Lambda_{b,r}(n)) \geq 1$ .

Moreover, if  $\beta \in \mathbb{Z}^r$ , then  $\lambda_p \leq -1$  and the functions  $\tilde{\Delta}_{r,t}^{p,\ell}$  are positive on  $[0, 1)$ . It follows that

$$v_p(\Lambda_{b,r}(n)) \geq -\lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n \left\{ \frac{\lambda_p}{p-1} \right\} \geq -\frac{\lambda_p}{p-1} + \left\{ \frac{\lambda_p}{p-1} \right\} \geq - \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor.$$

This completes the proof of Lemma 12.  $\square$

In the sequel, we set  $\mathcal{K}_b := \mathcal{A}_b^*$  if  $p$  does not divide  $d_{\alpha,\beta}$ . If  $p$  divides  $d_{\alpha,\beta}$ , we set

$$\mathcal{K}_b := \begin{cases} p^{-1-\lfloor \lambda_p/(p-1) \rfloor} \mathcal{A}_b & \text{if } \beta \in \mathbb{Z}^r; \\ \mathcal{A}_b & \text{if } \beta \notin \mathbb{Z}^r, \mathbf{m} \text{ is odd and } p = 2; \\ \mathcal{A}_b & \text{if } \beta \notin \mathbb{Z}^r \text{ and } p-1 \nmid \lambda_p; \\ \mathcal{A}_b^* & \text{otherwise.} \end{cases}.$$

By Lemma 12, for all  $r \in \mathbb{N}$ ,

$$(t \in \Omega_b \mapsto F_{\langle t^{(r)}\alpha \rangle, \langle t^{(r)}\beta \rangle}(C'z)) \in 1 + z\mathcal{K}_b[[z]]$$

is an invertible formal series in  $\mathcal{K}_b[[z]]$ . Hence, to prove Theorem 2, it is enough to prove that the function

$$t \in \Omega_b \mapsto G_{\langle t^{(1)}\alpha \rangle, \langle t^{(1)}\beta \rangle}(C'z^p)F_{\langle t\alpha \rangle, \langle t\beta \rangle}(C'z) - pG_{\langle t\alpha \rangle, \langle t\beta \rangle}(C'z)F_{\langle t^{(1)}\alpha \rangle, \langle t^{(1)}\beta \rangle}(C'z^p) \quad (6.1)$$

is in  $p\mathcal{K}_b[[z]]$ .

For all  $a \in \{0, \dots, p-1\}$  and all  $K \in \mathbb{N}$ , the  $a + Kp$ -th coefficient of the formal series (6.1) is

$$t \in \Omega_b \mapsto \Phi_t(a + Kp) := \sum_{i=1}^r (\Phi_{\alpha_i, t}(a + Kp) - \Phi_{\beta_i, t}(a + Kp)),$$

where

$$\Phi_{\alpha, t}(a + Kp) := \sum_{j=0}^K \mathcal{Q}_{0,t}(a + jp) \mathcal{Q}_{1,t}(K - j) (H_{\langle t^{(1)}\alpha \rangle}(K - j) - pH_{\langle t\alpha \rangle}(a + jp)).$$

It is sufficient to show that, for all terms  $\alpha$  of the sequences  $\alpha$  and  $\beta$ , for all  $a \in \{0, \dots, p-1\}$  and all  $K \in \mathbb{N}$ , we have

$$\Phi_{\alpha, \cdot}(a + Kp) \in p\mathcal{K}_b. \quad (6.2)$$

If  $a + Kp = 0$ , then  $\Phi_{\alpha, \cdot}(0)$  is obviously the null map. In the sequel, we assume that  $a + Kp \neq 0$ , so that for all  $j \in \{0, \dots, K\}$ , we have  $a + jp \geq 1$  or  $K - j \geq 1$ .

If  $p$  divides  $d_{\alpha,\beta}$  and if  $\alpha$  is a term of the sequences  $\alpha$  or  $\beta$  whose denominator is divisible by  $p$ , then for all  $n, r \in \mathbb{N}$  and all  $t \in \Omega_b$ , we have

$$H_{\langle t^{(r)}\alpha \rangle}(n) = \sum_{k=0}^{n-1} \frac{d(\alpha)}{d(\alpha)(\langle t^{(r)}\alpha \rangle + k)},$$

yielding  $(t \in \Omega_b \mapsto H_{\langle t^{(r)}\alpha \rangle}(n)) \in p\mathcal{A}_b$ . By Lemma 12, for all  $n, r \in \mathbb{N}$ ,  $n \geq 1$ , we have  $\mathcal{Q}_{r,\cdot}(n) \in \Lambda_{b,r}(n)\mathcal{A}_b$  with

$$\Lambda_{b,r}(n) \in \begin{cases} p^{-\lfloor \lambda_p/(p-1) \rfloor} \mathbb{Z}_p & \text{if } \beta \in \mathbb{Z}^r; \\ p\mathbb{Z}_p & \text{otherwise.} \end{cases}.$$

Hence, we have  $(t \in \Omega_b \mapsto \Phi_{\alpha,t}(a + Kp)) \in p^2\mathcal{K}_b \subset p\mathcal{K}_b$ , as expected.

It remains to deal with the case when the denominator of  $\alpha$  is not divisible by  $p$ . We fix an element  $\alpha \in \mathbb{Z}_p$  of the sequences  $\alpha$  or  $\beta$  in the proof of (6.2). We recall that  $\langle t\alpha \rangle$  is independent of  $t \in \Omega_b$  because  $\alpha \in \mathbb{Z}_p$ . By [12, Lemma 4.1], for all  $j \in \{0, \dots, K\}$ , we have

$$pH_{\langle t\alpha \rangle}(a + jp) \equiv pH_{\langle t\alpha \rangle}(jp) + \frac{\rho(a, \langle t\alpha \rangle)}{\mathfrak{D}_p(\langle t\alpha \rangle) + j} \pmod{p\mathbb{Z}_p},$$

where we recall that for all  $x \in \mathbb{Q} \cap \mathbb{Z}_p$ , we have

$$\rho(a, x) = \begin{cases} 0 & \text{if } a \leq p\mathfrak{D}_p(x) - x; \\ 1 & \text{if } a > p\mathfrak{D}_p(x) - x. \end{cases}.$$

Moreover,

$$H_{\langle t\alpha \rangle}(jp) = \sum_{k=0}^{jp-1} \frac{1}{\langle t\alpha \rangle + k} = \frac{1}{p} \sum_{k=0}^{j-1} \frac{1}{\mathfrak{D}_p(\langle t\alpha \rangle) + k} + \sum_{\substack{i=0 \\ i \neq p\mathfrak{D}_p(\langle t\alpha \rangle) - \langle t\alpha \rangle}}^{p-1} \sum_{k=0}^{j-1} \frac{1}{\langle t\alpha \rangle + i + kp},$$

so that  $pH_{\langle t\alpha \rangle}(jp) \equiv H_{\mathfrak{D}_p(\langle t\alpha \rangle)}(j) \pmod{p\mathbb{Z}_p}$ . Writing  $\langle \alpha \rangle = k/N$  as an irreducible fraction, we obtain

$$\mathfrak{D}_p(\langle t\alpha \rangle) = \frac{\varpi_N(Np^{-1}\langle t\alpha \rangle)}{N} = \frac{\varpi_N(p^{-1}\varpi_N(bk))}{N} = \frac{\varpi_N(p^{-1}bk)}{N} = \langle t^{(1)}\alpha \rangle. \quad (6.3)$$

Hence,

$$pH_{\langle t\alpha \rangle}(a + jp) \equiv H_{\langle t^{(1)}\alpha \rangle}(j) + \frac{\rho(a, \langle t\alpha \rangle)}{\mathfrak{D}_p(\langle t\alpha \rangle) + j} \pmod{p\mathbb{Z}_p}. \quad (6.4)$$

We now use the following fact (6.5), to be proved in Section 6.2.1. For all  $j \in \{0, \dots, K\}$ , we have

$$\left( t \in \Omega_b \mapsto \frac{\rho(a, \langle t\alpha \rangle)}{\mathfrak{D}_p(\langle t\alpha \rangle) + j} \mathcal{Q}_{0,t}(a + jp) \mathcal{Q}_{1,t}(K - j) \right) \in p\mathcal{K}_b. \quad (6.5)$$

The notation  $f(t) \equiv g(t) \pmod{I}$  where  $I$  is an ideal of  $\mathcal{A}_b$  means that there exists  $h \in I$  such that  $f - g = h|_{\mathbb{Q}_p}$  with  $f : t \in \Omega_b \mapsto f(t) \in \mathbb{Q}_p$  and  $g : t \in \Omega_b \mapsto g(t) \in \mathbb{Q}_p$ . Using (6.4) and (6.5) in the definition of  $\Phi_{\alpha, \cdot}(a + Kp)$ , we obtain

$$\begin{aligned} \Phi_{\alpha, t}(a + Kp) &\equiv \sum_{j=0}^K \mathcal{Q}_{0, t}(a + jp) \mathcal{Q}_{1, t}(K - j) (H_{\langle t(1) \alpha \rangle}(K - j) - H_{\langle t(1) \alpha \rangle}(j)) \\ &\equiv - \sum_{j=0}^K H_{\langle t(1) \alpha \rangle}(j) \left( \mathcal{Q}_{0, t}(a + jp) \mathcal{Q}_{1, t}(K - j) - \mathcal{Q}_{0, t}(a + (K - j)p) \mathcal{Q}_{1, t}(j) \right), \end{aligned}$$

modulo  $p\mathcal{K}_b$ .

6.2.1. *Proof of Equation (6.5).* For this, we prove several results that will be used again in the proof of Theorem 2.

**Lemma 13.** *Let  $a \in \{0, \dots, p-1\}$ ,  $m \in \mathbb{N}$  and  $x \in \mathbb{Z}_p \cap \mathbb{Q} \cap (0, 1]$ . If  $\rho(a, x) = 1$ , then for all  $\ell \in \{1, \dots, 1 + v_p(\mathfrak{D}_p(x) + m)\}$ , we have  $\{(a + mp)/p^\ell\} \geq \mathfrak{D}_p^\ell(x)$ .*

*Proof.* We write  $m = \sum_{j=0}^{\infty} m_j p^j$  with  $m_j \in \{0, \dots, p-1\}$  and we fix some  $\ell$  in  $\{1, \dots, 1 + v_p(\mathfrak{D}_p(x) + m)\}$ . Then,

$$\left\{ \frac{a + mp}{p^\ell} \right\} = \frac{a + p \sum_{j=0}^{\ell-2} m_j p^j}{p^\ell}.$$

We have  $\mathfrak{D}_p(x) + m \in p^{\ell-1} \mathbb{Z}_p$  and thus

$$\mathfrak{D}_p(x) + m - \sum_{j=\ell-1}^{\infty} m_j p^j \in p^{\ell-1} \mathbb{Z}_p,$$

so that

$$p \mathfrak{D}_p(x) + p \sum_{j=0}^{\ell-2} m_j p^j - p^\ell \mathfrak{D}_p^\ell(x) \in p^\ell \mathbb{Z},$$

because  $p \mathfrak{D}_p(x) - p^\ell \mathfrak{D}_p^\ell(x) \in \mathbb{Z}$ . We obtain

$$\frac{p \mathfrak{D}_p(x) + p \sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} - \mathfrak{D}_p^\ell(x) \in \mathbb{Z}.$$

Moreover  $\mathfrak{D}_p^\ell(x) \in (0, 1]$  and

$$0 < \frac{p \mathfrak{D}_p(x) + p \sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} \leq \frac{p + p(p^{\ell-1} - 1)}{p^\ell} \leq 1,$$

so that

$$\frac{p \mathfrak{D}_p(x) + p \sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} - \mathfrak{D}_p^\ell(x) = 0.$$



We have  $\rho(a, x) = 1$  hence  $a > p\mathfrak{D}_p(x) - x$  i.e.  $a \geq p\mathfrak{D}_p(x) - x + 1$  and  $a \geq p\mathfrak{D}_p(x)$ . It follows that

$$\frac{a + p \sum_{j=0}^{\ell-2} m_j p^j}{p^\ell} \geq \mathfrak{D}_p^\ell(x).$$

□

For all  $c \in \{1, \dots, p^\nu\}$  not divisible by  $p$  and all  $\ell, r \in \mathbb{N}$ ,  $\ell \geq 1$ , we define  $\tau(r, \ell)$  as the smallest of the numbers  $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha \rangle)$ , where  $\alpha$  runs through the elements of the sequences  $\alpha$  and  $\beta$  whose denominator is not divisible by  $p$ . Since  $\langle t^{(r)}\alpha \rangle \in \mathbb{Z}_p$ , the number  $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha \rangle)$  does not depend on  $c$  and thus  $\tau(r, \ell)$  neither. Moreover, since  $\alpha \in \mathbb{Z}_p$ , the rational number  $\langle t^{(r)}\alpha \rangle$  does not depend on  $t \in \Omega_b$  and thus  $\tau(r, \ell)$  neither. We define  $\mathbf{1}_{r,\ell}$  as the characteristic function of the interval  $[\tau(r, \ell), 1)$ . For all  $m, r \in \mathbb{N}$ , we set

$$\mu_r(m) := \sum_{\ell=1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{m}{p^\ell} \right\} \right) \in \mathbb{N} \quad \text{and} \quad g_r(m) := p^{\mu_r(m)}.$$

Similarly, the function  $g_r$  does not depend on  $t \in \Omega_b$ .

**Lemma 14.** *Let  $r, \ell, n \in \mathbb{N}$ ,  $\ell \geq 1$ , be such that  $\{n/p^\ell\} \geq \tau(r, \ell)$ . Then for all  $t \in \Omega_b$  and all  $c \in \{1, \dots, p^\nu\}$  not divisible by  $p$ , we have  $\Delta_{r,t}^{c,\ell}(\{n/p^\ell\}) \geq 1$ . In particular for all  $n \in \mathbb{N}$ , we have*

$$v_p(\Lambda_{b,r}(n)) \geq v_p(g_r(n)) + n \left\{ \frac{\lambda_p}{p-1} \right\}.$$

If  $\beta \in \mathbb{Z}^r$ , then for all  $n \in \mathbb{N}$ ,  $n \geq 1$ , we have

$$v_p(\Lambda_{b,r}(n)) \geq v_p(g_r(n)) - \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor.$$

*Proof.* Let  $r, \ell, n \in \mathbb{N}$ ,  $\ell \geq 1$ , such that  $\{n/p^\ell\} \geq \tau(r, \ell)$ . Let  $c \in \{1, \dots, p^\nu\}$  not divisible by  $p$ . There exists an element  $\alpha_c$  of the sequences  $\alpha$  or  $\beta$  such that  $\Delta_{r,t}^{c,\ell}(\{n/p^\ell\}) = \Delta_{r,t}^{c,\ell}(\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha_c \rangle))$  with  $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha_c \rangle) \leq \{n/p^\ell\} < 1$ . Hence  $\langle t^{(r)}\alpha_c \rangle < 1$ . By Lemma 5, we obtain

$$\Delta_{r,t}^{c,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) = \xi_{\langle t^{(r)}\alpha \rangle, \langle t^{(r)}\beta \rangle} (a, a \langle t^{(r)}\alpha_c \rangle),$$

where  $a \in \{1, \dots, d_{\alpha,\beta}\}$  satisfies  $p_{c,\ell}a \equiv 1 \pmod{d_{\alpha,\beta}}$ . We also have  $m_{\langle t^{(r)}\alpha \rangle, \langle t^{(r)}\beta \rangle}(a) \preceq a \langle t^{(r)}\alpha_c \rangle \prec a$  and by Lemma 2, Assertion  $H_{\langle t^{(r)}\alpha \rangle, \langle t^{(r)}\beta \rangle}$  holds. Hence,  $\Delta_{r,t}^{c,\ell}(\{n/p^\ell\}) \geq 1$ . By Lemma 12, we have

$$v_p(\Lambda_{b,r}(n)) = \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + n \left\{ \frac{\lambda_p}{p-1} \right\},$$

so that

$$v_p(\Lambda_{b,r}(n)) \geq v_p(g_r(n)) + n \left\{ \frac{\lambda_p}{p-1} \right\}.$$

Let us now assume that  $\beta \in \mathbb{Z}^r$ . If we have  $1 > \{n/p^\ell\} \geq \tau(r, \ell)$ , then there exists an element  $\alpha$  of  $\alpha$  whose denominator is not divisible by  $p$  and such that  $\{n/p^\ell\} \geq \mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha \rangle)$  for some  $c \in \{1, \dots, p^r\}$  not divisible by  $p$ . The denominator of  $\langle t^{(r)}\alpha \rangle$  divides  $D$  and  $p_{c,\ell} \equiv p^\ell \pmod{D}$  hence we have  $\mathfrak{D}_{p_{c,\ell}}(\langle t^{(r)}\alpha \rangle) = \mathfrak{D}_p^\ell(\langle t^{(r)}\alpha \rangle)$ , which yields  $\tilde{\Delta}_{r,t}^{p,\ell}(\{n/p^\ell\}) \geq 1$ . By Lemma 12, for all  $n \in \mathbb{N}$ ,  $n \geq 1$ , we have

$$\begin{aligned} v_p(\Lambda_{b,r}(n)) &= \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(n)}{p-1} + n \left\{ \frac{\lambda_p}{p-1} \right\} \\ &\geq \mu_r(n) - \frac{\lambda_p}{p-1} + \left\{ \frac{\lambda_p}{p-1} \right\} \geq v_p(g_r(n)) + \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor, \end{aligned}$$

because  $\lambda_p \leq 0$ . This proves Lemma 14.  $\square$

We are now in position to prove (6.5).

*Proof of (6.5).* If  $\rho(a, \langle t\alpha \rangle) = 0$  then (6.5) holds. We can thus assume that  $\rho(a, \langle t\alpha \rangle) = 1$ , i.e. that  $a > p\mathfrak{D}_p(\langle t\alpha \rangle) - \langle t\alpha \rangle$ . In particular, we have  $\langle t\alpha \rangle < 1$  and  $a \geq 1$ . For all  $j \in \{0, \dots, K\}$ , we have  $a + jp \geq 1$  hence by Lemma 14,

$$\mathcal{Q}_{0,\cdot}(a + jp) \in g_0(a + jp)\mathcal{K}_b.$$

It follows that it is sufficient to show that

$$\frac{\rho(a, \langle t\alpha \rangle)}{\mathfrak{D}_p(\langle t\alpha \rangle) + j} g_0(a + jp) \in p\mathbb{Z}_p. \quad (6.6)$$

By Lemma 13 with  $\langle t\alpha \rangle$  instead of  $x$  and  $j$  instead  $m$ , we obtain, for all  $j \in \{0, \dots, K\}$  and all  $\ell \in \{1, \dots, 1 + v_p(\mathfrak{D}_p(\langle t\alpha \rangle) + j)\}$ , that  $\{(a + jp)/p^\ell\} \geq \mathfrak{D}_p^\ell(\langle t\alpha \rangle) = \mathfrak{D}_{p_{c,\ell}}(\langle t\alpha \rangle)$  because  $\langle t\alpha \rangle \in \mathbb{Z}_p$ . We obtain  $\{(a + jp)/p^\ell\} \geq \tau(0, \ell)$ , thus

$$v_p(g_0(a + jp)) = \sum_{\ell=1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{a + jp}{p^\ell} \right\} \right) \geq v_p(\mathfrak{D}_p(\langle t\alpha \rangle) + j) + 1,$$

and this completes the proof of (6.6) and also that of (6.5).  $\square$

6.2.2. *A combinatorial lemma.* We now use a combinatorial identity due to Dwork (see [12, Lemma 4.2, p. 308]) that enables us to write

$$\begin{aligned} \sum_{j=0}^K H_{\langle t^{(1)}\alpha \rangle}(j) \left( \mathcal{Q}_{0,t}(a + jp) \mathcal{Q}_{1,t}(K - j) - \mathcal{Q}_{1,t}(j) \mathcal{Q}_{0,t}(a + (K - j)p) \right) \\ = \sum_{s=0}^r \sum_{m=0}^{p^{r+1-s}-1} W_t(a, K, s, p, m), \end{aligned}$$

where  $r$  is such that  $K < p^r$ ,

$$W_t(a, K, s, p, m) := \left( H_{\langle t^{(1)}\alpha \rangle}(mp^s) - H_{\langle t^{(1)}\alpha \rangle}(\lfloor m/p \rfloor p^{s+1}) \right) S_t(a, K, s, p, m)$$

and

$$S_t(a, K, s, p, m) = \sum_{j=mp^s}^{(m+1)p^s-1} \left( \mathcal{Q}_{0,t}(a+jp) \mathcal{Q}_{1,t}(K-j) - \mathcal{Q}_{1,t}(j) \mathcal{Q}_{0,t}(a+(K-j)p) \right),$$

where, for all  $r \in \mathbb{N}$ , we set  $\mathcal{Q}_{r,t}(n) = 0$  if  $n < 0$ . Thus, to complete the proof, it is enough to show that for all  $s, m \in \mathbb{N}$ , we have  $(t \in \Omega_b \mapsto W_t(a, K, s, p, m)) \in p\mathcal{K}_b$ . If  $m = 0$ , this is obvious. We now assumes that  $m \geq 1$ .

We write  $m = k + qp$  with  $k \in \{0, \dots, p-1\}$  and  $q \in \mathbb{N}$ , so that  $mp^s = kp^s + qp^{s+1}$  and  $\lfloor m/p \rfloor p^{s+1} = qp^{s+1}$ . By [12, Lemma 4.1], we obtain

$$H_{\langle t^{(1)}\alpha \rangle}(mp^s) - H_{\langle t^{(1)}\alpha \rangle}(\lfloor m/p \rfloor p^{s+1}) \equiv \frac{1}{p^{s+1}} \frac{\rho(k, \mathfrak{D}_p^s(\langle t^{(1)}\alpha \rangle))}{\mathfrak{D}_p^{s+1}(\langle t^{(1)}\alpha \rangle) + q} \pmod{\frac{1}{p^s}\mathbb{Z}_p}.$$

Let us show that for all  $s, m \in \mathbb{N}$ ,  $m \geq 1$ , we have

$$g_{s+1}(m) \frac{\rho(k, \mathfrak{D}_p^s(\langle t^{(1)}\alpha \rangle))}{\mathfrak{D}_p^{s+1}(\langle t^{(1)}\alpha \rangle) + q} \in p\mathbb{Z}_p. \quad (6.7)$$

If  $\rho(k, \mathfrak{D}_p^s(\langle t^{(1)}\alpha \rangle)) = 0$ , this is clear. Let us assume that  $\rho(k, \mathfrak{D}_p^s(\langle t^{(1)}\alpha \rangle)) = 1$ . Since  $\langle t^{(1)}\alpha \rangle \in \mathbb{Z}_p$ , Eq. (6.3) yields  $\mathfrak{D}_p^s(\langle t^{(1)}\alpha \rangle) = \langle t^{(s+1)}\alpha \rangle$  and  $\mathfrak{D}_p^{s+1}(\langle t^{(1)}\alpha \rangle) = \mathfrak{D}_p(\langle t^{(s+1)}\alpha \rangle)$ .

Using Lemma 13 with  $\langle t^{(s+1)}\alpha \rangle$  for  $x$ ,  $k$  for  $a$  and  $q$  for  $m$ , we get that, for all  $\ell \in \{1, \dots, 1 + v_p(\mathfrak{D}_p(\langle t^{(s+1)}\alpha \rangle) + q)\}$ , we have  $\{m/p^\ell\} \geq \mathfrak{D}_p^\ell(\langle t^{(s+1)}\alpha \rangle) = \mathfrak{D}_{p_c, \ell}(\langle t^{(s+1)}\alpha \rangle)$  because  $\langle t^{(s+1)}\alpha \rangle \in \mathbb{Z}_p$ . We obtain  $\{m/p^\ell\} \geq \tau(s+1, \ell)$  and

$$v_p(g_{s+1}(m)) = \sum_{\ell=1}^{\infty} \mathbf{1}_{s+1, \ell} \left( \left\{ \frac{m}{p^\ell} \right\} \right) \geq v_p(\mathfrak{D}_p(\langle t^{(s+1)}\alpha \rangle) + q) + 1,$$

which finishes the proof of (6.7).

By (6.7), for all  $s, m \in \mathbb{N}$ ,  $m \geq 1$ , we have

$$(H_{\langle t^{(1)}\alpha \rangle}(mp^s) - H_{\langle t^{(1)}\alpha \rangle}(\lfloor m/p \rfloor p^{s+1})) p^{s+1} g_{s+1}(m) \in p\mathbb{Z}_p.$$

Hence, to complete the proof of Theorem 2, it is enough to show that, for all  $s, m \in \mathbb{N}$ ,  $m \geq 1$ , we have

$$(t \in \Omega_b \mapsto S_t(a, K, s, p, m)) \in p^{s+1} g_{s+1}(m) \mathcal{K}_b. \quad (6.8)$$

We do this in the next section.

**6.2.3. Application of Theorem 4.** To prove (6.8), we will use Theorem 4 with the ring  $\mathbb{Z}_p$  for  $\mathcal{Z}$  and the  $\mathbb{Z}_p$ -algebra  $\mathcal{A}$  defined as follows:

- $\mathcal{A} = \mathcal{A}_b$  if  $(\beta \in \mathbb{Z}^r$  or  $p-1 \nmid \lambda_p)$  or if  $(p=2$  and  $\mathbf{m}$  is odd);
- $\mathcal{A} = \mathcal{A}_b^*$  otherwise.

A map  $f \in \mathcal{A}_b^*$  is regular if and only if for all  $t \in \Omega_b$ , we have  $f(t) \neq 0$ . Moreover, we have  $\mathcal{A}_b \subset \mathcal{A}_b^*$ .

In particular, by Lemma 12 and Assertion (2) of Lemma 11, for all  $r, m \in \mathbb{N}$ , the map  $\mathcal{Q}_{r,\cdot}(m)$  is a regular element of  $\mathcal{A}_b$ . In the sequel, for all  $r, m \in \mathbb{N}$ , we set  $\mathbf{A}_r(m) := \mathcal{Q}_{r,\cdot}(m)$  and we define a function  $\mathbf{g}_r$  as follows:

- If  $\beta \in \mathbb{Z}^r$  and  $p \mid d_{\alpha,\beta}$ , then  $\mathbf{g}_r(0) = 1$  and  $\mathbf{g}_r(m) = \frac{1}{p} \Lambda_{b,r}(m)$  for  $m \geq 1$ ;
- If  $\beta \notin \mathbb{Z}^r$  or  $p \nmid d_{\alpha,\beta}$ , then  $\mathbf{g}_r = g_r$ .

We recall that if  $m \geq 1$  and if  $p \mid d_{\alpha,\beta}$ , then for all  $r \in \mathbb{N}$ , we have  $\Lambda_{b,r}(m) \in p\mathbb{Z}_p$ . Hence, the maps  $\mathbf{g}_r$  take their values in  $\mathbb{Z}_p$ .

We will show in the next sections that the sequences  $(\mathbf{A}_r)_{r \geq 0}$  and  $(\mathbf{g}_r)_{r \geq 0}$  satisfy Hypothesis (i), (ii) and (iii) of Theorem 4. Thus, for all  $m, s \in \mathbb{N}$ ,  $m \geq 1$ , we will obtain that

$$S.(a, K, s, p, m) \in \begin{cases} p^s \Lambda_{b,s+1}(m) \mathcal{A}_b & \text{if } \beta \in \mathbb{Z}^r \text{ and } p \mid d_{\alpha,\beta}; \\ p^{s+1} g_{s+1}(m) \mathcal{A}_b & \text{if } \beta \notin \mathbb{Z}^r \text{ and } p-1 \nmid \lambda_p; \\ p^{s+1} g_{s+1}(m) \mathcal{A}_b & \text{if } \beta \notin \mathbb{Z}^r, p=2 \text{ and } \mathbf{m} \text{ is odd}; \\ p^{s+1} g_{s+1}(m) \mathcal{A}_b^* & \text{otherwise.} \end{cases}$$

because if  $p \nmid d_{\alpha,\beta}$ , then  $\mathcal{A}_b = \mathcal{A}_b^*$ .

Proceeding this way, we will obtain (6.8). Indeed, the only non-obvious case is the one for which  $\beta \in \mathbb{Z}^r$  and  $p \mid d_{\alpha,\beta}$ . But in this case, by Lemma 14, we have

$$p^s \Lambda_{b,s+1}(m) \mathcal{A}_b \in p^{s+1} p^{-1 - \lfloor \frac{\lambda_p}{p-1} \rfloor} g_{s+1}(m) \mathcal{A}_b = p^{s+1} g_{s+1}(m) \mathcal{K}_b.$$

In the next sections, we check the various hypothesis of Theorem 4.

**6.2.4. Verification of Conditions (i) and (ii) of Theorem 4.** For all  $r \geq 0$ , the map  $\mathcal{Q}_{r,\cdot}(0)$  is constant on  $\Omega_b$  with value 1, and thus it is invertible in  $\mathcal{A}_b$ .

By Lemmas 12 and 14, for all  $m \in \mathbb{N}$ , we have  $\mathcal{Q}_{r,\cdot}(m) \in g_r(m) \mathcal{A}_b$  and  $\mathcal{Q}_{r,\cdot}(m) \in \Lambda_{b,r}(m) \mathcal{A}_b$  so that in all these cases we have  $\mathcal{Q}_{r,\cdot}(m) \in \mathbf{g}_r(m) \mathcal{A}_b$ . This shows that Conditions (i) and (ii) of Theorem 4 hold.

**6.2.5. Verification of Condition (iii) of Theorem 4.** For all  $r \in \mathbb{N}$ , we set

$$\mathcal{N}_r := \bigcup_{t \geq 1} \left( \left\{ n \in \{0, \dots, p^t - 1\} : \forall \ell \in \{1, \dots, t\}, \{n/p^\ell\} \geq \tau(r, \ell) \right\} \times \{t\} \right).$$

We apply Theorem 4 with the sequence  $\mathcal{N} := (\mathcal{N}_r)_{r \geq 0}$ . We observe that for all  $r, \ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have  $\tau(r, \ell) > 0$  and hence if  $(n, t) \in \mathcal{N}_r$ , then  $n \geq 1$ . Moreover, in the sequel, we will often use the fact that for all  $h \in \mathbb{N}$ , all  $c \in \{1, \dots, p^\nu\}$  not divisible by  $p$  and all  $t \in \Omega_b$ , we have

$$\tau(r, \ell + h) = \tau(r + h, \ell), \quad \tilde{\Delta}_{r,t}^{p,\ell+h} = \tilde{\Delta}_{r+h,t}^{p,\ell} \quad \text{and} \quad \Delta_{r,t}^{c,\ell+h} = \Delta_{r+h,t}^{c,\ell}. \quad (6.9)$$

Indeed, let  $\alpha$  be a term of the sequences  $\alpha$  or  $\beta$ . Writing  $\langle \alpha \rangle = k/N$  as an irreducible fraction, we obtain

$$\mathfrak{D}_{p_{c,\ell+h}}(\langle t^{(r)}\alpha \rangle) = \frac{\varpi_N(p_{c,\ell+h}^{-1}t^{(r)}k)}{N} = \frac{\varpi_N(p_{c,\ell}^{-1}t^{(r+h)}k)}{N} = \mathfrak{D}_{p_{c,\ell}}(\langle t^{(r+h)}\alpha \rangle),$$

so that  $\tau(r, \ell+h) = \tau(r+h, \ell)$  and  $\Delta_{r,t}^{c,\ell+h} = \Delta_{r+h,t}^{c,\ell}$ . Furthermore, if  $\alpha \in \mathbb{Z}_p$ , then by (6.3), we have

$$\mathfrak{D}_p^{\ell+h}(\langle t^{(r)}\alpha \rangle) = \mathfrak{D}_p^\ell(\mathfrak{D}_p^h(\langle t^{(r)}\alpha \rangle)) = \mathfrak{D}_p^\ell(\langle t^{(r+h)}\alpha \rangle),$$

which yields  $\tilde{\Delta}_{r,t}^{p,\ell+h} = \tilde{\Delta}_{r+h,t}^{p,\ell}$ .

6.2.6. *Verification of Condition (b) of Theorem 4.* Let  $r, m \in \mathbb{N}$  and  $(n, u) \in \mathcal{N}_r$ . We want to show that  $\mathfrak{g}_r(n+mp^u) \in p^u \mathfrak{g}_{r+u}(m)\mathbb{Z}_p$ . We need to distinguish two cases.

- If  $\beta \in \mathbb{Z}^r$  and  $p \mid d_{\alpha,\beta}$ , then

$$\begin{aligned} v_p(\Lambda_{b,r}(n+mp^u)) &= \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n+mp^u}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(n+mp^u)}{p-1} + (n+mp^u) \left\{ \frac{\lambda_p}{p-1} \right\} \\ &> \sum_{\ell=1}^u \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + \sum_{\ell=u+1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n+mp^u}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + m \left\{ \frac{\lambda_p}{p-1} \right\}, \end{aligned}$$

because  $\lambda_p \leq -1$  and  $n \geq 1$ . Since  $(n, u) \in \mathcal{N}_r$ , for all  $\ell \in \{1, \dots, u\}$ , we have  $\{n/p^\ell\} \geq \tau(r, \ell)$  and thus

$$v_p(\Lambda_{b,r}(n+mp^u)) > u + \sum_{\ell=u+1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{n+mp^u}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + m \left\{ \frac{\lambda_p}{p-1} \right\}.$$

We set  $m = \sum_{k=0}^{\infty} m_k p^k$ , where  $m_k \in \{0, \dots, p-1\}$  is 0 for all but a finite number of  $k$ 's. For all  $\ell \geq u+1$ ,

$$\left\{ \frac{n+mp^u}{p^\ell} \right\} = \frac{n+p^u \left( \sum_{k=0}^{\ell-u-1} m_k p^k \right)}{p^\ell} \geq \frac{p^u \left( \sum_{k=0}^{\ell-u-1} m_k p^k \right)}{p^\ell} = \left\{ \frac{m}{p^{\ell-u}} \right\}.$$

Moreover, since  $\langle \beta \rangle = (1, \dots, 1)$ , the map  $\tilde{\Delta}_{r,t}^{p,\ell}$  is non-decreasing on  $[0, 1)$  and we obtain that

$$v_p(\Lambda_{b,r}(n+mp^u)) > u + \sum_{\ell=u+1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{m}{p^{\ell-u}} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + m \left\{ \frac{\lambda_p}{p-1} \right\}.$$

But we have

$$\sum_{\ell=u+1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell} \left( \left\{ \frac{m}{p^{\ell-u}} \right\} \right) = \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r,t}^{p,\ell+u} \left( \left\{ \frac{m}{p^\ell} \right\} \right) = \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r+u,t}^{p,\ell} \left( \left\{ \frac{m}{p^\ell} \right\} \right),$$

which yields  $v_p(\Lambda_{b,r}(n+mp^u)) > u + v_p(\Lambda_{b,r+u}(m))$  and thus

$$v_p(\Lambda_{b,r}(n+mp^u)) \geq u + v_p(\Lambda_{b,r+u}(m)) + 1.$$

Since  $n \geq 1$ , we have  $\mathbf{g}_r(n + mp^u) = \frac{1}{p}\Lambda_{b,r}(n + mp^u)$  and we obtain

$$v_p(\mathbf{g}_r(n + mp^u)) \geq u + v_p(\Lambda_{b,r+u}(m)) \geq u + v_p(\mathbf{g}_{r+u}(m)),$$

as expected.

- If  $\beta \notin \mathbb{Z}^r$  or  $p \nmid d_{\alpha,\beta}$ , then we have to show that  $g_r(n + mp^u) \in p^u g_{r+u}(m) \mathbb{Z}_p$ . We have

$$\begin{aligned} v_p(g_r(n + mp^u)) &= \sum_{\ell=1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{n + mp^u}{p^\ell} \right\} \right) \\ &\geq \sum_{\ell=1}^u \mathbf{1}_{r,\ell} \left( \left\{ \frac{n}{p^\ell} \right\} \right) + \sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{n + mp^u}{p^\ell} \right\} \right) \\ &\geq u + \sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{n + mp^u}{p^\ell} \right\} \right), \end{aligned} \quad (6.10)$$

because  $(n, u) \in \mathcal{N}_r$ . Hence, for all  $\ell \in \{1, \dots, u\}$ , we have  $\{n/p^\ell\} \geq \tau(r, \ell)$ . Furthermore, for all  $h \in \mathbb{N}$ , we have  $\tau(r, \ell + h) = \tau(r + h, \ell)$  and consequently

$$\sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{n + mp^u}{p^\ell} \right\} \right) \geq \sum_{\ell=u+1}^{\infty} \mathbf{1}_{r,\ell} \left( \left\{ \frac{m}{p^{\ell-u}} \right\} \right) = \sum_{\ell=1}^{\infty} \mathbf{1}_{r+u,\ell} \left( \left\{ \frac{m}{p^\ell} \right\} \right) = v_p(g_{r+u}(m)).$$

Together with (6.10), we obtain  $g_r(n + mp^u) \in p^u g_{r+u}(m) \mathbb{Z}_p$ .

6.2.7. *Verification of Condition (a<sub>2</sub>) of Theorem 4.* Let  $r, s, m \in \mathbb{N}$ ,  $u \in \Psi_{\mathcal{N}}(r, s)$  and  $v \in \{0, \dots, p-1\}$  be such that  $v + up \notin \Psi_{\mathcal{N}}(r-1, s+1)$ . It is enough to show that

$$\mathbf{g}_r(v + up) \frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}_b. \quad (6.11)$$

We will first provide a few important properties concerning the set  $\Psi_{\mathcal{N}}(r, s)$ .

**Lemma 15.** *Let  $r \in \mathbb{Z}$ ,  $r \geq -1$  and  $s \in \mathbb{N}$ . Then  $\Psi_{\mathcal{N}}(r, s)$  is the set of the  $u \in \{0, \dots, p^s - 1\}$  such that  $\{u/p^s\} < \tau(r+1, s)$ . Moreover, for all  $u \in \Psi_{\mathcal{N}}(r, s)$  and all  $\ell \geq s$ , we have  $\{u/p^\ell\} < \tau(r+1, \ell)$  and, for all  $m \in \mathbb{N}$ , we have*

$$\frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in 2^{tmp^s} p^{\left\{ \frac{\lambda_p}{p-1} \right\} m(p^s-1)} \Lambda_{b,r+s+1}(m) \mathcal{A}_b.$$

By Lemma 15, to show (6.11) and thus to complete the verification of Condition (a<sub>2</sub>), it is enough to show that  $v_p(\mathbf{g}_r(v + up)) \geq s + 1$ .

We have  $v + up \notin \Psi_{\mathcal{N}}(r-1, s+1)$  hence there exist  $(n, t) \in \mathcal{N}_{r+s-t+1}$ ,  $t \leq s+1$  and  $j \in \{0, \dots, p^{s+1-t} - 1\}$  such that  $v + up = j + p^{s+1-t}n$ . Since  $u \in \Psi_{\mathcal{N}}(r, s)$ , we necessarily have  $s+1-t = 0$ , so that  $(v + up, s+1) \in \mathcal{N}_r$ , i.e., for all  $\ell \in \{1, \dots, s+1\}$ , we have  $\{(v + up)/p^\ell\} \geq \tau(r, \ell)$  and thus  $g_r(v + up) \in p^{s+1} \mathbb{Z}_p$ . Furthermore, if  $\beta \in \mathbb{Z}^r$  and  $p \mid d_{\alpha,\beta}$ , then since  $v + up \geq 1$ , we have  $\mathbf{g}_r(v + up) = \frac{1}{p} \Lambda_{b,r}(v + up) \mathbb{Z}_p$  and by Lemma 14, we obtain

$$v_p(\mathbf{g}_r(v + up)) \geq v_p(g_r(v + up)) - 1 - \left\lfloor \frac{\lambda_p}{p-1} \right\rfloor \geq s + 1,$$

because  $\lambda_p \leq -1$ . This completes the verification modulo Lemma 15.

*Proof of Lemma 15.* We first show that  $\Psi_{\mathcal{N}}(r, s)$  is the set of the  $u \in \{0, \dots, p^s - 1\}$  such that  $\{u/p^s\} < \tau(r + 1, s)$ . If  $s = 0$ , then  $\Psi_{\mathcal{N}}(r, 0) = \{0\}$  and  $\tau(r + 1, 0) > 0$  thus this is obvious. We can then assume that  $s \geq 1$ . Let  $u \in \{0, \dots, p^s - 1\}$ ,  $u = \sum_{k=0}^{s-1} u_k p^k$ , with  $u_k \in \{0, \dots, p - 1\}$ . It is sufficient to prove that the following assertions are equivalent.

- (1) We have  $\{u/p^s\} \geq \tau(r + 1, s)$ .
- (2) There exist  $(n, t) \in \mathcal{N}_{r+s-t+1}$ ,  $t \leq s$  and  $j \in \{0, \dots, p^{s-t} - 1\}$  such that  $u = j + p^{s-t}n$ .

Proof of (2)  $\Rightarrow$  (1): we have

$$\left\{ \frac{u}{p^s} \right\} = \frac{u}{p^s} = \frac{j + p^{s-t}n}{p^s} \geq \frac{n}{p^t} = \left\{ \frac{n}{p^t} \right\}.$$

Moreover, by definition of the sequence  $\mathcal{N}$ , we have  $\{n/p^t\} \geq \tau(r + s - t + 1, t) = \tau(r + 1, s)$  and hence  $\{u/p^s\} \geq \tau(r + 1, s)$ .

Proof of (1)  $\Rightarrow$  (2): for all  $s \geq 1$ , we denote by  $\mathcal{B}_s$  the assertion : “For all  $u \in \{0, \dots, p^s - 1\}$  and all  $r \in \mathbb{N}$  such that  $\{u/p^s\} \geq \tau(r + 1, s)$ , there exists  $i \in \{0, \dots, s - 1\}$  such that  $(\sum_{k=i}^{s-1} u_k p^{k-i}, s - i) \in \mathcal{N}_{r+i+1}$ .” It is enough to show by induction on  $s$  that for all  $s \geq 1$ ,  $\mathcal{B}_s$  holds.

If  $s = 1$ , then, for all  $u \in \{0, \dots, p - 1\}$  and all  $r \in \mathbb{N}$  such that  $\{u/p\} \geq \tau(r + 1, 1)$ , we have  $(u, 1) \in \mathcal{N}_{r+1}$ . Hence,  $\mathcal{B}_1$  holds.

Let  $s \geq 2$  be such that  $\mathcal{B}_1, \dots, \mathcal{B}_{s-1}$  hold, let  $u \in \{0, \dots, p^s - 1\}$  and  $r \in \mathbb{N}$  be such that  $\{u/p^s\} \geq \tau(r + 1, s)$ . We further assume that  $(u, s) \notin \mathcal{N}_{r+1}$ . Hence, there exists  $\ell \in \{1, \dots, s\}$  such that

$$a_\ell := \frac{\sum_{k=0}^{\ell-1} u_k p^k}{p^\ell} = \left\{ \frac{u}{p^\ell} \right\} < \tau(r + 1, \ell).$$

We necessarily have  $\ell \in \{1, \dots, s - 1\}$ . We write

$$\left\{ \frac{u}{p^s} \right\} = \frac{u}{p^s} = \frac{p^\ell a_\ell + p^\ell \sum_{k=\ell}^{s-1} u_k p^{k-\ell}}{p^s} = \frac{a_\ell}{p^{s-\ell}} + \frac{\sum_{k=\ell}^{s-1} u_k p^{k-\ell}}{p^{s-\ell}}.$$

Since  $\{u/p^s\} \geq \tau(r + 1, s)$ , we obtain that

$$\sum_{k=\ell}^{s-1} u_k p^{k-\ell} \geq p^{s-\ell} \tau(r + 1, s) - a_\ell > p^{s-\ell} \tau(r + 1, s) - \tau(r + 1, \ell),$$

or

$$\sum_{k=\ell}^{s-1} u_k p^{k-\ell} > p^{s-\ell} \tau(r + \ell + 1, s - \ell) - \tau(r + \ell + 1, 0).$$

Let  $\alpha$  be an element of the sequences  $\tilde{\alpha}$  or  $\tilde{\beta}$  such that  $\tau(r+\ell+1, s-\ell) = \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha \rangle)$ . Then, we have  $\tau(r+\ell+1, 0) \leq \langle t^{(r+\ell+1)}\alpha \rangle$  and thus

$$\sum_{k=\ell}^{s-1} u_k p^{k-\ell} > p^{s-\ell} \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha \rangle) - \langle t^{(r+\ell+1)}\alpha \rangle. \quad (6.12)$$

Both sides of inequality (6.12) are integers, so that

$$\sum_{k=\ell}^{s-1} u_k p^{k-\ell} \geq p^{s-\ell} \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha \rangle) - \langle t^{(r+\ell+1)}\alpha \rangle + 1 \geq p^{s-\ell} \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha \rangle).$$

It follows that

$$\frac{\sum_{k=\ell}^{s-1} u_k p^{k-\ell}}{p^{s-\ell}} \geq \mathfrak{D}_p^{s-\ell}(\langle t^{(r+\ell+1)}\alpha \rangle) = \tau(r+\ell+1, s-\ell).$$

By  $\mathcal{B}_{s-\ell}$ , there exists  $i \in \{0, \dots, s-\ell-1\}$  such that  $(\sum_{k=\ell+i}^{s-1} u_k p^{k-\ell-i}, s-\ell-i) \in \mathcal{N}_{r+\ell+i+1}$ . Hence there exists  $j \in \{\ell, \dots, s-1\}$  such that  $(\sum_{k=j}^{s-1} u_k p^{k-j}, s-j) \in \mathcal{N}_{r+j+1}$ , which proves Assertion  $\mathcal{B}_s$  and finishes the induction on  $s$ .

The equivalence of Assertions (1) and (2) is now proved and we have

$$\Psi_{\mathcal{N}}(r, s) = \{u \in \{0, \dots, p^s - 1\} : \{u/p^s\} < \tau(r+1, s)\}.$$

Let  $u \in \Psi_{\mathcal{N}}(r, s)$ . Let us prove that for all  $\ell \geq s$ , we have  $\{u/p^\ell\} < \tau(r+1, \ell)$ .

To get a contradiction, let us assume that there exists  $\ell \geq s$  such that  $\{u/p^\ell\} \geq \tau(r+1, \ell)$ . Let  $\alpha$  be an element of the sequences  $\tilde{\alpha}$  or  $\tilde{\beta}$  such that  $\tau(r+1, \ell) = \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle)$ . We obtain that

$$\begin{aligned} \left\{ \frac{u}{p^s} \right\} &= p^{\ell-s} \left\{ \frac{u}{p^\ell} \right\} \geq p^{\ell-s} \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) \\ &\geq p^{\ell-s} \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) - \mathfrak{D}_p^s(\langle t^{(r+1)}\alpha \rangle) + \mathfrak{D}_p^s(\langle t^{(r+1)}\alpha \rangle) \\ &\geq \mathfrak{D}_p^s(\langle t^{(r+1)}\alpha \rangle) \\ &\geq \tau(r+1, s), \end{aligned}$$

which is a contradiction. Hence, for all  $\ell \geq s$ , we have  $\{u/p^\ell\} < \tau(r+1, \ell)$ .

To complete the proof of Lemma 15, it remains to prove that, for all  $u \in \Psi_{\mathcal{N}}(r, s)$  and all  $m \in \mathbb{N}$ , we have

$$\frac{\mathcal{Q}_{r+1, \cdot}(u + mp^s)}{\mathcal{Q}_{r+1, \cdot}(u)} \in 2^{imp^s} p^{\left\{ \frac{\lambda_p}{p-1} \right\} m(p^s-1)} \Lambda_{b, r+s+1}(m) \mathcal{A}_b. \quad (6.13)$$

By Lemma 12, we have

$$\frac{\mathcal{Q}_{r+1, \cdot}(u + mp^s)}{\mathcal{Q}_{r+1, \cdot}(u)} \in 2^{imp^s} \frac{\Lambda_{b, r+1}(u + mp^s)}{\Lambda_{b, r+1}(u)} \mathcal{A}_b^\times,$$



with

$$\begin{aligned}
v_p \left( \frac{\Lambda_{b,r+1}(u + mp^s)}{\Lambda_{b,r+1}(u)} \right) &= \sum_{\ell=1}^{\infty} \left( \tilde{\Delta}_{r+1,t}^{p,\ell} \left( \left\{ \frac{u + mp^s}{p^\ell} \right\} \right) - \tilde{\Delta}_{r+1,t}^{p,\ell} \left( \left\{ \frac{u}{p^\ell} \right\} \right) \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + mp^s \left\{ \frac{\lambda_p}{p-1} \right\} \\
&= \sum_{\ell=s+1}^{\infty} \tilde{\Delta}_{r+1,t}^{p,\ell} \left( \left\{ \frac{u + mp^s}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + mp^s \left\{ \frac{\lambda_p}{p-1} \right\}, \quad (6.14)
\end{aligned}$$

because, for all  $\ell \in \{1, \dots, s\}$ , we have  $\{u/p^\ell\} = \{(u + mp^s)/p^\ell\}$  and, for all  $\ell \geq s+1$ , we have  $\{u/p^\ell\} < \tau(r+1, \ell)$ , thus  $\tilde{\Delta}_{r+1,t}^{p,\ell}(\{u/p^\ell\}) = 0$ . Let us show that, for all  $\ell \geq s+1$ , we have

$$\tilde{\Delta}_{r+1,t}^{p,\ell} \left( \left\{ \frac{u + mp^s}{p^\ell} \right\} \right) = \tilde{\Delta}_{r+s+1,t}^{p,\ell-s} \left( \left\{ \frac{m}{p^{\ell-s}} \right\} \right). \quad (6.15)$$

Let  $\alpha$  be an element of the sequences  $\alpha$  or  $\beta$  whose denominator is not divisible by  $p$ . To prove (6.15), it is enough to show that, for all  $\ell \geq s+1$ , we have

$$\left\{ \frac{u + mp^s}{p^\ell} \right\} \geq \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) \iff \left\{ \frac{m}{p^{\ell-s}} \right\} \geq \mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha \rangle). \quad (6.16)$$

We write  $m = \sum_{k=0}^{\infty} m_k p^k$  with  $m_k \in \{0, \dots, p-1\}$ . Then, we have

$$\left\{ \frac{u + mp^s}{p^\ell} \right\} = \frac{u + \sum_{k=0}^{\ell-s-1} m_k p^{k+s}}{p^\ell} = \frac{u}{p^\ell} + \left\{ \frac{m}{p^{\ell-s}} \right\}.$$

We observe that  $\mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha \rangle) = \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle)$ , so that

$$\left\{ \frac{m}{p^{\ell-s}} \right\} \geq \mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha \rangle) \implies \left\{ \frac{u + mp^s}{p^\ell} \right\} \geq \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle).$$

Moreover, we have

$$\begin{aligned}
\left\{ \frac{u + mp^s}{p^\ell} \right\} \geq \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) &\implies \frac{u}{p^\ell} + \left\{ \frac{m}{p^{\ell-s}} \right\} \geq \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) \\
&\implies p^{\ell-s} \left\{ \frac{m}{p^{\ell-s}} \right\} \geq p^{\ell-s} \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) - \frac{u}{p^s} \\
&\implies p^{\ell-s} \left\{ \frac{m}{p^{\ell-s}} \right\} > p^{\ell-s} \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) - \mathfrak{D}_p^s(\langle t^{(r+1)}\alpha \rangle) \\
&\implies p^{\ell-s} \left\{ \frac{m}{p^{\ell-s}} \right\} \geq p^{\ell-s} \mathfrak{D}_p^\ell(\langle t^{(r+1)}\alpha \rangle) - \mathfrak{D}_p^s(\langle t^{(r+1)}\alpha \rangle) + 1 \\
&\implies \left\{ \frac{m}{p^{\ell-s}} \right\} \geq \mathfrak{D}_p^{\ell-s}(\langle t^{(r+s+1)}\alpha \rangle).
\end{aligned}$$

Equivalence (6.16) is thus proved and we have (6.15). Using (6.15) in (6.14), we obtain

$$\begin{aligned}
v_p \left( \frac{\Lambda_{b,r+1}(u + mp^s)}{\Lambda_{b,r+1}(u)} \right) &= \sum_{\ell=s+1}^{\infty} \tilde{\Delta}_{r+s+1,t}^{p,\ell-s} \left( \left\{ \frac{m}{p^{\ell-s}} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + mp^s \left\{ \frac{\lambda_p}{p-1} \right\} \\
&= \sum_{\ell=1}^{\infty} \tilde{\Delta}_{r+s+1,t}^{p,\ell} \left( \left\{ \frac{m}{p^\ell} \right\} \right) - \lambda_p \frac{\mathfrak{s}_p(m)}{p-1} + mp^s \left\{ \frac{\lambda_p}{p-1} \right\} \\
&= v_p(\Lambda_{b,r+s+1}(m)) + m(p^s - 1) \left\{ \frac{\lambda_p}{p-1} \right\}.
\end{aligned}$$

This completes the proof of Lemma 15.  $\square$

6.2.8. *Verification of Conditions (a) and (a<sub>1</sub>) of Theorem 4.* Let us fix  $r \in \mathbb{N}$ . For all  $s \in \mathbb{N}$ , all  $v \in \{0, \dots, p-1\}$  and all  $u \in \Psi_{\mathcal{N}}(r, s)$ , we set  $\theta_{r,s}(v + up) := \mathcal{Q}_{r,\cdot}(v + up)$  if  $v + up \notin \Psi_{\mathcal{N}}(r-1, s+1)$ , and  $\theta_{r,s}(v + up) := \mathfrak{g}_r(v + up)$  otherwise.

The aim of this section is to prove the following fact: for all  $s, m \in \mathbb{N}$ , all  $v \in \{0, \dots, p-1\}$  and all  $u \in \Psi_{\mathcal{N}}(r, s)$ , we have

$$\theta_{r,s}(v + up) \left( \frac{\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v + up)} - \frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \right) \in p^{s+1} \mathfrak{g}_{r+s+1}(m) \mathcal{A}. \quad (6.17)$$

This will prove Conditions (a) and (a<sub>1</sub>) of Theorem 4. Indeed, by Lemmas 12 and 14, for all  $v \in \{0, \dots, p-1\}$  and all  $u \in \Psi_{\mathcal{N}}(r, s)$ , we have

$$\mathcal{Q}_{r,\cdot}(v + up) \in \Lambda_{b,r}(v + up) \mathcal{A} \subset \mathfrak{g}_r(v + up) \mathcal{A}.$$

Hence, Congruence (6.17) implies Condition (a) of Theorem 4. Moreover, by definition of  $\theta_{r,s}$ , when  $v + up \in \Psi_{\mathcal{N}}(r-1, s+1)$ , Congruence (6.17) implies Condition (a<sub>1</sub>) of Theorem 4.

If  $m = 0$ , then we have (6.17). In the sequel, we assume that  $m \geq 1$  and we split the proof of (6.17) into four distinct cases.

- Case 1: we assume that  $v + up \notin \Psi(r-1, s+1)$ .

We then have  $\theta_{r,s}(v + up) = \mathcal{Q}_{r,\cdot}(v + up) \in \Lambda_{b,r}(v + up) \mathcal{A}_b$ . Let us show that  $\Lambda_{b,r}(v + up) \in p^{s+1} \mathbb{Z}_p$ . We have

$$v_p(\Lambda_{b,r}(v + up)) = \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{v + up}{p^\ell} \right\} \right) + (v + up) \left\{ \frac{\lambda_p}{p-1} \right\}.$$

Since  $v + up \notin \Psi(r-1, s+1)$  and  $u \in \Psi(r, s)$ , we obtain that  $(v + up, s+1) \in \mathcal{N}_r$  and, for all  $\ell \in \{1, \dots, s+1\}$ , we have  $\{(v + up)/p^\ell\} \geq \tau(r, \ell)$ . It follows that

$$\frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{s+1} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{v + up}{p^\ell} \right\} \right) \geq s + 1$$

and  $v_p(\Lambda_{b,r}(v + up)) \geq s + 1$  because the functions  $\Delta_{r,t}^{c,\ell}$  are positive on  $[0, 1)$ .

Since  $u \in \Psi(r, s)$ , Lemma 15 yields

$$\mathcal{Q}_{r,\cdot}(v + up) \frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{s+1} \Lambda_{b,r+s+1}(m) \mathcal{A}_b \subset p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}_b.$$

Thus, to show (6.17), it is enough to show

$$\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1}) \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}_b. \quad (6.18)$$

By Lemma 12, we have

$$\begin{aligned} v_p(\Lambda_{b,r}(v + up + mp^{s+1})) &= \frac{1}{\varphi(p^\nu)} \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{v + up + mp^{s+1}}{p^\ell} \right\} \right) \\ &\quad + (v + up + mp^{s+1}) \left\{ \frac{\lambda_p}{p-1} \right\}, \end{aligned}$$

hence

$$\begin{aligned} v_p(\Lambda_{b,r}(v + up + mp^{s+1})) &\geq s + 1 + \frac{1}{\varphi(p^\nu)} \sum_{\ell=s+2}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{v + up + mp^{s+1}}{p^\ell} \right\} \right) \\ &\quad + m \left\{ \frac{\lambda_p}{p-1} \right\}. \end{aligned}$$

If  $\beta \in \mathbb{Z}^r$ , then the functions  $\Delta_{r,t}^{c,\ell}$  are non-decreasing on  $[0, 1)$  and, by (6.9), for all  $\ell \geq s + 2$ , we obtain

$$\begin{aligned} \sum_{\ell=s+2}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{v + up + mp^{s+1}}{p^\ell} \right\} \right) &\geq \sum_{\ell=s+2}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{mp^{s+1}}{p^\ell} \right\} \right) \\ &\geq \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell+s+1} \left( \left\{ \frac{m}{p^\ell} \right\} \right) \\ &\geq \sum_{\ell=1}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r+s+1,t}^{c,\ell} \left( \left\{ \frac{m}{p^\ell} \right\} \right). \end{aligned}$$

Consequently if  $\beta \in \mathbb{Z}^r$ , then

$$v_p(\Lambda_{b,r}(v + up + mp^{s+1})) \geq s + 1 + v_p(\Lambda_{b,r+s+1}(m)) \geq s + 1 + v_p(\mathbf{g}_{r+s+1}(m)),$$

as expected.

On the other hand, if  $\beta \notin \mathbb{Z}^r$ , then we observe that, for all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have

$$\begin{aligned} \left\{ \frac{m}{p^\ell} \right\} \geq \tau(r+s+1, \ell) &\implies \left\{ \frac{mp^{s+1}}{p^{\ell+s+1}} \right\} \geq \tau(r, \ell+s+1) \\ &\implies \left\{ \frac{v+up+mp^{s+1}}{p^{\ell+s+1}} \right\} \geq \tau(r, \ell+s+1) \\ &\implies \frac{1}{\varphi(p^\nu)} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell+s+1} \left( \left\{ \frac{v+up+mp^{s+1}}{p^{\ell+s+1}} \right\} \right) \geq 1, \end{aligned}$$

so that

$$\frac{1}{\varphi(p^\nu)} \sum_{\ell=s+2}^{\infty} \sum_{\substack{c=1 \\ \gcd(c,p)=1}}^{p^\nu} \Delta_{r,t}^{c,\ell} \left( \left\{ \frac{v+up+mp^{s+1}}{p^\ell} \right\} \right) \geq v_p(g_{r+s+1}(m))$$

and thus  $v_p(\Lambda_{b,r}(v+up+mp^{s+1})) \geq s+1 + v_p(\mathbf{g}_{r+s+1}(m))$ , as expected. Hence (6.18) is proved, which finishes the proof of (6.17) when  $v+up \notin \Psi(r-1, s+1)$ .

- Case 2: we assume that  $v+up \in \Psi(r-1, s+1)$  and that  $p-1 \nmid \lambda_p$ .

We have  $\theta_{r,s}(v+up) = \mathbf{g}_r(v+up)$ ,  $\mathcal{A} = \mathcal{A}_b$  and we have to show that

$$\mathbf{g}_r(v+up) \left( \frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} - \frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \right) \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}_b.$$

By Lemma 15,

$$\frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{\left\{ \frac{\lambda_p}{p-1} \right\} m(p^s-1)} \Lambda_{b,r+s+1}(m) \mathcal{A}_b$$

and

$$\frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in p^{\left\{ \frac{\lambda_p}{p-1} \right\} m(p^{s+1}-1)} \Lambda_{b,r+s+1}(m) \mathcal{A}_b.$$

Since  $p-1 \nmid \lambda_p$  and  $m \geq 1$ , we have

$$\left\{ \frac{\lambda_p}{p-1} \right\} m(p^s-1) \geq m \frac{p^s-1}{p-1} \geq s \quad \text{and} \quad \left\{ \frac{\lambda_p}{p-1} \right\} m(p^{s+1}-1) \geq s+1.$$

Thus, we obtain

$$\mathbf{g}_r(v+up) \frac{\mathcal{Q}_{r,\cdot}(v+up+mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v+up)} \in p^{s+1} \Lambda_{b,r+s+1}(m) \mathcal{A}_b \subset p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}_b,$$

because  $\mathbf{g}_r(v+up) \in \mathbb{Z}_p$ . It remains to show that

$$\mathbf{g}_r(v+up) \frac{\mathcal{Q}_{r+1,\cdot}(u+mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in p^{s+1} \mathbf{g}_{r+s+1}(m) \mathcal{A}_b. \quad (6.19)$$

By Lemma 14,

$$v_p(\Lambda_{b,r+s+1}(m)) \geq v_p(g_{r+s+1}(m)) + m \left\{ \frac{\lambda_p}{p-1} \right\}$$

and thus, since  $p - 1 \nmid \lambda_p$  and  $m \geq 1$ , we obtain that  $\Lambda_{b,r+s+1}(m) \in pg_{r+s+1}(m)\mathbb{Z}_p$ . Hence, we have  $\Lambda_{b,r+s+1}(m) \in p\mathbf{g}_{r+s+1}(m)\mathbb{Z}_p$ , as well as (6.19) because  $\mathbf{g}_r(v + up) \in \mathbb{Z}_p$ .

- Case 3: we assume that  $v + up \in \Psi(r - 1, s + 1)$ ,  $\beta \notin \mathbb{Z}^r$ ,  $p = 2$  and that  $\mathbf{m}$  is odd.

We have  $\theta_{r,s}(v + up) = \mathbf{g}_r(v + up) = g_r(v + up)$ ,  $\mathcal{A} = \mathcal{A}_b$  and we have to show

$$g_r(v + up) \left( \frac{\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v + up)} - \frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \right) \in p^{s+1}g_{r+s+1}(m)\mathcal{A}_b. \quad (6.20)$$

By Lemma 15, we have

$$\frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r+1,\cdot}(u)} \in 2^{mp^s} \Lambda_{b,r+s+1}(m)\mathcal{A}_b$$

and

$$\frac{\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v + up)} \in 2^{mp^{s+1}} \Lambda_{b,r+s+1}(m)\mathcal{A}_b.$$

Moreover, we have  $m2^s \geq s + 1$  and  $m2^{s+1} \geq s + 1$  because  $m \geq 1$ . Since  $\Lambda_{b,r+s+1}(m) \in g_{r+s+1}(m)\mathbb{Z}_p$  and  $g_r(v + up) \in \mathbb{Z}_p$ , we get (6.20).

- Case 4: we assume that  $v + up \in \Psi(r - 1, s + 1)$ ,  $p - 1$  divides  $\lambda_p$  and that, if  $p = 2$  and  $\beta \notin \mathbb{Z}^r$ , then  $\mathbf{m}$  is even.

We set

$$X_{r,s}(v, u, m) := \frac{\mathcal{Q}_{r,\cdot}(v + up)}{\mathcal{Q}_{r+1,\cdot}(u)} \frac{\mathcal{Q}_{r+1,\cdot}(u + mp^s)}{\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1})}.$$

Assertion (6.17) is satisfied if and only if for all  $s, m \in \mathbb{N}$ , all  $v \in \{0, \dots, p - 1\}$  and all  $u \in \Psi_{\mathcal{N}}(r, s)$ , we have

$$\mathbf{g}_r(v + up)(X_{r,s}(v, u, m) - 1) \frac{\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v + up)} \in p^{s+1}\mathbf{g}_{r+s+1}(m)\mathcal{A}. \quad (6.21)$$

The following lemma will give the conclusion.

**Lemma 16.** *We assume that  $p - 1$  divides  $\lambda_p$  and that, if  $p = 2$  and  $\beta \notin \mathbb{Z}^r$ , then  $\mathbf{m}$  is even. Then,*

- (1) *For all  $r, s \in \mathbb{N}$ , all  $v \in \{0, \dots, p - 1\}$ , all  $u \in \Psi_{\mathcal{N}}(r, s)$  and all  $m \in \mathbb{N}$ , there exists  $Y_{r,s}(v, u, m) \in \mathbb{Z}_p$  independent of  $t \in \Omega_b$  such that*

$$X_{r,s}(v, u, m) \in \begin{cases} Y_{r,s}(v, u, m)(1 + p^s \mathcal{A}_b) & \text{if } \beta \in \mathbb{Z}^r \text{ and } p \mid d_{\alpha, \beta}; \\ Y_{r,s}(v, u, m)(1 + p^{s+1} \mathcal{A}_b^*) & \text{otherwise;} \end{cases}.$$

- (2) *If there exists  $j \in \{1, \dots, s + 1\}$  such that  $\{(v + up)/p^j\} < \tau(r, j)$ , then we have  $Y_{r,s}(v, u, m) \in 1 + p^{s-j+2}\mathbb{Z}_p$ .*

Since  $v + up \in \Psi(r - 1, s + 1)$ , Lemma 15 implies that  $\{(v + up)/p^{s+1}\} < \tau(r, s + 1)$ . Let  $j_0$  be the smallest  $j \in \{1, \dots, s + 1\}$  such that  $\{(v + up)/p^j\} < \tau(r, j)$ . By Lemma 16 applied with  $j_0$ , we obtain that  $Y_{r,s}(v, u, m) \in 1 + p^{s-j_0+2}\mathbb{Z}_p$  and that

$$X_{r,s}(v, u, m) \in \begin{cases} 1 + p^{s-j_0+1}\mathcal{A}_b & \text{if } \beta \in \mathbb{Z}^r \text{ et } p \mid d_{\alpha,\beta}; \\ 1 + p^{s-j_0+2}\mathcal{A}_b^* & \text{otherwise.} \end{cases}$$

Hence, Lemma 15 yields

$$(X_{r,s}(v, u, m) - 1) \frac{\mathcal{Q}_{r,\cdot}(v + up + mp^{s+1})}{\mathcal{Q}_{r,\cdot}(v + up)} \in p^{s-j_0+2} \mathbf{g}_{r+s+1}(m) \times \begin{cases} \mathcal{A}_b & \text{if } \beta \in \mathbb{Z}^r \text{ and } p \mid d_{\alpha,\beta}; \\ \mathcal{A}_b^* & \text{otherwise.} \end{cases}$$

Therefore to prove (6.21), it is enough to show that  $\mathbf{g}_r(v + up) \in p^{j_0-1}\mathbb{Z}_p$ . If  $v + up = 0$ , then we have  $j_0 = 1$  and the conclusion is clear. We can thus assume that  $v + up \geq 1$ . But for all  $j \in \{1, \dots, j_0 - 1\}$ , we have  $\{(v + up)/p^j\} \geq \tau(r, j)$ , hence  $v_p(g_r(v + up)) \geq j_0 - 1$ . Furthermore, if  $\beta \in \mathbb{Z}^r$  and if  $p \mid d_{\alpha,\beta}$ , we have  $\lambda_p \leq -1$  and, by Lemma 14, we have

$$\mathbf{g}_r(v + up) = \frac{\Lambda_{b,r}(v + up)}{p} \in g_r(v + up)\mathbb{Z}_p \subset p^{j_0-1}\mathbb{Z}_p,$$

as expected.

To complete the proof of (6.21) and that of Theorem 2, it remains to prove Lemma 16.

*Proof of Lemma 16.* We will show that Lemma 16 holds with

$$Y_{r,s}(v, u, m) := \frac{\prod_{\beta_i \in \mathbb{Z}_p} \left(1 + \frac{mp^s}{\langle t^{(r+1)}\beta_i \rangle + u}\right)^{\rho(v, \langle t^{(r)}\beta_i \rangle)}}{\prod_{\alpha_i \in \mathbb{Z}_p} \left(1 + \frac{mp^s}{\langle t^{(r+1)}\alpha_i \rangle + u}\right)^{\rho(v, \langle t^{(r)}\alpha_i \rangle)}}.$$

By Lemma 1 of [13, Dwork], if  $\alpha$  is an element of the sequences  $\alpha$  or  $\beta$  whose denominator is not divisible by  $p$ , then for all  $v \in \{0, \dots, p - 1\}$ , all  $s, m \in \mathbb{N}$  and all  $u \in \{0, \dots, p^s - 1\}$ , we have

$$\frac{(\alpha)_{v+up+mp^{s+1}}(\mathfrak{D}_p(\alpha))_u}{(\mathfrak{D}_p(\alpha))_{u+mp^s}(\alpha)_{v+up}} \in ((-p)^{p^s} \varepsilon_{p^s})^m \left(1 + \frac{mp^s}{\mathfrak{D}_p(\alpha) + u}\right)^{\rho(v, \alpha)} (1 + p^{s+1}\mathbb{Z}_p), \quad (6.22)$$

where  $\varepsilon_k = -1$  if  $k = 2$ , and  $\varepsilon_k = 1$  otherwise.

Similarly, using Dwork's method, we will show that if  $\alpha$  is an element of the sequences  $\alpha$  or  $\beta$  whose denominator is divisible by  $p$ , then for all  $v \in \{0, \dots, p - 1\}$ , all  $r, s, m \in \mathbb{N}$  and all  $u \in \{0, \dots, p^s - 1\}$ , we have

$$\left(t \in \Omega_b \mapsto d(\alpha)^{m\varphi(p^{s+1})} \frac{\langle t^{(r)}\alpha \rangle_{v+up+mp^{s+1}} \langle t^{(r+1)}\alpha \rangle_u}{\langle t^{(r+1)}\alpha \rangle_{u+mp^s} \langle t^{(r)}\alpha \rangle_{v+up}}\right) \in \varepsilon'_{p^s}(\alpha)^m (1 + p^{s+1}\mathcal{A}_b^*), \quad (6.23)$$

where  $\varepsilon'_k(\alpha) = \varepsilon_k$  if  $v_p(d(\alpha)) = 1$  and  $\varepsilon'_k(\alpha) = 1$  otherwise.

We first show that (6.22) and (6.23) imply the validity of Assertion (1) of Lemma 16. Indeed, by (6.22), we obtain

$$\frac{\Lambda_{b,r}(v + up + mp^{s+1})\Lambda_{b,r+1}(u)}{\Lambda_{b,r+1}(u + mp^s)\Lambda_{b,r}(v + up)} \in \left( C \frac{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)} \right)^{m\varphi(p^{s+1})} \frac{((-p)^{p^s} \varepsilon_{p^s})^{m\lambda_p}}{Y_{r,s}(v, u, m)} (1 + p^{s+1}\mathbb{Z}_p). \quad (6.24)$$

We write

$$C \frac{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)} = \sigma p^{-\lfloor \frac{\lambda_p}{p-1} \rfloor} = \sigma p^{-\frac{\lambda_p}{p-1}},$$

with  $\sigma \in \mathbb{Z}_p^\times$ , so that

$$\left( C \frac{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)} \right)^{m\varphi(p^{s+1})} \in p^{-mp^s \lambda_p} (1 + p^{s+1}\mathbb{Z}_p).$$

We thus have

$$\left( C \frac{\prod_{\beta_i \notin \mathbb{Z}_p} d(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} d(\alpha_i)} \right)^{m\varphi(p^{s+1})} ((-p)^{p^s} \varepsilon_{p^s})^{m\lambda_p} \in (-1)^{mp^s \lambda_p} \varepsilon_{p^s}^{m\lambda_p} (1 + p^{s+1}\mathbb{Z}_p) \subset \varepsilon_{p^s}^{m\lambda_p} (1 + p^{s+1}\mathbb{Z}_p), \quad (6.25)$$

because  $-1 \in \mathbb{Z}_p^\times$  and  $\varphi(p^{s+1}) = p^s(p-1)$  divides  $mp^s \lambda_p$ . Using (6.25) in (6.24), we obtain that

$$\frac{\Lambda_{b,r+1}(u)\Lambda_{b,r}(v + up + mp^{s+1})}{\Lambda_{b,r}(v + up)\Lambda_{b,r+1}(u + mp^s)} \in \frac{\varepsilon_{p^s}^{m\lambda_p}}{Y_{r,s}(v, u, m)} (1 + p^{s+1}\mathbb{Z}_p). \quad (6.26)$$

By (6.23), we also obtain

$$\frac{\mathfrak{R}_{r+1}(u + mp^s, \cdot)\mathfrak{R}_r(v + up, \cdot)}{\mathfrak{R}_r(v + up + mp^{s+1}, \cdot)\mathfrak{R}_{r+1}(u, \cdot)} \in \left( \frac{\prod_{\beta_i \notin \mathbb{Z}_p} \varepsilon'_{p^s}(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} \varepsilon'_{p^s}(\alpha_i)} \right)^m (1 + p^{s+1}\mathcal{A}_b^*).$$

If  $p^s \neq 2$ , then, for any element  $\alpha \notin \mathbb{Z}_p$  of  $\boldsymbol{\alpha}$  or  $\boldsymbol{\beta}$ , we have  $\varepsilon'_{p^s}(\alpha) = \varepsilon_{p^s} = 1$ . If  $p^s = 2$  and if the number of elements  $\alpha$  of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  that satisfy  $v_2(d(\alpha)) \geq 2$  is even, then since  $r = s$ , we have

$$\frac{\prod_{\beta_i \notin \mathbb{Z}_p} \varepsilon'_{p^s}(\beta_i)}{\prod_{\alpha_i \notin \mathbb{Z}_p} \varepsilon'_{p^s}(\alpha_i)} = (-1)^{\lambda_2} = \varepsilon_2^{\lambda_2}.$$

Moreover, we have  $p\mathcal{A}_b^* \subset \mathcal{A}_b$  and  $\varepsilon_{p^s}, \varepsilon'_{p^s}(\alpha) \in 1 + p^s\mathbb{Z}_p$ . It follows that we obtain

$$\frac{\mathfrak{R}_{r+1}(u + mp^s, \cdot)\mathfrak{R}_r(v + up, \cdot)}{\mathfrak{R}_r(v + up + mp^{s+1}, \cdot)\mathfrak{R}_{r+1}(u, \cdot)} \in \begin{cases} 1 + p^s\mathcal{A}_b & \text{if } \boldsymbol{\beta} \in \mathbb{Z}^r \text{ and } p \mid d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}; \\ \varepsilon_{p^s}^{m\lambda_p} (1 + p^{s+1}\mathcal{A}_b^*) & \text{otherwise.} \end{cases} \quad (6.27)$$

By (6.26) and (6.27), we obtain

$$X_{r,s}(v, u, m) \in Y_{r,s}(v, u, m) \times \begin{cases} (1 + p^s \mathcal{A}_b) & \text{if } \beta \in \mathbb{Z}^r \text{ and } p \mid d_{\alpha, \beta}; \\ (1 + p^{s+1} \mathcal{A}_b^*) & \text{otherwise.} \end{cases}$$

To finish the proof of Assertion (1) of Lemma 16, we have to prove (6.23).

Let  $\alpha$  be an element of  $\alpha$  or  $\beta$  whose denominator is divisible by  $p$ . For all  $s, m \in \mathbb{N}$  and all  $u \in \{0, \dots, p^s - 1\}$ , we set

$$\mathfrak{q}_r(u, s, m) := t \in \Omega_b \mapsto d(\alpha)^{mp^s} \frac{(\langle t^{(r)} \alpha \rangle)_{u+mp^s}}{(\langle t^{(r)} \alpha \rangle)_u} = \prod_{k=0}^{mp^s-1} (d(\alpha) \langle t^{(r)} \alpha \rangle + d(\alpha)u + d(\alpha)k).$$

Hence, proving (6.23) amounts to proving that

$$\frac{\mathfrak{q}_r(v + up, s + 1, m)}{\mathfrak{q}_{r+1}(u, s, m)} \in \varepsilon'_{p^s}(\alpha)^m (1 + p^{s+1} \mathcal{A}_b^*).$$

As functions of  $t$ , we have

$$\begin{aligned} \mathfrak{q}_r(u, s, m)(t) &= \prod_{i=0}^{p^s-1} \prod_{j=0}^{m-1} (d(\alpha) \langle t^{(r)} \alpha \rangle + d(\alpha)u + d(\alpha)i + d(\alpha)jp^s) \\ &\equiv \prod_{i=0}^{p^s-1} (d(\alpha) \langle t^{(r)} \alpha \rangle + d(\alpha)u + d(\alpha)i)^m \pmod{p^{s+1} \mathcal{A}_b} \\ &\equiv \prod_{i=0}^{p^s-1} (d(\alpha) \langle t^{(r)} \alpha \rangle + d(\alpha)i)^m \pmod{p^{s+1} \mathcal{A}_b}. \end{aligned}$$

Since  $d(\alpha)$  is divisible by  $p$ , we obtain that, for all  $i \in \{0, \dots, p^s - 1\}$ , the map  $t \in \Omega_b \mapsto d(\alpha) \langle t^{(r)} \alpha \rangle + d(\alpha)i$  is invertible in  $\mathcal{A}_b$  and thus

$$\mathfrak{q}_r(u, s, m) \in \mathfrak{q}_r(0, s, 1)^m (1 + p^{s+1} \mathcal{A}_b).$$

Hence proving (6.23) amounts to proving that, for all  $s \in \mathbb{N}$ , we have

$$\frac{\mathfrak{q}_r(0, s + 1, 1)}{\mathfrak{q}_{r+1}(0, s, 1)} \in \varepsilon'_{p^s}(\alpha) (1 + p^{s+1} \mathcal{A}_b^*). \quad (6.28)$$

- Case 1: we assume that  $s = 0$ .

As functions of  $t$ , we have

$$\frac{\mathfrak{q}_r(0, 1, 1)(t)}{\mathfrak{q}_{r+1}(0, 0, 1)(t)} \in \frac{(d(\alpha) \langle t^{(r)} \alpha \rangle)^p}{d(\alpha) \langle t^{(r+1)} \alpha \rangle} (1 + p \mathcal{A}_b)$$

and

$$t^{(r)} \equiv \varpi_{p^\nu} \left( \frac{t}{D} \right) D + \varpi_D \left( \frac{b}{p^{\nu+r}} \right) p^\nu \pmod{p^\nu D}.$$



Hence with  $\langle \alpha \rangle := \kappa/d(\alpha)$ , we obtain the existence of  $\eta(r, t) \in \mathbb{Z}$  such that

$$d(\alpha)\langle t^{(r)}\alpha \rangle = \varpi_{p^\nu} \left( \frac{t\kappa}{D} \right) D + \varpi_D \left( \frac{b\kappa}{p^{\nu+r}} \right) p^\nu + d(\alpha)\eta(r, t).$$

Moreover by Assertions (2), (4) and (5) of Lemma 11, the maps  $t \in \Omega_b \mapsto d(\alpha)\langle t^{(r)}\alpha \rangle$  and  $f : t \in \Omega_b \mapsto \varpi_{p^\nu}(t\kappa/D)D$  are in  $\mathcal{A}_b^*$ . Thus  $t \in \Omega_b \mapsto d(\alpha)\eta(r, t)$  is in  $\mathcal{A}_b$  and  $t \in \Omega_b \mapsto d(\alpha)\eta(r, t)/p$  is in  $\mathcal{A}_b^*$  because  $p$  divides  $d(\alpha)$ . It follows that

$$(t \in \Omega_b \mapsto d(\alpha)\langle t^{(r)}\alpha \rangle) \in f(1 + p\mathcal{A}_b^*). \quad (6.29)$$

We obtain

$$\frac{\mathfrak{q}_r(0, 1, 1)}{\mathfrak{q}_{r+1}(0, 0, 1)} \in f^{p-1}(1 + p\mathcal{A}_b^*) \subset (1 + p(\mathfrak{E}_1 \circ f))(1 + p\mathcal{A}_b^*) \subset 1 + p\mathcal{A}_b^*,$$

as expected, where the final inclusion is obtained *via* Assertion (3) of Lemma 10.

- Case 2: we assume that  $s \geq 1$ .

If  $s \geq 1$ , then

$$\prod_{i=0}^{p^s-1} (d(\alpha)\langle t^{(r)}\alpha \rangle + d(\alpha)i) = \prod_{j=0}^{p^{s-1}-1} \prod_{a=0}^{p-1} (d(\alpha)\langle t^{(r)}\alpha \rangle + d(\alpha)j + d(\alpha)ap^{s-1}) \quad (6.30)$$

$$\equiv \prod_{j=0}^{p^{s-1}-1} (d(\alpha)\langle t^{(r)}\alpha \rangle + d(\alpha)j)^p \pmod{p^s \mathcal{A}_b}. \quad (6.31)$$

Using (6.31) with  $s+1$  for  $s$ , we obtain

$$\mathfrak{q}_r(0, s+1, 1) \in \mathfrak{q}_r(0, s, 1)^p (1 + p^{s+1} \mathcal{A}_b)$$

and thus

$$\mathfrak{q}_r(0, s+1, 1) \in (d(\alpha)\langle t^{(r)}\alpha \rangle)^{p^{s+1}} (1 + p^{s+1} \mathcal{A}_b). \quad (6.32)$$

We set  $P(x) := x^p - x \in \mathbb{Z}_p[x]$ . For all  $a \in \{0, \dots, p-1\}$ , we have  $a^p - a \equiv 0 \pmod{p\mathbb{Z}_p}$ . Since  $P'(x) = px^{p-1} - 1$ , for all  $a \in \{0, \dots, p-1\}$ , we have  $v_p(P'(a)) = 0$  and, by Hensel's lemma (see [30]), there exists a root  $w_a$  of  $P$  in  $\mathbb{Z}_p$  such that  $w_a \equiv a \pmod{p\mathbb{Z}_p}$ . Consequently, for all  $x \in \mathbb{Z}_p$  and all  $s \in \mathbb{N}$ ,  $s \geq 1$ , we have

$$\begin{aligned} \prod_{a=0}^{p-1} (x + d(\alpha)ap^{s-1}) &\equiv \prod_{i=0}^{p-1} (x - d(\alpha)w_i p^{s-1}) \pmod{p^{s+1}\mathbb{Z}_p} \\ &\equiv x^p - (d(\alpha)p^{s-1})^{p-1} x \pmod{p^{s+1}\mathbb{Z}_p}. \end{aligned} \quad (6.33)$$

If  $p \neq 2$ , then  $(d(\alpha)p^{s-1})^{p-1} x \in p^{s+1}\mathbb{Z}_p$  thus, by (6.30), for all  $s \in \mathbb{N}$ ,  $s \geq 1$ , we obtain

$$\mathfrak{q}_{r+1}(0, s, 1) \in \prod_{j=0}^{p^{s-1}-1} (d(\alpha)\langle t^{(r+1)}\alpha \rangle + d(\alpha)j)^p (1 + p^{s+1} \mathcal{A}_b),$$

hence  $\mathfrak{q}_{r+1}(0, s, 1) \in \mathfrak{q}_{r+1}(0, s-1, 1)^p(1 + p^{s+1}\mathcal{A}_b)$  and

$$\mathfrak{q}_{r+1}(0, s, 1) \in (d(\alpha)\langle t^{(r+1)}\alpha \rangle)^{p^s}(1 + p^{s+1}\mathcal{A}_b).$$

By (6.32) and (6.29), we obtain the existence of  $f_1, f_2 \in \mathcal{A}_b^*$  such that

$$\begin{aligned} \frac{\mathfrak{q}_r(0, s+1, 1)}{\mathfrak{q}_{r+1}(0, s, 1)} &\in f^{\varphi(p^{s+1})} \frac{(1 + pf_1)^{p^{s+1}}}{(1 + pf_2)^{p^s}} (1 + p^{s+1}\mathcal{A}_b) \\ &\subset (1 + p^{s+1}(\mathfrak{E}_{s+1} \circ f))(1 + p^{s+1}\mathcal{A}_b^*) \subset 1 + p^{s+1}\mathcal{A}_b^*, \end{aligned}$$

which proves (6.28) when  $p \neq 2$  because in this case we have  $\varepsilon'_{p^s}(\alpha) = 1$ .

Let us now assume  $p = 2$ . Then by (6.30) and (6.33), for all  $s \in \mathbb{N}$ ,  $s \geq 1$ , we obtain

$$\mathfrak{q}_{r+1}(0, s, 1) \in \prod_{j=0}^{2^{s-1}-1} (d(\alpha)\langle t^{(r+1)}\alpha \rangle + d(\alpha)j)^2 \left(1 - \frac{d(\alpha)2^{s-1}}{d(\alpha)\langle t^{(r+1)}\alpha \rangle + d(\alpha)j}\right) (1 + 2^{s+1}\mathcal{A}_b).$$

Since 2 divides  $d(\alpha)$ , we have

$$\begin{aligned} \prod_{j=0}^{2^{s-1}-1} \left(1 - \frac{d(\alpha)2^{s-1}}{d(\alpha)\langle t^{(r+1)}\alpha \rangle + d(\alpha)j}\right) &= \prod_{j=0}^{2^{s-1}-1} \left(1 - \frac{d(\alpha)2^{s-1}}{1 + 2\mathfrak{E}_1(d(\alpha)\langle t^{(r+1)}\alpha \rangle + d(\alpha)j)}\right) \\ &\equiv \prod_{j=0}^{2^{s-1}-1} (1 - d(\alpha)2^{s-1}) \pmod{2^{s+1}\mathcal{A}_b^*} \\ &\equiv 1 - d(\alpha)2^{2s-2} \pmod{2^{s+1}\mathcal{A}_b^*}, \end{aligned}$$

with  $1 - d(\alpha)2^{2s-2} \equiv 1 \pmod{2^{s+1}}$  if  $s \geq 2$  or  $v_2(d(\alpha)) \geq 2$ , and  $1 - d(\alpha)2^{2s-2} \equiv -1 \pmod{4}$  if  $s = v_2(d(\alpha)) = 1$ . It follows that

$$\mathfrak{q}_{r+1}(0, s, 1) \in \varepsilon'_{2^s}(\alpha) \prod_{j=0}^{2^{s-1}-1} (d(\alpha)\langle t^{(r+1)}\alpha \rangle + d(\alpha)j)^2 (1 + 2^{s+1}\mathcal{A}_b^*),$$

*i.e.*  $\mathfrak{q}_{r+1}(0, s, 1) \in \varepsilon'_{2^s}(\alpha)\mathfrak{q}_{r+1}(0, s-1, 1)^2(1 + 2^{s+1}\mathcal{A}_b^*)$  and thus

$$\mathfrak{q}_{r+1}(0, s, 1) \in \varepsilon'_{2^s}(\alpha)(d(\alpha)\langle t^{(r+1)}\alpha \rangle)^{2^s}(1 + 2^{s+1}\mathcal{A}_b^*).$$

By (6.32) and (6.29), we obtain the existence of  $f_1, f_2 \in \mathcal{A}_b^*$  such that

$$\begin{aligned} \frac{\mathfrak{q}_r(0, s+1, 1)}{\mathfrak{q}_{r+1}(0, s, 1)} &\in \frac{1}{\varepsilon'_{2^s}(\alpha)} f^{\varphi(2^{s+1})} \frac{(1 + 2f_1)^{2^{s+1}}}{(1 + 2f_2)^{2^s}} (1 + 2^{s+1}\mathcal{A}_b^*) \\ &\subset \varepsilon'_{2^s}(\alpha)(1 + 2^{s+1}(\mathfrak{E}_{s+1} \circ f))(1 + 2^{s+1}\mathcal{A}_b^*) \subset \varepsilon'_{2^s}(\alpha)(1 + 2^{s+1}\mathcal{A}_b^*), \end{aligned}$$

which proves (6.28) and completes the proof of (1) of Lemma 16.

Let us now prove Assertion (2) of Lemma 16. We have

$$Y_{r,s}(v, u, m) = \frac{\prod_{\beta_i \in \mathbb{Z}_p} \left(1 + \frac{mp^s}{\langle t^{(r+1)} \beta_i \rangle + u}\right)^{\rho(v, \langle t^{(r)} \beta_i \rangle)}}{\prod_{\alpha_i \in \mathbb{Z}_p} \left(1 + \frac{mp^s}{\langle t^{(r+1)} \alpha_i \rangle + u}\right)^{\rho(v, \langle t^{(r)} \alpha_i \rangle)}}.$$

Let  $j \in \{1, \dots, s+1\}$  be such that  $\{(v+up)/p^j\} < \tau(r, j)$ . We set  $u = \sum_{k=0}^{\infty} u_k p^k$ . For all elements  $\alpha \in \mathbb{Z}_p$  of the sequences  $\boldsymbol{\alpha}$  or  $\boldsymbol{\beta}$ , we have

$$\begin{aligned} \left\{ \frac{v+up}{p^j} \right\} < \tau(r, j) &\implies v + p \sum_{k=0}^{j-2} u_k p^k < p^j \mathfrak{D}_p^j(\langle t^{(r)} \alpha \rangle) \\ &\implies v + p \sum_{k=0}^{j-2} u_k p^k \leq p^j \mathfrak{D}_p^j(\langle t^{(r)} \alpha \rangle) - \langle t^{(r)} \alpha \rangle \\ &\implies v + p \sum_{k=0}^{j-2} u_k p^k \leq \sum_{k=0}^{j-1} p^k (p \mathfrak{D}_p^{k+1}(\langle t^{(r)} \alpha \rangle) - \mathfrak{D}_p^k(\langle t^{(r)} \alpha \rangle)) \\ &\implies \left( \rho(v, \langle t^{(r)} \alpha \rangle) = 0 \quad \text{or} \quad \sum_{k=0}^{j-2} u_k p^k < p^{j-1} \mathfrak{D}_p^j(\langle t^{(r)} \alpha \rangle) - \mathfrak{D}_p(\langle t^{(r)} \alpha \rangle) \right) \\ &\implies \left( \rho(v, \langle t^{(r)} \alpha \rangle) = 0 \quad \text{or} \quad \sum_{k=0}^{j-2} u_k p^k < p^{j-1} \mathfrak{D}_p^{j-1}(\langle t^{(r+1)} \alpha \rangle) - \langle t^{(r+1)} \alpha \rangle \right) \\ &\implies (\rho(v, \langle t^{(r)} \alpha \rangle) = 0 \quad \text{or} \quad v_p(u + \langle t^{(r+1)} \alpha \rangle) \leq j-2) \\ &\implies \left(1 + \frac{mp^s}{\langle t^{(r+1)} \alpha \rangle + u}\right)^{\rho(v, \langle t^{(r)} \alpha \rangle)} \in 1 + p^{s-j+2} \mathbb{Z}_p, \end{aligned}$$

as expected. This completes the proof of Lemma 16 and that of Theorem 2.  $\square$

## 7. PROOF OF ASSERTION (1) OF THEOREM 3

We shall prove the more precise following statement.

**Proposition 6.** *Let  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\langle \boldsymbol{\alpha} \rangle$  and  $\langle \boldsymbol{\beta} \rangle$  are disjoint. Let  $a \in \{1, \dots, d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\}$  coprime to  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  be such that, for all  $x \in \mathbb{R}$ , we have  $\xi_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(a, x) \geq 0$ . Then, all the Taylor coefficients at the origin of  $q_{(\boldsymbol{\alpha}), (a\boldsymbol{\beta})}(z)$  are positive, but its constant term which is 0.*

To prove Proposition 6, we follow the method used by Delaygue in [11, section 10.3], itself inspired by the work of Krattenthaler-Rivoal in [21]. We state three lemmas which enable us to prove Proposition 6.

**Lemma 17** (Lemma 2.1 in [21]). *Let  $a(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{R}[[z]]$ ,  $a_0 = 1$ , be such that all Taylor coefficients at the origin of  $\mathbf{a}(z) = 1 - 1/a(z)$  are nonnegative. Let*

$b(z) = \sum_{n=0}^{\infty} a_n h_n z^n$  where  $(h_n)_{n \geq 0}$  is a nondecreasing sequence of nonnegative real numbers. Then, all Taylor coefficients at the origin of  $b(z)/a(z)$  are non-negative.

Furthermore, if all Taylor coefficients of  $a(z)$  and  $\mathbf{a}(z)$  are positive (excepted the constant term of  $\mathbf{a}(z)$ ) and if  $(h_n)_{n \geq 0}$  is an increasing sequence, then all Taylor coefficients at the origin of  $b(z)/a(z)$  are positive, except its constant term if  $h_0 = 0$ .

The following lemma is a refined version of Kaluza's Theorem [15, Satz 3]. Initially, Satz 3 did not cover the case  $a_{n+1}a_{n-1} > a_n^2$ .

**Lemma 18** (Lemma 2.2 in [21]). *Let  $a(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{R}[[z]]$ ,  $a_0 = 1$ , be such that  $a_1 > 0$  and  $a_{n+1}a_{n-1} \geq a_n^2$  for all positive integers  $n$ . Then, all Taylor coefficients of  $\mathbf{a}(z) = 1 - 1/a(z)$  are nonnegative.*

*Furthermore, if we have  $a_{n+1}a_{n-1} > a_n^2$  for all positive integers  $n$ , then all Taylor coefficients of  $\mathbf{a}(z)$  are positive (except its constant term).*

By Lemmas 17 and 18, to prove Proposition 6, it suffices to prove the following result.

**Lemma 19.** *Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$  and  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\langle \boldsymbol{\alpha} \rangle$  and  $\langle \boldsymbol{\beta} \rangle$  are disjoint. Let  $a \in \{1, \dots, d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\}$  coprime to  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  be such that, for all  $x \in \mathbb{R}$ , we have  $\xi_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(a, x) \geq 0$ . Then, for all positive integers  $n$ , we have*

$$\mathcal{Q}_{\langle a\boldsymbol{\alpha} \rangle, \langle a\boldsymbol{\beta} \rangle}(n+1)\mathcal{Q}_{\langle a\boldsymbol{\alpha} \rangle, \langle a\boldsymbol{\beta} \rangle}(n-1) > \mathcal{Q}_{\langle a\boldsymbol{\alpha} \rangle, \langle a\boldsymbol{\beta} \rangle}(n)^2.$$

*Furthermore,  $(\sum_{i=1}^r H_{\langle a\alpha_i \rangle}(n) - \sum_{j=1}^s H_{\langle a\beta_j \rangle}(n))_{n \geq 0}$  is an increasing sequence.*

To prove Lemma 19, we first prove the following lemma that we also use in the proof of Assertion (2) of Theorem 3.

**Lemma 20.** *Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$  and  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\langle \boldsymbol{\alpha} \rangle$  and  $\langle \boldsymbol{\beta} \rangle$  are disjoint. Let  $a \in \{1, \dots, d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\}$  be coprime to  $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ . Let  $\gamma_1, \dots, \gamma_t$  be rational numbers such that  $\langle a\gamma_1 \rangle < \dots < \langle a\gamma_t \rangle$  and such that  $\{\langle a\gamma_1 \rangle, \dots, \langle a\gamma_t \rangle\}$  is the set of the numbers  $\langle a\gamma \rangle$  when  $\gamma$  describes all the elements of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$ . For all  $i \in \{1, \dots, t\}$ , we define  $m_i := \#\{1 \leq j \leq r : \langle a\alpha_j \rangle = \langle a\gamma_i \rangle\} - \#\{1 \leq j \leq s : \langle a\beta_j \rangle = \langle a\gamma_i \rangle\}$ .*

*Assume that, for all  $x \in \mathbb{R}$ , we have  $\xi_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(a, x) \geq 0$ . Then, for all  $i \in \{1, \dots, t\}$  and all  $b \in \mathbb{R}$ ,  $b \geq 0$ , we have*

$$\sum_{k=1}^i \frac{m_k}{\langle a\gamma_k \rangle + b} > 0 \quad \text{and} \quad \prod_{k=1}^i \left(1 + \frac{1}{\langle a\gamma_k \rangle + b}\right)^{m_k} > 1.$$

*Proof of Lemma 20.* First, observe that by Proposition 2, for all  $j \in \{1, \dots, t\}$ , we have

$$\sum_{i=1}^j m_i = \xi_{\langle a\boldsymbol{\alpha} \rangle, \langle a\boldsymbol{\beta} \rangle}(1, \langle a\gamma_j \rangle) \geq 0.$$

Furthermore, since  $\langle a\boldsymbol{\alpha} \rangle$  and  $\langle a\boldsymbol{\beta} \rangle$  are disjoint, for all  $i \in \{1, \dots, t\}$ , we have  $m_i \neq 0$ . In particular, we obtain that  $m_1 \geq 1$ . It follows that we have

$$\frac{m_1}{\langle a\gamma_1 \rangle + b} > 0 \quad \text{and} \quad \left(1 + \frac{1}{\langle a\gamma_1 \rangle + b}\right)^{m_1} > 1.$$

Now assume that  $t \geq 2$ . We shall prove by induction on  $i$  that, for all  $i \in \{2, \dots, t\}$ , we have

$$\sum_{k=1}^i \frac{m_k}{\langle a\gamma_k \rangle + b} > \frac{\sum_{k=1}^i m_k}{\langle a\gamma_i \rangle + b} \quad \text{and} \quad \prod_{k=1}^i \left(1 + \frac{1}{\langle a\gamma_k \rangle + b}\right)^{m_k} > \left(1 + \frac{1}{\langle a\gamma_i \rangle + b}\right)^{\sum_{k=1}^i m_k}. \quad (7.1)$$

We have  $\langle a\gamma_1 \rangle < \langle a\gamma_2 \rangle$  and  $m_1 > 0$  thus we get

$$\frac{m_1}{\langle a\gamma_1 \rangle + b} + \frac{m_2}{\langle a\gamma_2 \rangle + b} > \frac{m_1 + m_2}{\langle a\gamma_2 \rangle + b}$$

and

$$\left(1 + \frac{1}{\langle a\gamma_1 \rangle + b}\right)^{m_1} \left(1 + \frac{1}{\langle a\gamma_2 \rangle + b}\right)^{m_2} > \left(1 + \frac{1}{\langle a\gamma_2 \rangle + b}\right)^{m_1 + m_2},$$

so that (7.1) holds for  $i = 2$ . We now assume that  $t \geq 3$  and let  $i \in \{2, \dots, t-1\}$  be such that (7.1) holds. We obtain that

$$\sum_{k=1}^{i+1} \frac{m_k}{\langle a\gamma_k \rangle + b} > \frac{\sum_{k=1}^i m_k}{\langle a\gamma_i \rangle + b} + \frac{m_{i+1}}{\langle a\gamma_{i+1} \rangle + b} \quad (7.2)$$

and

$$\prod_{k=1}^{i+1} \left(1 + \frac{1}{\langle a\gamma_k \rangle + b}\right)^{m_k} > \left(1 + \frac{1}{\langle a\gamma_i \rangle + b}\right)^{\sum_{k=1}^i m_k} \left(1 + \frac{1}{\langle a\gamma_{i+1} \rangle + b}\right)^{m_{i+1}}. \quad (7.3)$$

Since  $\langle a\gamma_i \rangle < \langle a\gamma_{i+1} \rangle$  and  $\sum_{k=1}^i m_k \geq 0$ , we obtain that

$$\frac{\sum_{k=1}^i m_k}{\langle a\gamma_i \rangle + b} \geq \frac{\sum_{k=1}^i m_k}{\langle a\gamma_{i+1} \rangle + b} \quad \text{and} \quad \left(1 + \frac{1}{\langle a\gamma_i \rangle + b}\right)^{\sum_{k=1}^i m_k} \geq \left(1 + \frac{1}{\langle a\gamma_{i+1} \rangle + b}\right)^{\sum_{k=1}^i m_k},$$

which, together with (7.2) and (7.3), finishes the induction on  $i$ . By (7.1) together with  $\sum_{k=1}^t m_k \geq 0$ , this completes the proof of Lemma 20.  $\square$

We can now prove Lemma 19 and hence complete the proof of Proposition 6 and of Assertion (1) of Theorem 3.

*Proof of Lemma 19.* Throughout this proof, we use the notations defined in Lemma 20. For all nonnegative integers  $n$ , we have

$$\begin{aligned} \frac{\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(n+1)}{\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(1)\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(n)} &= \frac{1}{\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(1)} \cdot \frac{\prod_{i=1}^r (\langle a\alpha_i \rangle + n)}{\prod_{j=1}^s (\langle a\beta_j \rangle + n)} \\ &= \frac{\prod_{i=1}^r (1 + n/\langle a\alpha_i \rangle)}{\prod_{j=1}^s (1 + n/\langle a\beta_j \rangle)} \\ &= \prod_{k=1}^t \left(1 + \frac{n}{\langle a\gamma_k \rangle}\right)^{m_k}. \end{aligned}$$

We deduce that for all positive integers  $n$ , we obtain

$$\begin{aligned} \frac{\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(n+1)\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(n-1)}{\mathcal{Q}_{\langle a\alpha \rangle, \langle a\beta \rangle}(n)^2} &= \prod_{k=1}^t \left( \frac{1+n/\langle a\gamma_k \rangle}{1+(n-1)/\langle a\gamma_k \rangle} \right)^{m_k} \\ &= \prod_{k=1}^t \left( 1 + \frac{1}{\langle a\gamma_k \rangle + n - 1} \right)^{m_k} > 1, \end{aligned}$$

where the last inequality is obtained by Lemma 20 with  $n-1$  instead of  $b$ .

Furthermore, for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \sum_{i=1}^r H_{\langle a\alpha_i \rangle}(n+1) - \sum_{j=1}^s H_{\langle a\beta_j \rangle}(n+1) - \left( \sum_{i=1}^r H_{\langle a\alpha_i \rangle}(n) - \sum_{j=1}^s H_{\langle a\beta_j \rangle}(n) \right) \\ = \sum_{i=1}^r \frac{1}{\langle a\alpha_i \rangle + n} - \sum_{j=1}^s \frac{1}{\langle a\beta_j \rangle + n} \\ = \sum_{k=1}^t \frac{m_k}{\langle a\gamma_k \rangle + n} > 0, \end{aligned}$$

where the last inequality is obtained by Lemma 20 with  $n$  instead of  $b$ . It follows that  $(\sum_{i=1}^r H_{\alpha_i}(n) - \sum_{j=1}^s H_{\beta_j}(n))_{n \geq 0}$  is an increasing sequence and Lemma 19 is proved.  $\square$

## 8. PROOF OF ASSERTION (3) OF THEOREM 3

Throughout this section, we fix two tuples  $\alpha$  and  $\beta$  of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  with same length such that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint. Furthermore, we assume that  $H_{\alpha, \beta}$  holds, that is, for all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and all  $x \in \mathbb{R}$  satisfying  $m_{\alpha, \beta}(a) \preceq x \prec a$ , we have  $\xi_{\alpha, \beta}(a, x) \geq 1$ . We will also use the notations defined at the beginning of Section 6.2.

**8.1. A  $p$ -adic reformulation of Assertion (3) of Theorem 3.** To prove Assertion (3) of Theorem 3, we have to prove that

$$\exp \left( \frac{S_{\alpha, \beta}(C'_{\alpha, \beta} z)}{\mathfrak{n}_{\alpha, \beta}} \right) \in \mathbb{Z}[[z]]. \quad (8.1)$$

A classical method to prove the integrality of the Taylor coefficients of exponential of a power series is to reduce the problem to a  $p$ -adic one for all primes  $p$  and to use Dieudonné-Dwork's lemma as follows. Assertion (8.1) holds if and only if, for all primes  $p$ , we have

$$\exp \left( \frac{S_{\alpha, \beta}(C'_{\alpha, \beta} z)}{\mathfrak{n}_{\alpha, \beta}} \right) \in \mathbb{Z}_p[[z]]. \quad (8.2)$$

Let us recall that we have

$$S_{\alpha, \beta}(z) = \sum_{\substack{a=1 \\ \gcd(a, d)=1}}^d \frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)} \in z\mathbb{Q}[[z]],$$

with  $d = d_{\alpha,\beta}$ . By Corollary 1 applied to (8.2), we obtain that (8.1) holds if and only if, for all primes  $p$ , we have

$$S_{\alpha,\beta}(C'_{\alpha,\beta}z^p) - pS_{\alpha,\beta}(C'_{\alpha,\beta}z) \in p\mathfrak{n}_{\alpha,\beta}\mathbb{Z}_p[[z]]. \quad (8.3)$$

The map  $t \mapsto t^{(1)}$  is a permutation of the elements of  $\{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ . Hence, we have

$$S_{\alpha,\beta}(C'z^p) - pS_{\alpha,\beta}(C'z) = \sum_{\substack{t=1 \\ \gcd(t,d)=1}}^d \left( \frac{G_{\langle t^{(1)}\alpha \rangle, \langle t^{(1)}\beta \rangle}}{F_{\langle t^{(1)}\alpha \rangle, \langle t^{(1)}\beta \rangle}}(C'z^p) - p \frac{G_{\langle t\alpha \rangle, \langle t\beta \rangle}}{F_{\langle t\alpha \rangle, \langle t\beta \rangle}}(C'z) \right),$$

with  $d = d_{\alpha,\beta}$  and  $C' = C'_{\alpha,\beta}$ . By Theorem 2, we obtain

$$\begin{aligned} S_{\alpha,\beta}(C'_{\alpha,\beta}z^p) - pS_{\alpha,\beta}(C'_{\alpha,\beta}z) &= p \sum_{\substack{b=1 \\ \gcd(b,D)=1}}^D \sum_{t \in \Omega_b} \sum_{k=0}^{\infty} R_{k,b}(t) z^k \\ &= p \sum_{\substack{b=1 \\ \gcd(b,D)=1}}^D \sum_{k=0}^{\infty} \left( \sum_{t \in \Omega_b} R_{k,b}(t) \right) z^k, \end{aligned}$$

with  $R_{k,b} \in \mathcal{A}_b^*$  and, moreover if  $p$  divides  $d_{\alpha,\beta}$ , then we have

$$R_{k,b} \in \begin{cases} p^{-1 - \lfloor \lambda_p / (p-1) \rfloor} \mathcal{A}_b & \text{if } \beta \in \mathbb{Z}^r; \\ \mathcal{A}_b & \text{if } \beta \notin \mathbb{Z}^r \text{ and } p-1 \nmid \lambda_p; \\ \mathcal{A}_b & \text{if } \beta \notin \mathbb{Z}^r, \mathfrak{m}_{\alpha,\beta} \text{ is odd and } p=2. \end{cases}.$$

By point (7) of Lemma 11, we have

$$\sum_{t \in \Omega_b} R_{k,b}(t) \in \mathfrak{n}_{\alpha,\beta}\mathbb{Z}_p. \quad (8.4)$$

Indeed, if  $p$  does not divide  $d_{\alpha,\beta}$ , then  $p$  does not divide  $\mathfrak{n}_{\alpha,\beta}$  and  $R_{k,b}(t) \in \mathbb{Z}_p$ . Let us now assume that  $p$  divides  $d_{\alpha,\beta}$  so that  $\nu \geq 1$ .

If  $\beta \in \mathbb{Z}^r$ , then we have  $v_p(\mathfrak{n}_{\alpha,\beta}) = \nu - 2 - \lfloor \lambda_p / (p-1) \rfloor$ . If  $\beta \notin \mathbb{Z}^r$  and if  $p-1 \nmid \lambda_p$ , then we have  $p \neq 2$  and  $v_p(\mathfrak{n}_{\alpha,\beta}) = \nu - 1$ . Let us now assume that  $\beta \notin \mathbb{Z}^r$  and that  $p-1 \mid \lambda_p$ . If  $p \neq 2$  then  $v_p(\mathfrak{n}_{\alpha,\beta}) = 0$  or  $\nu - 2$ . On the other hand, if  $p = 2$ , then either  $\mathfrak{m}_{\alpha,\beta}$  is even and  $v_2(\mathfrak{n}_{\alpha,\beta}) = 0$  or  $\nu - 2$ , or  $\mathfrak{m}_{\alpha,\beta}$  is odd and  $v_2(\mathfrak{n}_{\alpha,\beta}) = \nu - 1$ .

It follows that in all cases, we have (8.3) and Assertion (3) of Theorem 3 is proved.

## 9. PROOF OF ASSERTION (2) OF THEOREM 3

Let  $\alpha$  and  $\beta$  be tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  such that  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint (this is equivalent to the irreducibility of  $\mathcal{L}_{\alpha,\beta}$ ) and such that  $F_{\alpha,\beta}$  is  $N$ -integral. Assertion (3) of Theorem 3 implies Assertion (iii)  $\Rightarrow$  (i) of Theorem 3. Indeed, it suffices to prove the following result.

**Proposition 7.** *Let  $f(z) \in 1 + z\mathbb{Q}[[z]]$  be an  $N$ -integral power series and let  $a$  be a positive integer. Then  $f(z)^{1/a}$  is an  $N$ -integral power series.*

*Proof.* We write  $f(z) = 1 + zg(z)$  with  $g(z) \in \mathbb{Q}[[z]]$ . Thus, we obtain that

$$f(z)^{1/a} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-1/a)_n}{n!} z^n g(z)^n.$$

Since  $f(z)$  is  $N$ -integral, there exists  $C \in \mathbb{N}$  such that  $g(Cz) \in \mathbb{Z}[[z]]$ . Furthermore, by Theorem A applied with  $\alpha = (-1/a)$  and  $\beta = (1)$ , we obtain that there exists  $K \in \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ , we have

$$K^n \frac{(-1/a)_n}{n!} \in \mathbb{Z}.$$

It follows that  $f(CKz)^{1/a} \in \mathbb{Z}[[z]]$ , *i.e.*  $f(z)^{1/a}$  is  $N$ -integral.  $\square$

Furthermore, by definition, we have  $(ii) \Rightarrow (i)$  of Theorem 3. Thus, we only have to prove that  $(i) \Rightarrow (iii)$ ,  $(i) \Rightarrow (ii)$  and that if  $(i)$  holds, then we have either  $\alpha = (1/2)$  and  $\beta = (1)$  or there are at least two elements equal to 1 in  $\langle \beta \rangle$ . Throughout this section, we assume that  $(i)$  holds, *i.e.* that  $q_{\alpha,\beta}$  is  $N$ -integral. Furthermore, for all  $n \in \mathbb{N}$ , we set

$$\mathcal{Q}_{\alpha,\beta}(n) := \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}.$$

**9.1. Proof of Assertion  $(iii)$  of Theorem 3.** The aim of this section is to prove that  $r = s$ , that  $H_{\alpha,\beta}$  holds and that, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have  $q_{\alpha,\beta}(z) = q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$ . Since  $F_{\alpha,\beta}$  and  $q_{\alpha,\beta}$  are  $N$ -integral, there exists  $C \in \mathbb{Q} \setminus \{0\}$  such that

$$F_{\alpha,\beta}(Cz) \in \mathbb{Z}[[z]] \quad \text{and} \quad q_{\alpha,\beta}(Cz) = \exp\left(\frac{G_{\alpha,\beta}(Cz)}{F_{\alpha,\beta}(Cz)}\right) \in \mathbb{Z}[[z]].$$

Thus, for almost all primes  $p$ , we have

$$F_{\alpha,\beta}(z) \in \mathbb{Z}_p[[z]] \quad \text{and} \quad \exp\left(\frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)}\right) \in \mathbb{Z}_p[[z]]. \quad (9.1)$$

We shall use Dieudonné-Dwork's lemma in order to get rid of the exponential map in (9.1).

Let  $p$  be a prime such that (9.1) holds. By Corollary 1 applied to (9.1), we obtain that

$$\frac{G_{\alpha,\beta}(z^p)}{F_{\alpha,\beta}(z^p)} - p \frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)} \in pz\mathbb{Z}_p[[z]].$$

Since  $F_{\alpha,\beta}(z) \in \mathbb{Z}_p[[z]]$ , we get

$$G_{\alpha,\beta}(z^p)F_{\alpha,\beta}(z) - pG_{\alpha,\beta}(z)F_{\alpha,\beta}(z^p) \in pz\mathbb{Z}_p[[z]]. \quad (9.2)$$

In the sequel of the proof of Assertion (2) of Theorem 3, we use several times that (9.2) holds for almost all primes  $p$ .



9.1.1. *Proof of  $r = s$ .* We give a proof by contradiction assuming that  $r \neq s$ . Since  $F_{\alpha,\beta}$  is  $N$ -integral, Christol's criterion ensures that, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$  and all  $x \in \mathbb{R}$ , we have  $\xi_{\alpha,\beta}(a, x) \geq 0$ . In particular, since  $r - s$  is the limit of  $\xi_{\alpha,\beta}(1, n)$  when  $n \in \mathbb{Z}$  tends to  $-\infty$ , we obtain that  $r - s \geq 1$ . For all  $n \in \mathbb{N}$ , we write  $A_n$  for the assertion

$$\sum_{i=1}^r H_{\alpha_i}(n) - \sum_{j=1}^s H_{\beta_j}(n) = 0.$$

First, we prove by induction on  $n$  that  $A_n$  is true for all  $n \in \mathbb{N}$ .

Assertion  $A_0$  holds. Let  $n$  be a positive integer such that, for all integer  $k$ ,  $0 \leq k < n$ ,  $A_k$  holds. The coefficient  $\Phi_p(np)$  of  $z^{np}$  in (9.2) belongs to  $p\mathbb{Z}_p$  and is equal to

$$\sum_{j=0}^n \mathcal{Q}_{\alpha,\beta}(jp) \mathcal{Q}_{\alpha,\beta}(n-j) \left( \sum_{i=1}^r (H_{\alpha_i}(n-j) - pH_{\alpha_i}(jp)) - \sum_{i=1}^s (H_{\beta_i}(n-j) - pH_{\beta_i}(jp)) \right).$$

By induction, we obtain that

$$\begin{aligned} \Phi_p(np) &= \mathcal{Q}_{\alpha,\beta}(n) \left( \sum_{i=1}^r H_{\alpha_i}(n) - \sum_{i=1}^s H_{\beta_i}(n) \right) \\ &\quad - p \sum_{j=1}^n \mathcal{Q}_{\alpha,\beta}(jp) \mathcal{Q}_{\alpha,\beta}(n-j) \left( \sum_{i=1}^r H_{\alpha_i}(jp) - \sum_{i=1}^s H_{\beta_i}(jp) \right). \end{aligned}$$

Furthermore, according to Lemma 4, there exists a constant  $M_{\alpha,\beta} > 0$  such that, for all  $x \in [0, 1/M_{\alpha,\beta}[$ , all primes  $p$  not dividing  $d_{\alpha,\beta}$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ , we have  $\Delta_{\alpha,\beta}^{p,\ell}(x) = 0$ . Hence, for almost all primes  $p$  and all  $j \in \{1, \dots, n\}$ , we have

$$v_p(\mathcal{Q}_{\alpha,\beta}(jp)) = \sum_{\ell=1}^{\infty} \Delta_{\alpha,\beta}^{p,\ell} \left( \frac{jp}{p^\ell} \right) = \Delta_{\alpha,\beta}^{p,1}(j) + \sum_{\ell=1}^{\infty} \Delta_{\alpha,\beta}^{p,\ell+1} \left( \frac{j}{p^\ell} \right) = \Delta_{\alpha,\beta}^{p,1}(j) = j(r-s). \quad (9.3)$$

According to Lemma 3, for almost all primes  $p$  and all the elements  $\alpha$  in  $\alpha$  or  $\beta$ , we have  $\mathfrak{D}_p(\alpha) = \mathfrak{D}_p(\langle \alpha \rangle)$ , so that  $\mathfrak{D}_p(\alpha) = \langle \omega \alpha \rangle$  where  $\omega \in \{1, \dots, d_{\alpha,\beta}\}$  satisfies  $\omega p \equiv 1 \pmod{d_{\alpha,\beta}}$ . Thus we get

$$\begin{aligned} pH_{\alpha}(jp) &= p \sum_{k=0}^{p-1} \sum_{i=0}^{j-1} \frac{1}{\alpha + k + ip} \\ &= H_{\mathfrak{D}_p(\alpha)}(j) + p \sum_{\substack{k=0 \\ k \neq p\mathfrak{D}_p(\alpha) - \alpha}}^{p-1} \sum_{i=0}^{j-1} \frac{1}{\alpha + k + ip} \in H_{\langle \omega \alpha \rangle}(j) + p\mathbb{Z}_p, \end{aligned}$$

which leads to

$$p \left( \sum_{i=1}^r H_{\alpha_i}(jp) - \sum_{i=1}^s H_{\beta_i}(jp) \right) \equiv \sum_{i=1}^r H_{\langle \omega \alpha_i \rangle}(j) - \sum_{i=1}^s H_{\langle \omega \beta_i \rangle}(j) \pmod{p\mathbb{Z}_p}. \quad (9.4)$$

Furthermore, for almost all primes  $p$ , we have

$$\left\{ \sum_{i=1}^r H_{\langle \omega \alpha_i \rangle}(j) - \sum_{i=1}^s H_{\langle \omega \beta_i \rangle}(j) : 1 \leq j \leq n, 1 \leq \omega \leq d_{\alpha, \beta}, \gcd(\omega, d_{\alpha, \beta}) = 1 \right\} \subset \mathbb{Z}_p,$$

which, together with (9.3) and (9.4), gives us that

$$-p \mathcal{Q}_{\alpha, \beta}(jp) \mathcal{Q}_{\alpha, \beta}(n-j) \left( \sum_{i=1}^r H_{\alpha_i}(jp) - \sum_{i=1}^s H_{\beta_i}(jp) \right) \in p^{r-s} \mathbb{Z}_p,$$

for almost all primes  $p$  and all  $j \in \{1, \dots, n\}$ . In addition, for almost all primes  $p$ , we have

$$\mathcal{Q}_{\alpha, \beta}(n) \left( \sum_{i=1}^r H_{\alpha_i}(n) - \sum_{j=1}^s H_{\beta_j}(n) \right) \in \mathbb{Z}_p^\times \cup \{0\} \quad \text{and} \quad \mathcal{Q}_{\alpha, \beta}(n) \neq 0.$$

Since  $\Phi_p(np) \in p\mathbb{Z}_p$  and  $r - s \geq 1$ , we obtain that  $A_n$  holds, which finishes the induction on  $n$ .

It follows that for all  $n \in \mathbb{N}$ , we obtain that

$$\sum_{i=1}^r \frac{1}{\alpha_i + n} - \sum_{i=1}^s \frac{1}{\beta_i + n} = \sum_{i=1}^r (H_{\alpha_i}(n+1) - H_{\alpha_i}(n)) - \sum_{i=1}^s (H_{\beta_i}(n+1) - H_{\beta_i}(n)) = 0,$$

contradicting that  $\alpha$  and  $\beta$  are disjoint since

$$\sum_{i=1}^r \frac{1}{\alpha_i + X} - \sum_{i=1}^s \frac{1}{\beta_i + X} \in \mathbb{Q}(X)$$

must be a non-trivial rational fraction in this case. Thus we have  $r = s$  as expected.  $\square$

9.1.2. *Proof of  $H_{\alpha, \beta}$ .* Let us recall that, since  $F_{\alpha, \beta}$  is  $N$ -integral, for all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and all  $x \in \mathbb{R}$ , we have  $\xi_{\alpha, \beta}(a, x) \geq 0$ . We give a proof by contradiction of  $H_{\alpha, \beta}$  assuming that there exist  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$  and  $x_0 \in \mathbb{R}$  such that  $m_{\alpha, \beta}(a) \preceq x_0 \prec a$  and  $\xi_{\alpha, \beta}(a, x_0) = 0$ . Let  $\alpha$  and  $\beta$  be such that

$$a\beta = \max(\{a\gamma : a\gamma \preceq x_0, \gamma \text{ is in } \alpha \text{ or } \beta\}, \preceq)$$

and

$$a\alpha = \min(\{a\gamma : x_0 \prec a\gamma, \gamma \text{ equals } 1 \text{ or is in } \alpha \text{ or } \beta\}, \preceq).$$

It follows that for all  $x \in \mathbb{R}$  satisfying  $a\beta \preceq x \prec a\alpha$ , we have  $\xi_{\alpha, \beta}(a, x) = 0$ . Observe that, since  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint,  $\langle a\alpha \rangle$  and  $\langle a\beta \rangle$  are also disjoint, thus  $\beta$  is a component of  $\beta$  and  $\alpha$  equals 1 or is an element of  $\alpha$  because  $\xi_{\alpha, \beta}(a, \cdot)$  is nonnegative on  $\mathbb{R}$ .

Let us write  $\mathfrak{P}_{\alpha, \beta}(a)$  for the set of all primes  $p$  such that  $ap \equiv 1 \pmod{d_{\alpha, \beta}}$ . For all large enough  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ , Lemma 3 gives us that  $\mathfrak{D}_p(\alpha) = \mathfrak{D}_p(\langle \alpha \rangle) = \langle a\alpha \rangle$  and  $\mathfrak{D}_p(\beta) = \langle a\beta \rangle$ . On the one hand, if  $\langle a\beta \rangle < \langle a\alpha \rangle$ , then, for almost all  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ , we obtain that

$$\mathfrak{D}_p(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p} - \mathfrak{D}_p(\beta) - \frac{\lfloor 1 - \beta \rfloor}{p} \geq \frac{1}{d_{\alpha, \beta}} + \frac{\lfloor 1 - \alpha \rfloor}{p} - \frac{\lfloor 1 - \beta \rfloor}{p} \geq \frac{1}{p}.$$

On the other hand, if  $\langle a\beta \rangle = \langle a\alpha \rangle$  and  $\beta > \alpha$ , then we have  $\langle \beta \rangle = \langle \alpha \rangle$  so  $\beta \geq 1 + \alpha$  and

$$\mathfrak{D}_p(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p} - \mathfrak{D}_p(\beta) - \frac{\lfloor 1 - \beta \rfloor}{p} = \frac{\lfloor 1 - \alpha \rfloor}{p} - \frac{\lfloor 1 - \beta \rfloor}{p} \geq \frac{1}{p}.$$

In both cases, we obtain that, for almost all  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ , there exists  $v_p \in \{0, \dots, p-1\}$  such that

$$\mathfrak{D}_p(\beta) + \frac{\lfloor 1 - \beta \rfloor}{p} \leq \frac{v_p}{p} < \mathfrak{D}_p(\alpha) + \frac{\lfloor 1 - \alpha \rfloor}{p},$$

which, together with Lemma 5, gives us that  $\Delta_{\alpha, \beta}^{p,1}(v_p/p) = 0$  for all large enough  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ . Furthermore, by Lemma 4, for almost all  $p \in \mathfrak{P}_{\alpha, \beta}(a)$  and all  $\ell \in \mathbb{N}$ ,  $\ell \geq 1$ ,  $\Delta_{\alpha, \beta}^{p, \ell}$  vanishes on  $[0, 1/p]$  so that

$$v_p(\mathcal{Q}_{\alpha, \beta}(v_p)) = \sum_{\ell=1}^{\infty} \Delta_{\alpha, \beta}^{p, \ell} \left( \frac{v_p}{p^\ell} \right) = \Delta_{\alpha, \beta}^{p,1} \left( \frac{v_p}{p} \right) = 0,$$

*i.e.*  $\mathcal{Q}_{\alpha, \beta}(v_p) \in \mathbb{Z}_p^\times$ . Now looking at the coefficient of  $z^{v_p}$  in (9.2), one obtains that

$$-p\mathcal{Q}_{\alpha, \beta}(v_p) \sum_{i=1}^r (H_{\alpha_i}(v_p) - H_{\beta_i}(v_p)) \in p\mathbb{Z}_p.$$

To get a contradiction, we shall prove that, for all large enough  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ , we have

$$p \left( \sum_{i=1}^r H_{\alpha_i}(v_p) - \sum_{i=1}^r H_{\beta_i}(v_p) \right) \in \mathbb{Z}_p^\times. \quad (9.5)$$

Indeed, for all elements  $\gamma$  of  $\alpha$  or  $\beta$  and all large enough  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ , we have

$$\begin{aligned} pH_\gamma(v_p) &= p \sum_{k=0}^{v_p-1} \frac{1}{\gamma+k} \equiv \frac{\rho(v_p, \gamma)}{\mathfrak{D}_p(\gamma)} \pmod{p\mathbb{Z}_p} \\ &\equiv \frac{\rho(v_p, \gamma)}{\langle a\gamma \rangle} \pmod{p\mathbb{Z}_p}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \rho(v_p, \gamma) = 1 &\iff v_p \geq p\mathfrak{D}_p(\gamma) - \gamma + 1 \iff v_p \geq p\mathfrak{D}_p(\gamma) + \lfloor 1 - \gamma \rfloor \\ &\iff \frac{v_p}{p} \geq \mathfrak{D}_p(\gamma) + \frac{\lfloor 1 - \gamma \rfloor}{p}, \end{aligned}$$

because  $p\mathfrak{D}_p(\gamma) - \gamma \in \mathbb{Z}$  which leads to  $v_p \geq p\mathfrak{D}_p(\gamma) + \lfloor 1 - \gamma \rfloor \Rightarrow v_p \geq p\mathfrak{D}_p + \lfloor 1 - \gamma \rfloor + \{1 - \gamma\}$ . Thus, by Lemma 5, for all large enough  $p \in \mathfrak{P}_{\alpha, \beta}(a)$ , we have  $\rho(v_p, \gamma) = 1$  if  $a\gamma \preceq a\beta$  and  $\rho(v_p, \gamma) = 0$  otherwise.

Now, let  $\gamma_1, \dots, \gamma_t$  be rational numbers such that  $\langle a\gamma_1 \rangle < \dots < \langle a\gamma_t \rangle$  and such that  $\{\langle a\gamma_1 \rangle, \dots, \langle a\gamma_t \rangle\}$  is the set of the numbers  $\langle a\gamma \rangle$  when  $\gamma$  describes all the elements of  $\alpha$  and  $\beta$  satisfying  $a\gamma \preceq a\beta$ . For all  $i \in \{1, \dots, t\}$ , we define

$$m_i := \#\{1 \leq j \leq r : \langle a\alpha_j \rangle = \langle a\gamma_i \rangle\} - \#\{1 \leq j \leq r : \langle a\beta_j \rangle = \langle a\gamma_i \rangle\}.$$

Then, we obtain that

$$\begin{aligned} p \left( \sum_{i=1}^r H_{\alpha_i}(v_p) - \sum_{j=1}^r H_{\beta_j}(v_p) \right) &\equiv \sum_{i=1}^r \frac{\rho(v_p, \alpha_i)}{\langle a\alpha_i \rangle} - \sum_{j=1}^r \frac{\rho(v_p, \beta_j)}{\langle a\beta_j \rangle} \pmod{p\mathbb{Z}_p} \\ &\equiv \sum_{i=1}^t \frac{m_i}{\langle a\gamma_i \rangle} \pmod{p\mathbb{Z}_p}. \end{aligned}$$

For almost all primes  $p$ , we have  $\sum_{i=0}^t (m_i / \langle a\gamma_i \rangle) \in \mathbb{Z}_p^\times \cup \{0\}$ , thus to prove (9.5), it suffices to prove that

$$\sum_{i=1}^t \frac{m_i}{\langle a\gamma_i \rangle} \neq 0,$$

which follows by Lemma 20 applied with  $b = 0$ . This finishes the proof of  $H_{\alpha, \beta}$ .  $\square$

9.1.3. *Last step in the proof of Assertion (iii) of Theorem 3.* To finish the proof of Assertion (iii) of Theorem 3, it remains to prove that, for all  $a \in \{1, \dots, d_{\alpha, \beta}\}$  coprime to  $d_{\alpha, \beta}$ , we have  $q_{\alpha, \beta}(z) = q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$ . For that purpose, we shall use Dwork's results presented in [12] on the integrality of Taylor coefficients at the origin of power series similar to  $q_{\alpha, \beta}$ . We remind the reader that, by Sections 9.1.1 and 9.1.2, we have  $r = s$  and  $H_{\alpha, \beta}$  holds.

More precisely, we prove the following lemma which shows that, under these assumptions, we can apply Dwork's result [12, Theorem 4.1] for almost all primes.

**Lemma 21.** *Let  $\alpha$  and  $\beta$  be two tuples of parameters in  $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$  with the same numbers of elements. If  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint (this is equivalent to the irreducibility of  $\mathcal{L}_{\alpha, \beta}$ ) and if  $H_{\alpha, \beta}$  holds, then for almost all primes  $p$  not dividing  $d_{\alpha, \beta}$ , we have*

$$\frac{G_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}(z^p)}{F_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}(z^p)} - p \frac{G_{\alpha, \beta}(z)}{F_{\alpha, \beta}(z)} \in p\mathbb{Z}_p[[z]].$$

*Remark.* Lemma 21 in combination with Lemma 2 gives us that  $S_{\alpha, \beta}(z) \in p\mathbb{Z}_p[[z]]$  for almost all primes  $p$ .

*Proof.* If  $p$  is a prime not dividing  $d_{\alpha, \beta}$ , then the elements of  $\alpha$  and  $\beta$  lie in  $\mathbb{Z}_p$  and

$$\frac{G_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}(z^p)}{F_{\mathfrak{D}_p(\alpha), \mathfrak{D}_p(\beta)}(z^p)} - p \frac{G_{\alpha, \beta}(z)}{F_{\alpha, \beta}(z)} \in \mathbb{Q}_p[[z]].$$

Furthermore,  $\alpha$  and  $\beta$  have the same number of elements so that Lemma 21 follows from the conclusion of Dwork's Theorem [12, Theorem 4.1]. In the sequel of this proof, we check that  $\alpha$  and  $\beta$  satisfy hypothesis of [12, Theorem 4.1] for almost all primes  $p$ , and we use the notations defined in Section 2.2.1. For a given fixed prime  $p$  not dividing  $d_{\alpha, \beta}$ , hypothesis of [12, Theorem 4.1] read

- (v) for all  $i \in \{1, \dots, r'\}$  and all  $k \in \mathbb{N}$ , we have  $\mathfrak{D}_p^k(\beta_i) \in \mathbb{Z}_p^\times$ ;
- (vi) for all  $a \in [0, p[$  and all  $k \in \mathbb{N}$ , we have either  $N_{p, \alpha}^k(a) = N_{p, \beta}^k(a+) = 0$  or  $N_{p, \alpha}^k(a) - N_{p, \beta}^k(a+) \geq 1$ .

If  $p$  is a large enough prime, then, by Lemma 3, for all  $i \in \{1, \dots, r'\}$ , we have  $\mathfrak{D}_p(\beta_i) = \mathfrak{D}_p(\langle \beta_i \rangle)$  so that

$$\mathfrak{D}_p(\beta_i) \in \left\{ \frac{1}{d_{\alpha, \beta}}, \dots, \frac{d_{\alpha, \beta} - 1}{d_{\alpha, \beta}}, 1 \right\} \subset \mathbb{Z}_p^\times. \quad (9.6)$$

Thus, for all large enough primes  $p$ ,  $\beta$  satisfies Assertion (v).

Let  $\alpha$  and  $\beta$  be elements of  $\alpha$  and  $\beta$ . First, we prove that, for all large enough primes  $p$  we have

$$p\mathfrak{D}_p(\alpha) - \alpha \leq p\mathfrak{D}_p(\beta) - \beta \iff \omega\alpha \preceq \omega\beta, \quad (9.7)$$

where  $\omega \in \{1, \dots, d_{\alpha, \beta}\}$  satisfies  $\omega p \equiv 1 \pmod{d_{\alpha, \beta}}$ . Assume that  $p$  is large enough so that, by Lemma 3, we get  $\mathfrak{D}_p(\alpha) = \langle \omega\alpha \rangle$  and  $\mathfrak{D}_p(\beta) = \langle \omega\beta \rangle$ . In particular, we obtain that

$$\mathfrak{D}_p(\alpha) = \mathfrak{D}_p(\beta) \quad \text{or} \quad |\mathfrak{D}_p(\alpha) - \mathfrak{D}_p(\beta)| \geq \frac{1}{d_{\alpha, \beta}}.$$

Thus, for all large enough primes  $p$ , we have

$$\begin{aligned} p\mathfrak{D}_p(\alpha) - \alpha \leq p\mathfrak{D}_p(\beta) - \beta &\iff \mathfrak{D}_p(\alpha) - \mathfrak{D}_p(\beta) \leq \frac{\alpha - \beta}{p} \\ &\iff \left( \mathfrak{D}_p(\alpha) < \mathfrak{D}_p(\beta) \quad \text{or} \quad (\mathfrak{D}_p(\alpha) = \mathfrak{D}_p(\beta) \quad \text{and} \quad \alpha \geq \beta) \right) \\ &\iff \omega\alpha \preceq \omega\beta, \end{aligned}$$

as expected. Now, we observe that if  $N_{p, \beta}^k(a+) = 0$ , then Assertion (vi) is trivial, so we can assume that  $N_{p, \beta}^k(a+) \geq 1$ . We set  $\beta' := (\beta_1, \dots, \beta_{r'})$ . Let us write  $\theta_p^k(x)$  for  $p\mathfrak{D}_p^{k+1}(x) - \mathfrak{D}_p^k(x)$  and let  $\gamma$  be the component of  $\alpha$  or  $\beta'$  such that  $\theta_p^k(\gamma)$  is the largest element of

$$\left\{ \theta_p^k(\alpha_i) : 1 \leq i \leq r, \theta_p^k(\alpha_i) < a \right\} \cup \left\{ \theta_p^k(\beta_j) : 1 \leq j \leq r', \theta_p^k(\beta_j) \leq a \right\}.$$

Since  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint,  $\mathfrak{D}_p^k(\alpha)$  and  $\mathfrak{D}_p^k(\beta')$  are also disjoint and, according to (9.7),  $\theta_p^k(\alpha)$  and  $\theta_p^k(\beta')$  are disjoint. It follows that  $N_{p, \alpha}^k(a) - N_{p, \beta}^k(a+)$  is equal to

$$\begin{aligned} &\#\{1 \leq i \leq r : \theta_p^k(\alpha_i) \leq \theta_p^k(\gamma)\} - \#\{1 \leq i \leq r' : \theta_p^k(\beta_j) \leq \theta_p^k(\gamma)\} \\ &= \#\{1 \leq i \leq r : \omega\mathfrak{D}_p^k(\alpha_i) \preceq \omega\mathfrak{D}_p^k(\gamma)\} - \#\{1 \leq i \leq r' : \omega\mathfrak{D}_p^k(\beta_j) \preceq \omega\mathfrak{D}_p^k(\gamma)\}. \end{aligned}$$

If  $k = 0$ , then we obtain that  $\omega\mathfrak{D}_p^k(\alpha) \preceq \omega\mathfrak{D}_p^k(\gamma) \iff \omega\alpha \preceq \omega\gamma$  with  $m_{\alpha, \beta}(\omega) \preceq \omega\gamma \prec \omega$  since  $\gamma \neq 1$ . Indeed, if  $\gamma$  is an element of  $\beta'$  then  $\gamma \neq 1$ , else  $\gamma$  is an element of  $\alpha$  and  $\theta_p^k(\gamma) < a$  so that  $\gamma \neq 1$ . Thus we have  $N_{p, \alpha}^0(a) - N_{p, \beta}^0(a+) = \xi_{\alpha, \beta}(\omega, \omega\gamma)$  and, by  $H_{\alpha, \beta}$ , we get  $N_{p, \alpha}^0(a) - N_{p, \beta}^0(a+) \geq 1$  as expected.

If  $k \geq 1$ , then, for all elements  $\alpha$  of  $\alpha$  and  $\beta'$ , we have  $\mathfrak{D}_p^k(\alpha) = \langle \omega^k \alpha \rangle$  and  $\langle \omega\mathfrak{D}_p^k(\alpha) \rangle = \langle \omega\langle \omega^k \alpha \rangle \rangle = \langle \omega^{k+1} \alpha \rangle$ . We deduce that we have  $\omega\mathfrak{D}_p^k(\alpha) \preceq \omega\mathfrak{D}_p^k(\gamma) \iff \langle \omega^{k+1} \alpha \rangle \leq \langle \omega^{k+1} \gamma \rangle$  because

$$\langle \omega^{k+1} \alpha \rangle = \langle \omega^{k+1} \gamma \rangle \iff \langle \alpha \rangle = \langle \gamma \rangle \iff \langle \omega^k \alpha \rangle = \langle \omega^k \gamma \rangle.$$

If  $\langle \gamma \rangle < 1$ , then  $\langle \omega^{k+1}\gamma \rangle < 1$  and we obtain that

$$N_{p,\alpha}^k(a) - N_{p,\beta}^k(a+) = \xi_{\alpha,\beta}(\omega^{k+1}, \langle \omega^{k+1}\gamma \rangle +) \geq 1.$$

On the contrary, if  $\langle \gamma \rangle = 1$ , then we get  $N_{p,\alpha}^k(a) - N_{p,\beta}^k(a+) = r - r'$ . Note that  $r' < r$  since there is at least one element of  $\beta$  equal to 1. Indeed, according to  $H_{\alpha,\beta}$ , if  $x \in \mathbb{R}$  satisfies  $m_{\alpha,\beta}(1) \preceq x \prec 1$ , then we have  $\xi_{\alpha,\beta}(1, x) \geq 1$ . Since  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint, we have  $\langle m_{\alpha,\beta}(1) \rangle < 1$  so that  $m_{\alpha,\beta}(1) \preceq 2 \prec 1$  and

$$1 \leq \xi_{\alpha,\beta}(1, 2) = \#\{1 \leq i \leq r : \alpha_i \neq 1\} - \#\{1 \leq j \leq r : \beta_j \neq 1\}.$$

We deduce that there is at least one  $j \in \{1, \dots, r\}$  such that  $\beta_j = 1$  and we obtain that

$$N_{p,\alpha}^k(a) - N_{p,\beta}^k(a+) = r - r' \geq 1,$$

as expected. Thus Assertion (vi) holds and Lemma 21 is proved.  $\square$

Now we fix  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ . For all large enough primes  $p \in \mathfrak{P}_{\alpha,\beta}(a)$  and all the elements  $\alpha$  of  $\alpha$  or  $\beta$ , we have  $\mathfrak{D}_p(\alpha) = \langle a\alpha \rangle$ . By Lemma 21, we obtain that, for almost all primes  $p \in \mathfrak{P}_{\alpha,\beta}(a)$ , we have

$$\frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z^p)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z^p)} - p \frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)} \in p\mathbb{Z}_p[[z]].$$

Furthermore, since  $q_{\alpha,\beta}(z)$  is  $N$ -integral, for almost all primes  $p$ , we have

$$\frac{G_{\alpha,\beta}(z^p)}{F_{\alpha,\beta}(z^p)} - p \frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)} \in p\mathbb{Z}_p[[z]].$$

Thus, for almost all primes  $p \in \mathfrak{P}_{\alpha,\beta}(a)$ , we obtain that

$$\frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z^p)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z^p)} - \frac{G_{\alpha,\beta}(z^p)}{F_{\alpha,\beta}(z^p)} \in p\mathbb{Z}_p[[z]],$$

which leads to

$$\frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)} - \frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)} \in p\mathbb{Z}_p[[z]].$$

By Dirichlet's theorem, there are infinitely many primes in  $\mathfrak{P}_{\alpha,\beta}(a)$  so that we have

$$\frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)} = \frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)},$$

which implies that  $q_{\alpha,\beta}(z) = q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)$  as expected. This finishes the proof of Assertion (iii) of Theorem 3.

**9.2. Proof of Assertion (ii) of Theorem 3.** We have to prove that  $q_{\alpha,\beta}(C'_{\alpha,\beta}z) \in \mathbb{Z}[[z]]$ . By Section 9.1, Assertion (iii) of Theorem 3 holds, *i.e.* we have  $r = s$ ,  $H_{\alpha,\beta}$  holds and, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have  $q_{\langle a\alpha \rangle, \langle a\beta \rangle}(z) = q_{\alpha,\beta}(z)$  so that

$$\frac{G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)}{F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)} = \frac{G_{\alpha,\beta}(z)}{F_{\alpha,\beta}(z)}. \quad (9.8)$$

By Theorem 2 in combination with (9.8), we obtain that

$$\frac{G_{\alpha,\beta}}{F_{\alpha,\beta}}(C'_{\alpha,\beta}z^p) - p \frac{G_{\alpha,\beta}}{F_{\alpha,\beta}}(C'_{\alpha,\beta}z) \in p\mathbb{Z}_p[[z]],$$

so that, according to Corollary 1, we have  $q_{\alpha,\beta}(C'_{\alpha,\beta}z) \in \mathbb{Z}_p[[z]]$ . Since  $p$  is an arbitrary prime; we get  $q_{\alpha,\beta}(C'_{\alpha,\beta}z) \in \mathbb{Z}[[z]]$ , as expected.

**9.3. Last step in the proof of Assertion (2) of Theorem 3.** To complete the proof of Assertion (2) of Theorem 3 and hence that of Theorem 3, we have to prove that we have either  $\alpha = (1/2)$  and  $\beta = (1)$ , or  $r \geq 2$  and there are at least two 1's in  $\beta$ . We shall distinguish two cases.

- Case 1: We assume that  $r = 1$ .

As already proved at the end of the proof of Lemma 21, there is at least one element of  $\beta$  equal to 1. Thus we obtain that  $\beta = (1)$ . We write  $\alpha = (\alpha)$ . Since Assertion (iii) of Theorem 3 holds, for all  $a \in \{1, \dots, d(\alpha)\}$  coprime to  $d(\alpha)$ , we have  $G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)/F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z) = G_{\alpha,\beta}(z)/F_{\alpha,\beta}(z)$ , *i.e.*

$$F_{\alpha,\beta}(z)G_{\langle a\alpha \rangle, \langle a\beta \rangle}(z) = F_{\langle a\alpha \rangle, \langle a\beta \rangle}(z)G_{\alpha,\beta}(z). \quad (9.9)$$

Now looking at the coefficient of  $z$  in power series involved in (9.9), one obtains that

$$\langle a\alpha \rangle \left( \frac{1}{\langle a\alpha \rangle} - 1 \right) = \alpha \left( \frac{1}{\alpha} - 1 \right).$$

We deduce that for all  $a \in \{1, \dots, d(\alpha)\}$  coprime to  $d(\alpha)$ , we have  $\langle a\alpha \rangle = \alpha$ . Thus we get that

$$\left\{ \frac{\kappa}{d(\alpha)} : 1 \leq \kappa \leq d(\alpha), \gcd(\kappa, d(\alpha)) = 1 \right\} = \{ \langle a\alpha \rangle : 1 \leq a \leq d(\alpha), \gcd(a, d(\alpha)) = 1 \} \\ = \{ \alpha \},$$

which implies that  $\alpha = 1/2$  as expected.

- Case 2: We assume that  $r \geq 2$ .

We already know that there is at least one element of  $\beta$  equal to 1. Since  $\langle \alpha \rangle$  and  $\langle \beta \rangle$  are disjoint, for all the elements  $\alpha$  of  $\alpha$ , we have  $\langle \alpha \rangle < 1$ . Furthermore, for all  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$ , we have

$$\xi_{\langle \alpha \rangle, \langle \beta \rangle}(a, 1-) = \#\{1 \leq i \leq r : \langle \alpha_i \rangle \neq 1\} - \#\{1 \leq i \leq r : \langle \beta_i \rangle \neq 1\} \\ = r - \#\{1 \leq i \leq r : \langle \beta_i \rangle \neq 1\}.$$

It follows that we have to prove that  $\xi_{\langle\alpha\rangle,\langle\beta\rangle}(a, 1-) \geq 2$ .

Let  $\gamma$  be an element of  $\alpha$  or  $\beta$  with the largest exact denominator. Then, there exists  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$  such that  $\langle a\gamma \rangle = 1/d(\gamma)$ . By  $H_{\alpha,\beta}$  in combination with Lemma 2, we obtain that  $H_{\langle\alpha\rangle,\langle\beta\rangle}$  holds. In addition, we have  $\langle a\langle\gamma\rangle \rangle = \langle a\gamma \rangle = 1/d(\gamma)$  so that  $\xi_{\langle\alpha\rangle,\langle\beta\rangle}(a, 1/d(\gamma) +) \geq 1$ . Since  $\langle a\alpha \rangle$  and  $\langle a\beta \rangle$  are disjoint and have elements larger than or equal to  $1/d(\gamma)$ , we obtain that  $\gamma$  is a component of  $\alpha$ .

Furthermore, there exists  $a \in \{1, \dots, d_{\alpha,\beta}\}$  coprime to  $d_{\alpha,\beta}$  such that

$$\langle a\gamma \rangle = \frac{d(\gamma) - 1}{d(\gamma)} =: \kappa.$$

Thus  $\kappa$  is the largest element distinct from 1 in  $\langle a\alpha \rangle$  and  $\langle a\beta \rangle$ , and we obtain that  $\xi_{\langle\alpha\rangle,\langle\beta\rangle}(a, \kappa+) = \xi_{\langle\alpha\rangle,\langle\beta\rangle}(a, 1-)$ . If  $\langle m_{\langle\alpha\rangle,\langle\beta\rangle}(a) \rangle = \kappa$ , then all the elements of  $\langle\beta\rangle$  are equal to 1 and the result is proved. Otherwise, we have  $\langle m_{\langle\alpha\rangle,\langle\beta\rangle}(a) \rangle < \kappa$  so that  $\xi_{\langle\alpha\rangle,\langle\beta\rangle}(a, \kappa-) \geq 1$ . Since  $\gamma$  is an element of  $\alpha$ , we obtain that  $\xi_{\langle\alpha\rangle,\langle\beta\rangle}(a, \kappa+) \geq 2$  as expected. This finishes the proof of Assertion (2) of Theorem 3 and thus the one of Theorem 3.

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E. Delaygue, Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 Novembre 1918, 69622 Villeurbanne cedex, France. Email: delaygue@math.univ-lyon1.fr

T. Rivoal, Institut Fourier, CNRS et Université Grenoble 1, CNRS UMR 5582, 100 rue des maths, BP 74, 38402 St Martin d'Hères cedex, France. Email: Tanguy.Rivoal@ujf-grenoble.fr

J. Roques, Institut Fourier, Université Grenoble 1, CNRS UMR 5582, 100 rue des maths, BP 74, 38402 St Martin d'Hères cedex, France. Email: Julien.Roques@ujf-grenoble.fr