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A survey on the pseudo-process driven by the high-order heat-type equation $\partial/\partial t = \pm\partial^N/\partial x^N$ concerning the hitting and sojourn times

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Abstract. Fix an integer $N > 2$ and let $X = (X(t))_{t \geq 0}$ be the pseudo-process driven by the high-order heat-type equation $\partial/\partial t = \pm\partial^N/\partial x^N$. The denomination “pseudo-process” means that X is related to a signed measure (which is not a probability measure) with total mass equal to 1.

In this survey, we present several explicit results and discuss some problems concerning the pseudo-distributions of various functionals of the pseudo-process X : the first or last overshooting times of a single barrier $\{a\}$ or a double barrier $\{a, b\}$ by X ; the sojourn times of X in the intervals $[a, +\infty)$ and $[a, b]$ up to a fixed time; the maximum or minimum of X up to a fixed time.

Keywords. Pseudo-process; pseudo-distribution; first hitting or overshooting time; sojourn time; up-to-date maximum.

1 Introduction

Consider the heat-type equation $\partial/\partial t = \kappa_N \partial^N/\partial x^N$ of order $N > 2$ where $\kappa_N = (-1)^{1+N/2}$ if N is even and $\kappa_N = \pm 1$ if N is odd. Let us introduce the corresponding kernel $p(t; x)$ which is characterized by

$$\int_{-\infty}^{+\infty} e^{iux} p(t; x) dx = \begin{cases} e^{-tu^N} & \text{if } N \text{ is even,} \\ e^{\kappa_N t(-iu)^N} & \text{if } N \text{ is odd.} \end{cases}$$

This kernel defines a pseudo-process $(X(t))_{t \geq 0}$ driven by a signed measure with total mass equal to 1 (which is not a probability measure) according as the usual Markov rules: we set for $t > 0$, $0 = t_0 < t_1 < \dots < t_m$ and $x = x_0, x_1, \dots, x_m, y \in \mathbb{R}$,

$$\mathbb{P}_x\{X(t) \in dy\} = p(t; x - y) dy$$

and

$$\mathbb{P}_x\{X(t_1) \in dx_1, \dots, X(t_m) \in dx_m\} = \prod_{i=1}^m p(t_i - t_{i-1}; x_{i-1} - x_i) dx_i.$$

Since we are dealing with a signed measure, it seems impossible to extend the definition of the pseudo-process over all the positive times. We can find in the literature two possible ad-hoc constructions: one over the set of times of the form kt/n , $k, n \in \mathbb{N}$ (depending on a fixed time t , see [6] and [8] for pioneering works related to this construction), the other one over the set of dyadic times $k/2^n$, $k, n \in \mathbb{N}$. (which do not depend on any particular time, see [17] for this last construction). For $N = 2$, this is the most well-known Brownian motion and for $N = 4$, $(X(t))_{t \geq 0}$ is the so-called biharmonic pseudo-process.

For the pseudo-process $(X(t))_{t \geq 0}$ started at a point x , we introduce:

- the first overshooting times of a one-sided barrier $\{a\}$ (or, equivalently, the first hitting time of the half-line $[a, +\infty)$) or a two-sided barrier $\{a, b\}$ (with the convention $\inf(\emptyset) = +\infty$):

$$\tau_a = \inf\{t \geq 0 : X(t) \geq a\} \text{ for } x \leq a, \quad \tau_{ab} = \inf\{t \geq 0 : X(t) \notin [a, b]\} \text{ for } x \in [a, b];$$

- the last overshooting times of such barriers before a fixed time t (with the convention $\sup(\emptyset) = 0$):

$$\sigma_a(t) = \sup\{s \in [0, t] : X(s) \geq a\}, \quad \sigma_{ab}(t) = \sup\{s \in [0, t] : X(s) \notin (a, b)\};$$

- the sojourn times in the intervals $[a, +\infty)$ and $[a, b]$ up to a fixed time t :

$$T_a(t) = \text{measure}\{s \in [0, t] : X(s) \geq a\}, \quad T_{ab}(t) = \text{measure}\{s \in [0, t] : X(s) \in [a, b]\};$$

- the maximum up to time t :

$$M(t) = \max_{0 \leq s \leq t} X(s).$$

In the foregoing rough definitions, the pseudo-distribution of the quantity $T_a(t)$ for instance is to be understood as the limit of $\mathbb{P}_x\{\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{[a, +\infty)}(X(kt/n)) \in ds\}$ when $n \rightarrow \infty$.

We could introduce the alternative first hitting time of $(-\infty, a]$, the alternative sojourn time in $(-\infty, a]$ and the up-to-date minimum $m(t) = \min_{0 \leq s \leq t} X(s)$. Actually, the pseudo-distributions of these three quantities are obviously related to the pseudo-distributions of the foregoing ones.

We shall also consider the pseudo-process with a drift $(X^b(t))_{t \geq 0}$ defined by $X^b(t) = X(t) + bt$ where b is a fixed real number. For this latter, we introduce:

- the first overshooting time of the threshold a :

$$\tau_a^b = \inf\{t \geq 0 : X^b(t) \geq a\} \text{ for } x \leq a$$

if the set $\{t \geq 0 : X^b(t) \geq a\}$ is not empty, else we set $\tau_a^b = +\infty$;

- the maximum functional up to time t :

$$M^b(t) = \max_{0 \leq s \leq t} X^b(s).$$

The aim of this survey is to provide a list of explicit results concerning the pseudo-distributions of $(X(t), T_a(t))$, $(X(t), M(t))$, $(\tau_a, X(\tau_a))$ and $\sigma_a(t)$, as well as those related to the pseudo-process with a drift. In particular, remarkable results hold for the pseudo-distributions of $T_0(t)$ and $X(\tau_a)$. We also provide some methods for deriving those of $T_{ab}(t)$, $\sigma_{ab}(t)$ and $(\tau_{ab}, X(\tau_{ab}))$.

A way consists in using the Feynman-Kac functional

$$\phi(t; x) = \mathbb{E}_x\left(e^{-\int_0^t f(X(s)) ds} g(X(t))\right) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mathbb{E}_x\left(e^{-\frac{t}{n} \sum_{k=0}^{n-1} f(X(\frac{kt}{n}))} g(X(t))\right)$$

which is a solution to the partial differential equation $\frac{\partial \phi}{\partial t}(t; x) = \kappa_N \frac{\partial^N \phi}{\partial x^N}(t; x) - f(x)\phi(t; x)$ with $\phi(0; x) = g(x)$. Its Laplace transform $\Phi(x) = \int_0^{+\infty} e^{-\lambda t} \phi(t; x) dt$ is a solution to the ordinary differential equation $\kappa_N \frac{d^N \Phi}{dx^N}(x) = (f(x) + \lambda)\Phi(x) - g(x)$. Another way consists in using Spitzer's identities which work actually when the starting point is 0 and N is even. Indeed, their validity holds thanks to the fact that the integral $\int_{-\infty}^{+\infty} |p(t; x)| dx$ is finite, which is true only when N is even. Additionally, Spitzer's identities hinge on a symmetry property which is fulfilled only when the starting point of the pseudo-process is 0. In the case $N = 4$, see [18] for many connections with fourth-order partial differential equations with various boundary value conditions.

Let us introduce the N^{th} roots of κ_N : $(\theta_\ell)_{1 \leq \ell \leq N}$ and $J = \{j \in \{1, \dots, N\} : \Re(\theta_j) > 0\}$, $K = \{k \in \{1, \dots, N\} : \Re(\theta_k) < 0\}$ that will be used for solving the above differential equation. The notations $\#J$ and $\#K$ stand for the cardinalities of the sets J and K . We have $\theta_\ell/\theta_m = e^{i(\ell-m)2\pi/N}$ for any $1 \leq \ell, m \leq N$.

Set, for $j, j' \in J$ and $k, k' \in K$,

$$A_j = \prod_{\ell \in J \setminus \{j\}} \frac{\theta_\ell}{\theta_\ell - \theta_j} \quad \text{and} \quad B_k = \prod_{\ell \in K \setminus \{k\}} \frac{\theta_\ell}{\theta_\ell - \theta_k},$$

$$C_{jj'k} = \prod_{j'' \in J} (\theta_j \theta_{j'} - \theta_{j''} \theta_k) \quad \text{and} \quad D_{jkk'} = \prod_{k'' \in K} (\theta_k \theta_{k'} - \theta_{k''} \theta_j).$$

Let us also introduce the $(N-1)^{\text{th}}$ roots of the complex number i : $(\tilde{\theta}_i)_{1 \leq i \leq N-1}$ and $\tilde{J} = \{j \in \{1, \dots, N-1\} : \Im(\tilde{\theta}_j) > 0\}$. We shall need to introduce the roots $(\omega_\ell^b(\lambda))_{1 \leq \ell \leq N}$ of the polynomial $u^N + ibu + \lambda$ (where $\Re(\lambda) > 0$). These last settings will be used for the pseudo-process with a drift. Finally, set for any integer ℓ such that $\ell \leq N-1$

$$I_\ell(t; \xi) = \frac{Ni}{2\pi} \left(e^{-i\frac{\ell}{N}\pi} \int_0^{+\infty} \lambda^{N-\ell-1} e^{-t\lambda^N + e^{i\frac{\ell}{N}\pi}\xi\lambda} d\lambda - e^{i\frac{\ell}{N}\pi} \int_0^{+\infty} \lambda^{N-\ell-1} e^{-t\lambda^N + e^{-i\frac{\ell}{N}\pi}\xi\lambda} d\lambda \right).$$

The functions I_ℓ satisfy $\int_0^{+\infty} e^{-\lambda t} I_\ell(t; \xi) dt = \lambda^{-\ell/N} e^{\sqrt[\ell]{\lambda}\xi}$ for $\lambda > 0$ and $\Re(\xi) \leq 0$. They will be useful for expressing several distributions.

The results are presented by topic and in certain topics we have chosen to exhibit them from the most particular to the most general thus following the chronology. Moreover, it is not easy sometimes to deduce the particular cases from the most general ones.

2 Distributions related to $T_a(t)$

See [1,3-5,7-10,15,19] for the chronology of the results concerning the distributions related to $T_a(t)$ as well as for the connections with the maximum and minimum functionals of $(X(t))_{t \geq 0}$.

2.1 Distribution of $T_a(t)$

Set $\Phi(x) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x(e^{-\mu T_a(t)}) dt$ for $\lambda, \mu > 0$ and $x \in \mathbb{R}$. The quantity $\Phi(x)$ should be understood as

$$\Phi(x) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x \left(e^{-\mu \frac{t}{n} \sum_{k=0}^{n-1} \mathbb{1}_{[a, +\infty)}(X(\frac{kt}{n}))} \right) dt.$$

Using the Feynman-Kac approach, it can be seen that the function Φ satisfies the system

$$\kappa_N \frac{d^N \Phi}{dx^N}(x) = \begin{cases} (\lambda + \mu) \Phi(x) - 1 & \text{for } x \in (a, +\infty), \\ \lambda \Phi(x) - 1 & \text{for } x \in (-\infty, a), \end{cases}$$

and

$$\forall k \in \{0, 1, \dots, N-1\}, \quad \frac{d^k \Phi}{dx^k}(a^+) = \frac{d^k \Phi}{dx^k}(a^-).$$

This system can be explicitly solved by computing Vandermonde determinants. In particular, for $x = a$, the following formula holds:

$$\Phi(a) = \frac{1}{\sqrt[\ell]{\lambda \#K} (\lambda + \mu)^{\#J}}$$

and this two-parameters Laplace transform can be inverted ([9]). The distribution of $T_a(t)$ under \mathbb{P}_a is the same as that of $T_0(t)$ under \mathbb{P}_0 .

Theorem 1 (Lachal, 2003). *The pseudo-distribution of $T_0(t)$ is a Beta law:*

$$\mathbb{P}_0\{T_0(t) \in ds\}/ds = \frac{1}{\pi} \sin\left(\frac{\#K}{N} \pi\right) \frac{1}{\sqrt[2]{s^{\#K}(t-s)^{\#J}}} \text{ for } s \in (0, t).$$

Example 1. If N is even, $T_0(t)$ obeys the famous Paul Lévy's Arcsine law:

$$\mathbb{P}_0\{T_0(t) \in ds\}/ds = \frac{1}{\pi \sqrt{s(t-s)}}.$$

In the history of pseudo-processes, this result was discovered by Krylov ([8]) when N is even. For $N = 3$, Orsingher obtained ([19])

$$\mathbb{P}_0\{T_0(t) \in ds\}/ds = \begin{cases} \frac{\sqrt{3}}{2\pi \sqrt[3]{s^2(t-s)}} & \text{when } \kappa_N = +1, \\ \frac{\sqrt{3}}{2\pi \sqrt[3]{s(t-s)^2}} & \text{when } \kappa_N = -1. \end{cases}$$

Using a similar method, the following simple results can be obtained ([9]).

Theorem 2 (Lachal, 2003). *The pseudo-distribution of $T_0(t)$ conditioned on $X(t) = 0$ is the uniform law on $(0, t)$: for $s \in (0, t)$,*

$$\mathbb{P}_0\{T_0(t) \in ds | X(t) = 0\}/ds = \frac{1}{t}.$$

The pseudo-distributions of $T_0(t)$ conditioned on $X(t) > 0$ and $X(t) < 0$ are Beta laws: for $s \in (0, t)$,

$$\begin{aligned} \mathbb{P}_0\{T_0(t) \in ds | X(t) > 0\}/ds &= \frac{N \sin(\frac{\#K}{N} \pi)}{(\#K \pi) t} \left(\frac{s}{t-s}\right)^{\frac{\#J}{N}}, \\ \mathbb{P}_0\{T_0(t) \in ds | X(t) < 0\}/ds &= \frac{N \sin(\frac{\#J}{N} \pi)}{(\#J \pi) t} \left(\frac{t-s}{s}\right)^{\frac{\#K}{N}}. \end{aligned}$$

Example 2. If N is even, for $s \in (0, t)$,

$$\mathbb{P}_0\{T_0(t) \in ds | X(t) > 0\}/ds = \frac{2}{\pi t} \sqrt{\frac{s}{t-s}}, \quad \mathbb{P}_0\{T_0(t) \in ds | X(t) < 0\}/ds = \frac{2}{\pi t} \sqrt{\frac{t-s}{s}}.$$

The results of Theorems 1 and 2 were found by Hochberg, Nikitin and Orsingher ([7,15,19]) in the cases $N = 3, 4, 5, 7$ and conjectured in the general case.

2.2 Distribution of $(X(t), T_a(t))$

Case $x = a$. Set $\Phi = \int_0^\infty e^{-\lambda t} \mathbb{E}_0(e^{i\mu X(t) - \nu T_0(t)}) dt$ for $\lambda, \nu > 0$ and $\mu \in \mathbb{R}$. The quantity Φ can be understood as

$$\begin{aligned} \Phi &\stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} \mathbb{E}_0\left(e^{i\mu X(k/2^n) - \frac{\nu}{2^n} \sum_{j=1}^k \mathbb{1}_{[0, +\infty)}(X(j/2^n))}\right) dt \\ &= \lim_{n \rightarrow \infty} \frac{1 - e^{-\lambda/2^n}}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda k/2^n} \mathbb{E}_0\left(e^{i\mu X(k/2^n) - \frac{\nu}{2^n} \sum_{j=1}^k \mathbb{1}_{[0, +\infty)}(X(j/2^n))}\right). \end{aligned}$$

A Spitzer's identity yields the following relationship which holds for $|z|, |\zeta| < 1$:

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbb{E}_0 \left(e^{i\mu X(k/2^n)} \zeta^{\sum_{j=1}^k \mathbb{1}_{[0,+\infty)}(X(j/2^n))} \right) z^k \\ = \frac{1}{1-z} \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E}_0 \left(e^{i\mu X(k/2^n)} \zeta^k \mathbb{1}_{[0,+\infty)}(X(k/2^n)) - 1 \right) z^k \right). \end{aligned}$$

With this identity at hand, it can be seen that

$$\Phi = \frac{1}{\prod_{j \in J} (\sqrt[N]{\lambda} + \nu - i\mu\theta_j) \prod_{k \in K} (\sqrt[N]{\lambda} - i\mu\theta_k)}.$$

This three-parameters Laplace-Fourier transform can be inverted ([4]).

Theorem 3 (Camarota & Lachal, 2010). *The pseudo-distribution of the vector $(X(t), T_0(t))$ is given, for $s \in (0, t)$ and $y \leq 0$, by*

$$\begin{aligned} \mathbb{P}_0 \{ X(t) \in dy, T_0(t) \in ds \} / dy ds \\ = -\frac{Ni}{2\pi} \sum_{m=0}^{\#K} \alpha_{-m} s^{\frac{m-\#K}{N}} \int_0^{\infty} \xi^{m+\#J} e^{-(t-s)\xi^N} \mathcal{K}_m(y\xi) E_{1, \frac{m+\#J}{N}}(-s\xi^N) d\xi \end{aligned}$$

and, for $s \in (0, t)$ and $y \geq 0$, by

$$\begin{aligned} \mathbb{P}_0 \{ X(t) \in dy, T_0(t) \in ds \} / dy ds \\ = \frac{Ni}{2\pi} \sum_{m=0}^{\#J} \beta_{-m} (t-s)^{\frac{m-\#J}{N}} \int_0^{\infty} \xi^{m+\#K} e^{-s\xi^N} \mathcal{J}_m(y\xi) E_{1, \frac{m+\#K}{N}}(-(t-s)\xi^N) d\xi \end{aligned}$$

where $\alpha_m = \sum_{j \in J} A_j \theta_j^m$, $\beta_m = \sum_{k \in K} B_k \theta_k^m$ for any integer m (in particular $\alpha_0 = \beta_0 = 1$, $\alpha_{-\#K} = 1, \beta_{-\#J} = (-1)^{\#J}$),

$$\begin{aligned} \mathcal{J}_m(z) &= e^{-i\frac{\#J-m-1}{N}\pi} \sum_{j \in J} A_j \theta_j^{m+1} e^{-\theta_j e^{i\frac{\pi}{N}} z} - e^{i\frac{\#J-m-1}{N}\pi} \sum_{j \in J} A_j \theta_j^{m+1} e^{-\theta_j e^{-i\frac{\pi}{N}} z}, \\ \mathcal{K}_m(z) &= e^{-i\frac{\#K-m-1}{N}\pi} \sum_{k \in K} B_k \theta_k^{m+1} e^{-\theta_k e^{i\frac{\pi}{N}} z} - e^{i\frac{\#K-m-1}{N}\pi} \sum_{k \in K} B_k \theta_k^{m+1} e^{-\theta_k e^{-i\frac{\pi}{N}} z}, \end{aligned}$$

and $E_{a,b}$ is the Mittag-Leffler function $E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}$.

Remark 1. By choosing $y = 0$ in the pseudo-distribution of $(X(t), T_0(t))$ in the foregoing theorem and next dividing the result by $\mathbb{P}_x \{ X(t) = 0 \}$, we could retrieve the pseudo-distribution of the corresponding sojourn time of the "pseudo-bridge" $(X(s) | X(0) = X(t) = 0)_{0 \leq s \leq t}$ displayed in Theorem 2.

Case $x \neq a$. Set $\Phi(x, y) = \int_0^{+\infty} e^{-\lambda t} (\mathbb{E}_x(e^{-\mu T_a(t)}, X(t) \in dy) / dy) dt$ for $\lambda, \mu > 0$ and $x, y \in \mathbb{R}$. It can be seen that Φ solves the differential equation

$$\kappa_N \frac{\partial^N \Phi}{\partial x^N}(x, y) = \begin{cases} (\lambda + \mu) \Phi(x, y) - \delta_y(x) & \text{for } x \in (a, +\infty), \\ \lambda \Phi(x, y) - \delta_y(x) & \text{for } x \in (-\infty, a), \end{cases}$$

with regularity conditions

$$\begin{cases} \forall k \in \{0, 1, \dots, N-1\}, \frac{\partial^k \Phi}{\partial x^k}(a^+, y) = \frac{\partial^k \Phi}{\partial x^k}(a^-, y), \\ \forall k \in \{0, 1, \dots, N-2\}, \frac{\partial^k \Phi}{\partial x^k}(y^+, y) = \frac{\partial^k \Phi}{\partial x^k}(y^-, y) \text{ and } \frac{\partial^{N-1} \Phi}{\partial x^{N-1}}(y^+, y) - \frac{\partial^{N-1} \Phi}{\partial x^{N-1}}(y^-, y) = \kappa_N. \end{cases}$$

This system can be explicitly solved by computing Vandermonde determinants and the inversion of the two-parameters Laplace transform can be performed ([5]).

Theorem 4 (Camarota & Lachal, 2010). *Set*

$$f(t; \theta) = \frac{\sin(\frac{\pi}{N})}{\pi N} \frac{t^{\frac{1}{N}-1}(\theta^2 - t^{\frac{2}{N}})}{(t^{\frac{2}{N}} - 2\theta \cos(\frac{\pi}{N})t^{\frac{1}{N}} + \theta^2)^2}.$$

1. Assume that $y \geq 0$. For $s \in (0, t)$, if $x \in (-\infty, 0]$,

$$\begin{aligned} & \mathbb{P}_x\{X(t) \in dy, T_0(t) \in ds\}/dy ds \\ &= \kappa_N \mathbf{1}_{\{\#J=\#K+1\}} \left(\sum_{j \in J} \theta_j A_j I_{\#K}(t-s; -\theta_j x) \right) \left(\sum_{j \in J} \theta_j^{\#J} A_j I_{\#K}(s; \theta_j y) \right) \\ &+ \kappa_N \sum_{j, j' \in J, k \in K} \frac{A_j \theta_{j'} A_{j'} \theta_k^{\#K-\#J} B_k C_{jj'k}}{\theta_j^{\#K-1} (\theta_{j'} - \theta_k)} \int_0^s \sigma^{\frac{1}{N}-1} I_{\#J-1}(s-\sigma; \theta_{j'} y) d\sigma \\ &\times \int_0^{t-s} I_{\#K}(\tau; -\theta_j x) f\left(t-s-\tau; \frac{\theta_j}{\theta_k} \sqrt[N]{\sigma}\right) d\tau \end{aligned}$$

and if $x \in [0, \infty)$,

$$\begin{aligned} & \mathbb{P}_x\{X(t) \in dy, T_0(t) \in ds\}/dy ds \\ &= \kappa_N \sum_{j, j' \in J, k \in K} \frac{\theta_j A_j A_{j'} B_k C_{jj'k}}{\theta_k^{\#J-1} (\theta_j - \theta_k)} \int_0^s \sigma^{\frac{1}{N}-1} I_{N-2}(s-\sigma; \theta_j y - \theta_k x) d\sigma \\ &\times \int_0^{t-s} \frac{(t-s-\tau)^{\frac{1}{N}-1}}{\Gamma(\frac{1}{N})} f\left(\tau; \frac{\theta_{j'}}{\theta_k} \sqrt[N]{\sigma}\right) d\tau. \end{aligned}$$

For $s = t$, if $x \in (-\infty, 0]$, $\mathbb{P}_x\{X(t) \in dy, T_0(t) = t\}/dy = 0$ and if $x \in [0, +\infty)$, there is an atom given by

$$\mathbb{P}_x\{X(t) \in dy, T_0(t) = t\}/dy = p(t; x-y) + \sum_{j \in J, k \in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j - \theta_k} I_{N-1}(t; \theta_j y - \theta_k x).$$

2. Assume that $y \leq 0$. For $s \in (0, t)$, if $x \in (-\infty, 0]$,

$$\begin{aligned} & \mathbb{P}_x\{X(t) \in dy, T_0(t) \in ds\}/dy ds \\ &= - \sum_{j \in J, k, k' \in K} \frac{A_j \theta_k B_k B_{k'} D_{jkk'}}{\theta_j^{\#K-2} \theta_{k'} (\theta_k - \theta_j)} s^{\frac{1}{N}-1} \int_0^{t-s} I_{N-1}(\tau; \theta_k y - \theta_j x) f\left(t-s-\tau; \frac{\theta_j}{\theta_{k'}} \sqrt[N]{s}\right) d\tau \end{aligned}$$

and if $x \in [0, \infty)$,

$$\mathbb{P}_x\{X(t) \in dy, T_0(t) \in ds\}/dy ds$$

$$\begin{aligned}
&= \mathbf{1}_{\{\#K=\#J+1\}} \left(\sum_{k \in K} \theta_k B_k I_{\#J}(s; -\theta_k x) \right) \left(\sum_{k \in K} \theta_k^{\#K} B_k I_{\#J}(t-s; \theta_k y) \right) \\
&+ \sum_{j \in J, k, k' \in K} \frac{\theta_j^{\#J-\#K+1} A_j B_k \theta_{k'} B_{k'} D_{jkk'}}{\theta_k^{\#J} (\theta_j - \theta_{k'})} \int_0^s \sigma^{\frac{1}{N}-1} I_{\#J-1}(s-\sigma; -\theta_k x) d\sigma \\
&\times \int_0^{t-s} I_{\#K}(\tau; \theta_{k'} y) f\left(t-s-\tau; \frac{\theta_j}{\theta_k} \sqrt[N]{\sigma}\right) d\tau.
\end{aligned}$$

For $s = 0$, if $x \in [0, \infty)$, $\mathbb{P}_x\{X(t) \in dy, T_0(t) = 0\}/dy = 0$ and if $x \in (-\infty, 0]$, there is an atom given by

$$\mathbb{P}_x\{X(t) \in dy, T_0(t) = 0\}/dy = p(t; x-y) + \sum_{j \in J, k \in K} \frac{\theta_j A_j \theta_k B_k}{\theta_j - \theta_k} I_{N-1}(t; \theta_k y - \theta_j x).$$

The following relationship between $T_0(t)$, $M(t)$ and τ_0 holds: for $x, y \leq 0$,

$$\mathbb{P}_x\{X(t) \in dy, T_0(t) = 0\}/dy = \mathbb{P}_x\{X(t) \in dy, M(t) < 0\}/dy = \mathbb{P}_x\{X(t) \in dy, \tau_0 > t\}/dy.$$

Remark 2. By integrating the joint pseudo-distribution of $(X(t), T_0(t))$ with respect to y in the foregoing theorem, we could derive the pseudo-distribution of $T_0(t)$. Actually, the result does not simplify so much.

3 Distribution related to $M(t)$

See [1,3,6,9–12,17] for references related to the various distributions related to $M(t)$.

3.1 Distribution of $M(t)$

The variables $T_a(t)$ and $M(t)$ are related together according to

$$\mathbb{P}_x\{M(t) \leq a\} = \mathbb{P}_x\{T_a(t) = 0\} = \lim_{\mu \rightarrow +\infty} \mathbb{E}_x(e^{-\mu T_a(t)}).$$

The quantity $\mathbb{P}_x\{M(t) \leq a\}$ should be understood as

$$\mathbb{P}_x\{M(t) \leq a\} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mathbb{P}_x\left\{ \max_{0 \leq k \leq n} X\left(\frac{kt}{n}\right) \leq a \right\}.$$

Thanks to this connection, it is possible to deduce that, for $x \leq a$ and $\lambda > 0$,

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{P}_x\{M(t) \leq a\} dt = \frac{1}{\lambda} \left(1 - \sum_{j \in J} A_j e^{\theta_k \sqrt[N]{\lambda} (x-a)} \right).$$

The foregoing Laplace transform can be inverted ([9]).

Theorem 5 (Lachal, 2003). *The pseudo-distribution of $M(t)$ is given by*

$$\begin{aligned}
\mathbb{P}_x\{M(t) \geq a\} &= \sum_{m=0}^{\#J-1} a_m \int_0^t \frac{\partial^m p}{\partial x^m}(s; x-a) \frac{ds}{(t-s)^{1-(m+1)/N}} \\
&= 2 \mathbb{P}_x\{X(t) \geq a\} - \sum_{m=0}^{\#J-1} b_m \int_0^t \frac{\partial^m p}{\partial x^m}(s; x-a) \frac{ds}{(t-s)^{1-(m+1)/N}}
\end{aligned}$$

where

$$a_m = \frac{(-1)^m N}{\Gamma(\frac{m+1}{N})} \sum_{j \in J} A_j^2 \sigma_{j, \#J-1-m} \quad \text{and} \quad b_m = \frac{2(-1)^m}{\Gamma(\frac{m+1}{N})} \sum_{j \in J} A_j \sigma_{j, \#J-1-m} - a_m.$$

The coefficients $\sigma_{j,p}$, $j \in J$, $0 \leq p \leq \#J - 1$, are given by $\sigma_{j,0} = 1$ and for $1 \leq p \leq \#J - 1$,

$$\sigma_{j,p} = \sum_{\substack{\ell_1, \dots, \ell_p \in J \setminus \{j\} \\ \ell_1 < \dots < \ell_p}} \theta_{\ell_1} \cdots \theta_{\ell_p}.$$

Remark 3. From the second displayed expression of $\mathbb{P}_x\{M(t) \geq a\}$, we can see that the famous reflection principle for Brownian motion does not hold any longer for pseudo-processes related to an order $N > 2$.

Example 3. For $N = 3$, Orsingher ([19]) derived the historical result

$$\begin{aligned} & \mathbb{P}_x\{M(t) \geq a\} \\ &= \begin{cases} \frac{3}{\Gamma(\frac{1}{3})} \int_0^t p(s; x-a) \frac{ds}{(t-s)^{\frac{2}{3}}} & \text{when } \kappa_N = +1, \\ \frac{2}{\Gamma(\frac{1}{3})} \int_0^t p(s; x-a) \frac{ds}{(t-s)^{\frac{2}{3}}} - \frac{1}{\Gamma(\frac{2}{3})} \int_0^t \frac{\partial p}{\partial x}(s; x-a) \frac{ds}{(t-s)^{\frac{1}{3}}} & \text{when } \kappa_N = -1. \end{cases} \end{aligned}$$

For $N = 4$, Hochberg ([6]) derived the historical result

$$\int_0^{+\infty} e^{-\lambda t} (\mathbb{P}_x\{M(t) \in da\}/da) dt = -\frac{\sqrt{2}}{\lambda^{3/4}} e^{\sqrt[4]{\lambda}(x-a)/\sqrt{2}} \sin\left(\frac{\sqrt[4]{\lambda}(x-a)}{\sqrt{2}}\right)$$

which was subsequently completed by Beghin, Orsingher and Ragozina ([3]):

$$\mathbb{P}_x\{M(t) \geq a\} = \frac{2\sqrt{2}}{\Gamma(\frac{1}{4})} \int_0^t p(s; x-a) \frac{ds}{(t-s)^{\frac{3}{4}}}.$$

3.2 Distribution of $(X(t), M(t))$

Set $\Phi(x) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x(e^{i\mu X(t) - \nu M(t)}) dt$ for $\lambda, \nu > 0$, $\mu \in \mathbb{R}$ and $x \in \mathbb{R}$. The quantity $\Phi(x)$ can be understood as

$$\begin{aligned} \Phi(x) &\stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t} \mathbb{E}_x\left(e^{i\mu X(k/2^n) - \nu \max_{0 \leq j \leq k} X(j/2^n)}\right) dt \\ &= \lim_{n \rightarrow \infty} e^{(i\mu - \nu)x} \frac{1 - e^{-\lambda/2^n}}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda k/2^n} \mathbb{E}_0\left(e^{i\mu X(k/2^n) - \nu \max_{0 \leq j \leq k} X(j/2^n)}\right). \end{aligned}$$

Another Spitzer's identity yields the following relationship which holds for $|z| < 1$:

$$\sum_{k=0}^{\infty} \mathbb{E}_0\left(e^{i\mu X(k/2^n) - \nu \max_{0 \leq j \leq k} X(j/2^n)}\right) z^k = \frac{1}{1-z} \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E}_0\left(e^{i\mu X(k/2^n) - \nu X(k/2^n)^+} - 1\right) z^k\right).$$

The Laplace-Fourier transform of the vector $(X(t), M(t))$ ensues:

$$\Phi(x) = \frac{e^{(i\mu - \nu)x}}{\prod_{j \in J} (\sqrt[N]{\lambda} - (i\mu - \nu)\theta_j) \prod_{k \in K} (\sqrt[N]{\lambda} - i\mu\theta_k)}.$$

This three-parameters transform can be progressively inverted ([11]). For $z \geq x \vee y$,

$$\int_0^{+\infty} e^{-\lambda t} [\mathbb{P}_x\{X(t) \in dy, M(t) \in dz\} / dy dz] dt = \frac{1}{\lambda} \chi_J(\lambda; x - z) \chi_K(\lambda; z - y),$$

with

$$\chi_J(\lambda; \xi) = \sqrt[N]{\lambda} \sum_{j \in J} \theta_j A_j e^{\theta_j \sqrt[N]{\lambda} \xi}, \quad \chi_K(\lambda; \xi) = -\sqrt[N]{\lambda} \sum_{k \in K} \theta_k B_k e^{\theta_k \sqrt[N]{\lambda} \xi}.$$

This last Laplace transform can also be inverted ([11]).

Theorem 6 (Lachal, 2006). *The joint pseudo-distribution of $(X(t), M(t))$ admits the representation below. For $z \geq x \vee y$,*

$$\begin{aligned} \mathbb{P}_x\{X(t) \leq y \leq z \leq M(t)\} &= \sum_{\substack{k \in K \\ 0 \leq m \leq \#J-1}} a_{km} \int_0^t \int_0^s \frac{\partial^m p}{\partial x^m}(\sigma; x - z) \frac{I_0(s - \sigma; \theta_k(z - y))}{(t - s)^{1-(m+1)/N}} ds d\sigma \\ &= \sum_{\substack{j \in J \\ 0 \leq m \leq \#K-1}} b_{jm} \int_0^t \int_0^s \frac{\partial^m p}{\partial x^m}(\sigma; z - y) \frac{I_0(s - \sigma; \theta_j(x - z))}{(t - s)^{1-(m+1)/N}} ds d\sigma \end{aligned}$$

where

$$a_{km} = \frac{(-1)^m N B_k}{\Gamma(\frac{m+1}{N})} \sum_{j \in J} \frac{\theta_j A_j^2 \sigma_{j, \#J-1-m}}{\theta_j - \theta_k} \quad \text{and} \quad b_{jm} = \frac{(-1)^m N \theta_j A_j}{\Gamma(\frac{m+1}{N})} \sum_{k \in K} \frac{B_k^2 \sigma_{k, \#K-1-m}}{\theta_k - \theta_j}.$$

Example 4. For $N = 3$, Beghin, Orsingher and Ragozina ([3]) derived the result

$$\mathbb{P}_x\{X(t) \leq y \leq z \leq M(t)\} = \begin{cases} \frac{1}{\Gamma(1/3)} \int_0^t \int_0^s p(\sigma; x - z) q(s - \sigma; z - y) \frac{ds d\sigma}{(t - s)^{2/3}} & \text{when } \kappa_N = +1, \\ \frac{1}{\Gamma(1/3)} \int_0^t \int_0^s p(\sigma; z - y) q(s - \sigma; x - z) \frac{ds d\sigma}{(t - s)^{2/3}} & \text{when } \kappa_N = -1, \end{cases}$$

with

$$p(t; \xi) = \frac{1}{\pi} \int_0^{+\infty} \cos(\xi \lambda - t \lambda^3) d\lambda$$

and

$$q(t; \xi) = \begin{cases} \frac{\xi}{\pi t} \int_0^{+\infty} e^{-t\lambda^3 + \frac{1}{2}\xi\lambda} \sin\left(\frac{\sqrt{3}}{2}\xi\lambda + \frac{\pi}{3}\right) d\lambda & \text{when } \kappa_N = +1, \\ \frac{\xi}{\pi t} \left[\sqrt{3} \int_0^{+\infty} e^{-t\lambda^3 + \xi\lambda} d\lambda + \int_0^{+\infty} e^{-t\lambda^3 - \frac{1}{2}\xi\lambda} \sin\left(\frac{\sqrt{3}}{2}\xi\lambda + \frac{\pi}{3}\right) d\lambda \right] & \text{when } \kappa_N = -1. \end{cases}$$

For $N = 4$, they derived

$$\begin{aligned} \mathbb{P}_x\{X(t) \leq y \leq z \leq M(t)\} &= \int_0^t \int_0^s p(\sigma; x - z) q_1(s - \sigma; z - y) \frac{ds d\sigma}{(t - s)^{3/4}} \\ &\quad + \int_0^t \int_0^s \frac{\partial p}{\partial x}(\sigma; x - z) q_2(s - \sigma; z - y) \frac{ds d\sigma}{\sqrt{t - s}} \end{aligned}$$

with

$$q_1(t; \xi) = \frac{\xi}{\pi\sqrt{2}\Gamma(1/4)t} \int_0^{+\infty} e^{-t\lambda^4} \cos(\xi\lambda) d\lambda,$$

$$q_2(t; \xi) = \frac{\xi}{2\pi^2 t} \int_0^{+\infty} e^{-t\lambda^4} [\cos(\xi\lambda) + \sin(\xi\lambda) - e^{-\xi\lambda}] d\lambda.$$

3.3 Distribution of $(X^b(t), M^b(t))$

The Laplace-Fourier transform of the vector $(X^b(t), M^b(t))$ is given, for $\lambda, \nu > 0$ and $\mu \in \mathbb{R}$, by

$$\mathbb{E}_x \left(\int_0^{+\infty} e^{-\lambda t + i\mu X^b(t) - \nu M^b(t)} dt \right) = \frac{e^{(i\mu - \nu)x}}{\prod_{j \in J} (\omega_j^b(\lambda) + \mu + i\nu) \prod_{k \in K} (\omega_k^b(\lambda) + \mu)}.$$

This three-parameters transform can be partially inverted for giving the following result ([12]).

Theorem 7 (Lachal, 2008). *The Laplace transform with respect to time t of the joint pseudo-distribution of $(X^b(t), M^b(t))$ is given, for $\lambda > 0$ and $z \geq x \vee y$, by*

$$\int_0^{+\infty} e^{-\lambda t} [\mathbb{P}_x \{X^b(t) \in dy, M^b(t) \in dz\} / dy dz] dt = \chi_J^b(\lambda; x - z) \chi_K^b(\lambda; z - y)$$

where

$$\chi_J^b(\lambda; \xi) = \sum_{j \in J} \frac{e^{-i\omega_j^b(\lambda)\xi}}{\prod_{\ell \in J \setminus \{j\}} (\omega_\ell^b(\lambda) - \omega_j^b(\lambda))} \quad \text{and} \quad \chi_K^b(\lambda; \xi) = \sum_{k \in K} \frac{e^{-i\omega_k^b(\lambda)\xi}}{\prod_{l \in K \setminus \{k\}} (\omega_l^b(\lambda) - \omega_k^b(\lambda))}.$$

3.4 Distribution of $\sigma_a(t)$

The variables $\sigma_a(t)$ and $M(t)$ are related together according as

$$\mathbb{P}_x \{\sigma_a(t) \leq s\} = \mathbb{P}_x \left\{ \max_{s \leq u \leq t} X(u) \leq a \right\} = \int_{-\infty}^a p(s; x - y) \mathbb{P}_y \{M(t - s) \leq a\} dy$$

The pseudo-distribution of $\sigma_a(t)$ under \mathbb{P}_a is the same as that of $\sigma_0(t)$ under \mathbb{P}_0 .

Theorem 8 (Lachal, 2003). *The iterated Laplace transform of $\sigma_0(t)$ under \mathbb{P}_0 is given, for $\lambda, \mu > 0$, by*

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0(e^{-\mu\sigma_0(t)}) dt = \frac{1}{\lambda + \mu} \left[1 - \frac{1}{N} \sum_{j \in J} \prod_{j' \in J} \left(1 - \frac{\theta_j}{\theta_{j'}} \sqrt{\frac{\lambda + \mu}{\lambda}} \right) \right].$$

4 Distribution related to τ_a

In this section, N is assumed to be an even integer. The reader is referred to [10–12,14,16–18].

4.1 Distribution of $(\tau_a, X(\tau_a))$

Using the definition

$$\mathbb{E}_x\left(e^{-\lambda\tau_a+i\mu X(\tau_a)}\right) = \lim_{n\rightarrow+\infty} \mathbb{E}_x\left(e^{-\lambda\frac{k}{2^n}+i\mu X(\frac{k}{2^n})} \mathbb{1}_{\{X(\frac{k-1}{2^n})<a\leq X(\frac{k}{2^n})\}}\right),$$

It can be seen that the Laplace-Fourier transform of the vector $(\tau_a, X(\tau_a))$ is related to the pseudo-distribution of the vector $(X(t), M(t))$ according as, for $\lambda > 0$, $\mu \in \mathbb{R}$ and $x \leq a$:

$$\mathbb{E}_x\left(e^{-\lambda\tau_a+i\mu X(\tau_a)}\right) = (\lambda + \mu^N) \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x\left(e^{i\mu X(t)} \mathbb{1}_{\{M(t)>a\}}\right) dt.$$

From this, it can be deduced that, for $x \leq a$,

$$\mathbb{E}_x\left(e^{-\lambda\tau_a+i\mu X(\tau_a)}\right) = \sum_{j \in J} A_j \prod_{\ell \in J \setminus \{j\}} \left(1 - \frac{i\mu}{\sqrt[N]{\lambda}} \bar{\theta}_\ell\right) e^{\theta_j \sqrt[N]{\lambda}(x-a)} e^{i\mu a}.$$

In particular,

$$\mathbb{E}_x\left(e^{-\lambda\tau_a}\right) = \lambda \int_0^{+\infty} e^{-\lambda t} \mathbb{P}_x\{M(t) > a\} dt = \sum_{j \in J} A_j e^{\theta_j \sqrt[N]{\lambda}(x-a)}.$$

The variables τ_a and $M(t)$ are related together according to $\mathbb{P}_x\{\tau_a \leq t\} = \mathbb{P}_x\{M(t) \geq a\}$. The two-parameters Laplace-Fourier transform can be inverted ([11]).

Theorem 9 (Lachal, 2006). *The joint pseudo-distribution of $(\tau_a, X(\tau_a))$ is given, for $x < a$, by*

$$\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt dz = \sum_{p=0}^{N/2-1} \mathcal{J}_p(t; x-a) \delta_a^{(p)}(z)$$

with $\mathcal{J}_p(t; \xi) = \sum_{j \in J} \overline{\sigma_{j,p}} A_j I_p(t; \theta_j \xi)$. In particular,

$$\mathbb{P}_x\{\tau_a \in dt\}/dt = \mathcal{J}_0(t; x-a), \quad \mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \sum_{p=0}^{N/2-1} (-1)^p \frac{(x-a)^p}{p!} \delta_a^{(p)}(z).$$

The $\delta_a^{(p)}$ are the successive derivatives of the Schwartz distribution δ_a , that is, for any test function ϕ , $\langle \delta_a^{(p)}, \phi \rangle = (-1)^p \phi^{(p)}(a)$. The pseudo-distribution of $X(\tau_a)$ is remarkable since it means that the pseudo-process $(X(t))_{t \geq 0}$ is formally concentrated at the site a at time τ_a in a “distributional” sense.

Example 5. In the case $N = 4$, Nishioka ([16,17]) obtained the remarkable result

$$\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \delta_a(z) - (x-a)\delta'_a(z).$$

Moreover,

$$\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt dz = \mathcal{J}_0(t; x-a) \delta_a(z) + \mathcal{J}_1(t; x-a) \delta'_a(z)$$

with

$$\begin{aligned} \mathcal{J}_0(t; \xi) &= \frac{\xi}{2\pi t} \int_0^{+\infty} (e^{\xi\lambda} - \cos(\xi\lambda) + \sin(\xi\lambda)) e^{-t\lambda^4} d\lambda, \\ \mathcal{J}_1(t; \xi) &= \frac{2}{\pi} \int_0^{+\infty} (\cos(\xi\lambda) + \sin(\xi\lambda) - e^{\xi\lambda}) \lambda^2 e^{-t\lambda^4} d\lambda. \end{aligned}$$

4.2 Distribution of $(\tau_a^b, X(\tau_a^b))$

The pseudo-distributions of the vectors $(\tau_a^b, X^b(\tau_a^b))$ and $(X^b(t), M^b(t))$ are related together according as, for $\lambda > 0$, $\mu \in \mathbb{R}$ and $x \leq a$,

$$\mathbb{E}_x\left(e^{-\lambda\tau_a^b + i\mu X^b(\tau_a^b)} \mathbb{1}_{\{\tau_a^b < \infty\}}\right) = (\lambda - ib\mu + \mu^N) \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x\left(e^{i\mu X^b(t)} \mathbb{1}_{\{M^b(t) > a\}}\right) dt$$

from which it comes that the Laplace-Fourier transform of the vector $(\tau_a^b, X^b(\tau_a^b))$ is given, for $\lambda > 0$, $\mu \in \mathbb{R}$ and $x \leq a$, by

$$\mathbb{E}_x\left(e^{-\lambda\tau_a^b + i\mu X^b(\tau_a^b)} \mathbb{1}_{\{\tau_a^b < \infty\}}\right) = \sum_{j \in J} \left(\prod_{\ell \in J \setminus \{j\}} \frac{\omega_\ell^b(\lambda) + \mu}{\omega_\ell^b(\lambda) - \omega_j^b(\lambda)} \right) e^{i\mu a - i\omega_j^b(\lambda)(x-a)}.$$

Example 6. In the case $N = 4$, Nakajima and Sato ([14]) derived the following result:

$$\mathbb{E}_x\left(e^{-\lambda\tau_a^b + i\mu X^b(\tau_a^b)} \mathbb{1}_{\{\tau_a^b < \infty\}}\right) = e^{i\mu a} \left(\frac{\mu + \omega_1}{\omega_2 - \omega_1} e^{-i\omega_1(x-a)} + \frac{\mu + \omega_2}{\omega_1 - \omega_2} e^{-i\omega_2(x-a)} \right)$$

where ω_1 and ω_2 are the two roots of the polynomial $X^4 + ibX + \lambda$ having positive imaginary part.

We can deduce the pseudo-distribution of the overshooting place $X^b(\tau_a^b)$ on the set $\{\tau_a^b < \infty\}$ and the pseudo-probability of eventually hitting the interval $[a, +\infty)$ ([12]).

Theorem 10 (Lachal, 2006). *The pseudo-distribution of the overshooting place $X^b(\tau_a^b) \times \mathbb{1}_{\{\tau_a^b < \infty\}}$ is given, for $x \leq a$, by*

$$\mathbb{P}_x\{X^b(\tau_a^b) \in dz, \tau_a^b < \infty\}/dz = \sum_{p=0}^{N/2-1} \frac{i^p}{|b|^{N-p-1}} \left(\sum_{j \in \tilde{J}} \frac{\tilde{\sigma}_{j, N/2-p-1}}{\prod_{\ell \in \tilde{J} \setminus \{j\}} (\tilde{\theta}_\ell - \tilde{\theta}_j)} e^{-i\tilde{\theta}_j |b|^{\frac{1}{N-1}}(x-a)} \right) \delta_a^{(p)}(z)$$

where the $\tilde{\sigma}_{j,p}$'s are given by $\tilde{\sigma}_{j,0} = 1$ and for $1 \leq p \leq N/2 - 1$,

$$\tilde{\sigma}_{j,p} = \sum_{\substack{\ell_1 < \dots < \ell_p \\ \ell_1, \dots, \ell_p \in \tilde{J} \setminus \{j\}}} \tilde{\theta}_{\ell_1} \dots \tilde{\theta}_{\ell_p}.$$

The pseudo-probability of eventually overshooting the level a is given by

$$\mathbb{P}_x\{\tau_a^b < \infty\} = \begin{cases} 1 & \text{if } b > 0, \\ \sum_{j \in \tilde{J}} \frac{e^{-i\tilde{\theta}_j |b|^{\frac{1}{N-1}}(x-a)}}{\prod_{\ell \in \tilde{J} \setminus \{j\}} (1 - e^{i\frac{2(j-\ell)}{N-1}\pi})} & \text{if } b < 0. \end{cases}$$

Example 7. For $N = 4$, the above results yield for $x \leq a$, in the case where $b > 0$,

$$\mathbb{P}_x\{\tau_a^b < \infty\} = 1, \quad \mathbb{P}_x\{X^b(\tau_a^b) \in dz, \tau_a^b < \infty\}/dz = \delta_a(z) + \frac{1}{\sqrt[3]{b}} (1 - e^{\sqrt[3]{b}(x-a)}) \delta'_a(z),$$

and, in the case where $b < 0$,

$$\mathbb{P}_x\{\tau_a^b < \infty\} = p_0^b(x-a), \quad \mathbb{P}_x\{X^b(\tau_a^b) \in dz, \tau_a^b < \infty\}/dz = p_0^b(x-a)\delta_a(z) + p_1^b(x-a)\delta'_a(z)$$

with

$$p_0^b(\xi) = \frac{2}{\sqrt{3}} e^{\frac{1}{2}\sqrt[3]{|b|}\xi} \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{|b|}\xi + \frac{\pi}{6}\right), \quad p_1^b(\xi) = -\frac{2}{\sqrt{3}\sqrt[3]{|b|}} e^{\frac{1}{2}\sqrt[3]{|b|}\xi} \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{|b|}\xi\right).$$

5 Works in progress

The problem of the two-sided barrier $\{a, b\}$ is much more difficult to tackle than the single one. The variables $T_{ab}(t)$, τ_{ab} and the maximum/minimum functionals are related together according as, for $x \in (a, b)$,

$$\mathbb{P}_x\{\tau_{ab} \geq t\} = \mathbb{P}_x\{a \leq m(t) \leq M(t) \leq b\} = \mathbb{P}_x\{T_{ab}(t) = t\} = \lim_{\mu \rightarrow +\infty} \mathbb{E}_x\left(e^{-\mu(T_{ab}(t)-t)}\right).$$

5.1 Distributions related to $T_{ab}(t)$

The variable $T_{ab}(t)$ is introduced in [2] for defining a local time for the pseudo-process $(X(t))_{t \geq 0}$. Set $\Phi(x) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x(e^{-\mu T_{ab}(t)}) dt$ for $\lambda, \mu > 0$ and $x \in \mathbb{R}$. The function Φ satisfies the system

$$\kappa_N \frac{d^N \Phi}{dx^N}(x) = \begin{cases} (\lambda + \mu) \Phi(x) - 1 & \text{for } x \in (a, b), \\ \lambda \Phi(x) - 1 & \text{for } x \notin (a, b), \end{cases}$$

and

$$\forall k \in \{0, 1, \dots, N-1\}, \frac{d^k \Phi}{dx^k}(a^+) = \frac{d^k \Phi}{dx^k}(a^-) \text{ and } \frac{d^k \Phi}{dx^k}(b^+) = \frac{d^k \Phi}{dx^k}(b^-).$$

It seems to be difficult to solve explicitly this system. In the same way, the problem of the joint pseudo-distribution of $(X(t), T_{ab}(t))$ should be more complicated.

The sojourn time within a strip is used by Beghin & Orsingher ([2]) for defining a local time at 0 for $(X(t))_{t \geq 0}$: $L(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{X(s) \in [-\varepsilon, \varepsilon]\}} ds$. They obtained the very simple formula

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_0(e^{-\mu L(t)}) dt = \frac{1}{\lambda + c\mu\sqrt[N]{\lambda}}$$

where

$$c = \frac{1}{\alpha N \sin \frac{\pi}{\alpha N}}, \quad \alpha = \begin{cases} 1 & \text{if } N \text{ is odd,} \\ 2 & \text{if } N \text{ is even.} \end{cases}$$

5.2 Distributions related to τ_{ab}

By introducing a pseudo-random walk as in [20] and studying the similar functional to τ_{ab} (first exit time from a finite interval), we can derive the pseudo-distribution of $(\tau_{ab}, X(\tau_{ab}))$ in the case where N is even. We have obtained the following result ([13]).

Theorem 11 (Lachal, 2010). *The pseudo-distribution of $X(\tau_{ab})$ has the form*

$$\mathbb{P}_x\{\tau_{ab} \in dt, X(\tau_{ab}) \in dz\}/dt dz = \sum_{p=0}^{N/2-1} \mathcal{J}_p(x) \delta_a^{(p)}(z) + \sum_{p=0}^{N/2-1} \mathcal{K}_p(x) \delta_b^{(p)}(z)$$

where \mathcal{J}_p and \mathcal{K}_p are some functions. The pseudo-distribution of $X(\tau_{ab})$ is given by

$$\mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz = \sum_{p=0}^{N/2-1} H_p^-(x) \delta_a^{(p)}(z) + \sum_{p=0}^{N/2-1} H_p^+(x) \delta_b^{(p)}(z)$$

where the functions H_p^- and H_p^+ , $0 \leq p \leq N/2 - 1$, are the interpolation Hermite polynomials such that

$$\frac{d^q H_p^-}{dx^q}(a) = \delta_{pq}, \quad \frac{d^q H_p^-}{dx^q}(b) = 0, \quad \frac{d^q H_p^+}{dx^q}(a) = 0, \quad \frac{d^q H_p^+}{dx^q}(b) = \delta_{pq} \text{ for } 0 \leq q \leq N/2 - 1.$$

In particular, the ‘‘ruin pseudo-probabilities’’ are given by

$$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = H_0^-(x) \text{ and } \mathbb{P}_x\{\tau_b^+ < \tau_a^-\} = H_0^+(x).$$

Example 8. In the case $N = 4$, the above results supply

$$\mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz = H_0^-(x) \delta_a^-(z) + H_1^-(x) \delta_a'(z) + H_0^+(x) \delta_b^-(z) + H_1^+(x) \delta_b'(z)$$

where

$$\begin{aligned} H_0^-(x) &= \frac{(x-b)^2(2x+b-3a)}{(b-a)^3}, & H_1^-(x) &= -\frac{(x-a)(x-b)^2}{(b-a)^2}, \\ H_0^+(x) &= -\frac{(x-a)^2(2x+a-3b)}{(b-a)^3}, & H_1^+(x) &= -\frac{(x-a)^2(x-b)}{(b-a)^2}. \end{aligned}$$

5.3 Distributions related to $\sigma_{ab}(t)$

By applying the pseudo-Markov property, it can be easily seen that the distributions of $\sigma_{ab}(t)$ and τ_{ab} are related together according as, for $x \in \mathbb{R}$ and $\sigma \in [0, t]$,

$$\mathbb{P}_x\{\sigma_{ab}(t) \leq \sigma\} = \mathbb{P}_x\{\forall s \in [\sigma, t], X(s) \in [a, b]\} = \mathbb{E}_x(\mathbb{P}_{X(\sigma)}\{\tau_{ab} \geq t - \sigma\} \mathbb{1}_{\{X(\sigma) \in [a, b]\}}).$$

This equality may be extended to an ‘‘excursion’’ between the thresholds a and b . Indeed, by introducing $\varsigma_{ab}(t) = \inf\{s \geq t : X(s) \notin (a, b)\}$, we have, for $x \in \mathbb{R}$ and $0 \leq \sigma \leq t \leq \varsigma$,

$$\begin{aligned} \mathbb{P}_x\{\sigma_{ab}(t) \leq \sigma, \varsigma_{ab}(t) \geq \varsigma, X(\varsigma_{ab}(t)) \in dy\} &= \mathbb{P}_x\{\forall s \in [\sigma, \varsigma], X(s) \in [a, b], X(\varsigma_{ab}(t)) \in dy\} \\ &= \mathbb{E}_x(\mathbb{P}_{X(\sigma)}\{\tau_{ab} \geq \varsigma - \sigma, X(\tau_{ab}) \in dy\} \mathbb{1}_{\{X(\sigma) \in [a, b]\}}). \\ &= \int_a^b p(\sigma; x - z) \mathbb{P}_z\{\tau_{ab} \geq \varsigma - \sigma, X(\tau_{ab}) \in dy\} dz. \end{aligned}$$

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