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BERNSTEIN PROCESSES DESCRIBED BY FORWARD AND BACKWARD HEAT EQUATIONS

MOHAMAD HOUDA

ABSTRACT. We present some results on Bernstein processes which are Brownian diffusions that appear in Euclidean Quantum Mechanics: We express the distributions of these processes with the help of those of Bessel processes. We then determine two solutions of the dual equation of the heat equation with potential. This continues the calculation of Lescot for the solution with an Ornstein-Uhlenbeck process [4].

Keywords: Bessel processes, Bernstein processes, interest rate models.

1. INTRODUCTION

Bernstein processes or reciprocal diffusions combine two dynamic behaviors : forward and backward. These processes are useful tools of stochastic quantum mechanics, and also find applications in various other fields.

We give in this paper an explicit expression for the distribution density of a special Bernstein process. This process is similar to the Cox-Ingersoll-Ross process of financial mathematics. Indeed, more recently it has appeared that each *one-factor affine interest rate model* (in the sense of Leblanc-Scaillet) could be described using such a Bernstein process.

The groundbreaking idea of replacing the complex Schrödinger equation by forward and backward heat equations in duality goes back to Schrödinger.

Professor Paul Lescot has computed two solutions of the dual equation by using the Gaussian character of Ornstein-Uhlenbeck process and Brownian motion. In this paper, this goal will be reached by using another class of diffusion which is that of *Bessel processes*.

Let us first give some definitions and recall some preliminary results.

Definition 1.1. A *Bernstein process* z is a process satisfying a stochastic differential equation of the form

$$dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt$$

relatively to the canonical filtration of the Brownian motion w and the dual stochastic differential equation

$$dz(t) = \theta dw_*(t) + \tilde{B}_*(t, z(t))dt$$

relatively to the canonical decreasing filtration of another Brownian motion w_* which is independent of w , where

$$\tilde{B} \equiv \theta^2 \frac{\partial \eta}{\partial q},$$

$$\tilde{B}_* \equiv -\theta^2 \frac{\partial \eta_*}{\partial q}$$

and, for each given $t > 0$, the law of $z(t)$ is $\eta(t, q)\eta_*(t, q)dq$. The function η is assumed to be an everywhere positive solution to

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V\eta \quad (\mathcal{C}_1^{(V)}).$$

Similarly, η_* is assumed to be everywhere positive and a solution to

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V\eta_* \quad (\mathcal{C}_2^{(V)}).$$

Definition 1.2. [1, p.454 (2.1)] The *Bessel function* J_λ with index $\lambda \in \mathbf{C}$ is defined by

$$J_\lambda(z) = \left(\frac{z}{2}\right)^\lambda \sum_{n=0}^{\infty} \frac{(-z^2)^n}{2^{2n} n! \Gamma(n + \lambda + 1)}.$$

This function satisfies the Bessel equation with parameter $\lambda \in \mathbf{C}$

$$(1.1) \quad z^2 \ddot{\omega} + z\dot{\omega} + (z^2 - \lambda^2)\omega = 0$$

Definition 1.3. The *Bessel modified function* I_ν with index $\nu \in \mathbf{R}$ is defined by

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

This function satisfies the linear differential equation of second order

$$(1.2) \quad z^2 \ddot{I}_\nu(z) + z\dot{I}_\nu(z) - (z^2 + \nu^2)I_\nu(z) = 0.$$

Indeed, we have $I_\nu(z) = i^{-\nu} J_\nu(iz)$, $\dot{I}_\nu(z) = i^{-\nu} i \dot{J}_\nu(iz)$ et $\ddot{I}_\nu(z) = i^{-\nu} i^2 \ddot{J}_\nu(iz)$. Equation (1.1) gives

$$(iz)^2 i^\nu i^{-2} \ddot{I}_\nu(z) + iz i^\nu i^{-1} \dot{I}_\nu(z) + (-z^2 - \nu^2) i^\nu I_\nu(z) = 0$$

$$z^2 \ddot{I}_\nu(z) + z\dot{I}_\nu(z) - (z^2 + \nu^2)I_\nu(z) = 0,$$

which yields (1.2).

We refer to [1] for details on Bessel functions.

Definition 1.4. For all $\delta \geq 0$ and $x_0 \geq 0$, the unique solution of

$$Y_t = x_0 + \delta t + 2 \int_0^t \sqrt{|Y_s|} dw_s.$$

starting from x_0 , is called *squared Bessel process of dimension* δ . This process will be denoted by $\text{BESQ}_{x_0}^\delta$.

We refer to [2] and [5] for a survey on Bessel processes.

Proposition 1.5. [5, p.441 (1.4)] *If $\mathbf{x}_0 = \mathbf{0}$, then the density q_t^δ of the law of Y_t is given by*

$$(1.3) \quad q_t^\delta(0, y) = (2t)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2t}\right), \quad t > 0.$$

If $\mathbf{x}_0 > \mathbf{0}$, then the density q_t^δ of the law of Y_t is given by

$$(1.4) \quad q_t^\delta(x_0, y) = \frac{1}{2t} \left(\frac{y}{x_0}\right)^{\frac{\nu}{2}} \exp\left(-\frac{x_0 + y}{2t}\right) I_\nu\left(\frac{\sqrt{x_0 y}}{t}\right), \quad t > 0$$

where $\nu := \frac{\delta}{2} - 1$.

2. ON AFFINE INTEREST RATE MODELS

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q)$ be a filtered probability space, and let $w(t)_{t \geq 0}$ be an $(\mathcal{F}_t)_{t \geq 0}$ standard Brownian motion.

A *one-factor affine interest rate model* is characterized by the instantaneous rate $r(t)$, satisfying the stochastic differential equation

$$dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt$$

under the risk-neutral probability Q ([3], p.351).

Assuming $\alpha \neq 0$, let us set, with the notations of Lescot [4],

$$\tilde{\phi} = \phi + \frac{\lambda\beta}{\alpha},$$

$$\delta = \frac{4\tilde{\phi}}{\alpha},$$

$$A = \frac{\alpha^4}{128} (\delta - 1)(\delta - 3),$$

and

$$B = \frac{\lambda^2}{8}.$$

Proposition 2.1. (Lescot [4]). *Let X_t be defined by*

$$X_t = \alpha r(t) + \beta.$$

The process X is called the Cox-Ingersoll-Ross process. Define $Z(t)$ by

$$Z(t) = \sqrt{\alpha r(t) + \beta}.$$

Then Z is a Bernstein process with

$$\theta = \frac{\alpha}{2}$$

and the potential

$$V(t, q) = \frac{A}{q^2} + Bq^2.$$

Lemma 2.2. [2, p.314 (2)]. *Let $\mathbf{X}_0 = \mathbf{x}_0$, then*

$$X_t = e^{-\lambda t} Y(s),$$

where $s = \frac{\alpha^2(e^{\lambda t} - 1)}{4\lambda}$ and Y is $BESQ_{x_0}^\delta$.

Our main result is the following :

We determine two solutions of the dual equation ($\mathcal{C}_2^{(V)}$), when the potential V is written as

$$V(t, q) = \frac{A}{q^2} + Bq^2.$$

Firstly we search $\rho_t(q)$ the density of the law of Z_t . Here, two cases arise : $\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_0 > \mathbf{0}$.

Proposition 2.3. *If $\mathbf{x}_0 = \mathbf{0}$, then the density $\rho_t(q)$ of the law of Z_t is written as*

$$\rho_t(q) = \alpha^{-\delta} 2^{\frac{\delta}{2}+1} \lambda^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} q^{\delta-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t} q^2}{\alpha^2(e^{\lambda t} - 1)}\right) \mathbb{1}_{\{q>0\}}.$$

Proof. For fixed t and for all g bounded continuous,

$$\begin{aligned} E(g(X_t)) &= E[g(\exp(-\lambda t)Y(s))] \\ &\stackrel{(1.3)}{=} \int_0^\infty g(\exp(-\lambda t)y) (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2s}\right) dy \\ &= (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} \int_0^\infty g(\exp(-\lambda t)y) y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2s}\right) dy. \end{aligned}$$

By the change of variable $x = \exp(-\lambda t)y$, we get

$$\begin{aligned} E(g(X_t)) &= (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} \int_0^\infty g(x) [x \exp(\lambda t)]^{\frac{\delta}{2}-1} \exp\left(-\frac{e^{\lambda t}x}{2s}\right) \exp(\lambda t) dx \\ &= \int_0^\infty g(x) (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\frac{\delta}{2}-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(-\frac{e^{\lambda t}x}{2s}\right) dx. \end{aligned}$$

Then the density ℓ_{X_t} of the law of X_t is given by

$$\ell_{X_t}(x) = \alpha^{-\delta} (2\lambda)^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\frac{\delta}{2}-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t}x}{\alpha^2(e^{\lambda t} - 1)}\right) \mathbb{1}_{\{x>0\}}.$$

For fixed t and for all bounded continuous φ , we have

$$\begin{aligned} E[\varphi(Z_t)] &= E[\varphi(\sqrt{X_t})] \\ &= \int_0^\infty \varphi(\sqrt{x}) \ell_{X_t}(x) dx. \end{aligned}$$

By the change of variable $q = \sqrt{x}$, we get

$$\begin{aligned} E[\varphi(Z_t)] &= \int_0^\infty \varphi(q) \alpha^{-\delta} (2\lambda)^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} q^{\delta-2} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t}q^2}{\alpha^2(e^{\lambda t} - 1)}\right) 2q dq \\ &= \int_0^\infty \varphi(q) \alpha^{-\delta} 2^{\frac{\delta}{2}+1} \lambda^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} q^{\delta-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t}q^2}{\alpha^2(e^{\lambda t} - 1)}\right) dq. \end{aligned}$$

This proves the result. \square

Let $\eta(t, q)$ be defined by

$$\eta(t, q) = \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2}\right) q^{\frac{\delta-1}{2}}.$$

The function η is a solution of

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V\eta \quad (\mathcal{C}_1^{(V)}),$$

with

$$V(t, q) = \frac{A}{q^2} + Bq^2.$$

Proposition 2.4. *The function $\eta_*(t, q)$ defined by*

$$\begin{aligned} \eta_*(t, q) &= \frac{\rho_t(q)}{\eta(t, q)} \\ &= \frac{2^{\frac{\delta}{2}+1}}{\alpha^\delta} \frac{\lambda^{\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} q^{\frac{\delta-1}{2}} \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})}\right), \end{aligned}$$

satisfies the dual equation

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V\eta_* \quad (C_2^{(V)}).$$

Proof. Let

$$C = \alpha^{-\delta} 2^{\frac{\delta}{2}+1} \lambda^{\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1}$$

and

$$C_1 = C(e^{\lambda t} - 1)^{-\frac{\delta}{2}}.$$

We have

$$\frac{\partial \eta_*}{\partial t} = \left(-\frac{\delta}{2} (\lambda e^{\lambda t}) + \frac{\lambda\delta}{4} - \frac{\lambda q^2}{\alpha^2} \left(-\frac{2\lambda e^{\lambda t}}{(e^{\lambda t} - 1)^2} \right) \right) C(e^{\lambda t} - 1)^{-\frac{\delta}{2}-1} q^{\frac{\delta-1}{2}} \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})}\right),$$

$$\frac{\partial \eta_*}{\partial q} = \left(\left(\frac{\delta-1}{2} \right) q^{\frac{\delta-3}{2}} + q^{\frac{\delta-1}{2}} \left(-\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) \right) C_1 \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})}\right),$$

$$\begin{aligned} \frac{\partial^2 \eta_*}{\partial q^2} &= \left(\left(\frac{\delta-1}{2} \right) \left(\frac{\delta-3}{2} \right) q^{\frac{\delta-5}{2}} + \left(\frac{\delta-1}{2} \right) q^{\frac{\delta-3}{2}} \left(-\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) + q^{\frac{\delta-1}{2}} \left(-\frac{2\lambda}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) \right. \\ &\quad \left. + \left(\frac{\delta-1}{2} \right) q^{\frac{\delta-3}{2}} \left(-\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) + q^{\frac{\delta-1}{2}} \left(-\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right)^2 \right) C_1 \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})}\right), \end{aligned}$$

and

$$\begin{aligned} V(t, q) \eta_*(t, q) &= \left(\frac{A}{q^2} + Bq^2 \right) \eta_*(t, q) \\ &= \left(\frac{\alpha^4}{27} (\delta-1)(\delta-3) q^{-2} + \frac{\lambda^2}{23} q^2 \right) \eta_*(t, q) \\ &= \left(\alpha^{-\delta+4} 2^{\frac{\delta}{2}-6} \lambda^{\frac{\delta}{2}} (\delta-1)(\delta-3) q^{\frac{\delta-5}{2}} + \alpha^{-\delta} 2^{\frac{\delta}{2}-2} \lambda^{\frac{\delta}{2}+2} q^{\frac{\delta+3}{2}} \right) \Gamma\left(\frac{\delta}{2}\right)^{-1} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \\ &\quad \times \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})}\right). \end{aligned}$$

It is then straightforward to see that, for all $(t, q) \in \mathbf{R}_+^* \times \mathbf{R}_+^*$,

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V\eta_* \quad (C_2^{(V)}).$$

□

Proposition 2.5. *If $\mathbf{x}_0 > \mathbf{0}$, then the density $\rho_t(q)$ of the law Z_t is given by*

$$\rho_t(q) = \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\lambda t(\frac{3}{4} + \frac{1}{2})}}{(e^{\lambda t} - 1)} I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) q^{\frac{3}{2}} \exp\left(-\frac{2\lambda}{\alpha^2} \left(\frac{z_0^2 + e^{\lambda t} q^2}{(e^{\lambda t} - 1)}\right)\right) \mathbb{1}_{\{q > 0\}}.$$

Proof. We have

$$X_t = e^{-\lambda t} Y(s), \quad s = \frac{\alpha^2}{4\lambda} (e^{\lambda t} - 1)$$

and

$$\begin{cases} Z_t &= \sqrt{X_t} \\ Z_0 &= \sqrt{x_0}. \end{cases}$$

For fixed t , for all g bounded continuous, using (1.4), we get

$$\begin{aligned} E(g(Z_t)) &= E\left(g\left(\sqrt{e^{-\lambda t} Y(s)}\right)\right) \\ &= \int_0^\infty g(\sqrt{e^{-\lambda t} y}) \frac{1}{2s} \left(\frac{y}{x_0}\right)^{\frac{\nu}{2}} \exp\left(-\frac{x_0 + y}{2s}\right) I_\nu\left(\frac{\sqrt{x_0 y}}{s}\right) dy. \end{aligned}$$

By the change of variable $z = \sqrt{e^{-\lambda t} y}$, we obtain

$$E(g(Z_t)) = \int_0^\infty g(z) \frac{1}{2s} \left(\frac{e^{\lambda t} z^2}{z_0^2}\right)^{\frac{\nu}{2}} \exp\left(-\frac{z_0^2 + e^{\lambda t} z^2}{2s}\right) I_\nu\left(\frac{z_0 z e^{\frac{\lambda t}{2}}}{s}\right) 2z e^{\lambda t} dz.$$

We then find the density $\rho_t(q)$ of the law de Z_t by replacing s and ν by their value. \square

Proposition 2.6. *The function $\eta_*(t, q)$ defined by*

$$\begin{aligned} \eta_*(t, q) &= \frac{\rho_t(q)}{\eta(t, q)} \\ &= \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) q^{\frac{1}{2}} \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \left(\frac{z_0^2 + e^{\lambda t} q^2}{(e^{\lambda t} - 1)}\right)\right) \end{aligned}$$

satisfies the dual equation $(\mathcal{C}_2^{(V)})$.

Proof. We have

$$\begin{aligned} \frac{\partial \eta_*}{\partial t} &= \left(\left(-\frac{2\lambda^2}{\alpha^2 z_0^\nu} e^{\frac{\lambda t}{2}} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)^2} q^{\frac{1}{2}} + \frac{8\lambda^3}{\alpha^4 z_0^\nu} \frac{e^{\frac{3\lambda t}{2}}}{(e^{\lambda t} - 1)^3} q^{\frac{5}{2}} + \frac{8\lambda^3}{\alpha^4 z_0^\nu} \frac{e^{\frac{3\lambda t}{2}}}{(e^{\lambda t} - 1)^3} z_0^2 q^{\frac{1}{2}}\right) I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right. \\ &\quad \left. - \frac{8z_0 \lambda^3}{\alpha^4 z_0^\nu} e^{\lambda t} \frac{e^{\lambda t} + 1}{(e^{\lambda t} - 1)^3} q^{\frac{3}{2}} I_\nu'\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right) \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right), \end{aligned}$$

$$\begin{aligned} \frac{\partial \eta_*}{\partial q} &= \left(\left(\frac{1}{2} q^{-\frac{1}{2}} - \frac{2\lambda}{\alpha^2} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)} q^{\frac{3}{2}}\right) I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) + \frac{4\lambda z_0}{\alpha^2} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} q^{\frac{1}{2}} I_\nu'\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right) \\ &\quad \times \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right), \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \eta_*}{\partial q^2} &= \left(-\frac{1}{4} q^{-\frac{3}{2}} - \frac{4\lambda}{\alpha^2} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)} q^{\frac{1}{2}} + \frac{4\lambda^2}{\alpha^4} \frac{(e^{\lambda t} + 1)^2}{(e^{\lambda t} - 1)^2} q^{\frac{5}{2}} \right) I_\nu \left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)} \right) \\
&+ \left(\frac{4\lambda z_0}{\alpha^2} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} q^{-\frac{1}{2}} - \frac{16\lambda^2 z_0}{\alpha^4} e^{\frac{\lambda t}{2}} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)^2} q^{\frac{3}{2}} \right) I'_\nu \left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)} \right) \\
&+ \frac{16\lambda^2 z_0^2}{\alpha^4} \frac{e^{\lambda t}}{(e^{\lambda t} - 1)^2} q^{\frac{1}{2}} I_\nu'' \left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)} \right) \frac{4\lambda}{\alpha^2 z_0'} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} \exp \left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1} \right),
\end{aligned}$$

and

$$\begin{aligned}
V(t, q) \eta_*(t, q) &= \left(\frac{A}{q^2} + Bq^2 \right) \eta_*(t, q) \\
&= \left(\frac{\alpha^4}{2^7} (\delta - 1)(\delta - 3) q^{-2} + \frac{\lambda^2}{2^3} q^2 \right) \eta_*(t, q) \\
&= \left(\lambda \frac{\alpha^2}{2^5} (\delta - 1)(\delta - 3) q^{-\frac{3}{2}} + \frac{\lambda^3}{2\alpha^2} q^{\frac{5}{2}} \right) \frac{e^{\frac{\lambda t}{2}}}{z_0' (e^{\lambda t} - 1)} I_\nu \left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)} \right) \\
&\times \exp \left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \left(\frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1} \right) \right).
\end{aligned}$$

Taking into account Equation (1.2) of the *Bessel* function I_ν , it is then straightforward to see that for all $(t, q) \in \mathbf{R}_+^* \times \mathbf{R}_+^*$,

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V \eta_* \quad (\mathcal{C}_2^{(V)}).$$

□

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