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# Reduction Algorithm for Random Abstract Simplicial Complexes

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## Abstract

In this paper, we present a reduction algorithm for abstract simplicial complexes that we apply to Vietoris-Rips complexes based on random point processes. Our algorithm aims at reducing the number of simplices of an abstract simplicial complex without modifying its topology, i.e. its Betti numbers. The vertices are removed in an optimal order for the abstract simplicial complex implementation complexity. Some mathematical properties of the reduction algorithm and its resulting abstract simplicial complex are derived. Then the complexity of the algorithm is investigated: we are reduced to compute the behavior of the size of the largest simplex in the abstract simplicial complex, which is known in graph theory as the clique number. We find its asymptotic behavior when the number of vertices goes to infinity depending on the percolation regime for the underlying random geometric graph on the torus.

## Index Terms

Simplicial homology, random configurations, reduction algorithm.

## I. INTRODUCTION

**A**LGEBRAIC topology is the field of mathematics that uses algebraic tools to study topological properties of spaces. Simplicial homology is the branch of algebraic topology dealing with topological spaces made up of  $k$ -simplices, the  $k$ -dimensional counterparts of triangles. First developed for the classification of manifolds, it is now heavily used in image processing and geometric data analysis. More recently, simplicial homology representation for wireless sensor networks was initiated in [1]. Consider a set of sensors deployed in a bounded domain of the plane, which can monitor some aspects (temperature, humidity, intrusion, etc) inside a limited region around each of them. The coverage problem is then to know whether the domain is fully monitored by the sensor network, which holds if there is no coverage hole. First, simplicial homology provides a mathematical tool to represent the wireless sensor network by the union of coverage disks centered on each sensor via the Čech complex. The method consists in calculating from the geometric data, a combinatorial object known as the simplicial complex. Basically an abstract simplicial complex is a list of vertices, edges, triangles, tetrahedra, etc, called  $k$ -simplices, satisfying one condition: every subset of a  $k$ -simplex of the complex must be in the list of  $l$ -simplices,  $l < k$ , of the complex. Then algebraic topology gives a way to compute the number of coverage holes in a simplicial complex. An algebraic structure on the list of simplices and linear maps between them are build. Some of the topological properties, such as the number of connected components and the number of coverage holes, are then given by the dimension of some quotient vector spaces: the so-called Betti numbers.

The coverage of a wireless sensor network is computed via homology in [1], [2]. However, the computational time can explode with the size of the simplicial complex. Persistent homology provides a way to compute the Betti numbers while avoiding the combinatorial explosion, as seen in [3]–[5]. Another way to avoid the combinatorial explosion is to reduce the size of the simplicial complex. For example, in [6], [7], the authors use reduction of chain complexes in order to compute the homology groups and the Betti numbers. Witness complexes are another example of simplicial complexes reduction: the simplicial complex is reduced to a given number of vertices in order to compute the various topological invariants, such as the Betti numbers. So reduction of a simplicial complex has been used in order to compute its homology. We intend to do the opposite: reduction of the simplicial complex become the aim, while the homology computation is the mean to do it.

In this paper, we present a reduction algorithm for abstract simplicial complexes: vertices are removed one by one from an abstract simplicial complex while its topology remains intact. The order in which the vertices are removed is such that the complexity of the abstract simplicial complex implementation is reduced as much as possible. When a vertex is removed from a simplicial complex, all the simplices it is a face of are subsequently removed. The algorithm and its properties are presented here for a particular random abstract simplicial complexes: the Vietoris-Rips complex built on either a binomial or a Poisson point process. However it can be applied to any type of abstract simplicial complexes. We show that the algorithm reaches a local optimum solution and we exhibit both an upper and a lower bound for the number of removed vertices. We also provide some characteristics of the resulting abstract simplicial complex. Finally we investigate the complexity of our reduction algorithm. We show that it depends on the size of the largest simplex of the abstract simplicial complex of which we compute the almost sure asymptotical behavior on the torus.

This is the first reduction algorithm for abstract simplicial complexes that uses homology to reduce the complex that we know of. The reduction problem can be seen as a dominating graph problem [8]. However, since there is no notion of coverage in graphs, algorithm for the dominating graph problem do not maintain the topology of the initial simplicial complex. Our problem has also been studied under a game-theoretic approach in [9], where authors define a coverage function. But they can only identify sub-optimal solutions that do not guarantee an unmodified coverage. When computing the complexity of the algorithm, we are reduced to compute the behavior of the size of the largest simplex, which is known as the clique number in graph theory. The clique number of the random geometric graph has been heavily studied in the literature, and its behavior described according to percolation regimes in [10]. In [11], the authors proved that monotone properties of random geometric graphs, such as the connectivity of the graph, have sharp thresholds. In [12], then in [13] for a weaker assumption, the authors prove a conjecture of Penrose [10] stating that, in the subcritical regime, the clique number becomes concentrated on two consecutive integers, as in the Erdős-Rényi model cf. [14]. Moreover, in the subcritical regime, weak laws of large numbers [15] and central limits theorems [16] have been found by Penrose and Yukich for some functionals, including the clique number, in random geometric graphs. Then in the supercritical regime, using the uniform norm, Appel and Russo [17], were able to find strong laws for the maximum vertex degree and for cliques. In particular, they found the behavior of the clique number in the supercritical regime via the behavior of the maximum vertex degree.

The remainder of the paper is organized as followed. First in Section II, we remind some simplicial homology and random configurations definitions and properties. Then the reduction algorithm is described in Section III, while its properties are exposed in Section IV. Finally the complexity of the algorithm is investigated in Section V.

## II. PRELIMINARIES

### A. Simplicial homology

For further reading on algebraic topology, see [18]. Graphs can be generalized to more generic combinatorial objects known as simplicial complexes. While graphs model binary relations, simplicial complexes represent higher order relations. A simplicial complex is a combinatorial object made up of vertices, edges, triangles, tetrahedra, and their  $n$ -dimensional counterparts. Given a set of vertices  $V$  and an integer  $k$ , a  $k$ -simplex is an unordered subset of  $k + 1$  vertices  $[v_0, v_1, \dots, v_k]$  where  $v_i \in V$  and  $v_i \neq v_j$  for all  $i \neq j$ . Thus, a 0-simplex is a vertex, a 1-simplex an edge, a 2-simplex a triangle, a 3-simplex a tetrahedron, etc. See Fig. 1 for instance.

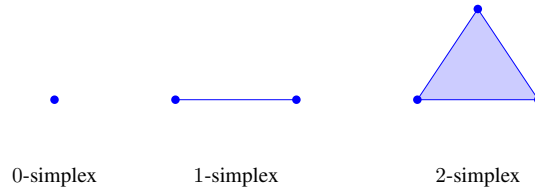


Fig. 1. Example of  $k$ -simplices.

Any subset of vertices included in the set of the  $k + 1$  vertices of a  $k$ -simplex is a face of this  $k$ -simplex. Thus, a  $k$ -simplex has exactly  $k + 1$   $(k - 1)$ -faces, which are  $(k - 1)$ -simplices. For example, a tetrahedron has four 3-faces which are triangles. The inverse notion of face is coface: if a simplex  $S_1$  is a face of a larger simplex  $S_2$ , then  $S_2$  is a coface of  $S_1$ . A simplicial complex is a collection of simplices which is closed with respect to the inclusion of faces, i.e. all faces of a simplex are in the set of simplices, and whenever two simplices intersect, they do so on a common face. An abstract simplicial complex is a purely combinatorial description of the geometric simplicial complex and therefore does not need the property of intersection of faces.

One can define an orientation for an abstract simplicial complex, where a change in the orientation corresponds to a change in the sign of the coefficient. For instance if one swaps vertices  $v_i$  and  $v_j$ :

$$[v_0, \dots, v_i, \dots, v_j, \dots, v_k] = -[v_0, \dots, v_j, \dots, v_i, \dots, v_k].$$

Then let us define vector spaces of  $k$ -simplices and a boundary map on them:

*Definition 1:* Given an abstract simplicial complex  $X$ , for each integer  $k$ ,  $C_k(X)$  is the vector space spanned by the set of oriented  $k$ -simplices of  $X$ .

*Definition 2:* The boundary map  $\partial_k$  is defined to be the linear transformation  $\partial_k : C_k \rightarrow C_{k-1}$  which acts on basis elements  $[v_0, \dots, v_k]$  of  $C_k$  via

$$\partial_k[v_0, \dots, v_k] = \sum_{i=0}^k (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k].$$

The boundary map on any  $k$ -simplex, is the cycle of its  $(k - 1)$ -faces. This map gives rise to a chain complex: a sequence of vector spaces and linear transformations.

$$\dots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Finally let us define:

*Definition 3:* The  $k$ -th boundary group of  $X$  is  $B_k(X) = \text{im } \partial_{k+1}$ .

*Definition 4:* The  $k$ -th cycle group of  $X$  is  $Z_k(X) = \text{ker } \partial_k$ .

The boundary map applied to a cycle gives the cycle of this cycle which is zero as can be seen in Fig. 2. Therefore a standard result asserts that for any integer  $k$ ,

$$\partial_k \circ \partial_{k+1} = 0.$$

It follows that  $B_k \subset Z_k$ .

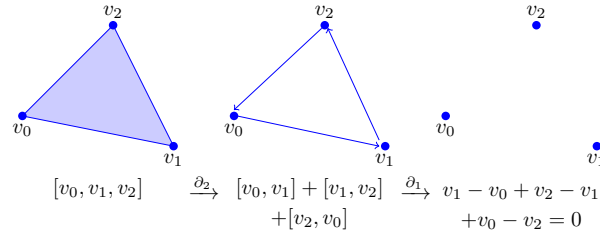


Fig. 2. Application of the boundary map to a 2-simplex.

We are now able to define the  $k$ -th homology group and its dimension:

*Definition 5:* The  $k$ -th homology group of  $X$  is the quotient vector space:

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}.$$

*Definition 6:* The  $k$ -th Betti number of  $X$  is the dimension:

$$\beta_k = \dim H_k = \dim Z_k - \dim B_k.$$

We can compute the Betti numbers in a simple case as an example. Let  $X$  be the abstract simplicial complex made up of 5 vertices  $[v_0], \dots, [v_4]$ , 6 edges  $[v_0, v_1], [v_0, v_2], [v_1, v_2], [v_1, v_4], [v_2, v_3]$  and  $[v_3, v_4]$ , and one triangle  $[v_0, v_1, v_2]$  represented in Fig. 3.

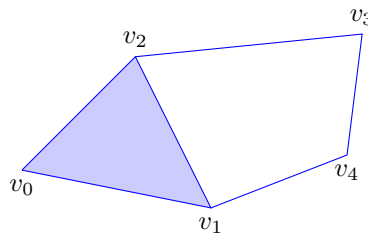


Fig. 3. A geometric representation of  $X$ .

The boundary maps associated to the abstract simplicial complex  $X$  are easy to obtain in matrix form:

$$\partial_1 = \begin{matrix} & v_0v_1 & v_0v_2 & v_1v_2 & v_1v_4 & v_2v_3 & v_3v_4 \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{matrix},$$

$$\partial_2 = \begin{matrix} & & & & v_0 v_1 v_2 \\ & & & & \\ & v_0 v_1 & & & \\ & v_0 v_2 & & & \\ & v_1 v_2 & & & \\ & v_1 v_4 & & & \\ & v_2 v_3 & & & \\ & v_3 v_4 & & & \end{matrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The boundary map  $\partial_0$  is the null function on the set of vertices. Then we can compute the Betti numbers:

$$\begin{aligned} \beta_0(X) &= \dim \ker \partial_0 - \dim \operatorname{im} \partial_1 \\ &= 5 - 4 \\ &= 1 \\ \beta_1(X) &= \dim \ker \partial_1 - \dim \operatorname{im} \partial_2 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

### B. Model

In this paper, we apply our reduction algorithm to Vietoris-Rips complexes based on a random point process that we define here. Let us first denote by  $\mathbb{T}_a^d$  the torus of side  $a$  in dimension  $d$ . The space of configurations on  $\mathbb{T}_a^d$ , is the set of locally finite simple point measures (see [19] for details):

$$\Omega^X = \left\{ \omega = \sum_{k=1}^n \delta(x_k) : (x_k)_{k=1}^n \subset X, n \in \mathbb{N} \cup \{\infty\} \right\},$$

where  $\delta(x)$  denotes the Dirac measure at  $x \in \mathbb{T}_a^d$ . It is often convenient to identify an element  $\omega$  of  $\Omega^X$  with the set corresponding to its support, i.e.  $\sum_{k=1}^n \delta(x_k)$  is identified with the unordered set  $\{x_1, \dots, x_n\}$ . For  $A \in \mathcal{B}(X)$ , we have  $\delta(x)(A) = 1_{[A]}(x)$ , so that

$$\omega(A) = \sum_{x \in \omega} 1_{[A]}(x),$$

counts the number of atoms in  $A$ . Simple measure means that  $\omega(\{x\}) \leq 1$  for any  $x \in X$ . Locally finite means that  $\omega(K) < \infty$  for any compact  $K$  of  $X$ . The configuration space  $\Omega^X$  is endowed with the vague topology and its associated  $\sigma$ -algebra denoted by  $\mathcal{F}^X$ .

Here, the set of points of the Vietoris-Rips complex is represented by a random point process: binomial or random:

*Definition 7 (Binomial point process):* Let  $f$  be the uniform probability density function on the torus  $\mathbb{T}_a^d$ , and  $n$  an integer. Then a point process  $\omega$  is a binomial point process of  $n$  points on  $\mathbb{T}_a^d$ , if the following two conditions hold:

- i) The process  $\omega$  has  $n$  points,
- ii) The vertices positions are drawn according to  $f$  independently of each other.

*Definition 8 (Poisson point process):* Let  $\lambda$  be a positive real number. Then a point process  $\omega$  is a Poisson point process of intensity  $\lambda$  on  $\mathbb{T}_a^d$  if the following two conditions hold:

- i) For any subset  $A$  of  $\mathbb{T}_a^d$ , the number of points of  $\omega$  in  $A$  has a Poisson distribution with parameter  $\lambda \|A\|$

$$\Pr(\omega(A) = k) = e^{-\lambda \|A\|} \frac{(\lambda \|A\|)^k}{k!},$$

- ii) For any disjoint subset  $A, A'$  of  $\mathbb{T}_a^d$ , the random variables  $\omega(A)$  and  $\omega(A')$  are independent.

Conditionally to the number of points, the points of a Poisson point process are drawn uniformly, that is to say according to  $f$  independently of each other. So the difference between a binomial and a Poisson point process lies in the fact that the number of points is set in the first one and drawn in the second one.

Given a set of points randomly drawn, we can now define the Vietoris-Rips complex which vertices are the set of points:

*Definition 9 (Vietoris-Rips complex):* Given  $(X, d)$  a metric space,  $\omega$  a finite set of points in  $X$ , and  $r$  a real positive number. The Vietoris-Rips complex of parameter  $r$  of  $\omega$ , denoted  $\mathcal{R}_r(\omega)$ , is the abstract simplicial complex whose  $k$ -simplices correspond to unordered  $(k+1)$ -tuples of vertices in  $\omega$  which are pairwise within distance less than  $2r$  of each other.

### III. REDUCTION ALGORITHM

In this section, we present the reduction algorithm which provides which vertices and associated simplices may be removed from an abstract simplicial complex without altering its topology. We use algebraic topology in order to compute the complex topology, but we also use this information to have the order in which we can remove the vertices so that the computation complexity is reduced as much as possible.

The algorithm needs two inputs. First it needs the abstract simplicial complex, with all its simplices specified. With only an abstract simplicial complex, the optimal reduction with no modification in the homology will always be to reduce the complex to a simple vertex. That is why the algorithm needs as a second input a list of vertices of the abstract simplicial complex that have to be kept by the reduction algorithm. We call these vertices *critical* vertices. Then the algorithm removes non-critical vertices and their faces one by one from the simplicial complex without changing its homology, i.e. its Betti numbers. At the end, we obtain a *final* simplicial complex, and a list of removed vertices. In this paper we reduce our examples to the Vietoris-Rips complex, even though the algorithm works with any abstract simplicial complex.

There are as many nonzero homology groups, hence nonzero Betti numbers, as sizes of simplices in the abstract simplicial complex. Therefore it is possible to define different algorithms following the number of Betti numbers that have to be kept unchanged. We will denote by  $k_0$  the number of Betti numbers that the algorithm takes into account.

In dimension  $d$ , for the Vietoris-Rips complex there are  $d$  Betti numbers that have a relevant geometric meaning. For simulations reasons we restrict our examples to maximum two dimensions. So in one dimension, the reduction algorithm can only consider one Betti number, whereas in two dimensions the algorithm can maintain only connectivity, or both connectivity and coverage.

Let us now define the full domain hypothesis when the reduction algorithm is applied a Vietoris-Rips complex in two or less dimensions:

*Definition 10 (Full domain hypothesis):* In dimension  $d \leq 2$ , we define if  $k_0 = d$ , for a Čech or a Vietoris-Rips complex with  $\beta_0 = 1$  and, if  $k_0 = 2$ ,  $\beta_1 = 0$ , the full domain hypothesis which holds when all the vertices of the abstract simplicial complex lie within the same geometric domain defined by the critical vertices.

In one dimension, this means that the critical vertices are two extremity vertices and all other vertices have to be in the same path linking the two critical vertices.

In two dimensions, this means that the critical vertices are boundary vertices and all other vertices have to lie in the area defined by the critical vertices.

We can note that it is always possible to satisfy the full domain hypothesis by removing, before the reduction algorithm, all vertices not in the path defined by the critical vertices for the connectivity algorithm in one dimension, or not lying in the convex hull of the critical vertices in two dimensions.

#### A. Degrees

The first step of the algorithm is the calculation of a number, which we call the *degree*, that we define for every  $k_0$ -simplex, where  $k_0$  is the number of Betti numbers to be kept unchanged. To connect vertices, we only need 1-simplices, and to cover an area, we only need 2-simplices. Generalizing this idea, we have that we only need  $k_0$ -simplices to maintain  $k_0$  Betti numbers. And larger simplices, i.e. simplices with more than  $k_0 + 1$  vertices, are superfluous for the  $k_0$ -th homology (first  $k_0$  Betti numbers). We intend to characterize the superfluosness of  $k_0$ -simplices with the following definition of degree:

*Definition 11:* For  $k_0$  an integer, the degree of a  $k_0$ -simplex  $[v_0, v_1, \dots, v_{k_0}]$  is the size of its largest coface:

$$D[v_0, v_1, \dots, v_{k_0}] = \max\{d \mid [v_0, v_1, \dots, v_{k_0}] \subset d\text{-simplex}\}.$$

By definition we have  $D[v_0, v_1, \dots, v_{k_0}] \geq k_0$ .

For the remainder of the thesis, let  $s_k(X)$ , or  $s_k$ , be the number of  $k$ -simplices of the simplicial complex  $X$ . Let  $D_1, \dots, D_{s_{k_0}}$  denote the  $s_{k_0}$  degrees of a simplicial complex; they are computed according to Algorithm 1.

---

#### Algorithm 1 Degree calculation

---

```

for  $i = 1 \rightarrow s_{k_0}$  do
  Get  $(v_0, \dots, v_{k_0})$  the vertices of the  $i$ -th  $k_0$ -simplex
   $k = k_0$ 
  while  $(v_0, \dots, v_{k_0})$  are vertices of a  $(k + 1)$ -simplex do
     $k = k + 1$ 
  end while
   $D_i = k$ 
end for
return  $D_1, \dots, D_{s_{k_0}}$ 

```

---

We can see an example of values of the degree for 2-simplices in Fig. 4: The degree of an alone 2-simplex is 2, when it is the face of 3-simplex it becomes 3.

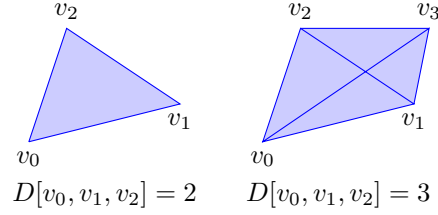


Fig. 4. Example of values of the degree for 2-simplices

### B. Indices

In order to reduce the abstract simplicial complex, we need to remove  $k$ -simplices for any  $k$ , including  $k < k_0$ . So, we remove the smallest simplices: 0-simplices, i.e. vertices, and their faces are subsequently removed. That is why we bring the superfluousness information of the  $k_0$ -simplices down to the vertices with an *index*. We consider a vertex sensitive if its removal leads to a change in the Betti numbers of the simplicial complex. Since a vertex is as sensitive as its most sensitive  $k_0$ -simplex, the index of a vertex is the minimum of the degrees of the  $k_0$ -simplices it is a vertex of:

*Definition 12:* The index of a vertex  $v$  is the minimum of the degrees of the  $k_0$ -simplices it is a vertex of:

$$I[v] = \min\{D[v_0, v_1, \dots, v_{k_0}] \mid v \in [v_0, v_1, \dots, v_{k_0}]\},$$

If a vertex  $v$  is not a vertex of any  $k_0$ -simplex then  $I[v] = 0$ .

Let  $v_1, v_2, \dots, v_{s_0}$  be the vertices of the simplicial complex, the computation of the  $s_0$  indices is done as shown in Algorithm 2.

---

#### Algorithm 2 Indices computation

---

```

for  $i = 1 \rightarrow s_0$  do
   $I[v_i] = 0$ 
  for  $j = 1 \rightarrow s_{k_0}$  do
    if  $v_i$  is vertex of  $k_0$ -simplex  $j$  then
      if  $I[v_i] == 0$  then
         $I[v_i] = D_j$ 
      else
         $I[v_i] = \min\{I[v_i], D_j\}$ 
      end if
    end if
  end for
end for
return  $I[v_1], \dots, I[v_{s_0}]$ 

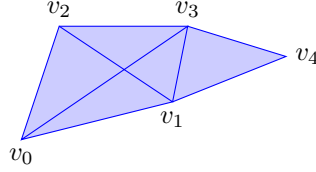
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We can see in Fig. 5 an example of index values in a simplicial complex. Vertices of a  $k_0$ -simplex are more sensitive than vertices of only largest simplices.

The index of a vertex is then an indicator of the density of vertices around the vertex: an index of  $k_0$  indicates that at least one of its  $k_0$ -coface has no  $(k_0 + 1)$ -cofaces, whereas a higher index indicates that each one of its  $k_0$ -cofaces is the face of at least one larger simplex. The main idea of the algorithm is thus to remove the vertices with the greatest indices.

*Remark 1:* An index of 0 indicates that the vertex has no  $k_0$ -coface: the vertex is isolated up to the  $k_0$ -th degree. For  $k_0 = 1$ , that means that the vertex is disconnected from any other vertices. For  $k_0 = 2$ , the vertex is only linked to other vertices by edges, therefore it is inside a coverage hole. Under the full domain hypothesis, these vertices should not exist.



$$\begin{aligned}
 D[v_0, v_1, v_2] &= D[v_0, v_1, v_3] = D[v_0, v_2, v_3] = D[v_1, v_2, v_3] = 3 \\
 D[v_1, v_3, v_4] &= 2 \\
 I[v_0] &= I[v_2] = 3 \text{ and } I[v_1] = I[v_3] = I[v_4] = 2
 \end{aligned}$$

Fig. 5. Example of values of the indices of vertices in a simplicial complex

### C. Optimized order for vertices removal

The algorithm now removes vertices from the initial abstract simplicial complex following an optimized order for the topology computation complexity. Indeed the computation of the Betti numbers as well as the implementation of the abstract simplicial complex can have their complexity explode with the size of the abstract simplicial complex. The removal of a vertex leads to the removal of all the simplices it is a face of. So the removal of a vertex with many  $k_0$ -faces leads to a significant decrease first in the complexity of the  $k_0$ -simplices degrees computations, then in the complexity of the Betti number  $\beta_{k_0-1}$  computation, since the boundary map  $\partial_{k_0}$  size decreases, but also in the abstract simplicial complex implementation itself. Indeed, an abstract simplicial complex is implemented via the list of all its simplices.

We begin by computing the first  $k_0$  Betti numbers. Then the degrees of all the  $k_0$ -simplices and the indices of all the vertices are computed as explained in the previous section. The critical vertices of the input list are given a negative index, to flag them as unremovable. Then the indices give us an order for the removal of vertices: the greater the index of a vertex, the more likely it is superfluous for the  $k_0$ -th homology of its simplicial complex. Therefore, the vertices with the greatest index are candidates for removal: one is chosen randomly. The removal of a vertex leads to the degradation of all its  $k$ -faces to  $(k-1)$ -simplices for every integer  $k \geq 1$ .

At every vertex removal, we need to ensure that the homology is unchanged. We compute the first  $k_0$  Betti numbers thanks to the boundary maps every time a vertex is removed. This computation is instantaneous since the complex is already built, and only adjacency matrices defining the complex are needed. If the removal changes the homology, the vertex is put back in the simplicial complex. Moreover its index is assigned a temporary negative value so that the vertex is not candidate for the following draw. The temporary values are recalculated at the next effective removal of a vertex.

Otherwise, if the removal does not change the homology, the removal is confirmed. And the modified degrees of the  $k_0$ -simplices and the indices of the vertices are recalculated. We can note that only the vertices of maximum index can have their indices changed, as explained in Lemma 1. Moreover, in order to improve the algorithm performance it is possible to only compute the modified degrees of  $k_0$ -simplices. It suffices to flag the  $k$ -simplices,  $k > k_0$ , which are the largest cofaces of  $k_0$ -simplices. Then when one of them is degraded, the degrees of its  $k_0$ -faces are decreased by one.

*Lemma 1:* When a vertex of index  $I_{\max}$  is removed, only the vertices sharing an  $I_{\max}$ -simplex with it, and of index  $I_{\max}$  can have their index changed.

*Proof:* Let  $w$  be the removed vertex of index  $I_{\max}$ , and  $v$  any vertex of the simplicial complex.

If  $v$  does not share any simplex with  $w$ , none of the degrees of its  $k_0$ -simplex will change, and neither will its index.

Thus let us consider that the maximum common simplex of  $v$  and  $w$  is a  $k$ -simplex. If  $k < k_0$ , then the removal of  $w$  and this  $k$ -simplex has no incidence at all on the degree of  $v$  by definition. Next, if  $k_0 \leq k < I_{\max}$  then  $w$  has an index  $k < I_{\max}$ , which is absurd. We can thus assume that  $k \geq I_{\max}$ . Either the index of  $v$  is strictly less than  $I_{\max}$  and thus comes from a simplex not shared with  $w$ , therefore it is unmodified by the removal of  $w$ . Or if the index of  $v$  is  $I_{\max}$ , it either still comes from a  $I_{\max}$ -simplex not shared with  $w$  and remains unmodified, or it comes from a common  $I_{\max}$ -simplex. This latter case is the only case where the index of  $v$  can change. ■

The algorithm goes on removing vertices until every remaining vertex is unremovable, thus achieving optimal result. Every vertex is unremovable when all indices are strictly below  $k_0$ . By the definition of indices, that means when all indices are equal to zero or negative (temporarily or not). Some things can be refined with the full domain hypothesis in the wireless sensor network application case. First the stopping condition can be improved to  $I_{\max}$  being strictly greater than  $k_0$ :

*Lemma 2:* Under the full domain hypothesis, the algorithm can stop when all vertex indices are below or equal to  $k_0$ .

*Proof:* Let us suppose the input data satisfies the full domain hypothesis. Let  $v$  be a vertex of index  $I(v) = k_0$ , which means that at least one of its  $k_0$ -cofaces has no  $(k_0+1)$ -coface. The removal of this vertex would lead to the removal of this particular  $k_0$ -simplex. Since we need to maintain the homology on the entire simplicial complex that is defined by the critical vertices without shrinking its covered domain, this would lead to a  $k_0$ -dimensional hole, and an increment of  $\beta_{k_0-1}$ .

For the connectivity algorithm, the removal of an edge which is not the side of a triangle leads to a disconnectivity in the path linking the critical vertices. For the coverage algorithm, the removal of a triangle which is not the face of a tetrahedron



leads to the creation of a coverage hole inside the area defined by the critical vertices. ■

Then under the full domain hypothesis in the wireless sensor network application, all temporarily unremovable vertices (negative index vertices) stay unremovable:

*Lemma 3:* Under the full domain hypothesis, when the removal of a vertex modify the homology, it will always modify it.

*Proof:* In the full domain hypothesis, the distance between the extremity vertices can not be decreased, or the covered domain can not have its area decreased. Since the domain size is unchanged, as in the proof of Lemma 2, the removal of a vertex that has led to a change in a Betti number will always lead to the same change. ■

We give in Algorithm 3 the whole algorithm for the conservation of the first  $k_0$  Betti numbers.

---

### Algorithm 3 Reduction algorithm

---

**Require:** Simplicial complex  $X$ , list  $L_C$  of critical vertices.

Computation of  $\beta_0(X), \dots, \beta_{k_0-1}(X)$

Computation of  $D_1(X), \dots, D_{s_{k_0}}(X)$

Computation of  $I[v_1(X)], \dots, I[v_{s_0}(X)]$

**for all**  $v \in L_C$  **do**

$I[v] = -1$

**end for**

$I_{\max} = \max\{I[v_1(X)], \dots, I[v_{s_0}(X)]\}$

**while**  $I_{\max} \geq k_0$  **do**

Draw  $w$  a vertex of index  $I_{\max}$

$X' = X \setminus \{w\}$

Computation of  $\beta_0(X'), \dots, \beta_{k_0-1}(X')$

**if**  $\beta_i(X') \neq \beta_i(X)$  for some  $i = 0, \dots, k_0 - 1$  **then**

$I[w] = -1$

**else**

Computation of  $D_1(X'), \dots, D_{s'_{k_0}}(X')$

**for**  $i = 1 \rightarrow s'_0$  **do**

**if**  $I[v_i(X')] == I_{\max}$  **then**

Recomputation of  $I[v_i(X')]$

**end if**

**if**  $I[v_i(X')] == -1$  &&  $v_i \notin L_C$  **then**

Recomputation of  $I[v_i(X')]$

**end if**

**end for**

$I_{\max} = \max\{I[v_1(X'), \dots, I[v_{s'_0}(X')]\}$

$X = X'$

**end if**

**end while**

**return**  $X$

---

#### D. Figures

We can see in Fig. 6 one realization of the connectivity algorithm on a Vietoris-Rips simplicial complex of parameter  $\epsilon = 1$  based on a Poisson point process of intensity  $\lambda = 4$  drawn on a square of side length 4. The critical vertices are chosen at random: a vertex is critical with probability  $p_c = 0.5$  independently from every other vertices. In the figure, critical vertices are circled, while non-critical ones which are kept to maintain the connectivity between critical vertices are represented by stars.

Then, we can see in Fig. 7 one realization of the coverage algorithm on a Vietoris-Rips complex of parameter  $\epsilon = 1$  based on a Poisson point process of intensity  $\lambda = 4.2$  drawn on a square of side length  $a = 2$ . To the vertices drawn with the Poisson point process, we add a fixed boundary of vertices on the square perimeter. The critical vertices are then defined to be the boundary vertices, thus we satisfy satisfying the full domain hypothesis for the coverage algorithm. The boundary/critical vertices are circled on the figure.

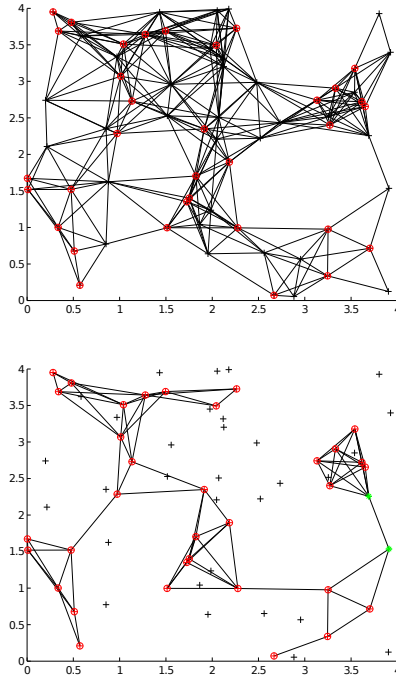


Fig. 6. A Vietoris-Rips complex before and after the connectivity algorithm.

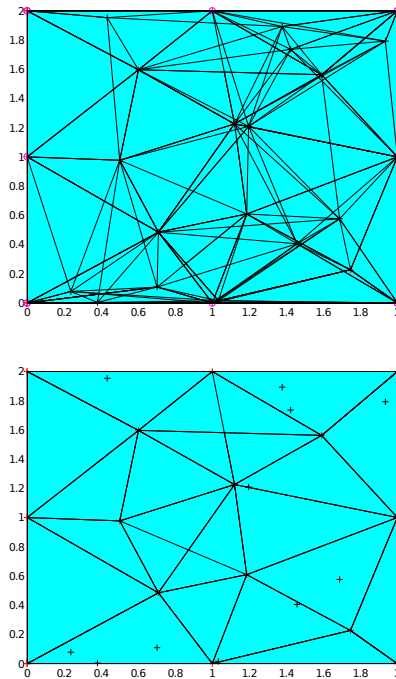


Fig. 7. A Vietoris-Rips complex before and after the coverage algorithm.

#### IV. PROPERTIES

The first property of our algorithm is that the reached solution is optimal. It may not be the optimum solution if there are multiple optima but it is a local optimum. In game theory vocabulary that means that the algorithm reaches a Nash equilibrium:

*Theorem 1 (Nash equilibrium):* The reduction algorithm reaches a Nash equilibrium as defined in [9]: every vertex in the final simplicial complex is needed to maintain its homology.

*Proof:* In the final simplicial complex, every vertex is of index strictly smaller than  $k_0$  in the general case. By the definition of the indices, we then differentiate two types of vertices: vertices of index  $-1$  and  $0$ .

First, negative indices are given to vertices to flag them as unremovable. Either a vertex is of negative index because it is a critical vertex, in which case it is required to stay in the complex. Or a vertex is of negative index if its removal leads to a change in the Betti numbers. Since there has been no other removal that has changed the complex, that fact is still true.

Then a vertex of null index is an isolated vertex. If it is isolated up to the  $k_0$ -th degree, its removal will decrease the  $k_0$ -th Betti number. For example, the removal of a disconnected vertex would decrement  $\beta_0$ . As well the removal of a vertex inside of a hole would lead to the union of 2 or more holes.

Finally, if the input data satisfies the full domain hypothesis, the proof of Lemma 2 shows that the algorithm still reaches a Nash equilibrium. ■

Secondly, we are able to find both a lower and an upper bound the number of removed vertices. The number of removed vertices is at least one vertex removed by index value, and at most every vertex of non minimal index before the algorithm:

*Theorem 2 (Upper and lower bounds):* Let  $E_k$  be the set of vertices that have indices  $k$  before the algorithm. The number of removed vertices  $M$  is bounded by:

$$\sum_{k=k_0+1}^{I_{\max}} 1_{[E_k \neq \emptyset]} \leq M \leq \sum_{k=k_0+1}^{I_{\max}} |E_k|.$$

with  $|E_k|$  being the cardinality of  $E_k$ .

*Proof:* Let us begin with the upper bound. The maximum number of vertices the can be removed by the algorithm is the number of vertices that initially have their index strictly greater than  $k_0$ . This is an optimal upper bound since this number of removed vertices is reached in the following case:

Let a  $k$ -simplex, with  $k > k_0$ , be the initial abstract simplicial complex, and  $n_C$  of its vertices be the initial critical vertices, necessarily  $n_C \leq k + 1$ . The  $n_C$  critical vertices have negative indices, the  $k + 1 - n_C$  other vertices have an index of  $k$ , and they are all removed.

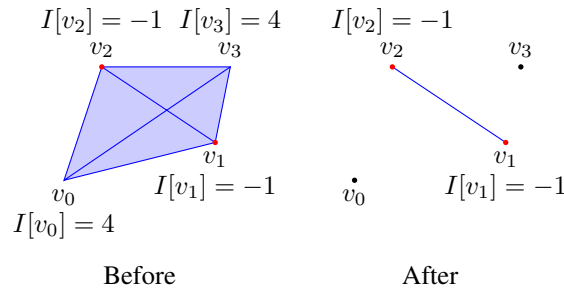


Fig. 8. Example of this case with  $k = 3$  and  $n_C = 2$ , the two vertices  $v_0$  and  $v_3$  are removed by the algorithm.

For the lower bound, we have seen in Lemma 1 that the removal of a vertex of index  $I_{\max}$  can only change the index of vertices of index  $I_{\max}$ . In the worst case, it decreases all indices  $I_{\max}$  and the value of  $I_{\max}$  changes, not necessarily to  $I_{\max} - 1$  depending on the critical vertices. Thus we can see, that at least one vertex per index value can be removed, hence the result.

The lower bound is reached in the previous case if  $n_C = k$ . ■

Then if we use the following definition of a dominating set:

*Definition 13 (Dominating set):* As defined in [8], a set  $S \subseteq V$  of vertices of a graph  $G = (V, E)$  is called a dominating set if every vertex  $v \in V$  is either an element of  $S$  or is adjacent to an element of  $S$ .

Under the full domain hypothesis, the final set of vertices is a dominating set of the initial set:

*Theorem 3:* In two dimensions for the coverage algorithm applied to a Vietoris-Rips complex under the full domain hypothesis, the set of remaining vertices of the final complex is a dominating set of the set of vertices of the initial complex.

*Proof:* Under the full domain hypothesis, initial vertices are all in the geometric domain defined by the critical vertices. For the coverage algorithm that means that initial vertices lie in the area defined by the critical vertices. The homology of the complex is unmodified by the algorithm therefore there is no coverage hole in the final complex. The area is still covered. Then, each point of this area is inside a 2-simplex. This is true for every vertex of the initial complex, it is thus adjacent to three remaining vertices. ■

## V. COMPLEXITY

In this section, we investigate the complexity of our algorithm for the Vietoris-Rips complex based on a binomial point process. If we consider the number of vertices  $n = s_0$  to be the parameter, then the complexity of the implementation of the abstract simplicial complex is potentially exponential. Indeed, the number of simplices in an abstract simplicial complex of  $n$  vertices is at most  $2^n - 1$ . Therefore the data implementation is at most of complexity  $O(2^n)$  compared to the number of vertices  $n$ .

*Theorem 4 (Complexity):* The complexity of the algorithm to keep  $k_0$  Betti numbers for an abstract simplicial complex of  $s_k$   $k$ -simplices and  $n = s_0$  vertices, is upper bounded by:

$$n^2 s_{k_0} + (n + s_{k_0}) \sum_{k=0}^{C-1} s_k$$

with  $C$  being the size of the largest simplex in the initial abstract simplicial complex.

*Proof:* At the beginning we suppose that the entry data is the simplicial complex represented by the list of all its simplices. So all simplices are already implemented.

Let  $C - 1$  be the size of the largest simplex in the abstract simplicial complex,  $C$  being known as the clique number of the underlying graph.

For the computation of the degrees of every  $k_0$ -simplex, the algorithm traverses at most all the  $k$ -simplices for  $k_0 < k \leq C - 1$  to check if the  $k_0$ -simplex is included in it. Since there is  $s_k$   $k$ -simplices, that means that the degrees computation complexity is upper bounded by  $s_{k_0} \sum_{k=k_0+1}^{C-1} s_k$ .

For the computation of the indices, the algorithm traverses, for every one of the  $n$  vertices, its  $k_0$ -cofaces, which is at most all the  $k_0$ -simplices. The complexity of the computation of the indices has therefore an upper bound of  $n s_{k_0}$ .

For the computation of the Betti numbers, we need to implement the boundary map matrices which are of sizes  $n \times s_1$  and  $s_1 \times s_2$  respectively. These computations do not appear on the algorithm complexity since their complexity is strictly less than the complexity of the computations of the degrees.

These three computations are done at the beginning of the algorithm. Then at each removal of vertices, at most  $n$  removed vertices, the following is done:

- The simplices of the removed vertex are deleted: complexity which upper bound is  $\sum_{k=0}^{C-1} s_k$ .
- The Betti numbers are recomputed via the adjacency matrices which already exist.
- The modified degrees are recomputed automatically with the deletions of simplices.
- The at most  $n - 1$  modified indices are recomputed: complexity which upper bound is  $(n - 1) s_{k_0}$ .

*Remark 2:* Since the simplicial complex is of size  $O(2^n)$ , the complexity of the algorithm is polynomial relatively to the size of the input data.

One can easily see that the size of the largest simplex of a Vietoris-Rips complex based on a binomial point process is the clique number of the underlying random geometric graph. And this parameter, denoted  $C$ , as one can see in the complexity formula of Theorem 4, is of great importance in the complexity: it can make the latter go from polynomial to exponential. That is why we investigate its behavior in more details.

In [20], the authors provide expressions for moments of random variables of the Čech complex by means of Malliavin calculus. Thanks to their use of the uniform norm the so-called Čech complex is the exact same as the Vietoris-Rips complex. Then we are able to apply the results of [20] to our case, especially in the expressions of the expectation and the variance of the number of  $(k - 1)$ -simplices.

In order to use the results from [20], we first need to make a few assumptions:

- 1) First, we use the uniform norm to calculate the distance between two vertices. Note that [20] also provides the needed results for the Euclidean norm, however their expressions are not as tractable as the ones for the uniform norm.
- 2) Let us take  $\theta = \left(\frac{r}{a}\right)^d$ , then we must have  $\theta \leq \left(\frac{1}{2}\right)^d$ . This insures that the graph is small on the torus, and that no vertex is its own neighbor.

We can note that the parameter  $\theta$  acts as a coverage parameter: if one puts vertices along a regular grid of side  $r$ , one needs  $\frac{1}{\theta}$  vertices to cover the area of  $\mathbb{T}_a^d$ . These assumptions hold for the remainder of the section.

Thus, let  $N_k$  denote the number of  $(k - 1)$ -simplices. Then  $N_1 = n$  is the number of vertices, and the results of [20] state that:

*Theorem 5 ([20]):* The expectation and variance of the number of  $(k - 1)$ -simplices, for  $k > 1$ , in a Vietoris-Rips simplex based on a binomial point process on the torus  $\mathbb{T}_a^d$  are given by:

$$\mathbf{E}[N_k] = \binom{n}{k} k^d \theta^{k-1}, \quad (1)$$

and,

$$\mathbf{V}[N_k] = \sum_{i=1}^k \binom{n}{2k-i} \binom{2k-i}{k} \binom{k}{i} \theta^{2k-i-1} \left(2k-i + 2\frac{(k-i)^2}{i+1}\right)^d. \quad (2)$$

In this section, we are interested in the almost sure asymptotic behavior of the complexity of our algorithm as  $n$  tends to infinity. Let us first define almost sure:

*Definition 14:* We say that  $G(n, r)$  asymptotically almost surely has property  $P$  if  $\mathbf{P}[G(n, r) \in P] \rightarrow 1$  when  $n$  tends to infinity.

Then throughout this section, we use Bachman-Landau notations. For non-negative functions  $f$  and  $g$  we write as  $n$  tends to infinity:

- $f(n) = o(g(n))$  if for every  $\epsilon > 0$  there exists  $n_0$  such that for  $n \geq n_0$ , we have  $f(n) \leq \epsilon g(n)$ . We say that  $f$  is dominated by  $g$  asymptotically.
- $f(n) = O(g(n))$  if there exists  $k > 0$  and  $n_0$  such that for  $n \geq n_0$ , we have  $f(n) \leq kg(n)$ . We say that  $f$  is bounded by  $g$  asymptotically.
- $f(n) \sim g(n)$  if  $f(n) = O(g(n))$  and  $g(n) = O(f(n))$ . We say that  $f$  and  $g$  are equal asymptotically.
- $f(n) \ll g(n)$  if  $\frac{f(n)}{g(n)} = o(1)$ . We say that  $f$  is small compared to  $g$  asymptotically.

We study the asymptotical behavior of the size of the largest simplex under percolation regimes: these regimes are defined following the variations of  $\theta$  compared to  $\frac{1}{n}$ .

#### A. Subcritical regime

In this subsection, we consider that  $\theta = o(\frac{1}{n})$ . In the subcritical regime for the percolation, the Vietoris-Rips complex mainly consists of disconnected components. The size of the largest simplex is small compared to the number of vertices  $n$  when the latter tends to infinity. Therefore we can focus on  $k$ -simplices for  $k$  asymptotically small compared to  $n$ .

From the exact expressions we had from [20], we can derive asymptotical expressions for the moments of the number of  $(k-1)$ -simplices:

*Lemma 4:* For  $k \geq 1$  small compared to  $n$ , according to Theorem 5,

$$\mathbf{E}[N_k] \sim n^k \theta^{k-1} k^d, \text{ and } \mathbf{V}[N_k] \sim n^k \theta^{k-1} k^d.$$

*Proof:* This is a direct consequence of the subcritical regime hypothesis applied to Equations 1 and 2. ■

The size of the largest simplex in this regime has sharp thresholds [12], [13], that means that it grows by steps, but we still need to find the regimes of  $\theta$  where we can bind the expectation of the number of  $(k-1)$ -simplices:

*Definition 15:* We define for  $\eta > 0$  and for  $k \geq 1$ :

$$\theta'_k = \frac{k^{\frac{1+\eta-d}{k-1}}}{n^{\frac{k}{k-1}}}, \text{ and } \theta_k = \frac{k^{-\frac{1+\eta+d}{k-1}}}{n^{\frac{k}{k-1}}}.$$

Then for  $\theta > \theta'_k$ , thanks to the approximations of Lemma 4 we have:

$$\mathbf{E}[N_k] \geq n^k \left( \frac{k^{\frac{1+\eta-d}{k-1}}}{n^{\frac{k}{k-1}}} \right)^{k-1} k^d \geq k^{1+\eta}.$$

And for  $\theta < \theta_k$ ,

$$\mathbf{E}[N_k] \leq n^k \left( \frac{k^{-\frac{1+\eta+d}{k-1}}}{n^{\frac{k}{k-1}}} \right)^{k-1} k^d \leq k^{-(1+\eta)}$$

This leads to:

*Theorem 6:* In the subcritical regime, for  $k \geq 1$  small compared to  $n$ , and  $\theta'_k < \theta < \theta_{k+1}$ , the size of the largest simplex is asymptotically almost surely  $C = k$ .

*Proof:* For  $k \geq 1$  small compared to  $n$  when  $n$  tends to infinity, we can easily check that if  $n > k^{2(1+\eta)k}$ . This holds in particular when  $n$  tends to infinity and  $k$  is fixed. Then we have that  $\theta_{k+1} > \theta'_k$ .

We can now consider  $\theta$  such that  $\theta'_k < \theta < \theta_{k+1}$ . Thanks to the approximations of Lemma 4, we can, on one hand, find an upper bound for the probability of the non-existence of  $(k-1)$ -simplices:

$$\mathbf{P}[N_k = 0, \theta > \theta'_k] \leq \frac{\mathbf{V}[N_k]}{\mathbf{E}[N_k]^2} \sim \frac{1}{\mathbf{E}[N_k]} \leq \frac{1}{k^{1+\eta}}.$$

On the other hand, we can find an upper bound for the probability of existence of  $k$ -simplices:

$$\mathbf{P}[N_{k+1} > 0, \theta < \theta_{k+1}] \leq \mathbf{E}[N_{k+1}] \leq \frac{1}{(k+1)^{1+\eta}}.$$

Finally we have:

$$\mathbf{P}[\exists \theta, \theta'_k < \theta < \theta_{k+1}, C \neq k] < \frac{1}{k^{1+\eta}} + \frac{1}{(k+1)^{1+\eta}}.$$

As the sum  $\sum_{k=1}^{\infty} k^{1+\eta}$  converges, the Borel-Cantelli theorem implies that with the exception of finitely many  $k$ 's, for all  $\theta$  such that  $\theta'_k < \theta < \theta_{k+1}$ , one has  $C = k$ . Then when  $n$  goes to infinity, we have asymptotically almost surely that  $C = k$ :

$$\mathbf{P}[C = k, \theta'_k < \theta < \theta_{k+1}] \rightarrow 1,$$

concluding the proof.  $\blacksquare$

Finally we can conclude about the complexity of our algorithm in the subcritical regime:

*Theorem 7 (Subcritical regime complexity):* In the subcritical regime for  $\theta'_k < \theta < \theta_{k+1}$ , the complexity of the reduction algorithm to keep  $k_0$  Betti numbers for the Vietoris-Rips complex on a binomial point process on the torus  $\mathbb{T}_a^d$ , is in  $O(n^{k_0+2})$ .

*Proof:* This is a direct consequence of Lemma 4, Theorem 4 and Theorem 6 with the subcritical regime assumptions.  $\blacksquare$

### B. Critical regime

In the critical regime, where  $\theta \sim \frac{1}{n}$ , percolation occurs: disconnected components of the complex begin to connect into one sole connected component. The size of the largest simplex is still rather small compared to  $n$ , allowing us to consider only the  $k$ -simplices for  $k = O(n)$  when  $n$  goes to infinity.

In this regime we have a direct approximation of our variable  $\theta$ , allowing us to compute an approximation of the expected number of  $(k-1)$ -simplices integer from the Equation 1.

*Lemma 5:* For  $k = O(n)$ , and according to Theorem 5, we have:

$$\mathbf{E}[N_k] \sim \frac{1}{\sqrt{2\pi}} n k^{d-k-\frac{1}{2}}, \text{ and } \mathbf{V}[N_k] \sim \frac{1}{\sqrt{2\pi}} n k^{d-k-\frac{1}{2}}.$$

*Proof:* We have from Equation 1 that  $\mathbf{E}[N_k] = \binom{n}{k} \theta^{k-1} k^d$ .

After some calculations via Stirling's approximation  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ , we obtain the result for the expectation.

Thanks to the fact that  $k = O(n)$ , we can still approximate the variance of  $N_k$  by its dominating term in  $i = k$ , and  $\mathbf{V}[N_k] \sim \mathbf{E}[N_k]$ .  $\blacksquare$

From that approximation and using the same process as in the previous subsection, we can write:

*Theorem 8:* In the critical regime, the size of the largest simplex grows asymptotically almost surely slower than  $\ln n$  with an arbitrarily small distance:

$$(\ln n)^{1-\eta} < C < \ln n, \quad \forall \eta > 0.$$

*Proof:* First, for  $k > \ln n$ , we can find an upper bound for the expectation approximation of Lemma 5 by:

$$\mathbf{E}[N_k] < n(\ln n)^{d-\frac{1}{2}-\ln n}.$$

One can easily check that  $n(\ln n)^{d-\frac{1}{2}-\ln n} \rightarrow 0$ . Since  $\mathbf{P}[N_k > 0] \leq \mathbf{E}[N_k]$ , the probability that there exists  $(k-1)$ -simplices tends to 0 and:

$$\mathbf{P}[C > k] = \mathbf{P}[N_k > 0] \rightarrow 0 \quad \forall k > \ln n,$$

and  $C < \ln n$  asymptotically almost surely.

Then, for  $k < (\ln n)^{1-\eta}$  with  $\eta > 0$ , we can now find a lower bound for the expectation approximation by:

$$\mathbf{E}[N_k] > n(\ln n)^{(1-\eta)(d-\frac{1}{2}-(\ln n)^{1-\eta})}$$

And one can check that  $n(\ln n)^{(1-\eta)(d-\frac{1}{2}-(\ln n)^{1-\eta})} \rightarrow +\infty$ . Then, thanks to the asymptotic equivalence of the variance and the expectation of  $N_k$ , we have  $\mathbf{P}[N_k = 0] \leq \frac{1}{\mathbf{E}[N_k]}$ : the probability that there exists no  $(k-1)$ -simplices tends to 0, and:

$$\mathbf{P}[C < k] = \mathbf{P}[N_k = 0] \rightarrow 0 \quad \forall k < (\ln n)^{1-\eta}.$$

Thus,  $C > (\ln n)^{1-\eta}$  asymptotically almost surely.  $\blacksquare$

Finally we can conclude about the complexity of our algorithm in the critical regime:

*Theorem 9 (Critical regime complexity):* In the critical regime, the complexity of the reduction algorithm to keep  $k_0$  Betti numbers for the Vietoris-Rips complex on a binomial point process on the torus  $\mathbb{T}_a^d$ , is in  $O(n^{k_0+2} \ln n)$ .

*Proof:* This is a direct consequence of Lemma 5, Theorem 4 and Theorem 8 using Stirling's approximation.  $\blacksquare$

### C. Supercritical regime

In the supercritical regime,  $\frac{1}{n} = o(\theta)$ , the Vietoris-Rips complex on a binomial point process is connected and tends to become the complete complex. The asymptotic behavior of the clique number has already been studied in this regime in [17] by Appel and Russo. They first find the almost sure asymptotic rate for the maximum vertex degree. Then by squeezing the clique number between two values of the maximum vertex degree, they obtain its asymptotic behavior. We propose here an alternative approach.

In this regime, percolation has occurred, that is to say that the complex is connected and the size of the simplices is not asymptotically small anymore compared to  $n$ . Therefore an upper bound via the expected number of  $k$ -simplices is not a good enough approach anymore. Instead, we came back to the definition of the Vietoris-Rips complex.

For the first step of our exploration, we use a similar argument as in [17]. To cover the torus  $\mathbb{T}_a^d$ , one needs at least  $\lceil \frac{1}{\theta} \rceil$  balls of diameter  $r$  in dimension  $d$ . If one places these balls along a lattice square grid with spacing  $r$ , one can denote  $B_i$ , for  $1 \leq i \leq \lceil \frac{1}{\theta} \rceil$ , the  $\lceil \frac{1}{\theta} \rceil$  needed balls centered on the points of the grid and of radius  $\frac{r}{2}$ . Then the number of vertices  $n$  is smaller than the sum of the number of vertices in each ball  $B_i$ :

$$n < \sum_{i=1}^{\lceil \frac{1}{\theta} \rceil} \#\{B_i\}$$

where  $\#\{B_i\}$  is the number of vertices in  $B_i$ .

Moreover, vertices in the same ball  $B_i$  are within distance  $r$  of each other, therefore they form a simplex. By definition, for every  $i$  we have  $\#\{B_i\} \leq C$ . Thus we have:

$$C > \frac{n}{\lceil \frac{1}{\theta} \rceil} \geq \frac{n\theta}{1+\theta}$$

When  $\theta$  tends to 0, the size of the largest simplex is asymptotically greater than  $n\theta$ . We can now write:

*Theorem 10:* In the supercritical regime, the size of the largest simplex  $C$  grows as  $n\theta$  asymptotically almost surely.

*Proof:* We still have to prove that  $C$  is asymptotically almost surely smaller than  $n\theta$ .

A  $(k-1)$ -simplex occurs when  $k$  vertices are in the same ball of diameter  $r$ . Without loss of generality we can center the ball on one of the vertex of the complex. We have:

$$\begin{aligned} \mathbf{P}[C > n\theta] &= \mathbf{P}[N_{n\theta} > 0] \\ &= \mathbf{P}[\exists x, \#\{B(x, \frac{r}{2})\} \geq n\theta - 1], \end{aligned}$$

where  $x$  is a vertex of the complex, and  $\#\{B(x, \frac{r}{2})\}$  the number of vertices in the ball centered in  $x$  and of radius  $\frac{r}{2}$ .

Let  $x_1, \dots, x_n$  denote the  $n$  vertices, their positions are independent, thus we can write:

$$\begin{aligned} \mathbf{P}[\exists i \in \{1, \dots, n\}, \#\{B(x_i, \frac{r}{2})\} \geq n\theta - 1] \\ &\leq \mathbf{P}[\bigcup_{i=1}^n \#\{B(x_i, \frac{r}{2})\} \geq n\theta - 1] \\ &\leq \sum_{i=1}^n \mathbf{P}[\#\{B(x_i, \frac{r}{2})\} \geq n\theta - 1] \\ &\leq n\mathbf{P}[\#\{B(x_1, \frac{r}{2})\} \geq n\theta - 1]. \end{aligned}$$

The number of vertices in the ball  $B(x_1, \frac{r}{2})$  follows a binomial distribution  $\text{Binom}(n-1, \theta)$ . Therefore Hoeffding's inequality implies that:

$$\begin{aligned} \mathbf{P}[\#\{B(x_1, \frac{r}{2})\} > n\theta] \\ &\leq \mathbf{P}[\#\{B(x_1, \frac{r}{2})\} > (n-1)(\theta + \frac{\theta}{n-1})] \\ &\leq \exp(-2\frac{\theta^2}{n-1}). \end{aligned}$$

Then the size of the largest simplex  $C$  is asymptotically almost surely smaller than  $n\theta$ , concluding the proof. ■

Finally we can conclude about the complexity of our algorithm in the supercritical regime:

*Theorem 11 (Supercritical regime complexity):* When in the supercritical regime, the complexity of the reduction algorithm to keep  $k_0$  Betti numbers for the Vietoris-Rips complex on a binomial point process on the torus  $\mathbb{T}_a^d$ , is in  $O(n^{k_0+1}2^n)$ .

*Proof:* This is a direct consequence of Theorem 4 and Theorem 10. ■

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