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Clique number of random geometric graphs

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Abstract

The clique number C of a graph is the largest clique size in the graph. For a random geometric graph of n vertices, taken uniformly at random, including an edge between two vertices if their distance, taken with the uniform norm, is less than a parameter r on a torus \mathbb{T}_a^d , we find the asymptotic behaviour of the clique number. Setting $\theta = (\frac{r}{a})^d$, in the subcritical regime where $\theta = o(\frac{1}{n})$, we exhibit the intervals of θ where C takes the same value asymptotically almost surely. In the critical regime, $\theta \sim \frac{1}{n}$, we show that C is growing slightly slower than $\ln n$ asymptotically almost surely. Finally, in the supercritical regime, $\frac{1}{n} = o(\theta)$, we prove that C grows as $n\theta$ asymptotically almost surely. We also investigate the behaviour of related graph characteristics: the chromatic number, the maximum vertex degree, and the independence number.

1 Introduction

In graph theory, a clique of a graph is a subset of its vertices such that every two vertices of the subset are connected by an edge. Cliques are one of the basis concepts of graph theory and are the subjects of many research articles since the middle of the twentieth century [7]. A maximum clique is a clique of the largest possible size in a given graph. The clique number of a graph is the number of vertices in a maximum clique of this graph. Then the clique number, as the chromatic number, is a graph characteristic and occurs in various problems. Such as the NP-complete clique problem in computer science, which is finding a particular clique such as a maximum clique or whether there exists a clique larger than a given size, had been highly documented, see [15] for instance. The clique number also appears outside graph theory, in simplicial homology, where cliques are named simplices and the clique number is thus the size of implementation of a simplicial complex. It is then a key factor in the implementation complexity of simplicial homology algorithms, see [16] for instance.

One of the most famous random graph is the Erdős-Rényi model $G(n, p)$ with n vertices and each edge is added with probability p . In this model, the

probability distribution of the clique number is known when the number of vertices n goes to infinity and p is fixed [2]. This result was derived from the investigation of Matula in [8], who proved the 2-point concentration of the clique number distribution. However such results are not yet available for the random geometric graph $G(n, r)$ of n vertices sampled uniformly and including an edge between two vertices if their distance is less than r . Indeed, it is necessary to consider the variations of r depending on n , thus consider different regimes, incrementing the complexity of the problem. Regimes were also investigated for the Erdős-Rényi model, considering the variations of p depending on n . In [6], the authors proved the phase transition at $p = \frac{1}{n}$ for the connectivity. For the random geometric graph case, Penrose in [10] distinguished two types of regime transition. One can consider regimes transition through percolation, or through connectivity. In [5], Kahle studied the homology of the clique complex of the random geometric graph in the three regimes delimited by percolation (subcritical, critical and supercritical), and in the connected regime.

In this paper, we find the asymptotic behaviour of the clique number of a random geometric graph $G(n, r)$ with the uniform norm when n goes to infinity. Derived from the stochastic analysis made in [14], the authors of [3] give the explicit moments of the number of cliques of size k for the random geometric graph on the torus. Thanks to these results, we are able to find the asymptotic behaviour of the clique number for the three regimes defined by percolation. In the subcritical regime, we find the intervals where the clique number takes asymptotically almost surely a given finite value. In the critical regime, we show that the clique number grows slightly slower than $\ln n$ asymptotically almost surely. Then in the supercritical regime, we prove the growth of the clique number asymptotically almost surely. We also investigate the behaviours of the related quantities: the chromatic number, the maximum vertex degree and the independence number.

To our knowledge, this is the first result for the behaviour of the clique number of a random geometric graph for all three regimes. In [4], the authors proved that monotone properties of random geometric graphs, such as the connectivity of the graph, have sharp thresholds. In [11], then in [9] for a weaker assumption, the authors prove a conjecture of Penrose [10] stating that, in the subcritical regime, the clique number becomes concentrated on two consecutive integers, as in the Erdős-Rényi model [8]. Moreover, in the subcritical regime, weak laws of large numbers [13] and central limits theorems [12] have been found by Penrose and Yukich for some functionals, including the clique number, in random geometric graphs. Then in the supercritical regime, using the uniform norm, Appel and Russo [1], were able to find strong laws for the maximum vertex degree and for cliques. In particular, they found the behaviour of the clique number in the supercritical regime via the behaviour of the maximum vertex degree. We propose here the opposite approach and find the same result.

The remainder of this paper is organized as follows. First in the following section, we present our model with the definitions necessary to its construction, and the previous results that have allowed us to write this article. Then the third section is devoted to the subcritical regime results. We find critical regime

results in section 4. We expose the results obtained in the supercritical regime in section 5. Finally, the last section is devoted to the investigation on the related quantities to the clique number in graph theory.

2 Model

In order to describe our model of random geometric graph we need some preliminary definitions from graph theory and probability.

Definition 1 *Let f be the uniform probability density function on the torus \mathbb{T}_a^d , let x_1, x_2, \dots be a set of independent and identically distributed d -dimensional random variables with common density f , and let $X_n = \{x_1, \dots, x_n\}$. For the set of n points X_n and the positive distance r , let us define the random geometric graph $G(n, r)$ as the graph with n vertices $V(G) = X_n$ and edges $E(G) = \{[x, y] \mid d(x, y) \leq r\}$.*

Definition 2 *In a graph, a clique is a subset of vertices such that every two vertices in the clique are connected by an edge.*

Definition 3 *The clique number of a graph, that we denote C , is the number of vertices in a clique of the largest possible size in the graph.*

Definition 4 *We say that $G(n, r)$ asymptotically almost surely has property P if $\mathbf{P}[G(n, r) \in P] \rightarrow 1$ when n tends to infinity.*

For the remainder of this paper, we consider a random geometric graph $G(n, r)$ of n vertices sampled following a uniform distribution on the torus of size a in dimension d , and including an edge between two vertices if their distance is less than r . Taking the torus \mathbb{T}_a^d instead of the cube $[0, a]^d$ allows us to not considerate boundaries effect. We note C its clique number.

In [3], the authors provide expressions for moments of random variables of a Čech complex by means of Malliavin calculus. Thanks to their use of the uniform norm the so-called Čech complex is the exact same as the Vietoris-rips complex, which is the clique complex of the random geometric graph. Indeed in simplicial homology, instead of considering only vertices and edges as in graph theory, cliques of n connected vertices are also considered and called simplices. And the Vietoris-Rips simplicial complex is the complexe whose simplices are the cliques of the random geometric graph. Then we are able to apply the results of [3] to random geometric graphs, in especially the expressions of the expectation and the variance of the number of cliques of size k .

In order to use the results from [3], we first need to make a few assumptions:

1. First, we use the uniform norm to calculate the distance between two vertices. Note that [3] also provides the needed results for the Euclidean norm, however their expression are not as tractable as the ones for the uniform norm.

2. Let us denote $\theta = \left(\frac{r}{a}\right)^d$, then we must have $\theta \leq \left(\frac{1}{2}\right)^d$. This insures that the graph is small on the torus, and that there is no vertex which is its own neighbor.

These assumptions hold for the remainder of the paper.

Thus, let us denote N_k the number of clique of k vertices, then $N_1 = n$ is the number of vertices, and the results of [3] state that:

Theorem 1 ([3]) *The expectation and variance of the number of k -vertex cliques, for $k > 1$, in a random geometric graph $G(n, r)$ on a torus \mathbb{T}_a^d are given by:*

$$\mathbf{E}[N_k] = \binom{n}{k} k^d \theta^{k-1}, \quad (1)$$

and,

$$\mathbf{V}[N_k] = \sum_{i=1}^k \binom{n}{2k-i} \binom{2k-i}{k} \binom{k}{i} \theta^{2k-i-1} \left(2k-i + 2\frac{(k-i)^2}{i+1}\right)^d. \quad (2)$$

In this paper, we are interested in the asymptotic behaviour of the clique number as n tends to infinity. Throughout the article, we use Bachman-Landau notations. For non-negative functions f and g we write as n tends to infinity:

- $f(n) = o(g(n))$ if for every $\epsilon > 0$ there exists n_0 such that for $n \geq n_0$, we have that $f(n) \leq \epsilon g(n)$. We say that f is dominated by g asymptotically.
- $f(n) = O(g(n))$ if there exists $k > 0$ and n_0 such that for $n \geq n_0$, we have that $f(n) \leq k g(n)$. We say that f is bounded by g asymptotically.
- $f(n) \sim g(n)$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. We say that f and g are equal asymptotically.
- $f(n) \ll g(n)$ if $\frac{f(n)}{g(n)} = o(1)$. We say that f is small compared to g asymptotically.

3 Subcritical regime

In this section, we consider that $\theta = o\left(\frac{1}{n}\right)$. In the subcritical regime for the percolation, the random geometric graph is mainly disconnected components, the clique number is small compared to the number of vertices n . Therefore we can focus on k -vertex cliques for k asymptotically small compared to n .

Lemma 1 *For $k \geq 1$ small compared to n , according to Theorem 1,*

$$\mathbf{E}[N_k] \sim n^k \theta^{k-1} k^d, \text{ and } \mathbf{V}[N_k] \sim n^k \theta^{k-1} k^d.$$

Proof This is a direct consequence of the subcritical regime hypothesis applied to Equations 1 and 2.

In this regime, the random geometric graph $G(n, r)$ is similar to the Erdős-Rényi graph $G(n, p)$ which we define:

Definition 5 *Let n be an integer, and $0 < p < 1$. Then the Erdős-Rényi random graph $G(n, p)$ is the graph of n vertices where each edge occurs with probability p .*

When n tends to infinity with p unchanged, the graph $G(n, p)$ is composed of many disconnected components as the random geometric graph in the subcritical regime. Therefore our investigation of the asymptotic behaviour of the clique number in this regime is similar to the one exposed in [2] by Bollobás and Erdős. The authors first define n_k and n'_k such that the expectation of the number of cliques of size k is respectively upper bounded by $k^{-(1+\epsilon)}$ and lower bounded by $k^{1+\epsilon}$ for $\epsilon > 0$. The values of n_k and n'_k are approximated by $p^{-k/2}$ and $(1 + \frac{3 \log k}{k})p^{-k/2}$ respectively. This leads to the result:

Theorem 2 ([2]) *For almost every graph $G(n, p)$ there is a constant c such that if $n'_k \leq n \leq n_{k+1}$ for some $k > c$ then the clique number C is $C = k$.*

In the random geometric graph $G(n, r)$ case, we observe the variations of $\theta = (\frac{r}{a})^d$ instead of n when n tends to infinity. Indeed θ varies with n , and it makes more sens to find the interval of θ where the clique number take a finite value. The first step is to find the regimes of θ where we can bound the expectation of the number of cliques of size k integer in a similar way:

Definition 6 *We define for $\eta > 0$ and for $k \geq 1$:*

$$\theta'_k = \frac{k^{\frac{1+\eta-d}{k-1}}}{n^{\frac{k}{k-1}}}, \text{ and } \theta_k = \frac{k^{-\frac{1+\eta+d}{k-1}}}{n^{\frac{k}{k-1}}}.$$

Then for $\theta > \theta'_k$, thanks to the approximations of Lemma 1 we have that

$$\mathbf{E}[N_k] \geq n^k \left(\frac{k^{\frac{1+\eta-d}{k-1}}}{n^{\frac{k}{k-1}}} \right)^{k-1} k^d \geq k^{1+\eta}.$$

And for $\theta < \theta_k$,

$$\mathbf{E}[N_k] \leq n^k \left(\frac{k^{-\frac{1+\eta+d}{k-1}}}{n^{\frac{k}{k-1}}} \right)^{k-1} k^d \leq k^{-(1+\eta)}$$

This leads to our main theorem for the subcritical regime:

Theorem 3 *In the subcritical regime, for $k \geq 1$ small compared to n , and $\theta'_k < \theta < \theta_{k+1}$, the clique number is asymptotically almost surely $C = k$.*

Proof For $k \geq 1$ small compared to n when n tends to infinity, we can easily check that if $n > k^{2(1+\eta)k}$. This holds in particular when n tends to infinity and k is fixed. Then we have that $\theta_{k+1} > \theta'_k$.

We can now consider θ such that $\theta'_k < \theta < \theta_{k+1}$. Thanks to the approximations of Lemma 1, we can, on the one hand, upper bound the probability of the non-existence of clique of size k :

$$\mathbf{P}[N_k = 0, \theta > \theta'_k] \leq \frac{\mathbf{V}[N_k]}{\mathbf{E}[N_k]^2} \sim \frac{1}{\mathbf{E}[N_k]} \leq \frac{1}{k^{1+\eta}}.$$

On the other hand, we can upper bound the probability of existence of cliques of size $k+1$:

$$\mathbf{P}[N_{k+1} > 0, \theta < \theta_{k+1}] \leq \mathbf{E}[N_{k+1}] \leq \frac{1}{k^{1+\eta}}.$$

Finally we have that:

$$\mathbf{P}[\exists \theta, \theta'_k < \theta < \theta_{k+1}, C \neq k] < \frac{1}{k^{1+\eta}} + \frac{1}{(k+1)^{1+\eta}}.$$

As the sum $\sum_{k=1}^{\infty} k^{1+\eta}$ converges, the Borel-Cantelli theorem implies that with the exception of finitely many k 's, for all θ such that $\theta'_k < \theta < \theta_{k+1}$, one has $C = k$. Then when n goes to infinity, we have asymptotically almost surely that $C = k$:

$$\mathbf{P}[C = k, \theta'_k < \theta < \theta_{k+1}] \rightarrow 1,$$

concluding the proof.

4 Critical regime

In the critical regime, where $\theta \sim \frac{1}{n}$, percolation occurs: disconnected components of the random geometric graph begin to connect into one sole connected component. The clique number is still rather small compared to n , allowing us to consider only the k -vertex cliques for $k = O(n)$ when n goes to infinity.

In this regime we have a direct approximation of our variable θ , allowing us to compute an approximation of the expected number of cliques of size k integer from the Equation 1.

Lemma 2 *For $k = O(n)$, and according to Theorem 1, we have that:*

$$\mathbf{E}[N_k] \sim \frac{1}{\sqrt{2\pi}} n k^{d-k-\frac{1}{2}}, \text{ and } \mathbf{V}[N_k] \sim \frac{1}{\sqrt{2\pi}} n k^{d-k-\frac{1}{2}}.$$

Proof We have from Equation 1 that $\mathbf{E}[N_k] = \binom{n}{k} \theta^{k-1} k^d$.

After some calculations via the Stirling's approximation $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, we obtain the result for the expectation.

Thanks to the fact that $k = O(n)$, we can still approximate the variance of N_k by its dominating term in $i = k$, and $\mathbf{V}[N_k] \sim \mathbf{E}[N_k]$.

From that approximation and using the same process as in the previous section, we can write the main theorem of this section:

Theorem 4 *In the critical regime, the clique number grows asymptotically almost surely slower than $\ln n$ with an arbitrarily small distance:*

$$(\ln n)^{1-\eta} < C < \ln n, \quad \forall \eta > 0.$$

Proof First, for $k > \ln n$, we can upper bound the expectation approximation of Lemma 2 by:

$$\mathbf{E}[N_k] < n(\ln n)^{d-\frac{1}{2}-\ln n}.$$

One can easily check that $n(\ln n)^{d-\frac{1}{2}-\ln n} \rightarrow 0$. Since $\mathbf{P}[N_k > 0] \leq \mathbf{E}[N_k]$, the probability that there exists k -vertex cliques tends to 0 and:

$$\mathbf{P}[C > k] = \mathbf{P}[N_k > 0] \rightarrow 0 \quad \forall k > \ln n,$$

and $C < \ln n$ asymptotically almost surely.

Then, for $k < (\ln n)^{1-\eta}$ with $\eta > 0$, we can now lower bound the expectation approximation by:

$$\mathbf{E}[N_k] > n(\ln n)^{(1-\eta)(d-\frac{1}{2}-(\ln n)^{1-\eta})}$$

And one can check that $n(\ln n)^{(1-\eta)(d-\frac{1}{2}-(\ln n)^{1-\eta})} \rightarrow +\infty$. Then, thanks to the asymptotic equivalence of the variance and the expectation of N_k , we have $\mathbf{P}[N_k = 0] \leq \frac{1}{\mathbf{E}[N_k]}$: the probability that there exists no k -vertex cliques tends to 0, and:

$$\mathbf{P}[C < k] = \mathbf{P}[N_k = 0] \rightarrow 0 \quad \forall k < (\ln n)^{1-\eta}.$$

Thus, $C > (\ln n)^{1-\eta}$ asymptotically almost surely.

5 Supercritical regime

In the supercritical regime, $\frac{1}{n} = o(\theta)$, the random geometric graph $G(n, r)$ is connected and tends to become the complete graph. The asymptotic behaviour of the clique number has already been studied in this regime in [1] by Appel and Russo. They first find the almost sure asymptotic rate for the maximum vertex degree, then by squeezing the clique number between two values of the maximum vertex degree, they obtain its asymptotic behaviour. We propose here an alternative approach, where the clique number asymptotic rate is used to squeeze the other related quantities such as the maximum vertex degree.

In this regime, percolation has occurred, that is to say that the graph is connected and the number of k -cliques is not asymptotically small anymore

compared to n . Therefore an upperbounding via the expected number of k -vertex cliques is not a good enough approach anymore. Instead, we came back to the definition of the random geometric graph.

For the first step of our exploration, we use a similar argument as in [1]. To cover the torus \mathbb{T}_a^d , one needs at least $\lceil \frac{1}{\theta} \rceil$ balls of diameter r in dimension d . If one places these balls along a lattice square grid with spacing r , we can denote B_i , for $1 \leq i \leq \lceil \frac{1}{\theta} \rceil$, the $\lceil \frac{1}{\theta} \rceil$ needed balls centered on the points of the grid and of radius $\frac{r}{2}$. Then the number of vertices n is smaller than the sum of the number of vertices in each ball B_i :

$$n < \sum_{i=1}^{\lceil \frac{1}{\theta} \rceil} \#\{B_i\}$$

where $\#\{B_i\}$ is the number of vertices in B_i .

On the other hand, vertices in the same ball B_i are within distance r of each other, therefore form a clique. By definition, for every i we have $\#\{B_i\} \leq C$. Thus we have:

$$C > \frac{n}{\lceil \frac{1}{\theta} \rceil} \geq \frac{n\theta}{1 + \theta}$$

When θ tends to 0, the clique number is asymptotically greater than $n\theta$. We can now write the main result of this section:

Theorem 5 *In the supercritical regime, the clique number C grows as $n\theta$ asymptotically almost surely.*

Proof We have yet to prove that the clique number is asymptotically almost surely smaller than $n\theta$.

In the random geometric graph $G(n, r)$, a clique of size k occurs when k vertices are in the same ball of diameter r . Without loss of generality we can center the ball on one of the vertex of the graph. We have:

$$\begin{aligned} \mathbf{P}[C > n\theta] &= \mathbf{P}[N_{n\theta} > 0] \\ &= \mathbf{P}[\exists x, \#\{B(x, \frac{r}{2})\} \geq n\theta - 1], \end{aligned}$$

where x is a vertex of the graph $G(n, r)$, and $\#\{B(x, \frac{r}{2})\}$ the number of vertices of $G(n, r)$ in the ball centered in x and of radius $\frac{r}{2}$.

Let us denote x_1, \dots, x_n the n vertices of the graph $G(n, r)$, their positions are independant, thus we can write:

$$\begin{aligned} \mathbf{P}[\exists i \in \{1, \dots, n\}, \#\{B(x_i, \frac{r}{2})\} \geq n\theta - 1] &\leq \mathbf{P}[\bigcup_{i=1}^n \#\{B(x_i, \frac{r}{2})\} \geq n\theta - 1] \\ &\leq \sum_{i=1}^n \mathbf{P}[\#\{B(x_i, \frac{r}{2})\} \geq n\theta - 1] \\ &\leq n\mathbf{P}[\#\{B(x_1, \frac{r}{2})\} \geq n\theta - 1]. \end{aligned}$$

The number of vertices in the ball $B(x_1, \frac{r}{2})$ follows a binomial distribution $\text{Binom}(n-1, \theta)$. Therefore Hoeffding's inequality implies that:

$$\begin{aligned} \mathbf{P}[\#\{B(x_1, \frac{r}{2})\} > n\theta] &\leq \mathbf{P}[\#\{B(x_i, \frac{r}{2})\} > (n-1)(\theta + \frac{\theta}{n-1})] \\ &\leq \exp(-2\frac{\theta^2}{n-1}). \end{aligned}$$

Then the clique number C is asymptotically almost surely smaller than $n\theta$, concluding the proof.

6 Maximum vertex degree, chromatic number, and independence number

In this section, for the graph G , we will denote $C(G)$ its clique number. We find, in this section, some of the results of [1], found by a slightly different approach.

6.1 Maximum vertex degree

Let us first define the degree of a vertex, then the maximum vertex degree of a graph:

Definition 7 *The degree of a vertex of a graph is the number of edges incident to the vertex.*

Definition 8 *The maximum degree of a graph G , denoted $\Delta(G)$, is the maximum degree of its vertices.*

By its definition we have the following inequality between the maximum vertex degree and the clique number of any graph G :

$$C(G) - 1 \leq \Delta(G). \quad (3)$$

Then if we consider the random geometric graphs $G(n, r)$ and $G(n, 2r)$, by doubling the adjacency distance between two vertices, we ensure that:

$$\Delta(G(n, r)) \leq C(G(n, 2r)) - 1. \quad (4)$$

Finally we have, for any graph G of n vertices v_1, \dots, v_n and N_2 edges, the equality $2N_2 = \sum_{i=1}^n \deg v_i$. For the graph $G(n, r)$, taking the mean we have then, thanks to Equation 1:

$$2^d(n-1)\theta \leq \mathbf{E}[\Delta(G(n, r))]. \quad (5)$$

Theorem 6 *In the subcritical regime, for $k \geq 1$ small compared to n and θ such that $\theta'_k < \theta < \theta_{k+1}$, the maximum vertex degree $\Delta(G(n, r))$ is asymptotically almost surely greater than $k - 1$.*

In the critical regime, the maximum vertex degree $\Delta(G(n, r))$ grows asymptotically almost surely faster than $(\ln n)^{1-\eta}$ pour tout $\eta > 0$, and slower than $2^d n \theta$.

In the supercritical regime, the maximum vertex degree $\Delta(G(n, r))$ grows as $2^d n \theta$ asymptotically in mean. It is asymptotically almost sure that $\Delta(G(n, r))$ grows slower than $2^d n \theta$.

Proof For the first part of the theorem, this is a direct consequence of Theorem 3 and inequality 3.

In the critical regime, we use inequalities 3 and 4, and the result from Theorem 4 for $C(G(n, r))$. Then one has to observe that the graph $G(n, 2r)$ is in the supercritical regime, then $C(G(n, 2r))$ asymptotically almost surely grows as $2^d n \theta$ according to Theorem 3.

In the supercritical regime, using Theorem 5 in inequalities 4 and 5 concludes the proof.

6.2 Chromatic number

Definition 9 *The chromatic number of a graph G , denoted $\chi(G)$, is the smallest number of colors needed to color the vertices of G such that no two adjacent vertices share the same color.*

Since two vertices in the same clique can not have the same color, we have that for any graph G :

$$C(G) \leq \chi(G).$$

A greedy coloring, assigning for each vertex the first available color, shows that for any graph G , $\chi(G) \leq \Delta(G) + 1$. Then we can write:

$$C(G(n, r)) \leq \chi(G(n, r)) \leq C(G(n, 2r)). \quad (6)$$

Theorem 7 *In the subcritical regime, for $k \geq 1$ small compared to n and θ such that $\theta'_k < \theta < \theta_{k+1}$, the chromatic number $\chi(G(n, r))$ is asymptotically almost surely greater than k .*

In the critical regime, the chromatic number $\chi(G(n, r))$ grows asymptotically almost surely faster than $(\ln n)^{1-\eta}$ pour tout $\eta > 0$, and slower than $2^d n \theta$.

In the supercritical regime, the chromatic number $\chi(G(n, r))$ grows asymptotically almost surely faster than $n \theta$, and slower than $2^d n \theta$.

Proof This is a direct consequence of inequality 6 and our three main theorems 3, 4, and 5.

6.3 Independence number

Definition 10 An independent set of a graph is a set of its vertices of which no pair is adjacent.

Definition 11 The independence number of a graph G , denoted $\alpha(G)$ is the size of the largest independent set of G .

The independence number and the chromatic number of a graph G are related by:

$$\alpha(G)\chi(G) \geq n, \quad (7)$$

where n is the number of vertices of G .

Then, in the graph $G(n, r)$, to have an independent set of size k , k balls centered on the independent vertices and of radius r , must be disjoint on the torus \mathbb{T}_a^d .

$$kr^d \leq a^d.$$

This is true for the largest independent set:

$$\alpha(G(n, r)) \leq \frac{1}{\theta}. \quad (8)$$

Theorem 8 In the critical and the supercritical regime, the independence number $\alpha(G(n, r))$ decreases asymptotically almost surely slower than $\frac{1}{2d\theta}$, and faster than $\frac{1}{\theta}$.

Proof This is a direct consequences of inequalities 7 and 8 and of Theorem 7.

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