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► **To cite this version:**

Anaïs Vergne, Laurent Decreasefond, Philippe Martins. Random Abstract Simplicial Complexes Reduction. 2017. hal-00864303v4

HAL Id: hal-00864303

<https://hal.science/hal-00864303v4>

Preprint submitted on 4 Sep 2017 (v4), last revised 11 May 2018 (v5)

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Random Abstract Simplicial Complexes Reduction

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Abstract—Random abstract simplicial complex representation provides a mathematical description of wireless networks and their topology. In order to reduce the energy consumption in this type of network, we intend to reduce the number of network nodes without modifying neither the connectivity nor the coverage of the network. In this paper, we present a reduction algorithm that lower the number of points of an abstract simplicial complex in an optimal order while maintaining its topology. Then, we study the complexity of such an algorithm for a network simulated by a binomial point process and represented by a Vietoris-Rips complex.

Index Terms—Simplicial homology; algebraic topology; reduction algorithm; point processes; complexity; wireless networks.

I. INTRODUCTION

Wireless networks are everyday more present in our lives: WiFi is the main internet access in our homes, cellular systems such as 4G and soon 5G provide its access everywhere else. Moreover with IoT, every object in our kitchen or in our bathroom will in the near future be connected as well. The quality of service of this type of network is primarily its connectivity and its coverage, only after checking this first two characteristics can come the capacity of the network, that is the number of users or connected devices a network can accept. However whether a network of sensors is fully connected, or whether a set of base stations does cover a whole domain, is not that easy to determine since network nodes are often irregularly deployed. Indeed, recent works such as [1] or [2] show that cellular networks deployment can be approached by random point processes going from the repulsive Ginibre point process to the neither repulsive nor attractive Poisson point process depending on the type of area (rural or urban) or the type of systems (every systems or only 4G systems for example).

Algebraic topology, [3], turns out to provide the solutions to the problem of how to compute the topology of a random set of points. Based on the geometrical data of the network (network nodes locations, communication or coverage radii), it is possible to build a combinatorial object representing it: the simplicial complex. Basically a simplicial complex is the generalization of the concept of graph, it is made of k -simplices where 0-simplices are vertices, 1-simplices are edges, 2-simplices are triangles, 3-simplices are tetrahedron and so on. In particular, the Čech simplicial complex allows to represent exactly the coverage of the union of the coverage disks as stated in the Nerve lemma in [4]. Then algebraic topology is a tool to compute the number of connected components, of coverage holes, and of 3D voids, that are the

so-called Betti numbers of the simplicial complex representing the network, as detailed in [5]. However, the computational time to obtain the Betti numbers can explode with the size of the simplicial complex, then it is possible to compute them in a decentralized way as seen in [6], or using persistent homology in [7], [8].

In this paper, we present a reduction algorithm for abstract simplicial complexes. Points, called 0-simplices, are removed one-by-one from an abstract simplicial complex while its topology remain unchanged. The removal order in the algorithm is optimal for the complexity of the abstract simplicial complex implementation and of the algorithm. We use the Vietoris-Rips simplicial complex built on a binomial point process for representing a wireless network and illustrate our algorithm. A first version of this algorithm has been presented in [9] and it has been used for cellular networks applications in [10]–[12]. We investigate the complexity of our reduction algorithm and show that it depends on the size of the largest simplex of the abstract simplicial complex. We compute its almost sure asymptotical behavior for a Vietoris-Rips complex based on a binomial point process, in which case it is also known as the clique number in a random geometric graph.

This is the first reduction algorithm for abstract simplicial complexes that uses homology to reduce the complex that we know of. Usually reduction algorithms for simplicial complexes are used to reduce complexes prior to the computation of their topology in order to reduce its complexity. For example, in [13] and [14], the authors use reduction of chain complexes in order to compute the homology groups and the Betti numbers. Witness complexes of [15] are another example of simplicial complexes reduction: the simplicial complex is reduced to a given number of vertices in order to compute the various topological invariants, such as the Betti numbers. So reduction of a simplicial complex has been used in order to compute its topology, we intend to do the opposite: reduction of the simplicial complex becomes the aim, while the homology computation is the mean to do it. The reduction problem can also be seen as a dominating graph problem [16]. However, since there is no notion of coverage in graphs, algorithms for the dominating graph problem do not maintain the topology of the initial simplicial complex. Our problem has also been studied under a game-theoretic approach in [17], where the authors define a coverage function. But they can only identify sub-optimal solutions that do not guarantee an unmodified coverage.

When computing the complexity of the algorithm, we focus on Vietoris-Rips complexes based on binomial point processes that fall into the class of random geometric complexes. There exists known results for this class of complexes [18], especially the moments of the number of k -simplices are explicitly

known [19]. In the end, we are reduced to compute the behavior of the size of the largest simplex, which is known as the clique number in graph theory. The clique number of the random geometric graph has been heavily studied in the literature, and its behavior described according to percolation regimes in [20]. In [21], it is proved that monotone properties of random geometric graph have sharp thresholds. Hence, in [22], [23], the authors prove that, in the subcritical regime, the clique number becomes concentrated on two consecutive integers. Moreover, in the subcritical regime, weak laws of large numbers [24] and central limits theorems [25] have been found for some functionals, including the clique number. Then for the supercritical regime, in [26], the authors described the behavior of the clique number. In this paper, we intend to gather all these results in one place: we provide the almost sure asymptotical behavior of the clique number of the random geometric graph for every of the three regimes thanks to the exact formulas of the moments of the number of k -simplices computed in [19].

The remainder of the paper is organized as followed. First in Section II, we remind some simplicial homology and algebraic topology definitions and properties. Then the reduction algorithm and its properties are described in Section III. The complexity of the algorithm is investigated for a random set of points in Section IV. Finally we conclude in Section V.

II. MATHEMATICAL BACKGROUND

A. Simplicial Homology

Considering a set of points representing network nodes, the first idea to apprehend the topology of the network would be to look at the neighbors graph: if the distance between two points is less than a given parameter then an edge is drawn between them. An example of a neighbors graph can be seen in Fig. 1. However this representation is too limited to transpose the network's topology. First, only 2-by-2 relationships are represented in the graph, there is no way to grasp interactions between three or more nodes. Moreover, there is no concept of coverage in a graph. That is why we are interested in more complex objects.

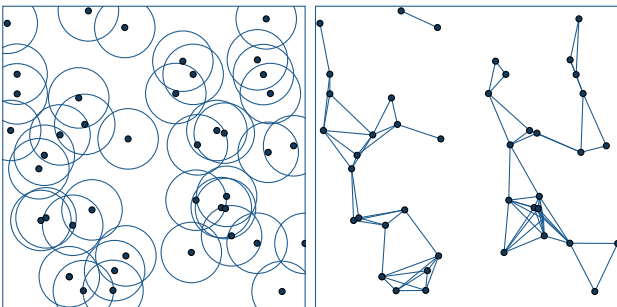


Fig. 1. A wireless network and its neighbors graph representation.

Indeed, graphs can be generalized to more generic combinatorial objects known as simplicial complexes. While graphs model binary relations, simplicial complexes can represent

higher order relations. A simplicial complex is thus a combinatorial object made up of vertices, edges, triangles, tetrahedra, and their n -dimensional counterparts.

Given a set of vertices X and an integer k , a k -simplex is an unordered subset of $k + 1$ vertices $\{x_0, x_1, \dots, x_k\}$ where $x_i \in X, \forall i \in \{0, \dots, k\}$ and $x_i \neq x_j$ for all $i \neq j$. Thus, a 0-simplex is a vertex, a 1-simplex an edge, a 2-simplex a triangle, a 3-simplex a tetrahedron, etc. See Fig. 2 for instance.

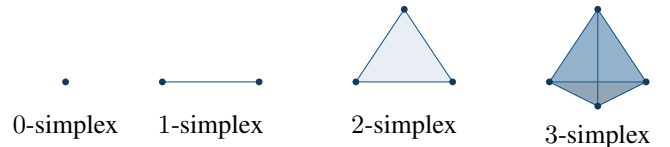


Fig. 2. Examples of k -simplices

Any subset of vertices included in the set of the $k + 1$ vertices of a k -simplex is a face of this k -simplex. A k -face is then a face that is a k -simplex. Thus, a k -simplex has exactly $k + 1$ $(k - 1)$ -faces, which are $(k - 1)$ -simplices. For example, a tetrahedron has four 3-faces which are triangles. The inverse notion of face is coface: if a simplex S_1 is a face of a larger simplex S_2 , then S_2 is a coface of S_1 . As for faces, a k -coface is a coface that is a k -simplex.

A simplicial complex is a collection of simplices which is closed with respect to the inclusion of faces, i.e. all faces of a simplex are in the set of simplices, and whenever two simplices intersect, they do so on a common face. An abstract simplicial complex is a purely combinatorial description of the geometric simplicial complex and therefore does not need the property of intersection of faces. In this article, we are only interested in the combinatorial description of a simplicial complex, that is why we will only consider abstract simplicial complexes, even if the adjective “abstract” may sometimes be dropped. Let us denote by x_0, x_1, \dots some vertices, and then write $[x_0, \dots, x_k]$ a k -simplex for any k integer. An example of an abstract simplicial complex with five 0-simplices x_0, \dots, x_4 , six 1-simplices $[x_0, x_1], [x_0, x_2], [x_1, x_2], [x_1, x_4], [x_2, x_3], [x_3, x_4]$, and one 2-simplex $[x_0, x_1, x_2]$ can be seen in Fig. 3.

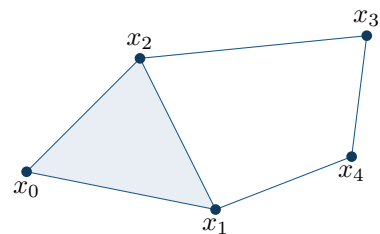


Fig. 3. Example of an abstract simplicial complex

The abstract simplicial complex that can exactly represent the topology of a wireless network (see the Nerve lemma in [4]) is the Čech complex whose definition is:

Definition 1 (Čech complex). Let (X, d) be a metric space, ω a finite set of points in X , and r a real positive number. The Čech complex of parameter r on the set of vertices ω , denoted $\mathcal{C}_r(\omega)$, is the abstract simplicial complex whose k -simplices are the unordered $(k+1)$ -tuples of vertices in ω for which the intersection of the $k+1$ balls of radius r centered at the $k+1$ vertices is non empty.

However, the Čech complex can be difficult to build since one must know the intersection of every three balls. Therefore, we are interested in the approximation of the Čech complex, the Vietoris-Rips complex:

Definition 2 (Vietoris-Rips complex). Let (X, d) be a metric space, ω a finite set of points in X , and r a real positive number. The Vietoris-Rips complex of parameter r of ω , denoted $\mathcal{R}_r(\omega)$, is the abstract simplicial complex whose k -simplices correspond to the unordered $(k+1)$ -tuples of vertices in ω which are pairwise within distance less than $2r$ of each other.

The Vietoris-Rips complex is easier to build than the Čech complex since it is build only based on the neighbors graph information, and it provides a good approximation for the network's topology if the network nodes are deployed according to a Poisson point process [27]. An example of a Vietoris-Rips complex can be seen in Fig. 4.

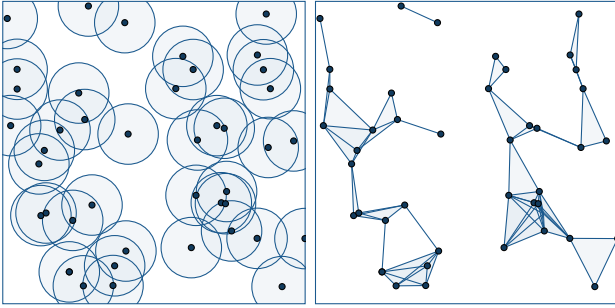


Fig. 4. A wireless network and its Vietoris-Rips complex.

B. Algebraic Topology

Given an abstract simplicial complex, one can define an orientation on the simplices by defining an order on the vertices, where a change in the orientation corresponds to a change in the sign:

$$\begin{array}{c} \begin{array}{ccc} x_0 & \xrightarrow{\quad} & x_1 \\ & \nearrow & \searrow \\ & x_1 & \\ & \nwarrow & \nearrow \\ x_0 & & x_2 \end{array} & [x_0, x_1] = -[x_1, x_0] \\ \\ \begin{array}{ccc} & \nearrow & \\ & x_1 & \\ & \nwarrow & \\ x_0 & & x_2 \end{array} & [x_0, x_1, x_2] = -[x_0, x_2, x_1] \end{array}$$

Then a change of sign corresponds to a swap between two vertices:

$$\begin{aligned} [x_0, \dots, x_i, \dots, x_j, \dots, x_k] = \\ - [x_0, \dots, x_j, \dots, x_i, \dots, x_k]. \end{aligned}$$

Then let us define the vector spaces of the k -simplices of a simplicial complex:

Definition 3. Let S be an abstract simplicial complex.

For any integer k , $\mathcal{C}_k(S)$ is the vector space spanned by the set of oriented k -simplices of S .

Then we can define a boundary map on these vector spaces:

Definition 4. Let S be an abstract simplicial complex and $\mathcal{C}_k(S)$ the vector space of its k -simplices for any k integer.

The boundary map ∂_k is defined as the linear transformation $\partial_k : \mathcal{C}_k(S) \rightarrow \mathcal{C}_{k-1}(S)$ which acts on the basis elements $[x_0, \dots, x_k]$ of $\mathcal{C}_k(S)$ via:

$$\partial_k[x_0, \dots, x_k] = \sum_{i=0}^k (-1)^i [x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k].$$

For example, for a 2-simplex we have:

$$\begin{array}{ccc} \begin{array}{ccc} x_1 & & \\ \nearrow & & \searrow \\ x_0 & & x_2 \end{array} & \begin{array}{ccc} x_1 & & \\ \nearrow & & \searrow \\ x_0 & & x_2 \end{array} \\ \partial_2([x_0, x_1, x_2]) & = [x_1, x_2] - [x_0, x_2] + [x_0, x_1] \end{array}$$

The boundary map on any k -simplex, is the cycle of its $(k-1)$ -faces. This map gives rise to a chain complex (a sequence of vector spaces and linear transformations):

$$\dots \xrightarrow{\partial_{k+2}} \mathcal{C}_{k+1} \xrightarrow{\partial_{k+1}} \mathcal{C}_k \xrightarrow{\partial_k} \mathcal{C}_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} 0.$$

We can see on our previous abstract simplicial complex example of Fig. 3 the computation of the three first boundary maps:

$$\begin{aligned} \partial_0 &= \begin{pmatrix} [x_0] & [x_1] & [x_2] & [x_3] & [x_4] \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \partial_1 &= \begin{array}{c} [x_0x_1] \ [x_0x_2] \ [x_1x_2] \ [x_1x_4] \ [x_2x_3] \ [x_3x_4] \\ \begin{pmatrix} [x_0] & -1 & -1 & 0 & 0 & 0 & 0 \\ [x_1] & 1 & 0 & -1 & -1 & 0 & 0 \\ [x_2] & 0 & 1 & 1 & 0 & -1 & 0 \\ [x_3] & 0 & 0 & 0 & 0 & 1 & -1 \\ [x_4] & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \end{aligned}$$

$$\partial_2 = \begin{array}{c} [x_0, x_1, x_2] \\ \begin{pmatrix} [x_0, x_1] & 1 \\ [x_0, x_2] & -1 \\ [x_1, x_2] & 1 \\ [x_1, x_4] & 0 \\ [x_2, x_3] & 0 \\ [x_3, x_4] & 0 \end{pmatrix} \end{array}$$

As its name indicates, the boundary map applied to a linear combination of simplices gives its boundary. The boundary of a boundary is the null application. Therefore this theorem can be easily demonstrated (see [3] for instance):

Theorem 1. For any k integer,

$$\partial_k \circ \partial_{k+1} = 0.$$

Let S be an abstract simplicial complex. Then we can denote the k -th boundary group of S as $B_k(S) = \text{im } \partial_{k+1}$, and the k -th cycle group of S as $Z_k(S) = \ker \partial_k$. Then, we have $B_k(S) \subset Z_k(S)$

We are now able to define the k -th homology group:

Definition 5. *The k -th homology group of an abstract simplicial complex S is the quotient vector space:*

$$H_k(S) = \frac{Z_k(S)}{B_k(S)}.$$

Then its dimension is:

Definition 6. *The k -th Betti number of the abstract simplicial complex S is:*

$$\beta_k(S) = \dim H_k(S).$$

According to its definition, the k -th Betti number counts the number of cycles of k -simplices that are not boundaries of $(k+1)$ -simplices, that are the k -th dimensional holes. In small dimensions, they have a geometrical interpretation:

- β_0 is the number of connected components,
- β_1 is the number of coverage holes,
- β_2 is the number of 3D-voids.

For any $k \geq d$ where d is the dimension, we can note that $\beta_k = 0$.

We can compute the Betti numbers of the abstract simplicial complex of Fig. 3:

$$\begin{aligned} \beta_0 &= \dim \ker \partial_0 - \dim \text{im } \partial_1 \\ &= 1 \\ \beta_1 &= \dim \ker \partial_1 - \dim \text{im } \partial_2 \\ &= 1. \end{aligned}$$

This complex indeed has one connected component and one coverage hole with 4 sides.

For further reading on algebraic topology, see [3].

III. REDUCTION ALGORITHM

In wireless networks, redundancy is frequent: sensors are cheap devices, adding too many sensors to a network creates reliability with reasonable cost. In cellular networks, dimensioning is done based on peak traffic hours, and is thus under-used during low-traffic periods. In order to conserve energy in both types of networks, some nodes can be turned off, may it be temporarily or until other nodes fail. But the topology of the network has to be maintained: connectivity is needed for the network nodes to communicate, and coverage maintenance assures the service to users. Simplicial homology representation of wireless networks provides a mathematical translation of the problem: remove vertices from an abstract simplicial complex without modifying its topology.

The main idea of our algorithm is to use the information from the topology of the network to reduce the number of vertices. First we use simplicial homology representation to compute the topology of the wireless network. Thanks to that representation, we are able to detect the vertices that are the more redundant. We then remove vertices in an optimal order

for the computation complexity, while the topology of the network, and in particular its Betti number, is unchanged. In the remainder of this section, we present in full details the reduction algorithm, that was first introduced in a limited form in [9].

A. Preliminaries

The reduction algorithm takes as input an abstract simplicial complex described by its list of simplices. But it also needs another information. Indeed, if we consider for example an abstract simplicial complex connected and without coverage holes, i.e. with Betti numbers $\beta_0 = 1$ and $\beta_1 = 0$, the optimal mathematical reduction of this complex is a single vertex. Its topology is in fact unchanged, there is still one connected component, and no cycle around any coverage hole. But that is not what we intend to do. Therefore we must designate *critical* vertices that define the limits of the reduction. They usually define the boundary of the area, they are then external boundary vertices. If there is a coverage hole whose size must not be increased, the vertices surrounding it, that are internal boundary vertices, have also to be listed as critical. But if the area covered is not essential, then critical vertices can be limited to access end points that have to stay connected for instance. In 3D, the critical vertices would define the limit surface of the volume. These critical vertices can not be removed by the algorithm. Then the list of critical vertices, whatever they are, internal or external boundary, access end points, etc, is given as input to the reduction algorithm, along with the abstract simplicial complex. Please note that for the algorithm to give adequate results, the critical vertices have to be correctly defined, that means that they must really define the limits of the abstract simplicial complex to reduce.

Then if the abstract simplicial complex is defined in dimension d , then there exists d nonzero homology groups. For $k \geq d$, the k -th homology group H_k does not exist and $\beta_k = 0$. Therefore it is possible to maintain the homology of the complex up to the $(d-1)$ -th degree. At the beginning of the algorithm, we must choose to which degree the homology has to be maintained. For example, in two dimensions, it is possible to maintain both connectivity and coverage, H_0 and H_1 , or to maintain only the connectivity (H_0). Note that there is no sense to maintain the k -th degree homology if the $(k-1)$ -th is not maintained since the former implies the latter, consider maintaining coverage without connectivity for example. The degree of homology that is to be maintained by the algorithm is denoted by k_0 , typically in two dimensions $k_0 = 1$, or sometimes $k_0 = 0$ to just maintain connectivity without considering coverage.

B. Characteristics

The algorithm works by calculating some characteristics for the simplices of the abstract simplicial complex it has to reduce. If the degree of homology that has to be maintained is k_0 , then the largest size of simplices that are concerned are (k_0+1) -simplices (in order to compute $B_{k_0} = \text{im } \partial_{k_0+1}$). We can conclude then that simplices larger than (k_0+1) -simplices are useless to the homology up to the k_0 -th degree.

For instance, in two dimensions, if we want to maintain the homology up to the first degree, that is the coverage, only 2-simplices are concerned, and larger simplices are useless.

Our idea is now to sort the $(k_0 + 1)$ -simplices in order to know which ones to conserve in the reduced abstract simplicial complex. To do that we characterize the superfluosness of each $(k_0 + 1)$ -simplex via a *degree* that we define below:

Definition 7 (Degree). Let $[x_0, \dots, x_k]$ be a k -simplex of an abstract simplicial complex S for k integer. Its degree is the size of its largest coface:

$$D[x_0, \dots, x_k] = \max\{d \mid [x_0, \dots, x_k] \subset d\text{-simplex}\}.$$

By definition, we can see that for any k -simplex, k integer, $D[x_0, \dots, x_k] \geq k$.

We can see an example of computation of degrees for the 2-simplices of an abstract simplicial complex in Fig. 5. A 2-simplex that is a maximum face, as $[x_2, x_6, x_7]$, has a degree of 2, whereas a 2-simplex that is the face of a 4-simplex, as $[x_0, x_1, x_7]$, has a degree of 4.

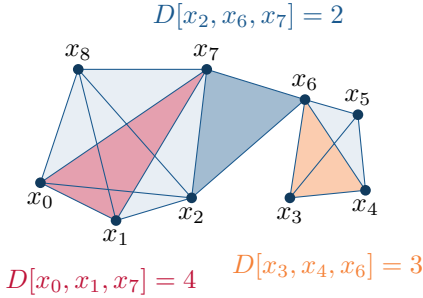


Fig. 5. Example of computation of degrees of 2-simplices

The greater the degree of a k -simplex is, the more superfluous this k -simplex is. Therefore, the greater the degree of a $(k_0 + 1)$ -simplex is, the more likely it is to be removed, and its removal is less likely to modify the k_0 -th homology.

But, in order to reduce an abstract simplicial complex, we do not remove directly $(k_0 + 1)$ -simplices, we remove 0-simplices, that lead to the removal of their cofaces. Therefore, we need to bring the information about the superfluosness of a $(k_0 + 1)$ -simplex that is contained in its degree, down to the 0-simplex level. To do that, we define an index for every 0-simplex, that characterizes its level of sensitivity for the k_0 -th homology.

Definition 8 (Index). Let x be a 0-simplex, its index is the minimum of its $(k_0 + 1)$ -cofaces degrees:

$$I(x) = \min\{D[x_0, \dots, x_{k_0+1}] \mid x \in [x_0, \dots, x_{k_0+1}]\}.$$

If x has no $(k_0 + 1)$ -coface then $I(x) = 0$.

We can see an example in Fig. 6 of computation of indices of the 0-simplices of the abstract simplicial complex of Fig. 5. In this example, we are interested in the 0-th and 1-st degrees of homology, i.e. connectivity and coverage, that means that $k_0 = 1$ and we compute the degrees on the 2-simplices.

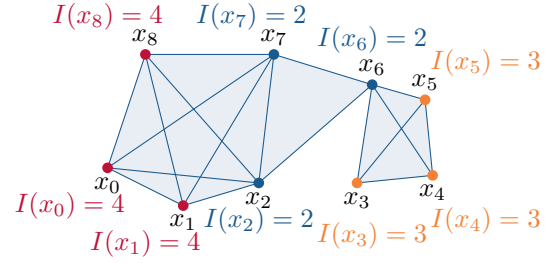


Fig. 6. Example of computation of indices of 0-simplices

With this definition of indices, we can see that a 0-simplex is as sensitive as its most sensitive coface, or in other terms as superfluous as its least superfluous coface. The index of a 0-simplex can also be viewed as an indicator of the density of 0-simplices “around” it, in the neighbor sense. For example, an index of $k_0 + 1$ indicates that at least one of its $(k_0 + 1)$ -coface has no $(k_0 + 2)$ -cofaces, whereas an index of $k > k_0 + 1$ indicates that each of its $(k_0 + 1)$ -cofaces are the face of simplices larger than k -simplices. The main idea of the algorithm is thus to remove the vertices with the greatest indices.

C. Algorithm description

As stated in the previous subsections, the reduction algorithm takes as input an abstract simplicial complex and the list of critical 0-simplices that can not be removed. We must also know the degree k_0 of homology that has to be maintained. The algorithm begins by computing the topology of the network up to the k_0 -th degree of homology, that is the first $k_0 + 1$ Betti numbers. Then, it computes the characteristics we need: the degrees of the $(k_0 + 1)$ -simplices and the indices of the 0-simplices. After that, the algorithm removes a 0-simplex with a maximal index. If there are more than one such index, one is chosen randomly uniformly among them. The removal of a 0-simplex leads to the removal of all of its cofaces. Then the algorithm goes on doing the same thing with the obtained reduced abstract simplicial complex.

There is one exception: if there is a difference between the Betti numbers computed on the newly reduced abstract simplicial complex and the original ones, the removal of the 0-simplex is cancelled. That means that this 0-simplex is put back in the simplicial complex. To prevent from trying to removing it again, that would lead to the same conclusion, the 0-simplex is flagged as critical, along with the input critical 0-simplices. Thus the list of critical 0-simplices evolve during the algorithm execution.

Then the algorithm goes on repeating the computation of the Betti numbers, degrees and indices to find a 0-simplex to remove. It removes 0-simplices one by one until the maximum index of a 0-simplex is equal to $k_0 + 1$. Indeed it is the minimum value for the degrees of $(k_0 + 1)$ -simplices, so it is the lower bound that it can reach.

We give in Algorithm 1 the whole reduction algorithm for the conservation of the k_0 -th homology. We use a negative index equal to -1 to flag critical 0-simplices as such.

Considering an abstract simplicial complex S , we denote by $s_k(S)$ its number of k -simplices and by $\beta_k(S)$ its k -th Betti number, for k integer. Then we denote by $x_1, \dots, x_{s_0(S)}$ its 0-simplices and by $y_1, \dots, y_{s_{k_0+1}(S)}$ its $(k_0 + 1)$ -simplices. We write $I(x)$ for the index of the 0-simplex x , and $D(y)$ for the degree of the $(k_0 + 1)$ -simplex y .

Algorithm 1 Reduction algorithm

Require: abstract simplicial complex S , list L of critical 0-simplices.

Computation of $\beta_0(S), \dots, \beta_{k_0}(S)$

Computation of $D(y_1), \dots, D(y_{s_{k_0+1}(S)})$

Computation of $I(x_1), \dots, I(x_{s_0(S)})$

for all $x \in L$ **do**

$I(x) = -1$

end for

$I_{\max} = \max\{I(x_1), \dots, I(x_{s_0(S)})\}$

while $I_{\max} > k_0 + 1$ **do**

Draw uniformly \hat{x} such that $I(\hat{x}) = I_{\max}$

$S' = S \setminus \{\hat{x}\}$ %Removal of \hat{x} and its cofaces

Computation of $\beta_0(S'), \dots, \beta_{k_0}(S')$

if $\beta_k(S') \neq \beta_k(S)$ for some $k = 0, \dots, k_0$ **then**

$I(\hat{x}) = -1$

else

Computation of $D(y_1), \dots, D(y_{s_{k_0+1}(S')})$

for all $x \in \{x_1, \dots, x_{s_0(S')}\}$ **do**

if $I(x) \neq -1$ **then**

Computation of $I(x)$

end if

end for

$I_{\max} = \max\{I(x_1), \dots, I(x_{s_0(S')})\}$

$S = S'$

end if

end while

return X

We can see in Fig. 7 an example of the reduction algorithm for the homology conservation up to the first degree on a Vietoris-Rips complex with a boundary of critical vertices along the square.

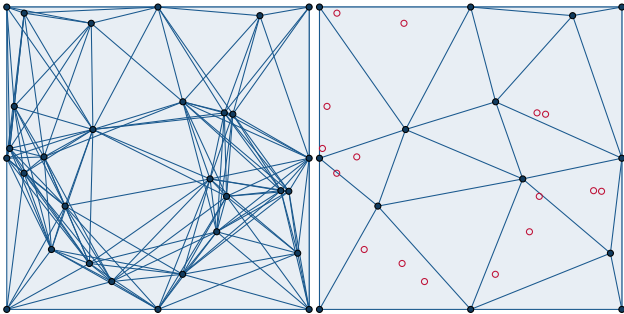


Fig. 7. Example of the reduction algorithm on a Vietoris-Rips complex.

We can note that it is possible to reduce the computations of the algorithm by remarking that when a 0-simplex of index I is removed then only the 0-faces of its I -cofaces with index

I can have their index impacted as proved in the following lemma. That means that we only need to re-compute the degrees needed for the indices of the 0-simplices that shared an I -simplex with the removed 0-simplex.

Lemma 1. *When a 0-simplex of index I is removed, only the 0-faces of its I -cofaces with index equal to I can have their index modified.*

Proof. Let \hat{x} be the removed 0-simplex, and x be any 0-simplex of the current abstract simplicial complex, we denote by $I(x)$ its index. We differentiate four cases:

- x and \hat{x} have no common coface.

Then none of the degrees of the $(k_0 + 1)$ -cofaces of x will change, and neither will its index.

- x and \hat{x} have a maximum common coface that is a k -simplex with $k < k_0$.

As in the previous case, none of the degrees of the $(k_0 + 1)$ -cofaces of x will change, and neither will its index.

- x and \hat{x} have a maximum common coface that is a k -simplex with $k_0 \leq k < I$.

Then \hat{x} should have an index of $k < I$, which is absurd.

- x and \hat{x} have a maximum common coface that is a k -simplex with $k \geq I$.

Either $I(x) < I$, then it comes from the degree of a $(k_0 + 1)$ -simplex not common with \hat{x} , and its index does not change with the removal of \hat{x} . Else, if $I(x) = I$, either its value comes from a I -simplex not shared with \hat{x} and remains unmodified. Or it comes from a common I -simplex. Only in this latter case, $I(x)$ is modified by the removal of \hat{x} . \square

D. Properties

1) *Homology invariance:* We have built our reduction algorithm to be homology invariant. That means that the initial abstract complex, the final reduced complex and every intermediary reduced complex are homotopy equivalent. They have the same Betti numbers and any basis element of the k -th homology group H_k in the initial complex can be mapped to a basis element of the k -th homology group in the final complex for any k integer.

Theorem 2. *The reduction algorithm stated in the algorithm described in Alg. 1 is homology invariant up to the k_0 -th degree.*

Proof. To verify that the algorithm is homology invariant up to the k_0 -th degree, we need only to check that each loop does not modify the k_0 -th homology. In each loop of the algorithm, we verify that the Betti numbers $\beta_0, \dots, \beta_{k_0}$ are unchanged. That means that the dimension of the homology groups H_0, \dots, H_{k_0} do not change. For any $k \in \{0, \dots, k_0\}$, if one or more cycle are added in H_k , by the removal of a 0-simplex, then the removal is cancelled by the algorithm. The same goes with one or more deletions of cycles in H_k . The only way that a change in the homology is undetected and allowed by the algorithm is if in a loop the same number of cycles of H_k are simultaneously added and deleted by a single 0-simplex removal.

For any $k \in \{0, \dots, k_0\}$, H_k is the k -th homology group of cycles of k -simplices (Z_k) that are not boundaries of $(k+1)$ -simplices (B_k), note that $B_k \subset Z_k$.

On the one hand, let us look what happens if a cycle is added in H_k . Since a new cycle of Z_k , that is a list of k -simplices, can not be created by removing simplices, the removal of a 0-simplex adds a cycle in H_k only if a cycle which was both in B_k and Z_k ceases to be in B_k , i.e. if the 0-simplex had a $(k+1)$ -coface that was not redundant.

On the other hand, we investigate the deletion of a cycle in H_k . The removal of a 0-simplex can not make a cycle only in Z_k to be in both B_k and Z_k since it does not create $(k+1)$ -simplices. So a deletion of a cycle in H_k , due to the removal of a 0-simplex, is necessarily a deletion in Z_k . Therefore, the removal of a 0-simplex deletes a cycle in H_k only if this 0-simplex had a k -coface which was in the cycles of Z_k and not the boundaries of B_K . That means that this 0-simplex was a boundary vertex of a k -th dimensional hole.

If a 0-simplex that is both a boundary vertex to a k -th dimensional hole and has a non-redundant $(k+1)$ -coface is removed, simultaneously a cycle is deleted and another is created in H_k . The deletion of the 0-simplex leads to an enlargement of the k -th dimensional hole. The $(k+1)$ -coface assures that the hole still has boundaries and exists. It is then possible to map the cycle of the smaller (previous to the 0-simplex removal) hole to the larger (post 0-simplex removal) hole. And the two abstract simplicial complexes (pre and post 0-simplex removal) are homotopy equivalent. \square

We can note that if we want to have a reduction algorithm that is homology invariant and that do not enlarge k -th dimensional holes, then all the hole boundary vertices must be defined as critical in the input.

2) *Optimal order for vertices removal:* In our reduction algorithm, we choose to compute the topology and use it to reduce an abstract simplicial complex in order to minimize the size of the complex in relation to its topology. In addition to keeping the homology invariant, we use the homology information to improve the algorithm performance.

Theorem 3. *The order in which 0-simplices and their cofaces are removed from the abstract simplicial complex by the reduction algorithm defined in Alg. 1 is optimal for its computation complexity.*

Proof. When a 0-simplex is removed from the abstract simplicial complex, all of its coface are subsequently removed too. In Alg. 1, the 0-simplices are removed by decreasing indices. An index of I indicates that every (k_0+1) -coface of the 0-simplex has I -faces at least. The bigger the index of a 0-simplex is, the bigger the degrees of its (k_0+1) -cofaces are. So the removal of the maximum index 0-simplex minimize at most the degrees computations that must be done in every loop.

Moreover we can see that the bigger the index of a 0-simplex is, the bigger the minimum common size of its cofaces greater than (k_0+1) -simplices is. An abstract simplicial complex is implemented by the list of all its simplices. And the homology up to the k_0 -th degree (i.e. (k_0+1) -simplices), is computed. By eliminating the greatest (k_0+1) -simplices

cofaces, the algorithm reduces also the size of the simplicial complex implementation, and its topology computation. \square

3) *Optimal solution:* The reduction algorithm reaches a local optimum, that may not be the global optimum if there are multiple local optima. In game theory vocabulary, that means that the algorithm reaches a Nash equilibrium as defined in [17]:

Theorem 4. *The reduction algorithm defined in Alg. 1 reaches a Nash equilibrium: the final complex can not be further reduced, no more 0-simplex can be removed.*

Proof. By definition of degrees and indices given in the previous subsection, indices computed on (k_0+1) -simplices degrees are greater or equal to k_0+1 . However, in the final complex, every 0-simplex is of index smaller or equal to k_0+1 , since its the ending condition on the “while” loop.

Then there are three possibility for the value $I(x)$ of the index of a 0-simplex x in the final complex:

- $I(x) = -1$ which means that x has been defined or flagged as critical.
- $I(x) = 0$ which means that x has no (k_0+1) -coface.
- $I(x) = k_0+1$.

First, a critical 0-simplex can be either defined as such, in which case its removal was forbidden, and it should be in the final complex. Or it as been flagged by the algorithm because its removal was tried and changed the Betti numbers of the abstract simplicial complex, which is forbidden. If the initial critical 0-simplices are well-defined as the limits of the complex, then the removal of a vertex 0-simplex that has led to a change in a Betti number would always lead to the same change. So a flagged critical 0-simplex stays as such.

Secondly, a 0-simplex of null index is an isolated vertex for the k_0 -th homology as it has no (k_0+1) -coface. Then, its removal would decrease one of the Betti numbers $\beta_0, \dots, \beta_{k_0}$. For example, the removal of a 0-simplex with no 1-cofaces will decrease β_0 , and the removal of a 0-simplex with 1-cofaces and no 2-cofaces would decrease either β_1 .

Finally, if x has a degree of $I(x) = k_0+1$. That means that at least one of its (k_0+1) -cofaces has no larger coface. Then the removal of x would lead to the removal of this (k_0+1) -simplex with no coface. This would create a k_0 -th dimensional hole, and β_{k_0} would be incremented. \square

4) *Bounds for the number of removed 0-simplices:* With the reduction algorithm, we go from an initial abstract simplicial complex to a final complex with an optimal number of 0-simplices. We are now interested in the number of 0-simplices that can be removed from the initial complex.

Theorem 5. *Let E_k be the set of 0-simplices that have index k in the initial complex, and $|E_k|$ be its cardinality. Then the number M of removed 0-simplices by the algorithm defined in Alg. 1 is bounded by:*

$$\sum_{k=k_0+2}^{I_{\max}} \mathbb{1}_{E_k \neq \emptyset} \leq M \leq \sum_{k=k_0+2}^{I_{\max}} |E_k|.$$

Proof. We begin by looking at the upper bound. First, let us state that a 0-simplex with an index equal or less than k_0+1 ,

can not have its index increased during the algorithm. It is a direct consequence of Lemma 1 and the fact that the stopping limit is when the maximum index is equal to $k_0 + 1$. Then 0-simplices of index equal or less than $k_0 + 1$ are never removed by the algorithm. Thus the number of other 0-simplices is an upper bound for M .

For the lower bound, according to Lemma 1, the removal of a 0-simplex of index I_{\max} can only modify the indices that were set to I_{\max} previously. In the worst case, all indices I_{\max} change and the value of I_{\max} is decreased. Thus, at least one 0-simplex per index value is removed. And the number of index values strictly above $k_0 + 1$ constitutes a lower bound for M . \square

Remark. We can note that the upper bound for the number of removed 0-simplices is optimal. Since it is reached for example in the case of an abstract simplicial complex limited to a n -simplex and its faces, with n integer. Whatever the definition of the limit with the critical 0-simplices, only the latter will stay in the final complex.

We can also note that the lower bound for the number of removed 0-simplices is optimal too. Indeed, it is reached for instance for a complex limited to a n -simplex with $n-1$ critical 0-simplices.

IV. COMPLEXITY

In this section, we investigate the complexity of the algorithm presented in Alg. 1 for the conservation of the k_0 -th homology. For S an abstract simplicial complex, let us denote by s_k the numbers of its k -simplices for any k integer. We also denote by K the integer such that the maximum simplex in S is a K -simplex. Then the size of the input data of the reduction algorithm depends on s_0, \dots, s_K .

To compute the complexity of the whole reduction algorithm, we must first compute the complexity of the computation of the $k_0 + 1$ first Betti numbers denoted $\beta_0, \dots, \beta_{k_0}$.

Proposition 1. *The complexity of the computation of the Betti numbers $\beta_0, \dots, \beta_{k_0}$ is in $O(\max_{k=0, \dots, k_0+1} (s_k^3))$.*

Proof. The computation of the Betti number β_k relies on the computation of the ranks of matrices ∂_k of size $s_{k-1} \times s_k$ and ∂_{k+1} of size $s_k \times s_{k+1}$.

Moreover, the computation of the rank of a matrix of size $n \times m$ is of complexity $O(nm \min(n, m))$. \square

Now, we look at the complexities of the computations of the degrees and the indices needed in the reduction algorithm:

Proposition 2. *The complexity of the computation of the degree of a $(k_0 + 1)$ -simplex is in $O(\sum_{k=k_0+2}^K s_k)$.*

Proof. To compute the degree of a k -simplex, we must explore at most all the larger simplices than k -simplices. \square

Proposition 3. *The complexity of the computation of the index of a 0-simplex is in $O(s_{k_0+1})$.*

Proof. The index of a 0-simplex is just the minimum of its $(k_0 + 1)$ -cofaces' degrees, that are at most s_{k_0+1} . \square

Then, we are able to write the whole reduction algorithm complexity:

Theorem 6 (Reduction algorithm complexity). *The reduction algorithm described in Alg. 1 has a complexity in:*

$$O(s_0(\max_{0 \leq k \leq k_0+1} s_k^3 + s_{k_0+1} \sum_{k=k_0+2}^K s_k)).$$

Proof. In the reduction algorithm, the $k_0 + 1$ first Betti numbers $\beta_0, \dots, \beta_{k_0}$, the s_{k_0+1} degrees and the s_0 indices are computed at each run that is at most s_0 times. The computation of the indices becomes negligible compared to the computation of the Betti numbers.

At the end, at most every simplex has been removed, then the removal of simplices is of overall complexity of $O(\sum_{k=0}^K s_k)$. This complexity is negligible for the whole algorithm.

The complexity to mark 0-simplices as critical and the complexity to compute I_{\max} are both in $O(s_0)$, and are also negligible. \square

Remark. *The complexity of the reduction algorithm is polynomial relatively to the size of the input data s_0, \dots, s_K .*

Traditionally we prefer to express the complexity of an algorithm just relatively to the number of points, that is s_0 the number of 0-simplices. But we can see that the number K is key here. Indeed every number s_k appears directly in the complexity formula. However, K appears as the limit of a sum, and since it is upper bounded only by s_0 , its behavior determines if the complexity is polynomial or exponential in s_0 .

Thus, we first keep the size of the largest simplex K as a variable in the complexity formula:

Corollary 7. *The reduction algorithm described in Alg. 1 has a complexity upper bounded by:*

$$O(s_0^{3k_0+7} + s_0^{k_0+3} \sum_{k=k_0+2}^K \binom{s_0}{k+1}).$$

Proof. The number of k -simplices s_k is upper bounded by $\binom{s_0}{k+1}$ for any k integer. Taking the expression of the complexity from Theorem 6, we have:

$$\begin{aligned} & s_0(\max_{0 \leq k \leq k_0+1} s_k^3 + s_{k_0+1} \sum_{k=k_0+2}^K s_k) \\ & \leq s_0(\max_{0 \leq k \leq k_0+1} \binom{s_0}{k+1}^3 + \binom{s_0}{k_0+2} \sum_{k=k_0+2}^K \binom{s_0}{k+1}) \\ & \leq s_0(\binom{s_0}{k_0+2}^3 + \binom{s_0}{k_0+2} \sum_{k=k_0+2}^K \binom{s_0}{k+1}) \end{aligned}$$

Since k_0 is a fixed small number compared to s_0 when s_0 goes to infinity, and $\binom{s_0}{k}$ is increasing for $k \leq k_0 + 2$.

Then the fact that $\binom{s_0}{k} \leq s_0^k$ concludes the proof \square

Then we express the complexity with only the number of 0-simplices s_0 as a parameter:

Corollary 8. *The reduction algorithm described in Alg. 1 has a complexity upper bounded by:*

$$O(s_0^{k_0+3} 2^{s_0}).$$

Proof. The size of the largest simplex K can only be upper bounded by s_0 in the general case. Then the fact that $\sum_{k=0}^n \binom{n}{k} = 2^n$ concludes the proof. \square

We can see that the complexity of the reduction algorithm relatively to s_0 can vary drastically. Indeed if the number of large simplices is negligible relatively to the number of smaller simplices then the computation of the Betti numbers will be the preponderant part and the complexity of the reduction algorithm will be polynomial in s_0 . However, if it is there are large simplices, then the computation of the degrees will be the longest, and the reduction algorithm will have a complexity that is exponential in s_0 .

We propose in the following of this section a thorough study on the behavior of the complex and the size of its largest simplex K that will determine the complexity of the reduction algorithm.

A. Model

In order to study the behavior of an abstract simplicial complex, we first must choose a particular complex. We select the Vietoris-Rips complex for its mathematical tractability and its wireless network representation capacity. We now have to decide on which set of points the complex is build.

When evaluating the behavior of a complex, we need to take into account side effect: indeed a point in the center of the plane area considered will have more neighbors, and thus be part of larger simplices that a point on the edge of the area. In order to avoid these side effects, we choose to consider a torus instead of a plane. Let us denote by \mathbb{T}_a^d the torus of side a in dimension d . Usually, we will have $d = 2$.

We now define the space of configurations:

Definition 9. *The space of configurations on \mathbb{T}_a^d is the set of locally finite simple point measures:*

$$\Omega^X = \left\{ \omega = \sum_{k=1}^n \delta(x_k) : (x_k)_{k=1}^{k=n} \subset X, n \in \mathbb{N} \cup \{\infty\} \right\},$$

where $\delta(x)$ denotes the Dirac measure for $x \in \mathbb{T}_a^d$.

It is convenient to identify an element ω of Ω^X with the set corresponding to its support, i.e. $\sum_{k=1}^n \delta(x_k)$ is identified with the unordered set $\{x_1, \dots, x_n\}$. For $A \subset X$, we have $\delta(x)(A) = \mathbb{1}_A(x)$, so that $\omega(A) = \sum_{x \in \omega} \mathbb{1}_A(x)$ counts the number of points in A . Simple measure means that there are no two points in the same place, that is $\omega(\{x\}) \leq 1$ for any $x \in X$. Locally finite means that $\omega(K) < \infty$ for any compact K of X . The configuration space Ω^X is endowed with the vague topology and its associated σ -algebra denoted by \mathcal{F}^X .

For further reading on random point processes, we refer to [28].

We now define the binomial point process that is a variation of the well-known Poisson point process but with a fixed number of points:

Definition 10 (Binomial point process). *Let f be the uniform probability density function on the torus \mathbb{T}_a^d , and n an integer. Then a point process ω is a binomial point process of n points on \mathbb{T}_a^d , if the following two conditions hold:*

- (i) *The process ω has n points,*
- (ii) *The points' positions are drawn according to f independently from each other.*

In order to have reasonable results we need to make two assumptions. First, we need to ensure that the ratio $\frac{r}{a}$ that is the ratio between the connexion distance of the Vietoris-Rips complex and the size of the torus, is not too big so that two points are able to be connected 2 times on both sides of the torus:

Assumption 1. *Let us denote by θ the ratio $(\frac{r}{a})^d$, then we assume that:*

$$\theta = \left(\frac{r}{a}\right)^d \leq \left(\frac{1}{2}\right)^d.$$

Then the behavior of the number of simplices is not easy to obtain. In [19], the authors provide expressions for the moments of the number of simplices for the Vietoris-Rips complex by means of Malliavin calculus. They obtain results for the classic Euclidean norm, however the expressions are not tractable. That is why we make Assumption 2:

Assumption 2. *For the construction of the Vietoris-Rips complex based on a binomial point process, we use the uniform norm.*

For $x \in \mathbb{T}_a^d$ of elements (x_1, \dots, x_d) the uniform norm of x is:

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_d|\}.$$

Then we can use the results presented in [19]:

Theorem 9 ([19]). *Let $k \geq 1$ be an integer. The expectation and variance of the number of k -simplices in a Vietoris-Rips complex based on a binomial point process of $n = s_0$ points on the torus \mathbb{T}_a^d are:*

$$\mathbf{E}[s_{k-1}] = \binom{n}{k} \theta^{k-1} k^d \quad (1)$$

$$\mathbf{V}[s_{k-1}] = \sum_{i=1}^{k+1} \binom{n}{2k-i} \binom{2k-i}{k} \binom{k}{i} \theta^{2k-i-1} \left(2k-i + 2\frac{(k-i)^2}{i+1}\right)^d. \quad (2)$$

For a better reading we will denote by n the number of points of the binomial point process, that is equal to s_0 the number of 0-simplices of the abstract simplicial complex. Throughout this section we will investigate the almost sure asymptotic behavior of the size of the largest simplex K and

the complexity of the reduction algorithm when n tends to infinity and with respect to the distribution of the n points according to a binomial point process.

Definition 11. We say that the property P is true asymptotically almost surely if $\mathbf{P}[P \text{ true}] \rightarrow 1$ when n tends to infinity.

B. Percolation regimes

One can easily see that the number K that is the size of the largest simplex in a complex is the equivalent of the clique number C in a graph. More precisely we have that $K = C - 1$, since a k -simplex has $k + 1$ points. Moreover the Vietoris-Rips complex is by definition the clique complex of the geometric graph. Then the Vietoris-Rips complex based on a binomial point process is the clique complex of the random geometric graph. That is why we use the same percolation regimes to study K as the ones described for random geometric graphs. For further reading random geometric graphs and the definition of percolation regimes, see [20]. We show in Figure 8 the three different percolation regimes.

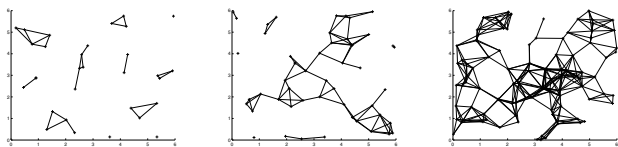


Fig. 8. Percolation regimes

The percolation regimes are defined with the use of the Bachman-Landau notations that we define here. For non-negative functions f and g we write as n tends to infinity:

- $f(n) = o(g(n))$ if for every $\varepsilon > 0$ there exists M such that for $n \geq M$, we have $f(n) \leq \varepsilon g(n)$. We say that f is dominated by g asymptotically.
- $f(n) = O(g(n))$ if there exists $k > 0$ and M such that for $n \geq M$, we have $f(n) \leq kg(n)$. We say that f is bounded by g asymptotically.
- $f(n) \sim g(n)$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. We say that f and g are equal asymptotically.
- $f(n) \ll g(n)$ if $\frac{f(n)}{g(n)} = o(1)$. We say that f is small compared to g asymptotically.

C. Subcritical regime

In this subsection, we consider that $\theta = o(\frac{1}{n})$. In the subcritical regime, the Vietoris-Rips complex is composed of various disconnected components. We can see an example of a Vietoris-Rips complex in the subcritical regime in Figure 9.

Therefore the size of the largest simplex K is expected to be rather small, n -simplices do not exist in this regime. That is why we focus on the number of k -simplices s_k for $k \geq 1$ small compared to n , i.e. $k \ll n$.

We are able to derive approximations from Theorem 9:

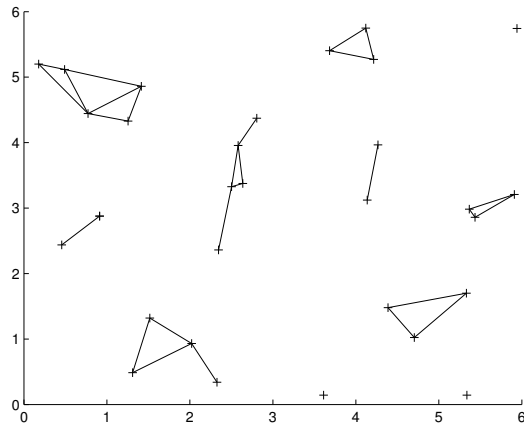


Fig. 9. Subcritical regime $\theta = o(\frac{1}{n})$

Lemma 2. For $k \geq 1$ and $k \ll n$ in the subcritical regime,

$$\mathbf{E}[s_k] \sim \frac{n^{k+1}}{(k+1)!} \theta^k (k+1)^d$$

$$\mathbf{V}[s_k] \sim \frac{n^{k+1}}{(k+1)!} \theta^k (k+1)^d.$$

Proof. This is a direct consequence of the subcritical regime hypothesis applied to Equations 1 and 2. \square

Meanwhile, in the subcritical regime, the random geometric graph shares similar properties with the Erdős-Rényi model, that is the graph with n points where each edge is chosen independently with probability p . The clique number of the Erdős-Rényi graph has been studied first in [29], and its almost sure behavior has been described in [30]:

Theorem 10 ([30]). Let us define $n_k \sim p^{-k/2}$ and $n'_k \sim (1 + \frac{3 \log k}{k}) p^{-k/2}$. When n goes to infinity, with a fixed probability p , for almost every graph, there is a constant c such that if $n'_k \leq n \leq n_{k+1}$ for some $k > c$, then the clique number is $K = k$.

That means that the size of the largest simplex K grows slowly step by step as the number of points n goes to infinity. We find similar results for the Vietoris-Rips complex:

Theorem 11. Let k be a non zero integer and η a strictly positive real number. We then define:

$$\theta'_k = \left(\frac{(k+1)!(k+1)^{1+\eta-d}}{n^{k+1}} \right)^{1/k}$$

$$\theta_k = \left(\frac{(k+1)!(k+1)^{-(1+\eta+d)}}{n^{k+1}} \right)^{1/k}.$$

Then for $k \ll n$ and $\theta'_k < \theta < \theta_{k+1}$ in the subcritical regime, the size of the largest simplex in a Vietoris-Rips complex based on a binomial point process of n points on the torus \mathbb{T}_a^d is asymptotically almost surely:

$$K = k.$$

Proof. For $\theta > \theta'_k$, thanks to the approximations of Lemma 2 we have:

$$\mathbf{E}[s_k] \geq \frac{n^{k+1}}{(k+1)!} \theta'^k (k+1)^d = (k+1)^{1+\eta}.$$

And for $\theta < \theta_k$,

$$\mathbf{E}[s_k] \leq n^{k+1} \theta_k^k (k+1)^d = (k+1)^{-(1+\eta)}.$$

Moreover, for $k \ll n$ when n goes to infinity, we consider n large enough so that $n > k!k^{k(1+2\eta)}$. Since $k^k > (k+1)^{k-1}$ for all $k \geq 1$, we have:

$$\begin{aligned} n &> k^{k(1+2\eta)} k! \\ \Rightarrow n &> k^{k(1+\eta-d)} k^{k(\eta+d)} k! \\ \Rightarrow n &> k^{k(1+\eta-d)} (k+1)^{(k-1)(\eta+d)} k! \\ \Rightarrow n &> k^{k(1+\eta-d)} (k+1)^{(k-1)(1+\eta+d)} \frac{1}{(k+1)^{k-1}} k! \\ \Rightarrow n &> k^{k(1+\eta-d)} (k+1)^{(k-1)(1+\eta+d)} \frac{(k!)^k}{((k+1)!)^{k-1}} \\ \Rightarrow n^{\frac{1}{k(k-1)}} &> \frac{(k^{1+\eta-d} k!)^{1/(k-1)}}{((k+1)^{-(1+\eta+d)} (k+1)!)^{1/k}} \\ \Rightarrow n^{\frac{k}{k-1}} &> \frac{(k^{1+\eta-d} k!)^{1/(k-1)}}{((k+1)^{-(1+\eta+d)} (k+1)!)^{1/k}} \\ \Rightarrow n^{\frac{k+1}{k}} &> \frac{(k^{1+\eta-d} k!)^{1/(k-1)}}{((k+1)^{-(1+\eta+d)} (k+1)!)^{1/k}} \\ \Rightarrow \theta_k &> \theta'_{k-1}. \end{aligned}$$

We can now consider θ such that $\theta'_k < \theta < \theta_{k+1}$.

On the one hand, we find an upper bound for the probability of the non-existence of k -simplices:

$$\mathbf{P}[s_k = 0, \theta > \theta'_k] \leq \frac{\mathbf{V}[s_k]}{\mathbf{E}[s_k]^2} \sim \frac{1}{\mathbf{E}[s_k]} \leq \frac{1}{(k+1)^{1+\eta}}.$$

On the other hand, we can find an upper bound for the probability of existence of $(k+1)$ -simplices:

$$\mathbf{P}[s_{k+1} > 0, \theta < \theta_{k+1}] \leq \mathbf{E}[s_{k+1}] \leq \frac{1}{(k+2)^{1+\eta}}.$$

Finally we have:

$$\begin{aligned} \mathbf{P}[\exists \theta, \theta'_k < \theta < \theta_{k+1}, K \neq k] \\ < \frac{1}{(k+1)^{1+\eta}} + \frac{1}{(k+2)^{1+\eta}}. \end{aligned}$$

As the sum $\sum_{k=1}^{\infty} k^{-(1+\eta)}$ converges, the Borel-Cantelli theorem implies that with the exception of finitely many k 's, for all θ such that $\theta'_k < \theta < \theta_{k+1}$, one has $K = k$. Then when n goes to infinity, we have asymptotically almost surely that $K = k$ as n goes to infinity:

$$\mathbf{P}[K = k, \theta'_k < \theta < \theta_{k+1}] \xrightarrow{n \rightarrow \infty} 1,$$

concluding the proof. \square

Finally we can conclude on the complexity of the reduction algorithm in this regime:

Theorem 12. *The reduction algorithm described in Alg. 1 for a Vietoris-Rips complex based on a binomial point process of*

n points on the torus \mathbb{T}_a^d in the subcritical regime such that $\theta = o(\frac{1}{n})$, has a complexity in:

$$O(n^{3k_0+7}).$$

Proof. This is a direct consequence of Corollary 7 and Theorem 11 as the first part of the complexity becomes preponderant. \square

D. Critical regime

We now consider that $\theta \sim \frac{1}{n}$ so that the abstract simplicial complex is in the critical regime. In this regime, percolation occurs: the disconnected components begin to connect into one sole connected component. We can see an example of a Vietoris-Rips complex in the critical regime in Figure 10.

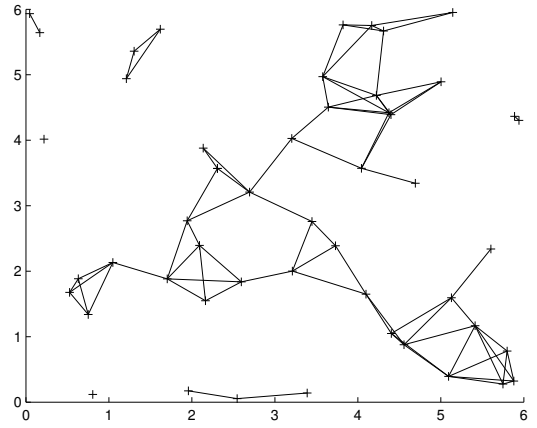


Fig. 10. Critical regime $\theta \sim \frac{1}{n}$

The size of the largest simplex K is still rather small compared to n as n goes to infinity, n -simplices still do not exist in this regime. This allows us to focus on the number of k -simplices s_k for $k \geq 1$ bounded by n asymptotically, i.e. $k = O(n)$.

We are able to derive new approximations from Theorem 9 in this regime:

Lemma 3. *For $k \geq 1$ and $k = O(n)$ in the critical regime,*

$$\mathbf{E}[s_k] \sim \frac{1}{\sqrt{2\pi}} n (k+1)^{d-k+\frac{1}{2}}$$

$$\mathbf{V}[s_k] \sim \frac{1}{\sqrt{2\pi}} n (k+1)^{d-k+\frac{1}{2}}.$$

Proof. The expectation approximation is a direct consequence of Equation 1, the critical regime approximation $\theta \sim \frac{1}{n}$ and Stirling's approximation: $n! \sim \sqrt{2\pi n} (\frac{n}{e})^n$.

Then, since $k = O(n)$, we can approximate the variance by its dominating term in Equation 2 that is $i = k$ that leads to $\mathbf{V}[s_k] \sim \mathbf{E}[s_k]$. \square

We then derive from these approximations, the almost sure asymptotical behavior of the size of the largest simplex K :

Theorem 13. *In the critical regime, the size of the largest simplex in a Vietoris-Rips complex based on a binomial point process of n points on the torus \mathbb{T}_a^d , grows asymptotically almost surely slower than $\ln n$ with an arbitrary small distance. That means that for all η strictly positive real number:*

$$(\ln n)^{1-\eta} < K < \ln n.$$

Proof. First for $k > \ln n$, we find an upper bound for the expectation of the number of k -simplices thanks to the approximation of Lemma 3:

$$\mathbf{E}[s_k] < \frac{1}{\sqrt{2\pi}} n(\ln n + 1)^{d+\frac{1}{2}-\ln n}.$$

One can easily check that this upper bound tends to 0 as n goes to infinity. Then, since $\mathbf{P}[s_k > 0] \leq \mathbf{E}[s_k]$, the probability that there exists k -simplices tends to 0:

$$\mathbf{P}[K > k] = \mathbf{P}[s_k > 0] \xrightarrow{n \rightarrow \infty} 0 \quad \forall k > \ln n,$$

and $K < \ln n$ asymptotically almost surely.

On the other hand, for $k < (\ln n)^{1-\eta}$ for all $\eta > 0$, we find a lower bound for the expectation of the number of k -simplices:

$$\mathbf{E}[s_k] > \frac{1}{\sqrt{2\pi}} n((\ln n)^{1-\eta} + 1)^{d+\frac{1}{2}-(\ln n)^{1-\eta}}.$$

This lower bound tends to infinity as n grows. Thanks to the asymptotic equivalence of the variance and the expectation of the number of k -simplices, we have that $\mathbf{P}[s_k = 0] \leq \frac{1}{\mathbf{E}[s_k]}$, and the probability that there exists no k -simplices tends to infinity:

$$\mathbf{P}[K < k] = \mathbf{P}[s_k = 0] \xrightarrow{n \rightarrow \infty} 0 \quad \forall k < (\ln n)^{1-\eta},$$

and $K > (\ln n)^{1-\eta}$ asymptotically almost surely. \square

We can now derive the complexity of the reduction algorithm in the critical regime:

Theorem 14. *The reduction algorithm described in Alg. 1 for a Vietoris-Rips complex based on a binomial point process of n points on the torus \mathbb{T}_a^d in the critical regime such that $\theta \sim \frac{1}{n}$ has a complexity in:*

$$O(n^{\ln n}).$$

Proof. The complexity from Corollary 7 is $O(n^{3k_0+7} + n^{k_0+3} \sum_{k=k_0+1}^K \binom{n}{k+1})$. Then Theorem 13 gives us an approximation for K .

Since $\ln n < \frac{n}{2}$ for every $n \geq 2$, the preponderant term of the sum is $\binom{n}{\ln n}$ that can only be upper bounded by $n^{\ln n}$. \square

E. Supercritical regime

In the supercritical regime, we have that $\frac{1}{n} = o(\theta)$. Percolation has occurred: the Vietoris-Rips complex is now connected and tends to become the complete complex, i.e. the complex with all simplices by analogy with the complete graph. We can see an instance of a Vietoris-Rips complex in this regime in Figure 11.

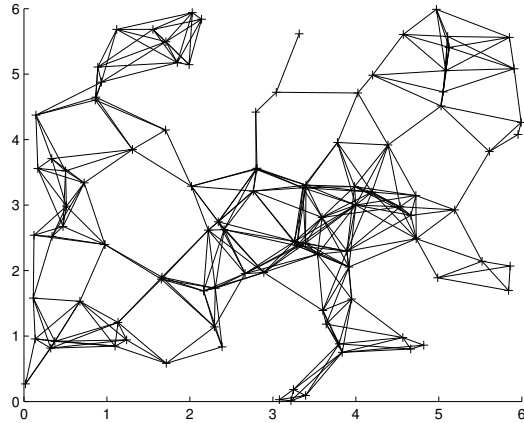


Fig. 11. Supercritical regime $\frac{1}{n} = o(\theta)$

In the supercritical regime, it is no more possible to obtain approximations from the exact formulas of Theorem 9 since K becomes equivalent to n as they tend to infinity.

However, the behavior of the size of the largest simplex for a binomial point process has already been studied by Appel and Russo in [26] as the clique number of a random geometric graph. They first find the almost sure asymptotic rate for the maximum vertex degree. Then by squeezing the clique number between two values of the maximum vertex degree, they obtain its asymptotic behavior. We propose here an alternative approach.

First, we state a fact true in any percolation regime:

Lemma 4. *In a Vietoris-Rips complex based on a binomial point process of n points we always have that:*

$$n \leq (K + 1) \lceil \frac{1}{\theta} \rceil.$$

Proof. Let us consider a lattice square grid of spacing r , the parameter of the Vietoris-Rips complex, on the torus \mathbb{T}_a^d . Then the number of little squares of side r is:

$$\lceil \left(\frac{a}{r}\right)^d \rceil = \lceil \frac{1}{\theta} \rceil.$$

All points that lie inside a same little square of side r are connected to each other, and are in the same simplex, by definition of the Vietoris-Rips complex. The number K is the size of the largest simplex, so there are at most $K + 1$ points in the same square.

The sum on all the squares concludes the proof. \square

We can now write the main theorem for the behavior of K in the supercritical regime:

Theorem 15. *In the supercritical regime, the size of the largest simplex in a Vietoris-Rips complex based on a binomial point process of n points on the torus \mathbb{T}_a^d grows asymptotically almost surely as $n\theta$:*

$$K \sim n\theta.$$

Proof. On the one hand, we know from Lemma 4 that $n \leq (K+1)\lceil \frac{1}{\theta} \rceil$. Then as $\lceil \frac{1}{\theta} \rceil \leq \frac{1}{\theta} + 1$, we have that:

$$K \geq \frac{n\theta}{1+\theta}.$$

As n goes to infinity, θ tends to 0. Then we have asymptotically almost surely that:

$$K \geq n\theta.$$

On the other hand, by definition of the Vietoris-Rips complex, a k -simplex occurs when $k+1$ points are in the same ball of diameter r . Without loss of generality, we can center the ball on one of the point. Then we can write:

$$\begin{aligned} \mathbf{P}[K > n\theta] &= \mathbf{P}[s_{n\theta} > 0] \\ &= \mathbf{P}[\exists x, |B(x, \frac{r}{2})| \geq n\theta], \end{aligned}$$

where x is a point of the binomial point process, and $|B(x, \frac{r}{2})|$ counts the number of points of the process that lie in the ball $B(x, \frac{r}{2})$ centered in x of radius $\frac{r}{2}$.

Let x_1, \dots, x_n denote the n points of the binomial point process,

$$\begin{aligned} \mathbf{P}[K > n\theta] &= \mathbf{P}[\exists 1 \leq i \leq n, |B(x_i, \frac{r}{2})| \geq n\theta] \\ &\leq \mathbf{P}[\bigcup_{i=1}^n |B(x_i, \frac{r}{2})| \geq n\theta] \\ &\leq \sum_{i=1}^n \mathbf{P}[|B(x_i, \frac{r}{2})| \geq n\theta]. \end{aligned}$$

Then by stationarity of the binomial point process, we have:

$$\begin{aligned} \mathbf{P}[K > n\theta] &\leq \sum_{i=1}^n \mathbf{P}[|B(x_i, \frac{r}{2})| \geq n\theta] \\ &\leq n\mathbf{P}[|B(x_1, \frac{r}{2})| \geq n\theta]. \end{aligned}$$

The number of points in the ball $B(x_1, \frac{r}{2})$ follows a binomial distribution of $n-1$ points and of probability θ : $\text{Binom}(n-1, \theta)$. Therefore Hoeffding's inequality implies that:

$$\begin{aligned} \mathbf{P}[K > n\theta] &\leq n\mathbf{P}[|B(x_1, \frac{r}{2})| \geq n\theta] \\ &\leq n\mathbf{P}[|B(x_1, \frac{r}{2})| \geq (n-1)(\theta + \frac{\theta}{n-1})] \\ &\leq n \exp(-2\frac{\theta^2}{n-1}). \end{aligned}$$

As this last upper bound tends to 0 as n tends to infinity, we have asymptotically almost surely:

$$K \leq n\theta. \quad \square$$

We are now able to derive the complexity of the reduction algorithm in the supercritical regime:

Theorem 16. *The reduction algorithm described in Alg. 1 for a Vietoris-Rips complex based on a binomial point process of*

n points on the torus \mathbb{T}_a^d in the supercritical regime such that $\frac{1}{n} = o(\theta)$ has a complexity in:

$$O(n^{k_0+3}2^n).$$

Proof. This is Corollary 8. □

We can see that in this regime, we were not able to improve the complexity via the behavior of K .

V. CONCLUSION

In this paper, we have presented a reduction algorithm for abstract simplicial complexes. We apply this algorithm to Vietoris-Rips complexes that can represent wireless networks and their topology, it then provides a solution for energy saving in redundant wireless networks.

We have proved that our reduction algorithm maintains the complex's homology, works in an optimal order for computation's complexity, and reaches an optimal solution. Finally we have investigated its complexity depending on the size of the input, and in a second approach depending only on the number of points.

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