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Abstract. The asymptotic behavior of the first eigenvalues of magnetic Laplacian operators with large magnetic fields in polyhedral domains is characterized by a hierarchy of model problems. We investigate properties of the model problems (continuity, semi-continuity, existence of generalized eigenfunctions). We prove estimates for the remainders of our asymptotic formula. Lower bounds are obtained with the help of a classical IMS partition based on adequate coverings of the polyhedral domain, whereas upper bounds are established by a novel construction of quasimodes, qualified as sitting or sliding according to spectral properties of local model problems.

1. Introduction. Main results

The Schrödinger operator with magnetic field (also called magnetic Laplacian) takes the form

$$(-i\nabla + \mathbf{A})^2$$

where $\mathbf{A}$ is a given vector field that will be assumed to be regular. When set on a domain Ω of $\mathbb{R}^n$ ($n = 2$ or 3) and completed by natural boundary conditions (Neumann), this operator is denoted by $H(\mathbf{A}, \Omega)$. If Ω is bounded with Lipschitz boundary, the form domain of $H(\mathbf{A}, \Omega)$ is the standard Sobolev space $H^1(\Omega)$ and $H(\mathbf{A}, \Omega)$ is positive self-adjoint with compact resolvent. The ground states of $H(\mathbf{A}, \Omega)$ are the eigenpairs (λ, ψ)

$$\begin{cases} (-i\nabla + \mathbf{A})^2\psi = \lambda\psi & \text{in } \Omega, \\ (-i\partial_n + \mathbf{n}\cdot\mathbf{A})\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

associated with the lowest eigenvalues λ. If Ω is simply connected, its eigenvalues only depend on the magnetic field $\mathbf{B}$ defined as

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{1.2}$$

The eigenvectors corresponding to two different instances of $\mathbf{A}$ for the same $\mathbf{B}$ are deduced from each other by a *gauge transform*.

Introducing a (small) parameter $h > 0$ and setting

$$H_h(\mathbf{A}, \Omega) = (-ih\nabla + \mathbf{A})^2 \quad \text{with Neumann b.c. on } \partial\Omega,$$

we get the relation

$$H_h(\mathbf{A}, \Omega) = h^2 H\Big(\frac{\mathbf{A}}{h}, \Omega\Big) \tag{1.3}$$

linking the problem with large magnetic field to the semiclassical limit $h \to 0$. We denote by $\lambda_h(\mathbf{B}, \Omega)$ (or λ_h if no confusion is possible) the smallest eigenvalue of $H_h(\mathbf{A}, \Omega)$ and by ψ_h an associated eigenvector, so that

$$\begin{cases} (-ih\nabla + \mathbf{A})^2 \psi_h = \lambda_h \psi_h & \text{in } \Omega\,, \\ (-ih\partial_n + \mathbf{n}\cdot\mathbf{A})\psi_h = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

The behavior of λ_h as $h \to 0$ clearly provide equivalent information about the lowest eigenvalue of $H(\breve{\mathbf{A}}, \Omega)$ when $\breve{\mathbf{B}}$ is large, especially in the parametric case when $\breve{\mathbf{B}} = B\mathbf{B}$ where the real number B tends to $+\infty$ and $\mathbf{B}$ is a chosen reference magnetic field.

From now on, we consider that $\mathbf{B}$ is fixed. We assume that it is smooth and does not vanishes[1] on $\overline{\Omega}$. We normalize $\mathbf{B}$ by the condition

$$\min_{\mathbf{x}\in\overline{\Omega}} |\mathbf{B}(\mathbf{x})| = 1. \tag{1.5}$$

The question of the semiclassical behavior of λ_h has been considered in many papers for a variety of domains, with constant or variable magnetic fields: Smooth domains [18, 13, 10, 1, 28] and polygons [3, 4, 5] in dimension $n = 2$, and mainly smooth domains [19, 14, 15, 29, 11] in dimension $n = 3$. Until now, three-dimensional non-smooth domains are only addressed in two particular configurations—rectangular cuboids [22] and lenses [24, 27], with special orientation of the (constant) magnetic field. We give more detail about the state of the art in section 2.

Let us briefly describe our main results in the three-dimensional setting.

Each point $\mathbf{x}$ in the closure of a polyhedral domain Ω is associated with a homogeneous tangent open set $\Pi_{\mathbf{x}}$, according to the following cases:

(1) If $\mathbf{x}$ is an interior point, $\Pi_{\mathbf{x}} = \mathbb{R}^3$,
(2) If $\mathbf{x}$ belongs to a *face* $\mathbf{f}$ (i.e., a connected component of the smooth part of $\partial\Omega$), $\Pi_{\mathbf{x}}$ is a half-space,
(3) If $\mathbf{x}$ belongs to an *edge* $\mathbf{e}$, $\Pi_{\mathbf{x}}$ is an infinite wedge,
(4) If $\mathbf{x}$ is a *vertex* $\mathbf{v}$, $\Pi_{\mathbf{x}}$ is an infinite polyhedral cone.

Let $\mathbf{B}_{\mathbf{x}}$ be the magnetic field frozen at $\mathbf{x}$. Let $E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ be the bottom of the spectrum (ground energy) of the tangent operator $H(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ where $\mathbf{A}_{\mathbf{x}}$ is the linear approximation of $\mathbf{A}$ at $\mathbf{x}$, so that

$$\operatorname{curl} \mathbf{A}_{\mathbf{x}} = \mathbf{B}_{\mathbf{x}}\,.$$

We introduce the quantity

$$\mathscr{E}(\mathbf{B}, \Omega) := \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}}). \tag{1.6}$$

[1] Should $\mathbf{B}$ cancel, the situation would be completely different [12, 9].

In this paper, we prove that this quantity provides the value of the limit of λ_h/h as $h \to 0$ with some control of the convergence rate as $h \to 0$, namely

$$-Ch^{5/4} \leq \lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega) \leq Ch^{5/4}, \tag{1.7}$$

where the constant C is bounded by the norm of $\mathbf{A}$ in $W^{2,\infty}(\Omega)$, as proved in Theorems 6.1 and 7.1. We can also control the constant C by the magnetic field $\mathbf{B}$ as established in Corollaries 6.2 and 7.2.

With the point of view of large magnetic fields in the parametric case $\check{\mathbf{B}} = B\mathbf{B}$, (1.7) yields obviously

$$-CB^{3/4} \leq \lambda(\check{\mathbf{B}},\Omega) - B\mathscr{E}(\mathbf{B},\Omega) \leq CB^{3/4}, \qquad \text{as } B \to +\infty. \tag{1.8}$$

Note that $B\mathscr{E}(\mathbf{B},\Omega) = \mathscr{E}(\check{\mathbf{B}},\Omega)$ by homogeneity (see Lemma 4.1).

This result is new in this generality. In view of [15] (smooth three-dimensional case) the upper bound is optimal. The lower bound coincides with the one obtained in the smooth case in dimensions 2 and 3 when no further assumptions are done. In the literature, improvements of the convergence rates are possible in certain cases when one knows more on $\mathscr{E}(\mathbf{B},\Omega)$, in particular whether the infimum is attained in some special points.

Our result does not need such extra assumptions, but our proofs have to distinguish cases whether the local ground energies $E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ are attained or not, and we have to understand the behavior of the function $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ when $\mathbf{x}$ spans the different regions of $\overline{\Omega}$. We have proved very general continuity and semi-continuity properties as described now.

Let $\mathfrak{F}$ be the set of faces $\mathbf{f}$, $\mathfrak{E}$ the set of edges $\mathbf{e}$ and $\mathfrak{V}$ the set of vertices of Ω. They form a partition of the closure of Ω, called stratification

$$\overline{\Omega} = \Omega \cup \Big(\bigcup_{\mathbf{f}\in\mathfrak{F}} \mathbf{f}\Big) \cup \Big(\bigcup_{\mathbf{e}\in\mathfrak{E}} \mathbf{e}\Big) \cup \Big(\bigcup_{\mathbf{v}\in\mathfrak{V}} \mathbf{v}\Big). \tag{1.9}$$

The sets Ω, $\mathbf{f}$, $\mathbf{e}$ and $\mathbf{v}$ are called the strata of $\overline{\Omega}$, compare with [20] and [21, Ch. 9]. We denote them by $\mathbf{t}$ and their set by $\mathfrak{T}$. For each stratum $\mathbf{t}$, let us denote by $\Lambda_\mathbf{t}$ the function

$$\Lambda_\mathbf{t} : \mathbf{t} \ni \mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x}). \tag{1.10}$$

We will show the following facts

a) The function $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ is lower semi-continuous on $\overline{\Omega}$.
b) For each stratum $\mathbf{t} \in \mathfrak{T}$, the function $\Lambda_\mathbf{t}$ is continuous on $\mathbf{t}$ and can be continuously extended to the closure $\bar{\mathbf{t}}$ of $\mathbf{t}$. Moreover, for each $\mathbf{x}_0 \in \bar{\mathbf{t}}$, $\Lambda_\mathbf{t}(\mathbf{x}_0)$ is the bottom of the spectrum $E(\mathbf{B}_\mathbb{X},\Pi_\mathbb{X})$ of a tangent magnetic operator $H(\mathbf{A}_\mathbb{X},\Pi_\mathbb{X})$ associated with a *singular chain* $\mathbb{X}$ originating at $\mathbf{x}_0$.

As a consequence, the infimum determining the limit $\mathscr{E}(\mathbf{B},\Omega)$ in (1.6) is a minimum

$$\mathscr{E}(\mathbf{B},\Omega) = \min_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x}). \tag{1.11}$$

Contents. In section 2 we place our results in the framework of existing literature for dimensions 2 and 3. In section 3 we introduce the wider class of corner domains, alongside with their tangent cones and singular chains. We particularize these notions in the case of polyhedral domains. In section 4 we introduce and classify magnetic model problems on tangent cones (taxonomy) and extract from the literature related facts. We show that to each point $x \in \overline{\Omega}$ is associated a singular chain $\mathbb{X}$ originating at $\mathbf{x}$ for which the tangent operator $H(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ possesses *admissible generalized eigenvectors* with energy $E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$. In section 5 we prove the semi-continuity and continuity properties of the functions $\mathbf{x} \mapsto E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ on $\overline{\Omega}$ and its strata. In section 6 we prove the upper bound $\lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + Ch^{5/4}$ by a construction of quasimodes based on admissible generalized eigenvectors for tangent problems. In section 7 we prove the lower bound $h\mathscr{E}(\mathbf{B}, \Omega) - Ch^{5/4} \leq \lambda_h(\mathbf{B}, \Omega)$ by a classical IMS formula.

Notation. We denote by $\mathfrak{S}(L)$ the spectrum of a self-adjoint operator L.

2. State of the art

In this section we review the literature about the semiclassical limit for the first eigenvalue $\lambda_h(\mathbf{B}, \Omega)$ of the magnetic Laplacian $H_h(\mathbf{A}, \Omega)$ depending on the dimension $n \in \{2, 3\}$ and the geometry of the domain Ω.

2.1. Dimension 2.

In dimension $n = 2$, two classes of domains are considered: the domains with a regular boundary and the polygonal domains.

2.1.1. *Regular domains.*

Let $\Omega \subset \mathbb{R}^2$ be a regular domain and B be a regular non-vanishing scalar magnetic field on $\overline{\Omega}$. To each $\mathbf{x} \in \overline{\Omega}$ is associated a tangent problem. According to whether $\mathbf{x}$ is an interior point or a boundary point, the tangent problem is the magnetic Laplacian on the plane $\mathbb{R}^2$ or the half-plane $\Pi_{\mathbf{x}}$ tangent to Ω at $\mathbf{x}$, with the constant magnetic field $B_{\mathbf{x}} \equiv B(\mathbf{x})$. The associated spectral quantities $E(B_{\mathbf{x}}, \mathbb{R}^2)$ and $E(B_{\mathbf{x}}, \Pi_{\mathbf{x}})$ are respectively equal to $|B_{\mathbf{x}}|$ and $|B_{\mathbf{x}}|\Theta_0$ where $\Theta_0 := E(1, \mathbb{R}^2_+)$ is a universal constant whose value is close to 0.59 (see [30]). With the quantities

$$b = \inf_{\mathbf{x} \in \Omega} |B(\mathbf{x})| \quad \text{and} \quad b' = \inf_{\mathbf{x} \in \partial\Omega} |B(\mathbf{x})|, \tag{2.1}$$

we find

$$\mathscr{E}(B, \Omega) = \min(b, b'\Theta_0) \ .$$

In this generality, the asymptotic limit

$$\lim_{h \to 0} \frac{\lambda_h(B, \Omega)}{h} = \mathscr{E}(B, \Omega) \tag{2.2}$$

is proven by Lu and Pan in [18]. Improvements of this result depend on the geometry and the variation of the magnetic field as we describe now.

• *Constant magnetic field.* If the magnetic field is constant and normalized to 1, then $\mathscr{E}(B,\Omega)=\Theta_0$. The following estimate is proved by Helffer and Morame:

$$\exists C>0,\quad -Ch^{3/2}\leq \lambda_h(1,\Omega)-h\Theta_0\leq Ch^{3/2}\ ,$$

for h small enough [13, §10], while the upper bound was already given by Bernoff and Sternberg [2]. This result is improved in [13, §11] in which a two-term asymptotics is proved, showing that a remainder in $\mathcal{O}(h^{3/2})$ is optimal. Under the additional assumption that the curvature of the boundary admits a unique and non-degenerate maximum, a complete expansion of $\lambda_h(1,\Omega)$ is provided by Fournais and Helffer [10].

• *Variable magnetic field.* Here we recall results from [13, §9] for variable magnetic fields (we use the notation (2.1))

$$\begin{aligned}
&\text{If } b<\Theta_0 b', \quad \exists C>0, \quad |\lambda_h(B,\Omega)-hb|\leq Ch^2,\\
&\text{If } b>\Theta_0 b', \quad \exists C>0, \quad -Ch^{5/4}\leq \lambda_h(B,\Omega)-h\Theta_0 b'\leq Ch^{3/2},\\
&\text{If } b=\Theta_0 b', \quad \exists C>0, \quad -Ch^{5/4}\leq \lambda_h(B,\Omega)-hb\leq Ch^2,
\end{aligned}$$

for h small enough. Under further assumptions, more precise asymptotic expansions are given by Aramaki [1] and Raymond [28]. They show that the upper bounds above are sharp.

2.1.2. *Polygonal domains.* Let Ω be a curvilinear polygon and let $\mathfrak{V}$ be the (finite) set of its vertices. In this case, new model operators appear on infinite sectors $\Pi_{\mathbf{x}}$ tangent to Ω at vertices $\mathbf{x}\in\mathfrak{V}$. By homogeneity $E(B_{\mathbf{x}},\Pi_{\mathbf{x}})=|B(\mathbf{x})|E(1,\Pi_{\mathbf{x}})$ and by gauge invariance, $E(1,\Pi_{\mathbf{x}})$ only depends on the opening $\alpha(\mathbf{x})$ of the sector $\Pi_{\mathbf{x}}$. Let $\mathcal{S}_\alpha$ be a model sector of opening $\alpha\in(0,2\pi)$. Then

$$\mathscr{E}(B,\Omega)=\min\left(b,\,b'\Theta_0,\,\min_{\mathbf{x}\in\mathfrak{V}}|B(\mathbf{x})|\,E(1,\mathcal{S}_{\alpha(\mathbf{x})})\right)\ .$$

In [3, §11], it is proved that

$$\exists C>0,\quad -Ch^{5/4}\leq \lambda_h(B,\Omega)-h\mathscr{E}(B,\Omega)\leq Ch^{9/8}.$$

This estimate can be improved under the assumption that

$$\mathscr{E}(B,\Omega)<\min(b,b'\Theta_0), \tag{2.3}$$

which means that at least one of the corners makes the energy lower than in the regular case: The asymptotic expansions provided in [4] then yield the sharp estimates

$$\exists C>0,\quad |\lambda_h(B,\Omega)-h\mathscr{E}(B,\Omega)|\leq Ch^{3/2}\ .$$

From [16, 3] follows that for all $\alpha\in(0,\frac{\pi}{2}]$ there holds

$$E(1,\mathcal{S}_\alpha)<\Theta_0. \tag{2.4}$$

Therefore condition (2.3) holds for constant magnetic fields as soon as there is an angle opening $\alpha_{\mathbf{x}}\leq\frac{\pi}{2}$. Finite element computations by Galerkin projection as presented in [5] suggest that (2.4) still holds for all $\alpha\in(0,\pi)$. Let us finally mention that if Ω has straight

sides and B is constant, the convergence of $\lambda_h(B,\Omega)$ to $h\mathscr{E}(B,\Omega)$ is exponential: Their difference is bounded by $C\exp(-\beta h^{-1/2})$ for suitable positive constants C and β (see [4]).

2.2. **Dimension 3.** In dimension $n=3$ we still distinguish the regular and singular domains.

2.2.1. *Regular domains.* Here $\Omega\subset\mathbb{R}^3$ is assumed to be regular. For a continuous magnetic field $\mathbf{B}$ it is known ([19] and [14]) that (2.2) holds. In that case

$$\mathscr{E}(\mathbf{B},\Omega)=\min\big(\inf_{\mathbf{x}\in\Omega}|\mathbf{B}(\mathbf{x})|,\inf_{\mathbf{x}\in\partial\Omega}|\mathbf{B}(\mathbf{x})|\sigma(\theta(\mathbf{x}))\big),$$

where $\theta(\mathbf{x})\in[0,\frac{\pi}{2}]$ denotes the angle between the magnetic field and the boundary at the point $\mathbf{x}\in\partial\Omega$, and the quantity $\sigma(\theta)$ is the bottom of the spectrum of a model problem, see section 4. Let us simply mention that σ is increasing on $[0,\frac{\pi}{2}]$ and that $\sigma(0)=\Theta_0$, $\sigma(\pi/2)=1$.

• *Constant magnetic field.* Here the magnetic field $\mathbf{B}$ is assumed to be constant and unitary. There exists a non-empty set Σ of $\partial\Omega$ on which $\mathbf{B}(\mathbf{x})$ is tangent to the boundary. In that case we have

$$\mathscr{E}(\mathbf{B},\Omega)=\Theta_0\ .$$

Theorem 1.1 of [15] states that

$$\exists C>0,\quad |\lambda_h(\mathbf{B},\Omega)-h\Theta_0|\leq Ch^{4/3},$$

for h small enough. Under some extra assumptions on Σ, Theorem 1.2 of [15] yields a two-term asymptotics for $\lambda_h(\mathbf{B},\Omega)$ showing the optimality of the previous estimate.

• *Variable magnetic field.* Let $\mathbf{B}$ be a smooth non-vanishing magnetic field. There holds [11, Theorem 9.1.1]

$$\exists C>0,\quad -Ch^{5/4}\leq\lambda_h(\mathbf{B},\Omega)-h\mathscr{E}(\mathbf{B},\Omega)\leq Ch^{5/4}\ .$$

The proof of this result was already sketched in [19]. In [15, Remark 6.2], the upper bound is improved to $\mathcal{O}(h^{4/3})$.

Under the following two extra assumptions

- The inequality $\inf_{\mathbf{x}\in\partial\Omega}|\mathbf{B}(\mathbf{x})|\,\sigma(\theta(\mathbf{x}))<\inf_{\mathbf{x}\in\Omega}|\mathbf{B}(\mathbf{x})|$ holds,
- The function $\mathbf{x}\mapsto|\mathbf{B}(\mathbf{x})|\sigma(\theta(\mathbf{x}))$ reaches its minimum at a point $\mathbf{x}_0$ where $\mathbf{B}$ is neither normal nor tangent to the boundary,

a two-term asymptotics is valid [29], providing the sharp estimate:

$$\exists C>0,\quad |\lambda_h(\mathbf{B},\Omega)-h\mathscr{E}(\mathbf{B},\Omega)|\leq Ch^{3/2}\,.$$

2.2.2. *Singular domains.* Until now, two examples of non-smooth domains have been addressed in the literature. In both cases, the magnetic field is assumed to be constant.

• *Rectangular cuboids.* There exists a vertex $\mathbf{v} \in \mathfrak{V}$ such that $\mathscr{E}(\mathbf{B}, \Omega) = E(\mathbf{B}, \Pi_{\mathbf{v}})$ and that (2.2) holds [22]. Note that $\Pi_{\mathbf{v}}$ is homeomorphic to the octant $(\mathbb{R}_+)^3$. In the case where the magnetic field is tangent to a face but is not tangent to any edge, Pan [22] shows

$$E(\mathbf{B}, \Pi_{\mathbf{v}}) < \inf_{\mathbf{x} \in \overline{\Omega} \setminus \mathfrak{V}} E(\mathbf{B}, \Pi_{\mathbf{x}})$$

and deduces that eigenfunctions associated to $\lambda_h(\mathbf{B}, \Omega)$ concentrate near corners as $h \to 0$.

• *Lenses.* The case where Ω has the shape of a lens is treated in [24] and [27]. The domain Ω is supposed to have two faces separated by an edge that is a regular loop Σ contained in the plane $x_3 = 0$. The magnetic field considered is $\mathbf{B} = (0, 0, 1)$.

It is proved in [24] that, if the opening angle α of the lens is constant and $\leq 0.38\pi$,

$$\inf_{\mathbf{x} \in \Sigma} E(\mathbf{B}, \Pi_{\mathbf{x}}) < \inf_{\mathbf{x} \in \overline{\Omega} \setminus \Sigma} E(\mathbf{B}, \Pi_{\mathbf{x}})$$

and that (2.2) holds with the following estimate:

$$\exists C > 0, \quad |\lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega)| \leq C h^{5/4} \ .$$

When the opening angle of the lens is variable and under some non-degeneracy hypotheses, a complete eigenvalue asymptotics is obtained in [27] resulting into the optimal estimate

$$\exists C > 0, \quad |\lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega)| \leq C h^{3/2} .$$

3. **Polyhedral domains and their singular chains**

For the sake of completeness and for ease of further discussion, in the same spirit as in [7, section 2], we introduce here a recursive definition of two intertwining classes of domains

a) $\mathfrak{P}_n$, a class of infinite open cones in $\mathbb{R}^n$.
b) $\mathfrak{D}(M)$, a class of bounded connected open subsets of a smooth manifold without boundary—actually, $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$, with $\mathbb{S}^n$ the unit sphere of $\mathbb{R}^{n+1}$,

3.1. **Domains and tangent cones.**

We call a *cone* any open subset Π of $\mathbb{R}^n$ satisfying

$$\forall \rho > 0 \ \text{ and } \ \mathbf{x} \in \Pi, \quad \rho\mathbf{x} \in \Pi,$$

and the *section* of the cone Π is its subset $\Pi \cap \mathbb{S}^{n-1}$. Note that $\mathbb{S}^0 = \{-1, 1\}$.

Initialization: $\mathfrak{P}_0$ has one element, $\{0\}$. $\mathfrak{D}(\mathbb{S}^0)$ is formed by all subsets of $\mathbb{S}^0$.

Recurrence: For $n \geq 1$,

(1) $\Pi \in \mathfrak{P}_n$ if and only if the section of Π belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$,

(2) $\Omega \in \mathfrak{D}(M)$ if and only if for any $\mathbf{x} \in \overline{\Omega}$, there exists a cone $\Pi_{\mathbf{x}} \in \mathfrak{P}_n$ and a local smooth diffeomorphism $\mathsf{U}_{\mathbf{x}}$ which maps a neighborhood $\mathcal{U}_{\mathbf{x}}$ of $\mathbf{x}$ in M onto a neighborhood $\mathcal{V}_{\mathbf{x}}$ of 0 in $\mathbb{R}^n$ and such that

$$\mathsf{U}_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}} \cap \Omega) = \mathcal{V}_{\mathbf{x}} \cap \Pi_{\mathbf{x}} \quad \text{and} \quad \mathsf{U}_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}} \cap \partial\Omega) = \mathcal{V}_{\mathbf{x}} \cap \partial\Pi_{\mathbf{x}}. \tag{3.1}$$

We assume without restriction that the differential of $\mathsf{U}_{\mathbf{x}}$ at the point $\mathbf{x}$ is the identity matrix $\mathbb{I}_n$. The cone $\Pi_{\mathbf{x}}$ is said to be tangent to Ω at $\mathbf{x}$.

Examples:

- The elements of $\mathfrak{P}_1$ are $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$.
- The elements of $\mathfrak{D}(\mathbb{S}^1)$ are $\mathbb{S}^1$ and all open intervals $\mathcal{I} \subset \mathbb{S}^1$ such that $\overline{\mathcal{I}} \neq \mathbb{S}^1$.
- The elements of $\mathfrak{P}_2$ are $\mathbb{R}^2$ and all sectors with opening $\alpha \in (0, 2\pi)$.
- The elements of $\mathfrak{D}(\mathbb{R}^2)$ are curvilinear polygons with piecewise smooth sides and opening angles $\neq 0, 2\pi$.

Let $\mathfrak{O}_n$ denote the group of orthogonal linear transformations of $\mathbb{R}^n$. We say that a cone Π is *equivalent* to another cone Π' and denote

$$\Pi \equiv \Pi'$$

if there exists $\underline{\mathsf{U}} \in \mathfrak{O}_n$ such that $\underline{\mathsf{U}}\Pi = \Pi'$. Let $\Pi \in \mathfrak{P}_n$. If Π is equivalent to $\mathbb{R}^{n-d} \times \Gamma$ with $\Gamma \in \mathfrak{P}_d$ and d is minimal for such an equivalence, Γ is said to be *a minimal reduced cone* associated with Π.

3.2. **Recursive definition of singular chains.** A singular chain $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k) \in \mathfrak{C}(\Omega)$ is a finite collection of points according to the following recursive definition.

Initialization: $\mathbf{x}_0 \in \overline{\Omega}$,

- $C_{\mathbf{x}_0}$ tangent cone to Ω at $\mathbf{x}_0$,
- $\Gamma_{\mathbf{x}_0} \in \mathfrak{P}_{d_0}$ its minimal reduced cone: $C_{\mathbf{x}_0} = \underline{\mathsf{U}}_0(\mathbb{R}^{\nu_0} \times \Gamma_{\mathbf{x}_0})$, with $\nu_0 = n - d_0$.
- Alternative:
  - If $k = 0$, stop here.
  - If $k > 0$, then $d_0 > 0$ and let $\Omega_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{d_0-1})$ be the section of $\Gamma_{\mathbf{x}_0}$

Recurrence: $\mathbf{x}_j \in \overline{\Omega}_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}} \in \mathfrak{D}(\mathbb{S}^{d_{j-1}-1})$. If $d_{j-1} = 1$, stop here ($k = j$). If not:

- $C_{\mathbf{x}_0,\ldots,\mathbf{x}_j}$ tangent cone to $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}}$ at $\mathbf{x}_j$,
- $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j} \in \mathfrak{P}_{d_j}$ its minimal reduced cone: $C_{\mathbf{x}_0,\ldots,\mathbf{x}_j} = \underline{\mathsf{U}}_j(\mathbb{R}^{\nu_j} \times \Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j})$.
- Alternative:
  - If $j = k$, stop here.
  - If $j < k$, then $d_j > 0$ and let $\Omega_{\mathbf{x}_0\ldots,\mathbf{x}_j} \in \mathfrak{D}(\mathbb{S}^{d_j-1})$ be the section of $\Gamma_{\mathbf{x}_0\ldots,\mathbf{x}_j}$.

Note that $n \geq d_0 > d_1 > \ldots > d_k$. Hence $k \leq n$. Note also that for $k = 0$, we obtain the trivial one element chain $(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \overline{\Omega}$.

While $\mathfrak{C}(\Omega)$ is the set of all singular chains, for any $\mathbf{x} \in \overline{\Omega}$, we denote by $\mathfrak{C}_{\mathbf{x}}(\Omega)$ the subset of chains originating at $\mathbf{x}$, i.e., the set of chains $\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_k)$ with $\mathbf{x}_0 = \mathbf{x}$. Note that the one element chain $(\mathbf{x})$ belongs to $\mathfrak{C}_{\mathbf{x}}(\Omega)$. We also set

$$\mathfrak{C}^*_{\mathbf{x}}(\Omega) = \{\mathbb{X} \in \mathfrak{C}_{\mathbf{x}}(\Omega),\ k > 0\} = \mathfrak{C}_{\mathbf{x}}(\Omega) \setminus \{(\mathbf{x})\}. \tag{3.2}$$

We set finally, with the notation $\langle \mathbf{y} \rangle$ for the vector space generated by $\mathbf{y}$,

$$\Pi_{\mathbb{X}} = \begin{cases} \mathbb{R}^n & \text{if} \quad d_{k-1} = 1 \\ \underline{\mathsf{U}}_0\big(\mathbb{R}^{\nu_0} \times \langle \mathbf{x}_1 \rangle \times \underline{\mathsf{U}}_1\big(\mathbb{R}^{\nu_1} \times \langle \mathbf{x}_2 \rangle \times \dots \underline{\mathsf{U}}_k(\mathbb{R}^{\nu_k} \times \Gamma_{\mathbf{x}_k})\ \big)\big) & \text{if} \quad d_{k-1} > 1 \end{cases} \tag{3.3}$$

Note that if $d_k = 0$, $\Pi_{\mathbb{X}} = \mathbb{R}^n$. Note also that if $\nu_0 = 0$, $\underline{\mathsf{U}}_0 = \mathbb{I}_n$.

Definition 3.1. Let $\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_k)$ and $\mathbb{X}' = (\mathbf{x}'_0, \dots, \mathbf{x}'_{k'})$ be two chains in $\mathfrak{C}(\Omega)$. We say that $\mathbb{X}$ is equivalent to $\mathbb{X}'$ if $\mathbf{x}_0 = \mathbf{x}'_0$ and $\Pi_{\mathbb{X}} = \Pi_{\mathbb{X}'}$.

Special subsets of $\overline{\Omega}$: For $d \in \{0, \dots, n\}$, let

$$\mathfrak{A}_d(\Omega) = \{\mathbf{x} \in \overline{\Omega}, \quad d_0(\mathbf{x}) = d\}. \tag{3.4}$$

The strata of $\overline{\Omega}$ are the connected components of $\mathfrak{A}_d(\Omega)$, for $d \in \{0, \dots, n\}$. They are denoted by $\mathbf{t}$ and their set by $\mathfrak{T}$.

Examples:

- $\mathfrak{A}_0(\Omega)$ coincides with Ω.
- $\mathfrak{A}_1(\Omega)$ is the subset of $\partial\Omega$ of the regular points of the boundary.
- If $n = 2$, $\mathfrak{A}_2(\Omega)$ is the set of corners.
- If $n = 3$, $\mathfrak{A}_2(\Omega)$ is the set of edge points.
- If $n = 3$, $\mathfrak{A}_3(\Omega)$ is the set of corners.

3.3. **Polyhedral domains.** Polyhedral domains and polyhedral cones form subclasses of $\mathfrak{D}(M)$ and $\mathfrak{P}_n$, denoted by $\overline{\mathfrak{D}}(M)$ and $\overline{\mathfrak{P}}_n$, respectively:

a) The cone $\Pi \in \mathfrak{P}_n$ is a polyhedral cone if its boundary is contained in a finite union of (hyper)surfaces. We write $\Pi \in \overline{\mathfrak{P}}_n$.
b) The domain $\Omega \in \mathfrak{D}(M)$ is a polyhedral domain if all its tangent cones $\Pi_{\mathbf{x}}$ are polyhedral. We write $\Omega \in \overline{\mathfrak{D}}(M)$.

This allows to make precise the definition of faces, edges and corners in dimension 3, in connection with singular chains.

(1) Interior point $\mathbf{x} \in \Omega$. Only one chain in $\mathfrak{C}_{\mathbf{x}}(\Omega)$: $\mathbb{X} = (\mathbf{x})$.
(2) The faces $\mathbf{f}$ are the connected components of $\mathfrak{A}_1(\Omega)$. The set of faces is denoted by $\mathfrak{F}$. Let $\mathbf{x}$ belong to a face. There are two chains in $\mathfrak{C}_{\mathbf{x}}(\Omega)$:
   (a) $\mathbb{X} = (\mathbf{x})$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}}$, the tangent half-space. $\Pi_{\mathbb{X}} \equiv \mathbb{R}^2 \times \mathbb{R}_+$.
   (b) $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1)$ where $\mathbf{x}_1 = 1$ is the only element in $\mathbb{R}_+ \cap \mathbb{S}^0$. Thus $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

(3) The edges $\mathbf{e}$ are the connected components of $\mathfrak{A}_2(\Omega)$. The set of edges is denoted by $\mathfrak{E}$. Let $\mathbf{x}$ belong to an edge. There are three types of chains in $\mathfrak{C}_{\mathbf{x}}(\Omega)$:
   (a) $\mathbb{X} = (\mathbf{x})$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}}$, the tangent wedge (which is not a half-plane). The reduced cone of $\Pi_{\mathbf{x}}$ is a sector $\Gamma_{\mathbf{x}}$ the section of which is an interval $\mathcal{I}_{\mathbf{x}} \subset \mathbb{S}^1$.
   (b) $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1)$ where $\mathbf{x}_1 \in \overline{\mathcal{I}}_{\mathbf{x}}$.
      (i) If $\mathbf{x}_1$ is interior to $\mathcal{I}_{\mathbf{x}}$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.
      (ii) If $\mathbf{x}_1$ is a boundary point of $\mathcal{I}_{\mathbf{x}}$, $\Pi_{\mathbb{X}}$ is a half-space, containing one of the two faces $\partial^{\pm}\Pi_{\mathbb{X}}$ of the wedge $\Pi_{\mathbf{x}}$.
   (c) $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 \in \partial\mathcal{I}_{\mathbf{x}}$, $\mathbf{x}_2 = 1$ and $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

   There are 4 equivalence classes in $\mathfrak{C}_{\mathbf{x}}(\Omega)$ in the case of an edge point $\mathbf{x}$:
   - $\mathbb{X} = (\mathbf{x})$
   - $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1^{\pm})$ with $\{\mathbf{x}_1^-, \mathbf{x}_1^+\} = \partial\mathcal{I}_{\mathbf{x}}$
   - $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1^{\circ})$ with $\mathbf{x}_1^{\circ}$ any chosen point in $\mathcal{I}_{\mathbf{x}}$.

(4) The corners $\mathbf{v}$ are the connected components of $\mathfrak{A}_3(\Omega)$. The set of corners is denoted by $\mathfrak{V}$. There are four types of chains in $\mathfrak{C}_{\mathbf{x}}(\Omega)$:
   (a) $\mathbb{X} = (\mathbf{x})$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}}$, the tangent cone (which is not a wedge). It coincides with its reduced cone. Its section $\Omega_{\mathbf{x}}$ is a polygonal domain in $\mathbb{S}^2$.
   (b) $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1)$ where $\mathbf{x}_1 \in \overline{\Omega}_{\mathbf{x}}$.
      (i) If $\mathbf{x}_1$ is interior to $\Omega_{\mathbf{x}}$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.
      (ii) If $\mathbf{x}_1$ is in a side of $\Omega_{\mathbf{x}}$, $\Pi_{\mathbb{X}}$ is a half-space, containing one of the faces of the cone $\Pi_{\mathbf{x}}$.
      (iii) If $\mathbf{x}_1$ is a corner of $\Omega_{\mathbf{x}}$, $\Pi_{\mathbb{X}}$ is a wedge. Its edge contains one of the edges of $\Pi_{\mathbf{x}}$.
   (c) $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 \in \partial\Omega_{\mathbf{x}}$
      (i) If $\mathbf{x}_1$ is in a side of $\Omega_{\mathbf{x}}$, $\mathbf{x}_2 = 1$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.
      (ii) If $\mathbf{x}_1$ is a corner of $\Omega_{\mathbf{x}}$, $C_{\mathbf{x},\mathbf{x}_1}$ is plane sector, and $\mathbf{x}_2 \in \overline{\mathcal{I}}_{\mathbf{x},\mathbf{x}_1}$ where the interval $\mathcal{I}_{\mathbf{x},\mathbf{x}_1}$ is its section. If $\mathbf{x}_2$ is an interior point, then $\Pi_{\mathbb{X}} = \mathbb{R}^3$.
   (d) $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ where $\mathbf{x}_1$ is a corner of $\Omega_{\mathbf{x}}$, $\mathbf{x}_2 \in \partial\mathcal{I}_{\mathbf{x},\mathbf{x}_1}$ and $\mathbf{x}_3 = 1$. Then $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

   Let $\mathbf{x}_1^j$, $1 \le j \le N$, be the corners of $\Omega_{\mathbf{x}}$, and $\mathcal{S}^j$ be its sides. There are $2N + 2$ equivalence classes in $\mathfrak{C}_{\mathbf{x}}(\Omega)$:
   - $\mathbb{X} = (\mathbf{x})$ (vertex)
   - $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1^j)$ with $1 \le j \le N$ (edge points limit)
   - $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1^{\circ,j})$ with $\mathbf{x}_1^{\circ,j}$ any chosen point inside $\mathcal{S}^j$ (face points limit)
   - $\mathbb{X} = (\mathbf{x}, \mathbf{x}_1^{\circ})$ with $\mathbf{x}_1^{\circ}$ any chosen point in $\Omega_{\mathbf{x}}$ (interior points limit).

Remark 3.2. For polyhedral domains Ω, it is a consequence of the description above that chains $(\mathbf{x}_0, \mathbf{x}_1)$ of length 2 are enough to describe all equivalence classes of the set of chains $\mathfrak{C}^*_{\mathbf{x}_0}(\Omega)$ (3.2). This does not hold anymore if general corner domains are considered. Besides, the notion of equivalence classes as introduced in Definition 3.1 is sufficient for the analysis of operators $H_h(\mathbf{A}, \Omega)$ in the case of magnetic fields $\mathbf{B}$ smooth in Cartesian

variables. Should $\mathbf{B}$ be smooth in polar variables only, the whole hierarchy of singular chains would be needed.

4. **Taxonomy of model problems**

4.1. **Tangent and model operators.** We recall that $\mathbf{A}$ is a magnetic potential associated with the magnetic field $\mathbf{B}$ on the polyhedral domain $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$. For each singular chain $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathfrak{C}(\Omega)$ we set

$$\mathbf{B}_{\mathbb{X}} = \mathbf{B}(\mathbf{x}_0) \quad \text{and} \quad \mathbf{A}_{\mathbb{X}} = \mathbf{A}(\mathbf{x}_0) + \big(\nabla \mathbf{A}(\mathbf{x}_0)\big)(\mathbf{x} - \mathbf{x}_0). \tag{4.1}$$

We have obviously

$$\operatorname{curl} \mathbf{A}_{\mathbb{X}} = \mathbf{B}_{\mathbb{X}}\,,$$

so that the tangent magnetic operator $H(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ and its ground energy $E(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ make sense.

The domain of $H(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ is

$$\begin{aligned}\operatorname{Dom}\big(H(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})\big) = \big\{&\psi \in L^2(\Pi_{\mathbb{X}}),\\ &(-i\nabla + \mathbf{A}_{\mathbb{X}})^2\psi \in L^2(\Pi_{\mathbb{X}}) \ \text{ and } \ (-i\partial_n + \mathbf{n}\cdot\mathbf{A}_{\mathbb{X}})\psi = 0 \text{ on } \partial\Pi_{\mathbb{X}}\big\}\,.\end{aligned} \tag{4.2}$$

The quadratic form associated is

$$q[\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}}](\psi) := \int_{\Pi_{\mathbb{X}}} |(-i\nabla + \mathbf{A}_{\mathbb{X}})\psi|^2$$

defined on $\{\psi \in L^2(\Pi_{\mathbb{X}}),\ (-i\nabla + \mathbf{A}_{\mathbb{X}})\psi \in L^2(\Pi_{\mathbb{X}})\}$.

By a change of variables we obtain

Lemma 4.1. *Let $\mathcal{O}$ be a domain in $\mathbb{R}^3$ and for $r > 0$, we denote by $r\mathcal{O}$ the domain $\{\mathbf{x} \in \mathbb{R}^3,\ \mathbf{x} = r\mathbf{x}'$ with $\mathbf{x}' \in \mathcal{O}\}$. Let $\mathbf{B}$ be a constant magnetic field. Then*

$$E(\mathbf{B}, \mathcal{O}) = \rho\, E\Big(\frac{\mathbf{B}}{\rho}, \sqrt{\rho}\,\mathcal{O}\Big)\,.$$

As a consequence there holds for tangent problems (setting $\rho = |\mathbf{B}_{\mathbb{X}}|$)

$$E(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}}) = |\mathbf{B}_{\mathbb{X}}|\, E\Big(\frac{\mathbf{B}_{\mathbb{X}}}{|\mathbf{B}_{\mathbb{X}}|}, \Pi_{\mathbb{X}}\Big)\,. \tag{4.3}$$

That is why we can reduce to consider *model problems* on cones $\Pi \in \overline{\mathfrak{P}}_3$ with unitary constant magnetic fields.

4.2. **Singular chains and generalized eigenvectors for model problems.** Let $\Pi \in \overline{\mathfrak{P}}_3$ be a polyhedral cone and $\mathbf{B}$ be a unitary constant magnetic field associated with a linear potential $\mathbf{A}$. Let $\Gamma \in \overline{\mathfrak{P}}_d$ be a minimal reduced cone associated with Π. We recall that this means that $\Pi \equiv \mathbb{R}^{3-d} \times \Gamma$ and that the dimension d is minimal for such an equivalence.

Let $\mathfrak{C}_0(\Pi)$ denote the singular chains of Π originating at its vertex 0 and let $\mathfrak{C}_0^*(\Pi)$ be the subset of chains of length ≥ 2 (see (3.2)). Note that $\mathfrak{C}_0^*(\Pi)$ is empty if and only if $\Pi = \mathbb{R}^3$, i.e., if $d = 0$. We introduce the quantity

$$\mathscr{E}^*(\mathbf{B}, \Pi) := \begin{cases} \inf_{\mathbb{X} \in \mathfrak{C}_0^*(\Pi)} E(\mathbf{B}, \Pi_{\mathbb{X}}) & \text{if } d > 0, \\ +\infty & \text{if } d = 0, \end{cases} \tag{4.4}$$

the infimum of the ground energy of the magnetic Laplacian over all the singular chains of length ≥ 2. If $d > 0$, let $\Omega_0 \in \overline{\mathfrak{D}}(\mathbb{S}^{d-1})$ be the section of Γ. Since Π is a polyhedral cone, we have (cf. Remark 3.2)

$$\mathscr{E}^*(\mathbf{B}, \Pi) = \inf_{\mathbf{x}_1 \in \overline{\Omega}_0} E(\mathbf{B}, \Pi_{(0,\mathbf{x}_1)}) \ , \tag{4.5}$$

i.e., among all chains $\mathbb{X} \in \mathfrak{C}_0^*(\Pi)$, we can restrict to those of length 2, $\mathbb{X} = (0, \mathbf{x}_1)$.

Since the cone Π is unbounded, it is relevant to define $\lambda_{\mathrm{ess}}(\mathbf{B}, \Pi)$ as the bottom of the essential spectrum of the operator $H(\mathbf{A}, \Pi)$. When $d \leq 2$, due to translation invariance we have $E(\mathbf{B}, \Pi) = \lambda_{\mathrm{ess}}(\mathbf{B}, \Pi)$. When $d = 3$, the operator $H(\mathbf{A}, \Pi)$ may have discrete spectrum.

With the aim of constructing quasimodes for our original problem on Ω, we need generalized eigenvectors for its tangent problems. We first need to introduce the space of the functions which are *locally*[2] in the domain of $H(\mathbf{A}, \Pi)$:

$$\begin{aligned} \mathrm{Dom}_{\mathrm{loc}}\left(H(\mathbf{A}, \Pi)\right) := \{\psi \in H^1_{\mathrm{loc}}(\overline{\Pi}), \\ (-i\nabla + \mathbf{A})^2 \psi \in H^0_{\mathrm{loc}}(\overline{\Pi}) \ \text{ and } \ (-i\partial_n + \mathbf{n} \cdot \mathbf{A})\psi = 0 \text{ on } \partial\Pi\}. \end{aligned} \tag{4.6}$$

We now make precise what we understand by generalized eigenvector.

Definition 4.2. Let $\Pi \in \overline{\mathfrak{P}}_3$ be a polyhedral cone and $\mathbf{A}$ a linear magnetic potential. We call *generalized eigenvector* for $H(\mathbf{A}, \Pi)$ a function $\Psi \in \mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A}, \Pi))$ associated with a real number λ, so that

$$\begin{cases} (-i\nabla + \mathbf{A})^2 \Psi = \lambda \Psi & \text{in } \Pi, \\ (-i\partial_n + \mathbf{n} \cdot \mathbf{A})\Psi = 0 & \text{on } \partial\Pi. \end{cases}$$

The function Ψ is said to be an *admissible generalized eigenvector* if in addition there exists a system of coordinates $(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{3-k} \times \mathbb{R}^k$ for some $k \in \{0, \dots, 3\}$, in which Π takes the form $\mathbb{R}^{3-k} \times \Upsilon$ with $\Upsilon \in \overline{\mathfrak{P}}_k$, and

$$\Psi(\mathbf{x}) = \mathrm{e}^{i\varphi(\mathbf{y},\mathbf{z})}\, \Phi(\mathbf{z}) \tag{4.7}$$

[2]Here $H^m_{\mathrm{loc}}(\overline{\Pi})$ denotes for $m = 0, 1$ the space of functions which are in $H^m(\Pi \cap \mathcal{B})$ for any ball $\mathcal{B}$.

with some exponentially decreasing function Φ and some real polynomial function φ of degree ≤ 2.

Lemma 4.3. *If Ψ is an admissible generalized eigenvector for $H(\mathbf{A}, \Pi)$ associated with λ, for any other linear magnetic potential $\mathbf{A}'$ such that $\operatorname{curl} \mathbf{A}' = \operatorname{curl} \mathbf{A}$, the operator $H(\mathbf{A}', \Pi)$ possesses an admissible generalized eigenvector Ψ' associated with the same value λ.*

Proof. If $\operatorname{curl} \mathbf{A} = \operatorname{curl} \mathbf{A}'$ and $\mathbf{A}$, $\mathbf{A}'$ are both linear, there exists a polynomial function ϕ of degree 2 such that $\mathbf{A} = \mathbf{A}' - \nabla\phi$. Using a change of gauge (Lemma A.1), we find that Ψ' defined as

$$\Psi'(\mathbf{x}) = \mathrm{e}^{i\phi(\mathbf{x})}\mathrm{e}^{i\varphi(\mathbf{y},\mathbf{z})}\,\Phi(\mathbf{z}), \quad \mathbf{x} \in \Pi,$$

is an admissible generalized eigenvector for $H(\mathbf{A}', \Pi)$. □

The main result which we prove in this section is a dichotomy statement, as follows.

Theorem 4.4. *Let $\Pi \in \overline{\mathfrak{P}}_3$ be a polyhedral cone and $\mathbf{A}$ be a linear magnetic potential ($\mathbf{B}$ is a constant magnetic field). Then*

$$E(\mathbf{B}, \Pi) \leq \mathscr{E}^*(\mathbf{B}, \Pi)\ . \tag{4.8}$$

Moreover there exists a singular chain $\mathbb{X} \in \mathfrak{C}_0(\Pi)$ such that $H(\mathbf{A}, \Pi_{\mathbb{X}})$ admits an admissible generalized eigenvector associated with the value $E(\mathbf{B}, \Pi)$ and we have the dichotomy:

(i) *If $E(\mathbf{B}, \Pi) < \mathscr{E}^*(\mathbf{B}, \Pi)$, then $\Pi_{\mathbb{X}} = \Pi$.*
(ii) *If $E(\mathbf{B}, \Pi) = \mathscr{E}^*(\mathbf{B}, \Pi)$, then $\mathbb{X} \in \mathfrak{C}_0^*(\Pi)$ and $\Pi_{\mathbb{X}} \neq \Pi$.*

Remark 4.5. In the case (ii), we note that the model cone $\Pi_{\mathbb{X}}$ associated with the chain $\mathbb{X}$ satisfies $E(\mathbf{B}, \Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbb{X}})$.

The following subsections are devoted to the proof of this statement according to increasing values of d, the dimension of the reduced cone of Π. For each value of d we also provide examples for the cases (i) and (ii) of the dichotomy. Owing to Lemma 4.3, in each case we may choose a suitable gauge to determine Ψ.

4.3. **Full space, $d = 0$. $\Pi = \mathbb{R}^3$.** By a change of variables and a change of gauge, the operator $H(\mathbf{A}, \mathbb{R}^3)$ is isospectral to $H(\underline{\mathbf{A}}, \mathbb{R}^3)$ with

$$\underline{\mathbf{A}} = (\tfrac{x_3}{2}, 0, -\tfrac{x_1}{2}) \quad \text{and} \quad \underline{\mathbf{B}} = (0, 1, 0).$$

Hence

$$H(\underline{\mathbf{A}}, \mathbb{R}^3) = (\mathsf{D}_1 + \tfrac{x_3}{2})^2 + \mathsf{D}_2^2 + (\mathsf{D}_3 - \tfrac{x_1}{2})^2 \quad \text{with} \quad \mathsf{D}_j = -i\partial_{x_j}.$$

It is classical (see [17]) that the spectrum of $H(\underline{\mathbf{A}}, \mathbb{R}^3)$ is $[1, +\infty)$. Therefore

$$E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1\ . \tag{4.9}$$

An admissible generalized eigenfunction associated to the ground energy is

$$\Psi(\mathbf{x}) = \mathrm{e}^{-(x_1^2 + x_3^2)/4}\,,$$

which has the form (4.7) with $\mathbf{y} = x_2$, $\mathbf{z} = (x_1, x_3)$, and $\varphi \equiv 0$.

4.4. **Half space,** $d = 1$**.** $\Pi = \mathbb{R}^3_+$. By a change of variables and a change of gauge, $H(\mathbf{A}, \Pi)$ is isospectral to $H(\underline{\mathbf{A}}, \mathbb{R}^3_+)$ with

$$\mathbb{R}^3_+ = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,\ x_1 > 0\} \quad \text{and} \quad \underline{\mathbf{A}} = (0, 0, b_1 x_2 - b_2 x_1) \text{ with } b_1^2 + b_2^2 = 1.$$

Hence

$$\underline{\mathbf{B}} = (b_1, b_2, 0) \quad \text{and} \quad H(\underline{\mathbf{A}}, \mathbb{R}^3_+) = \mathrm{D}_1^2 + \mathrm{D}_2^2 + (\mathrm{D}_3 + b_1 x_2 - b_2 x_1)^2.$$

We note that

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+) = E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1. \tag{4.10}$$

There exists $\theta \in [0, 2\pi)$ such that $b_1 = \sin\theta$ and $b_2 = \cos\theta$, so that θ is the angle between the magnetic field and the boundary of $\mathbb{R}^3_+$. Due to symmetries we can reduce to $\theta \in [0, \frac{\pi}{2}]$. Denote by $\mathcal{F}_3$ the Fourier transform in x_3-variable, τ the Fourier variable associated with x_3, and

$$\widehat{H}_\tau(\underline{\mathbf{A}}; \mathbb{R}^3_+) := \mathrm{D}_1^2 + \mathrm{D}_2^2 + (\tau + \sin\theta\, x_2 - \cos\theta\, x_1)^2,$$

there holds

$$\mathcal{F}_3\ H(\underline{\mathbf{A}}, \mathbb{R}^3_+)\ \mathcal{F}_3^* = \int_{\tau\in\mathbb{R}}^{\oplus} \widehat{H}_\tau(\underline{\mathbf{A}}; \mathbb{R}^3_+)\, \mathrm{d}\tau.$$

We discriminate three cases:

• *Tangent field.* $\theta = 0$, then $\widehat{H}_\tau(\underline{\mathbf{A}}; \mathbb{R}^3_+) := \mathrm{D}_1^2 + \mathrm{D}_2^2 + (\tau - x_1)^2$, let ξ be the partial Fourier variable associated with x_2 and define the new operators

$$\widehat{H}_{\xi,\tau}(\underline{\mathbf{A}}; \mathbb{R}^3_+) := \mathrm{D}_1^2 + \xi^2 + (\tau - x_1)^2, \quad L(\mathbb{R}_+; \tau) = \mathrm{D}_1^2 + (\tau - x_1)^2.$$

There holds

$$\inf \mathfrak{S}(L(\mathbb{R}_+; \tau)) = \mu(\tau), \quad \inf \mathfrak{S}(\widehat{H}_{\tau,\xi}(\underline{\mathbf{A}}; \mathbb{R}^3_+)) = \mu(\tau) + \xi^2,$$

in which the behavior of the first eigenvalue $\mu(\tau)$ is well-known (see [8]): The function μ admits a unique minimum denoted by $\Theta_0 \simeq 0.59$ for the value $\tau = \sqrt{\Theta_0}$. Hence

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) = \Theta_0 < \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+).$$

We are in case (i) of Theorem 4.4. If Φ denotes an associated eigenvector (function of $x_1 \in \mathbb{R}_+$), a corresponding admissible generalized eigenvector is

$$\Psi(\mathbf{x}) = \mathrm{e}^{i\sqrt{\Theta_0}\, x_3}\, \Phi(x_1). \tag{4.11}$$

which has the form (4.7) with $\mathbf{y} = (x_2, x_3)$, $\mathbf{z} = x_1$, and $\varphi(\mathbf{y}, \mathbf{z}) = y_2\sqrt{\Theta_0} = x_3\sqrt{\Theta_0}$.

• *Normal field.* $\theta = \frac{\pi}{2}$, then $\widehat{H}_\tau(\underline{\mathbf{A}}; \mathbb{R}^3_+) := \mathrm{D}_1^2 + \mathrm{D}_2^2 + (\tau + x_2)^2$. There holds for all $\tau \in \mathbb{R}$, $\inf \mathfrak{S}(\widehat{H}_\tau(\underline{\mathbf{A}}; \mathbb{R}^3_+)) = 1$, hence

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) = 1 = \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+).$$

We are in case (ii) of Theorem 4.4 and we can use the admissible generalized eigenvector on the full space associated with the chain $\mathbb{X} = (0, 1)$.

• *Neither tangent nor normal.* $\theta \in (0, \frac{\pi}{2})$. Then for any $\tau \in \mathbb{R}$, $\widehat{H}_\tau(\underline{\mathbf{A}}; \mathbb{R}^3_+)$ is isospectral to $\widehat{H}_0(\underline{\mathbf{A}}; \mathbb{R}^3_+)$ the ground energy of which is an eigenvalue $\sigma(\theta) < 1$ (see [14]). We deduce

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) = \sigma(\theta) \quad \text{with} \quad \sigma(\theta) < 1.$$

We are in case (i) of Theorem 4.4. We recall from [14, 19] the continuity properties:

Lemma 4.6. *Set $\sigma(0) = \Theta_0$ and $\sigma(1) = 1$. The function $\theta \mapsto \sigma(\theta)$ is continuous and increasing on $[0, \frac{\pi}{2}]$.*

The first eigenvalue of $\widehat{H}_0(\underline{\mathbf{A}}; \mathbb{R}^3_+)$ is associated with an exponentially decreasing eigenvector Φ which is a function of $(x_1, x_2) \in \mathbb{R}^2_+$. An admissible generalized eigenvector is given by

$$\Psi(\mathbf{x}) = \Phi(x_1, x_2), \tag{4.12}$$

which has the form (4.7) with $\mathbf{y} = x_3$, $\mathbf{z} = (x_1, x_2)$, and $\varphi \equiv 0$.

Thus Theorem 4.4 is proved for half-spaces.

4.5. **Wedges,** $d = 2$**.** Let Π be a wedge and let α denote its opening, $\alpha \in (0, \pi) \cup (\pi, 2\pi)$. Let

$$\mathcal{S}_\alpha = \begin{cases} \{x = (x_1, x_2),\ x_1 \tan\frac{\alpha}{2} > |x_2|\} & \text{if } \alpha \in (0, \pi) \\ \{x = (x_1, x_2),\ x_1 \tan\frac{\alpha}{2} > -|x_2|\} & \text{if } \alpha \in (\pi, 2\pi) \end{cases} \quad \text{and} \quad \mathcal{W}_\alpha = \mathcal{S}_\alpha \times \mathbb{R} \tag{4.13}$$

be the model sector and wedge. By a change of variables and a change of gauge, $H(\mathbf{A}, \Pi)$ is isospectral to $H(\underline{\mathbf{A}}, \mathcal{W}_\alpha)$ where

$$\underline{\mathbf{A}} = (0, b_3 x_1, b_1 x_2 - b_2 x_1) \quad \text{with} \quad b_1^2 + b_2^2 + b_3^2 = 1.$$

Hence

$$\underline{\mathbf{B}} = (b_1, b_2, b_3) \quad \text{and} \quad H(\underline{\mathbf{A}}, \mathcal{W}_\alpha) = \mathsf{D}_1^2 + (\mathsf{D}_2 + b_3 x_1)^2 + (\mathsf{D}_3 + b_1 x_2 - x_1 b_2)^2.$$

Denote by τ the Fourier variable associated with x_3, and

$$\widehat{H}_\tau(\underline{\mathbf{A}}; \mathcal{W}_\alpha) := \mathsf{D}_1^2 + (\mathsf{D}_2 + b_3 x_1)^2 + (\tau + b_1 x_2 - b_2 x_1)^2.$$

We introduce the notation:

$$s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau) := \inf \mathfrak{S}(\widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha)),$$

so that we have the direct Fourier integral decomposition

$$\mathcal{F}_3\ H(\underline{\mathbf{A}}, \mathcal{W}_\alpha)\ \mathcal{F}_3^* = \int_{\tau \in \mathbb{R}}^{\oplus} \widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha)\, \mathrm{d}\tau$$

and the relation

$$E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \inf_{\tau \in \mathbb{R}} s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau)\,. \tag{4.14}$$

The singular chains of $\mathfrak{C}_0^*(\mathcal{W}_\alpha)$ have three equivalence classes, cf. Definition 3.1 and section 3.3 (3), corresponding to three distinct model operators, associated to half-spaces $\Pi_\alpha^\pm$ corresponding to the faces $\partial^\pm\mathcal{W}_\alpha$ of $\mathcal{W}_\alpha$, and to the full space $\mathbb{R}^3$. Thus

$$\mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha) = \min\{E(\underline{\mathbf{B}},\Pi_\alpha^+),\ E(\underline{\mathbf{B}},\Pi_\alpha^-),\ E(\underline{\mathbf{B}},\mathbb{R}^3)\}.$$

Let $\theta^\pm \in [0,\frac{\pi}{2}]$ be the angle between $\underline{\mathbf{B}}$ and the face $\partial\Pi_\alpha^\pm$. We have, cf. Lemma 4.6,

$$\mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha) = \min\{\sigma(\theta^+),\sigma(\theta^-),\ 1\} = \sigma(\min\{\theta^+,\ \theta^-\}). \tag{4.15}$$

When $\Pi = \mathcal{W}_\alpha$, Theorem 4.4 relies on the following result [25, Theorem 3.5]:

Lemma 4.7. *We have $E(\underline{\mathbf{B}},\mathcal{W}_\alpha) \le \mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha)$.*
Moreover, if $E(\underline{\mathbf{B}},\mathcal{W}_\alpha) < \mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha)$, then the function $\tau \mapsto s(\underline{\mathbf{B}},\mathcal{S}_\alpha;\tau)$ reaches its infimum. Let τ^* be a minimizer. Then $E(\underline{\mathbf{B}},\mathcal{W}_\alpha)$ is a discrete eigenvalue for the operator $\widehat{H}_{\tau^*}(\underline{\mathbf{A}},\mathcal{W}_\alpha)$ and the associated eigenfunctions have exponential decay.

From the previous lemma we deduce

(i) If $E(\underline{\mathbf{B}},\mathcal{W}_\alpha) < \mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha)$, there exists τ^* such that the operator $\widehat{H}_{\tau^*}(\underline{\mathbf{A}},\mathcal{W}_\alpha)$ admits an exponential decaying eigenfunction Φ of $(x_1,x_2) \in \mathcal{S}_\alpha$ associated with $E(\underline{\mathbf{B}},\mathcal{W}_\alpha)$. The function

$$\Psi(\mathbf{x}) = \mathrm{e}^{i\tau^* x_3}\Phi(x_1,x_2)$$

is an admissible generalized eigenvector for the operator $H(\underline{\mathbf{A}},\mathcal{W}_\alpha)$ associated with $E(\underline{\mathbf{B}},\mathcal{W}_\alpha)$. It has the form (4.7) with $\mathbf{y} = x_3$, $\mathbf{z} = (x_1,x_2)$, and $\varphi(\mathbf{y},\mathbf{z}) = \tau^*\mathbf{y}$.

(ii) If $E(\underline{\mathbf{B}},\mathcal{W}_\alpha) = \mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha)$, let $\varepsilon \in \{-,+\}$ satisfy $\theta^\varepsilon = \min(\theta^-,\theta^+)$. We have $\theta^\varepsilon \in [0,\frac{\pi}{2})$ and $\mathscr{E}^*(\underline{\mathbf{B}},\mathcal{W}_\alpha) = \sigma(\theta^\varepsilon)$. We deduce from Subsection 4.4 that there exists an admissible generalized eigenvector for the operator $H(\underline{\mathbf{A}},\Pi_\alpha^\varepsilon)$ associated with the eigenvalue $\sigma(\theta^\varepsilon)$.

Thus Theorem 4.4 is proved for wedges. We extend the definition of $\mathcal{W}_\alpha$ to $\alpha = \pi$ by setting $\mathcal{W}_\pi := \mathbb{R}^3_+$. Let us quote now the continuity result of [25, Theorem 4.5]:

Lemma 4.8. *The function $(\mathbf{B},\alpha) \mapsto E(\mathbf{B},\mathcal{W}_\alpha)$ is continuous on $\mathbb{S}^2 \times (0,2\pi)$.*

We end this subsection by a few examples.

Example 4.9. Let $\mathbf{B} \in \mathbb{S}^2$ be a constant magnetic field. Let α be chosen in $(0,\pi)\cup(\pi,2\pi)$.

a) For α small enough $E(\mathbf{B},\mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha)$ (see [25] when the magnetic field is not tangent to the plane of symmetry of the wedge and [24, Ch. 7] otherwise).

b) Let $\mathbf{B} = (0,0,1)$ be tangent to the edge. Then $\mathscr{E}^*(\mathbf{B},\mathcal{W}_\alpha) = \Theta_0$ and $E(\mathbf{B},\mathcal{W}_\alpha) = E(1,\mathcal{S}_\alpha)$, cf. section 2.1.2. According to whether the ground energy $E(1,\mathcal{S}_\alpha)$ of the plane sector $\mathcal{S}_\alpha$ is less than Θ_0 or equal to Θ_0, we are in case (i) or (ii) of the dichotomy.

c) Let $\mathbf{B}$ be tangent to a face of the wedge and normal to the edge. Then $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$. It is proved in [26] that there holds $E(\mathbf{B}, \mathcal{W}_\alpha) < \Theta_0$ for α small enough, whereas $E(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$ for $\alpha \in [\frac{\pi}{2}, \pi)$.

4.6. **Polyhedral cones, $d = 3$.** The main result of this subsection is the characterization of the bottom of the essential spectrum of $H(\mathbf{A}, \Pi)$.

Proposition 4.10. *Let $\Pi \in \overline{\mathfrak{P}}_3$ be a polyhedral cone with $d = 3$, which means that Π is not a wedge, nor a half-space, nor the full space. Let $\mathbf{B}$ be a constant magnetic field. With the quantity $\mathscr{E}^*(\mathbf{B}, \Pi)$ introduced in (4.4), there holds*

$$\lambda_{\mathrm{ess}}(\mathbf{B}, \Pi) = \mathscr{E}^*(\mathbf{B}, \Pi)\,.$$

Before writing proof details, let us specify what is $\mathscr{E}^*(\mathbf{B}, \Pi)$ in the case of a polyhedral cone. Let Ω_0 be the section of Π, i.e., $\Omega_0 = \Pi \cap \mathbb{S}^2$. We recall from (4.5) that

$$\mathscr{E}^*(\mathbf{B}, \Pi) = \inf_{\mathbf{x}_1 \in \overline{\Omega}_0} E(\mathbf{B}, \Pi_{(0,\mathbf{x}_1)})\,. \tag{4.16}$$

In fact the set of equivalence classes (Definition 3.1) of the chains $\mathbb{X} = (0, \mathbf{x}_1)$ is finite. Let us describe this set, cf. section 3.3 (4). Let $\mathfrak{F}$ and $\mathfrak{E}$ be the set of faces $\mathbf{f}$ and edges $\mathbf{e}$ of Π. For $\mathbf{f} \in \mathfrak{F}$, let $\Pi_{\mathbf{f}}$ be the half-space whose boundary contains $\mathbf{f}$ and containing points of Π near any point of $\mathbf{f}$. For $\mathbf{e} \in \mathfrak{E}$, there are two faces $\mathbf{f}_{\mathbf{e}}^{\pm}$ adjacent to $\mathbf{e}$. Let $\Pi_{\mathbf{e}}$ be the wedge whose boundary contains $\mathbf{e} \cup \mathbf{f}_{\mathbf{e}}^{+} \cup \mathbf{f}_{\mathbf{e}}^{-}$ and containing points of Π near any point of $\mathbf{f}_{\mathbf{e}}^{+} \cup \mathbf{f}_{\mathbf{e}}^{-}$.

Let $\mathbf{x}_1 \in \overline{\Omega}_0$. There are three possibilities:

(1) $\mathbf{x}_1$ is interior to Ω_0. Then $\Pi_{(0,\mathbf{x}_1)} = \mathbb{R}^3$.
(2) $\mathbf{x}_1$ belongs to a side of Ω_0. This side is contained in a face $\mathbf{f}$ of Π. Then $\Pi_{(0,\mathbf{x}_1)} = \Pi_{\mathbf{f}}$.
(3) $\mathbf{x}_1$ belongs to a vertex of Ω_0. This vertex is contained in an edge $\mathbf{e}$ of Π. Then $\Pi_{(0,\mathbf{x}_1)} = \Pi_{\mathbf{e}}$.

We have that

$$\mathscr{E}^*(\mathbf{B}, \Pi) = \min\Big\{ \min_{\mathbf{e} \in \mathfrak{E}} E(\mathbf{B}, \Pi_{\mathbf{e}}),\ \min_{\mathbf{f} \in \mathfrak{F}} E(\mathbf{B}, \Pi_{\mathbf{f}}),\ 1 \Big\}\,. \tag{4.17}$$

Since (4.8) is proved for $d = 2$, we have $E(\mathbf{B}, \Pi_{\mathbf{e}}) \leq \min\{E(\mathbf{B}, \Pi_{\mathbf{f}_{\mathbf{e}}^{+}}), E(\mathbf{B}, \Pi_{\mathbf{f}_{\mathbf{e}}^{-}})\}$. Therefore equation (4.17) becomes

$$\mathscr{E}^*(\mathbf{B}, \Pi) = \min_{\mathbf{e} \in \mathfrak{E}} \{E(\mathbf{B}, \Pi_{\mathbf{e}})\}\,. \tag{4.18}$$

We recall the Persson Lemma that gives a characterization of the bottom of the essential spectrum (see [23]):

Lemma 4.11. *We have*

$$\lambda_{\mathrm{ess}}(\mathbf{B}, \Pi) = \lim_{R \to +\infty} \Sigma\,(\mathbf{B}, \Pi, R)$$

with

$$\Sigma\,(\mathbf{B},\Pi,R) := \inf_{\substack{u\in\mathscr{C}_0^\infty(\overline{\Pi}\cap\complement\mathcal{B}_R)\\ u\neq 0}} \frac{q[\mathbf{A},\Pi](u)}{\|u\|^2}$$

where $\mathcal{B}_R$ is the ball of radius R centered at the origin and $\complement\mathcal{B}_R$ its complementary in $\mathbb{R}^3$.

Proof. (of Proposition 4.10). Let $\mathbf{A}$ be a linear potential associated with $\mathbf{B}$.
Upper bound: We denote by $\mathbf{x}_{\mathbf{e}^*}$ a vertex of $\overline{\Omega}_0$ and $\mathbf{e}^*$ the associated edge such that $\mathscr{E}^*(\mathbf{B},\Pi) = E(\mathbf{B},\Pi_{\mathbf{e}^*})$, cf. (4.18). Let $\varepsilon > 0$, there exists $\psi_\varepsilon \in \mathscr{C}_0^\infty(\overline{\Pi_{\mathbf{e}^*}})$ a normalized function such that

$$q[\mathbf{A},\Pi_{\mathbf{e}^*}](\psi_\varepsilon) \leq E(\mathbf{B},\Pi_{\mathbf{e}^*}) + \varepsilon\,.$$

For $r > 0$ we define

$$\psi_\varepsilon^r(\mathbf{x}) := \mathrm{e}^{i\langle \mathbf{x},\,\mathbf{A}(r\mathbf{x}_{\mathbf{e}^*})\rangle}\,\psi_\varepsilon(\mathbf{x} - r\mathbf{x}_{\mathbf{e}^*})\,,$$

so that we have, due to gauge invariance and translation effect, cf. Lemma A.3,

$$\mathrm{supp}(\psi_\varepsilon^r) = \mathrm{supp}(\psi_\varepsilon) + r\mathbf{x}_{\mathbf{e}^*} \quad \text{and} \quad q[\mathbf{A},\Pi](\psi_\varepsilon^r) = q[\mathbf{A},\Pi_{\mathbf{e}^*}](\psi_\varepsilon)\,.$$

Let $R > 0$, for r large enough we have $\mathrm{supp}(\psi_\varepsilon^r) \subset \complement\mathcal{B}_R$ and $\psi_\varepsilon^r \in \mathrm{Dom}(q[\mathbf{A},\Pi])$. We get

$$q[\mathbf{A},\Pi](\psi_\varepsilon^r) = q[\mathbf{A},\Pi_{\mathbf{e}^*}](\psi_\varepsilon) \leq E(\mathbf{B},\Pi_{\mathbf{e}^*}) + \varepsilon\,.$$

We deduce

$$\forall\varepsilon > 0, \forall R > 0, \quad \Sigma(\mathbf{B},\Pi,R) \leq E(\mathbf{B},\Pi_{\mathbf{e}^*}) + \varepsilon$$

and Lemma 4.11 provides the upper bound of Proposition 4.10: $\lambda_{\mathrm{ess}}(\mathbf{B},\Pi) \leq \mathscr{E}^*(\mathbf{B},\Pi)$.

Lower bound: Let

$$\mathcal{U}_{\mathbf{0}} \cup \Big(\bigcup_{\mathbf{f}\in\mathfrak{F}} \mathcal{U}_{\mathbf{f}}\Big) \cup \Big(\bigcup_{\mathbf{e}\in\mathfrak{E}} \mathcal{U}_{\mathbf{e}}\Big)$$

a covering of $\overline{\Omega}_0$ according to its stratification, which means that

$$\mathcal{U}_{\mathbf{0}} \subset \Omega_0, \quad \mathcal{U}_{\mathbf{f}} \cap \overline{\Omega}_0 = \Pi_{\mathbf{f}} \cap \overline{\Omega}_0 \ (\forall \mathbf{f}\in\mathfrak{F}), \quad \text{and} \quad \mathcal{U}_{\mathbf{e}} \cap \overline{\Omega}_0 = \Pi_{\mathbf{e}} \cap \overline{\Omega}_0 \ (\forall \mathbf{e}\in\mathfrak{E}).$$

Let $\hat{\chi}_j$, $j \in \mathfrak{J} := \{\mathbf{0}\}\cup\mathfrak{F}\cup\mathfrak{E}$, be an associated partition of unity of the section $\overline{\Omega}_0$ such that

$$\sum_{j\in\mathfrak{J}} (\hat{\chi}_j)^2 = 1 \quad \text{and} \quad \mathrm{supp}(\hat{\chi}_j) \subset \mathcal{U}_j,\ \forall j \in \mathfrak{J}.$$

Let $\chi \in \mathscr{C}^\infty(\mathbb{R}_+)$ such that $\chi \equiv 0$ on $[0,\frac{1}{2}]$ and $\chi \equiv 1$ on $[1,+\infty)$. We now define a partition of the unity of $\complement\mathcal{B}_R \cap \overline{\Pi}$ by setting

$$\chi_j^R(\mathbf{x}) = \chi\Big(\frac{|\mathbf{x}|}{R}\Big)\,\hat{\chi}_j\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big), \quad j \in \mathfrak{J}.$$

We have $\sum_j (\chi_j^R)^2 = 1$ on $\complement\mathcal{B}_R \cap \overline{\Pi}$ and

$$\forall R > 0, \quad \sum_{j\in\mathfrak{J}} |\nabla\chi_j^R|^2 \leq C R^{-2}.$$

Moreover we have $\mathrm{supp}(\chi_j^R) \cap \overline{\Pi} \subset \overline{\Pi}_j$, $j \in \mathfrak{J} = \{\mathbf{0}\} \cup \mathfrak{F} \cup \mathfrak{E}$, where we have set $\Pi_{\mathbf{0}} = \mathbb{R}^3$. The IMS formula for quadratic forms (see Lemma A.4) provides

$$\begin{aligned}\forall \psi \in \mathscr{C}_0^\infty(\overline{\Pi} \cap \complement\mathcal{B}_R), \quad q[\mathbf{A}, \Pi](\psi) &\geq \sum_{j\in\mathfrak{J}} q[\mathbf{A}, \Pi](\chi_j^R \psi) - CR^{-2}\|\psi\|^2 \\ &= \sum_{j\in\mathfrak{J}} q[\mathbf{A}, \Pi_j](\chi_j^R \psi) - CR^{-2}\|\psi\|^2 \\ &\geq \sum_{j\in\mathfrak{J}} E(\mathbf{B}, \Pi_j)\|\chi_j^R \psi\|^2 - CR^{-2}\|\psi\|^2 \\ &\geq (\mathscr{E}^*(\mathbf{B}, \Pi) - CR^{-2})\|\psi\|^2.\end{aligned}$$

Thus we deduce the lower bound of Proposition 4.10 by using Lemma 4.11. □

Then it is clear that Theorem 4.4 in the case of polyhedral cones is a consequence of Proposition 4.10.

Example 4.12 (Octant). Let $\Pi = \mathcal{O}$ be an octant, e.g. $(\mathbb{R}_+)^3$. We quote from [22, §8]:

(i) If the magnetic field $\mathbf{B}$ is tangent to a face but not to an edge, there exists an edge $\mathbf{e}$ such that $\mathscr{E}^*(\mathbf{B}, \mathcal{O}) = E(\mathbf{B}, \Pi_{\mathbf{e}})$ and there holds $E(\mathbf{B}, \mathcal{O}) < E(\mathbf{B}, \Pi_{\mathbf{e}})$.
(ii) If the magnetic field $\mathbf{B}$ is tangent to an edge $\mathbf{e}$ of the octant, $\mathscr{E}^*(\mathbf{B}, \mathcal{O}) = E(\mathbf{B}, \Pi_{\mathbf{e}}) = E(\mathbf{B}, \mathcal{O})$.

5. **Continuity properties of the ground energy**

Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$ be a continuous magnetic field. In this section we investigate the continuity properties on $\overline{\Omega}$ of the application $\Lambda : \mathbf{x} \mapsto E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$. Let $\mathbf{t}$ be a stratum of $\overline{\Omega}$ (see (1.9)). We have denoted by $\Lambda_{\mathbf{t}}$ the restriction of Λ to $\mathbf{t}$ (see (1.10)). Combining (4.3), (4.9), Lemma 4.6 and Lemma 4.8 we get that $\Lambda_{\mathbf{t}}$ is continuous.

Let us assume that $\mathbf{t}$ is not reduced to a point. We now describe how we extend $\Lambda_{\mathbf{t}}$ to the boundary of $\mathbf{t}$. Let $\mathbf{x} \in \partial\mathbf{t}$ and $\Pi_{\mathbf{x}} \in \overline{\mathfrak{P}}_3$ be its tangent cone. Let $\mathcal{U}_{\mathbf{x}}$, $\mathcal{V}_{\mathbf{x}}$ and $\mathsf{U}_{\mathbf{x}}$ be the open sets and the diffeomorphism introduced in Subsection 3.1. Let $\tilde{\mathbf{t}}$ be the stratum of $\Pi_{\mathbf{x}}$ such that

$$\mathsf{U}_{\mathbf{x}}(\mathbf{t} \cap \mathcal{U}_{\mathbf{x}}) = \tilde{\mathbf{t}} \cap \mathcal{V}_{\mathbf{x}} \ .$$

To $\tilde{\mathbf{t}}$ is associated the singular chain $\mathbb{X} \in \mathfrak{C}_0^*(\Pi_{\mathbf{x}})$ such that $\Pi_{\mathbb{X}}$ is the tangent cone to $\Pi_{\mathbf{x}}$ at any point of $\tilde{\mathbf{t}}$.

We extend $\Lambda_{\mathbf{t}}$ in $\mathbf{x}$ by setting

$$\Lambda_{\mathbf{t}}(\mathbf{x}) = E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbb{X}}) \,. \tag{5.1}$$

Lemma 5.1. *Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$. Let $\mathbf{t}$ a stratum of Ω which is not a vertex. Then formula (5.1) defines a continuous extension of the function $\Lambda_{\mathbf{t}}$ to $\overline{\mathbf{t}}$.*

Proof. For $\mathbf{x} \in \partial\mathbf{t}$ we show that the extension defined by (5.1) is continuous in $\mathbf{x}$. Let $\mathbf{y} \in \mathcal{U}_\mathbf{x} \cap \mathbf{t}$ and $\Pi_\mathbf{y}$ be the tangent cone to Ω at $\mathbf{y}$. In the following we will prove that

$$\lim_{\mathbf{y}\to\mathbf{x}} E(\mathbf{B}_\mathbf{y}, \Pi_\mathbf{y}) = E(\mathbf{B}_\mathbf{x}, \Pi_\mathbb{X}) \,. \tag{5.2}$$

For a tangent cone Π we denote by $d(\Pi)$ the dimension of its reduced cone. Since $\mathbf{t}$ is not reduced to a point, we have $d(\Pi_\mathbb{X}) = d(\Pi_\mathbf{y}) \leq 2$ and we distinguish several cases:

- $d = 0$. $\Pi_\mathbb{X} = \Pi_\mathbf{y} = \mathbb{R}^3$. It follows from Subsection 4.3 that $E(\mathbf{B}_\mathbf{y}, \Pi_\mathbf{y}) = |\mathbf{B}_\mathbf{y}|$ and $E(\mathbf{B}_\mathbf{x}, \Pi_\mathbb{X}) = |\mathbf{B}_\mathbf{x}|$. Therefore (5.2).
- $d = 1$. $\Pi_\mathbb{X}$ and $\Pi_\mathbf{y}$ are half-spaces. We denote by $\theta_\mathbf{x}$ (respectively $\theta_\mathbf{y}$) the angle between $\mathbf{B}_\mathbf{x}$ and $\Pi_\mathbb{X}$ (respectively $\mathbf{B}_\mathbf{y}$ and $\Pi_\mathbf{y}$). We have $E(\mathbf{B}_\mathbf{y}, \Pi_\mathbf{y}) = |\mathbf{B}_\mathbf{y}|\sigma(\theta_\mathbf{y})$ and $E(\mathbf{B}_\mathbf{x}, \Pi_\mathbb{X}) = |\mathbf{B}_\mathbf{x}|\sigma(\theta_\mathbf{x})$ (see Subsection 4.4). Since $\theta_\mathbf{y}$ goes to $\theta_\mathbf{x}$ when $\mathbf{y}$ goes to $\mathbf{x}$, (5.2) follows from the continuity of the function σ, see Lemma 4.6.
- $d = 2$. $\Pi_\mathbb{X}$ and $\Pi_\mathbf{y}$ are wedges. We denote by $\alpha_\mathbf{x}$ and $\alpha_\mathbf{y}$ their openings. We denote by $\underline{\mathsf{U}}_\mathbf{x} \in \mathfrak{O}_3$ (respectively $\underline{\mathsf{U}}_\mathbf{y} \in \mathfrak{O}_3$) the linear orthogonal transformation which maps $\Pi_\mathbb{X}$ on $\mathcal{W}_{\alpha_\mathbf{x}}$ (respectively $\Pi_\mathbf{y}$ on $\mathcal{W}_{\alpha_\mathbf{y}}$). We have

$$E(\mathbf{B}_\mathbf{x}, \Pi_\mathbb{X}) = E(\underline{\mathbf{B}}_\mathbf{x}, \mathcal{W}_{\alpha_\mathbf{x}}) \quad \text{and} \quad E(\mathbf{B}_\mathbf{y}, \Pi_\mathbf{y}) = E(\underline{\mathbf{B}}_\mathbf{y}, \mathcal{W}_{\alpha_\mathbf{y}}) \,. \tag{5.3}$$

  where we have denoted $\underline{\mathbf{B}}_\mathbf{x} = \underline{\mathsf{U}}_\mathbf{x}(\mathbf{B}_\mathbf{x})$ and $\underline{\mathbf{B}}_\mathbf{y} = \underline{\mathsf{U}}_\mathbf{y}(\mathbf{B}_\mathbf{y})$. We have

$$\lim_{\mathbf{y}\to\mathbf{x}} \|\underline{\mathsf{U}}_\mathbf{x} - \underline{\mathsf{U}}_\mathbf{y}\| = 0 \quad \text{and} \quad \lim_{\mathbf{y}\to\mathbf{x}} |\alpha_\mathbf{x} - \alpha_\mathbf{y}| = 0 \,,$$

  therefore we deduce (5.2) from (5.3) and Lemma 4.8.

Hence we have proved (5.2) in all cases. □

Let $\mathbf{x} \in \partial\mathbf{t}$, we deduce from (4.8) that

$$\Lambda_\mathbf{t}(\mathbf{x}) = E(\mathbf{B}_\mathbf{x}, \Pi_\mathbb{X}) \geq E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}) = \Lambda(\mathbf{x}) \,.$$

Combining this with Lemma 5.1, we obtain the following:

Theorem 5.2. *Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$ be a continuous magnetic field. Then the function $\Lambda : \mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is lower semi-continuous on $\overline{\Omega}$.*

6. Upper bound for first eigenvalues

In this section we give upper bounds for the first eigenvalue $\lambda_h(\mathbf{B}, \Omega)$. Our first result is general and concerns polyhedral domains:

Theorem 6.1. *Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$ be a polyhedral domain, $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential such that the associated magnetic field $\mathbf{B}$ does not vanish on $\overline{\Omega}$. Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that*

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{5/4} \,. \tag{6.1}$$

We recall that the quantity $\mathscr{E}(\mathbf{B}, \Omega)$ is the lowest local energy defined in (1.6).

It is possible to obtain an upper bound in (6.1) depending on the magnetic field $\mathbf{B}$ and not on the magnetic potential. For this, we consider $\mathbf{B}$ as a datum and associate a potential $\mathbf{A}$ with it. Operators $\mathscr{A} : \mathbf{B} \mapsto \mathbf{A}$ lifting the curl (i.e. such that $\operatorname{curl} \circ \mathscr{A} = \mathbb{I}$) and satisfying suitable estimates do exist in the literature. We quote [6] in which it is proved that such lifting can be constructed as a pseudo-differential operator of order -1. As a consequence $\mathscr{A}$ is continuous between Hölder classes of non integer order:

$$\forall \alpha \in (0,1), \quad \exists C_\alpha > 0, \quad \|\mathscr{A}\mathbf{B}\|_{W^{2+\alpha,\infty}(\Omega)} \leq C_\alpha \|\mathbf{B}\|_{W^{1+\alpha,\infty}(\Omega)} .$$

Choosing $\mathbf{A} = \mathscr{A}\mathbf{B}$ in Theorem 6.1, we deduce the following.

Corollary 6.2. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ be a polyhedral domain, $\mathbf{B} \in W^{1+\alpha,\infty}(\overline{\Omega})$ be a non-vanishing Hölder continuous magnetic field of order $1+\alpha$ with some $\alpha \in (0,1)$. Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that*

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + C(\Omega)(1 + \|\mathbf{B}\|^2_{W^{1+\alpha,\infty}(\Omega)})h^{5/4} . \tag{6.2}$$

Theorem 6.1 is proved in subsections 6.1–6.3. Let us give here the main ideas of the proof. Since the energy $\mathbf{x} \mapsto E(B_\mathbf{x}, \Pi_\mathbf{x})$ is lower semi-continuous (see Theorem 5.2), it reaches its infimum over the compact $\overline{\Omega}$. We denote by $\mathbf{x}_0 \in \overline{\Omega}$ a point such that

$$E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = \mathscr{E}(\mathbf{B}, \Omega) \tag{6.3}$$

where $\Pi_{\mathbf{x}_0}$ is the tangent cone at $\mathbf{x}_0$. Using the diffeomorphism (3.1) $\mathrm{U}_{\mathbf{x}_0}$ in a small neighborhood of $\mathbf{x}_0$ and making a Taylor approximation of the metric associated with the change of variable $\mathbf{x} \to \mathrm{U}_{\mathbf{x}_0}(\mathbf{x})$, we are led to study the magnetic Laplacian $H_h(\widetilde{\mathbf{A}}, \Pi_{\mathbf{x}_0})$ where $\widetilde{\mathbf{A}}$ is the magnetic potential given by the change of variable associated with $\mathrm{U}_{\mathbf{x}_0}$.

Let $\mathbf{A}_0$ be the linear part of $\widetilde{\mathbf{A}}$ at $\mathbf{x}_0$. It satisfies $\operatorname{curl} \mathbf{A}_0 = \mathbf{B}_{\mathbf{x}_0}$. Theorem 4.4 provides a singular chain $\mathbb{X} \in \mathfrak{C}_0(\Pi_{\mathbf{x}_0})$ and an associated cone $\Pi_\mathbb{X}$ such that the operator $H(\mathbf{A}_0, \Pi_\mathbb{X})$ has a generalized eigenfunction associated to the energy $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$. This generalized eigenfunction will be scaled, truncated and translated in order to give a quasimode for the operator $H_h(\mathbf{A}_0, \Pi_{\mathbf{x}_0})$. Using the diffeomorphism $\mathrm{U}_{\mathbf{x}_0}$ we finally construct a quasimode for $H_h(\mathbf{A}, \Omega)$ localized in a neighborhood of $\mathbf{x}_0$. The estimation of the associated Rayleigh quotient and the min-max principle will provide Theorem 6.1.

6.1. **Change of variables.** In this subsection we describe how the operator $H_h(\mathbf{A}, \Omega)$ acting on functions with support in a small neighborhood of a point $\mathbf{x} \in \overline{\Omega}$ is transformed using the change of variable associated with $\mathrm{U}_\mathbf{x}$ into an operator on the tangent cone $\Pi_\mathbf{x}$. Note that the results below are valid for any point $\mathbf{x} \in \overline{\Omega}$ and will be useful in establishing the lower bound for the first eigenvalue, see Section 7.

• *Magnetic Laplacian with a metric.* Let G be a metric of $\mathbb{R}^3$, that is a 3×3 positive symmetric matrix with regular coefficients. We define the magnetic Laplacian with metric

G as the Friedrichs extension of the quadratic form

$$q_h[\mathbf{A},\Omega,G](\psi)=\int_\Omega (hD_{\mathbf{x}}-\mathbf{A})\psi\cdot G(\overline{(hD_{\mathbf{x}}-\mathbf{A})\psi})\,|G|^{-1/2}\,\mathrm{d}\mathbf{x}$$

where $\mathbf{A}$ is a magnetic potential associated with the magnetic field $\mathbf{B}$ and $|G|=\det G$. The Friedrichs extension is taken from the space of the square-integrable functions for the weight $|G|^{-1/2}$ denoted by $L^2_G(\Omega)$ with natural Neumann boundary conditions associated to the metric G, see [15, §5] for more details. The associated scalar product is $\langle f,g\rangle_{L^2_G(\Omega)}:=\int_\Omega f\overline{g}\,|G|^{-1/2}\,\mathrm{d}\mathbf{x}$ and we denote by $\|\cdot\|_{L^2_G(\Omega)}$ the associated norm.

We also denote by $q_h[\mathbf{A},\Omega]=q_h[\mathbf{A},\Omega,\mathbb{I}_3]$ the quadratic form associated with the operator $H_h(\mathbf{A},\Omega)$. These definitions are still valid if we take for the domain a cone $\Pi\in\mathfrak{D}_3$.

• *Effect of a change of variable.* Let $\mathcal{U}_{\mathbf{x}}$, $\mathcal{V}_{\mathbf{x}}$, and $\mathsf{U}_{\mathbf{x}}$ be the open sets and the diffeomorphism described in Subsection 3.1. We denote by

$$J_{\mathbf{x}}:=\mathrm{d}(\mathsf{U}_{\mathbf{x}}^{-1})$$

the jacobian matrix of the inverse of $\mathsf{U}_{\mathbf{x}}$. Let f be a function of $H^1(\Omega)$ compactly supported in $\mathcal{U}_{\mathbf{x}}$ and $\psi:=f\circ\mathsf{U}_{\mathbf{x}}^{-1}$ be the associated function supported in $\mathcal{V}_{\mathbf{x}}$. We have

$$q_h[\mathbf{A},\Omega](f)=q_h[\widetilde{\mathbf{A}},\Pi_{\mathbf{x}},G_{\mathbf{x}}](\psi)\quad\text{and}\quad\|f\|_{L^2(\Omega)}=\|\psi\|_{L^2_{G_{\mathbf{x}}}(\Pi_{\mathbf{x}})}\tag{6.4}$$

where the new magnetic potential and the metric are respectively given by

$$\widetilde{\mathbf{A}}:=J_{\mathbf{x}}^T\cdot\mathbf{A}\circ\mathsf{U}_{\mathbf{x}}^{-1}\quad\text{and}\quad G_{\mathbf{x}}:=J_{\mathbf{x}}^{-1}(J_{\mathbf{x}}^{-1})^T\,.\tag{6.5}$$

We deduce:

Lemma 6.3. *Let $\Omega\in\overline{\mathfrak{D}}(\mathbb{R}^3)$ be a polyhedral domain. Let $\mathbf{x}\in\overline{\Omega}$ and $\mathcal{U}_{\mathbf{x}}$, $\mathcal{V}_{\mathbf{x}}$, $\mathsf{U}_{\mathbf{x}}$ be the open sets and diffeomorphism associated with $\mathbf{x}$, see (3.1). Let $\mathbf{A}$ be a magnetic potential. Let $f\in H^1(\Omega)$ be a function compactly supported in $\mathcal{U}_{\mathbf{x}}$. We denote by $\psi:=f\circ\mathsf{U}_{\mathbf{x}}^{-1}$ the associated function on $\Pi_{\mathbf{x}}$ and $\widetilde{\mathbf{A}}$ the magnetic potential given by (6.5). Let $r_0>0$ be such that $\mathcal{V}_{\mathbf{x}}$ contains the ball $\mathcal{B}(0,r_0)$ of center 0 and radius r_0. Then there exists a constant $C(\Omega)$ such that for all $r\in(0,r_0]$, if $\mathrm{supp}(\psi)\subset\mathcal{B}(0,r)$ we have the two estimates*

$$\big|q_h[\widetilde{\mathbf{A}},\Pi_{\mathbf{x}}](\psi)-q_h[\mathbf{A},\Omega](f)\big|\le C(\Omega)\,r\,q_h[\widetilde{\mathbf{A}},\Pi_{\mathbf{x}}](\psi),\tag{6.6}$$

$$\big|\|\psi\|_{L^2(\Pi_{\mathbf{x}})}-\|f\|_{L^2(\Omega)}\big|\le C(\Omega)\,r\,\|\psi\|_{L^2(\Pi_{\mathbf{x}})}.\tag{6.7}$$

Proof. Recall that $J_{\mathbf{x}}(\mathbf{x})=\mathbb{I}_3$ and therefore $G_{\mathbf{x}}(\mathbf{x})=\mathbb{I}_3$. We deduce

$$\|G_{\mathbf{x}}-\mathbb{I}_3\,\|_{L^\infty(\mathcal{B}(0,r))}\le r\|G_{\mathbf{x}}\|_{W^{1,\infty}(\mathcal{V}_{\mathbf{x}})}.\tag{6.8}$$

Since Ω is assumed to be polyhedral, its curvature (curvature of the faces and curvature of the edges) is bounded, therefore

$$\sup_{x\in\overline{\Omega}}\|G_{\mathbf{x}}\|_{W^{1,\infty}(\mathcal{V}_{\mathbf{x}})}<+\infty\,.$$

We deduce the Lemma by using (6.8) in (6.4). □

Therefore we are reduced to study the Laplacian with magnetic potential $\widetilde{\mathbf{A}}$ on the cone $\Pi_{\mathbf{x}}$ with the identity metric.

6.2. **Construction of quasimodes.** Let $\mathbf{x}_0 \in \overline{\Omega}$ be a point satisfying (6.3). Thus $\mathbf{x}_0$ minimizes the local ground energy. For shortness we denote by λ this energy:

$$\lambda = E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}). \tag{6.9}$$

In order to prove Theorem 6.1, we are going to construct a family of quasimodes $f_h \in H^1(\Omega)$ satisfying the estimate for $h \leq h_0$ (with some chosen positive h_0)

$$\frac{q_h[\mathbf{A}, \Omega](f_h)}{\|f_h\|^2} \leq h\lambda + C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{5/4}. \tag{6.10}$$

Let $\mathbf{A}_0$ be the linear part of $\mathbf{A}$ at $\mathbf{x}_0$. Let $\widetilde{\mathbf{A}}$ be the magnetic potential in the tangent cone $\Pi_{\mathbf{x}_0} \cap \mathcal{V}_{\mathbf{x}_0}$ given by (6.5). Since $\mathrm{d}\mathsf{U}_{\mathbf{x}_0}(\mathbf{x}_0) = \mathbb{I}_3$, the linear part of $\widetilde{\mathbf{A}}$ at $\mathbf{x}_0$ is still $\mathbf{A}_0$. Let $\mathbb{X} \in \mathfrak{C}_0(\Pi_{\mathbf{x}_0})$ be a singular chain given by Theorem 4.4 and $\Pi_{\mathbb{X}}$ be the associated tangent cone. Let Ψ be an admissible generalized eigenvector for $H(\mathbf{A}_0, \Pi_{\mathbb{X}})$ associated with λ.

Up to a rotation, we can assume that the coordinates in $\Pi_{\mathbb{X}}$ are $\mathbf{x} = (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{3-k} \times \Upsilon$, with $\Upsilon \in \overline{\mathfrak{P}}_k$, and that in these coordinates, the generalized eigenfunction writes:

$$\Psi(\mathbf{x}) = \mathrm{e}^{i\varphi(\mathbf{y},\mathbf{z})}\,\Phi(\mathbf{z}), \quad (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{3-k} \times \Upsilon, \tag{6.11}$$

where Φ is an exponentially decreasing function (see Theorem 4.4). The function Ψ satisfies

$$\begin{cases} (-i\nabla + \mathbf{A}_0)^2\Psi = \lambda\Psi & \text{in } \Pi_{\mathbb{X}}, \\ (-i\partial_n + \mathbf{n}\cdot\mathbf{A}_0)\Psi = 0 & \text{on } \partial\Pi_{\mathbb{X}}. \end{cases} \tag{6.12}$$

Then the scaled function

$$\Psi_h(\mathbf{x}) := \Psi\Big(\frac{\mathbf{x}}{\sqrt{h}}\Big) \tag{6.13}$$

defines a generalized eigenfunction for the operator $H_h(\mathbf{A}_0, \Pi_{\mathbb{X}})$ associated with $h\lambda$.

For any $R > 0$, let $\underline{\chi}_R$ be a cut-off function in $\mathscr{C}^\infty(\mathbb{R}^+)$ such that

$$\underline{\chi}_R(r) = \begin{cases} 1 \text{ if } r \leq R, \\ 0 \text{ if } r \geq 2R. \end{cases} \tag{6.14}$$

We define the cut-off function

$$\chi_h(\mathbf{x}) = \underline{\chi}_R\left(\frac{|\mathbf{x}|}{h^\delta}\right) \quad \text{with} \quad 0 \leq \delta < \frac{1}{2} \tag{6.15}$$

and set

$$\psi_h(\mathbf{x}) := \chi_h(\mathbf{x})\Psi_h(\mathbf{x}) \tag{6.16}$$

which provides a quasimode for $q_h[\mathbf{A}_0, \Pi_{\mathbb{X}}]$ satisfying $(-ih\partial_n + \mathbf{n}\cdot\mathbf{A}_0)\Psi_h = 0$ on $\partial\Pi_{\mathbb{X}}$.

At this point, let us emphasize that, in order to obtain better cut-off estimates, cf. Lemma A.5, we need that our quasimodes on the tangent cone $\Pi_{\mathbf{x}_0}$ satisfy the Neumann

boundary conditions. Here the dichotomy in Theorem 4.4 comes into play: According as $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ is less or equal to $\mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$, we are in case (i) or (ii) of the dichotomy and we define our quasimode in distinct ways as follows.

(i) $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) < \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$. Then $\mathbb{X}$ is the trivial chain $(\mathbf{x}_0)$ and $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0}$. The function ψ_h will be our quasimode.

(ii) $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$. Then $\mathbb{X} \in \mathfrak{C}^*_{\mathbf{x}_0}(\Omega)$. Let $\underline{\mathsf{U}} \in \mathfrak{O}_3$ such that $\underline{\mathsf{U}}\Pi_{\mathbf{x}_0} = \mathbb{R}^{3-d} \times \Gamma$ where Γ is the reduced cone of $\Pi_{\mathbf{x}_0}$. Let Ω_0 be the section of Γ. According to Remark 3.2, there exists $\mathbf{x}_1 \in \overline{\Omega_0}$ so that $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$. Let $\underline{\boldsymbol{\tau}} := (0, \mathbf{x}_1) \in \mathbb{R}^{3-d} \times \Gamma$. We take $\boldsymbol{\tau} = \underline{\mathsf{U}}^{-1}(\underline{\boldsymbol{\tau}})$. and we define our quasimode by

$$\psi_h^{\boldsymbol{\tau}}(\mathbf{x}) = \mathrm{e}^{i\langle \frac{\mathbf{x}}{h}, \mathbf{A}_0(\boldsymbol{\tau} h^\delta)\rangle}\, \psi_h(\mathbf{x} - \boldsymbol{\tau} h^\delta), \quad \mathbf{x} \in \Pi_{\mathbf{x}_0}\,. \tag{6.17}$$

In case (ii), $\boldsymbol{\tau}$ gives a direction associated to the cone $\Pi_{\mathbb{X}}$ starting from the origin of $\Pi_{\mathbf{x}_0}$:

(1) If $\Pi_{\mathbb{X}} = \mathbb{R}^3$, $\boldsymbol{\tau}$ belongs to the interior of $\Pi_{\mathbf{x}_0}$ and $\psi_h^{\boldsymbol{\tau}}$ is centered on the interior.
(2) If $\Pi_{\mathbb{X}} = \mathbb{R}^3_+$, $\boldsymbol{\tau}$ belongs to a face of $\Pi_{\mathbf{x}_0}$ and $\psi_h^{\boldsymbol{\tau}}$ is centered on the same face.
(3) If $\Pi_{\mathbb{X}} = \mathcal{W}_\alpha$, $\boldsymbol{\tau}$ belongs to an edge of $\Pi_{\mathbf{x}_0}$ and $\psi_h^{\boldsymbol{\tau}}$ is centered on the same edge.

Note that unless we are in the last case ($\Pi_{\mathbf{x}_0}$ is a wedge), the choice of $\boldsymbol{\tau}$ is not unique.

In case (i), to unify notation we set $\boldsymbol{\tau} = 0$ and $\psi_h^{\boldsymbol{\tau}} = \psi_h$. We choose $R = 1$ in (6.14).

In case (ii), we choose $R > 0$ in (6.14) such that $\mathcal{B}(\boldsymbol{\tau}, 2R) \cap \Pi_{\mathbf{x}_0} = \mathcal{B}(\boldsymbol{\tau}, 2R) \cap \Pi_{\mathbb{X}}$. Hence

$$\mathrm{supp}\left(\underline{\chi}_R(\cdot - \boldsymbol{\tau})\right) \cap \Pi_{\mathbf{x}_0} = \mathrm{supp}\left(\underline{\chi}_R(\cdot - \boldsymbol{\tau})\right) \cap \Pi_{\mathbb{X}}\,. \tag{6.18}$$

Note that R depends only on the geometry of Ω near $\mathbf{x}_0$. It follows by scaling that

$$\forall h > 0, \quad \mathrm{supp}(\psi_h^{\boldsymbol{\tau}}) \cap \Pi_{\mathbf{x}_0} = \mathrm{supp}(\psi_h^{\boldsymbol{\tau}}) \cap \Pi_{\mathbb{X}}\,.$$

Therefore $\psi_h^{\boldsymbol{\tau}}$ satisfies Neumann boundary conditions on $\partial\Pi_{\mathbf{x}_0}$ and we have

$$q_h[\mathbf{A}_0, \Pi_{\mathbf{x}_0}](\psi_h^{\boldsymbol{\tau}}) = q_h[\mathbf{A}_0, \Pi_{\mathbb{X}}](\psi_h^{\boldsymbol{\tau}}) = q_h[\mathbf{A}_0, \Pi_{\mathbb{X}}](\psi_h)\,,$$

the last equality coming from Lemma A.3.

Definition 6.4. The functions $\psi_h^{\boldsymbol{\tau}} \in \mathrm{Dom}(H_h(\mathbf{A}_0, \Pi_{\mathbf{x}_0}))$ constructed above along formulas (6.11)-(6.18) are called

(i) *sitting quasimodes* when $\boldsymbol{\tau} = 0$: Their supports contain the vertex 0 of $\Pi_{\mathbf{x}_0}$,
(ii) *sliding quasimodes* when $\boldsymbol{\tau} \neq 0$: Their supports do not contain 0 but get closer as $h \to 0$.

6.3. **Estimation of the quasimodes.** We separately estimate the cut-off errors, the linearization errors, and error due to the change of metric.

6.3.1. *Cut-off effect.* Applying Lemma A.5 with $\mathbf{A}_0$, $\mathrm{G} = \mathrm{Id}$, $\chi = \chi_h$ and $\psi = \Psi_h$, we obtain for the Rayleigh quotient of ψ_h^τ:

$$\frac{q_h[\mathbf{A}_0, \Pi_{\mathbf{x}_0}](\psi_h^\tau)}{\|\psi_h^\tau\|^2} = \frac{q_h[\mathbf{A}_0, \Pi_{\mathbb{X}}](\chi_h \Psi_h)}{\|\chi_h \Psi_h\|^2} = h\lambda + h^2 \rho_h \quad \text{with} \quad \rho_h = \frac{\|\nabla \chi_h \, \Psi_h\|^2}{\|\chi_h \Psi_h\|^2}. \tag{6.19}$$

The following lemma estimates the remainder due to the cut-off effect:

Lemma 6.5. *Let Ψ be an admissible generalized eigenvector given by (6.11), Ψ_h the rescaled associated function given by (6.13) and χ_h a cut-off function defined by (6.15). Then there exists a constant $C(\Omega) > 0$ such that*

$$\rho_h = \frac{\|\nabla \chi_h \, \Psi_h\|^2}{\|\chi_h \Psi_h\|^2} \leq \begin{cases} C(\Omega)\, h^{-2\delta} & \text{if } k < 3, \\ C(\Omega)\, \mathrm{e}^{-c h^{\delta - \frac{1}{2}}} & \text{if } k = 3. \end{cases}$$

Proof. By assumption, there exist positive constants c, C such that

$$\int_\Upsilon \mathrm{e}^{c|\mathbf{z}|} |\Phi(\mathbf{z})|^2 \mathrm{d}\mathbf{z} \leq C \|\Phi\|^2_{L^2(\Upsilon)}.$$

Let us first give a upper bound for $\|\nabla \chi_h \Psi_h\|$:
If $k < 3$, then

$$\begin{aligned} \|\nabla \chi_h \Psi_h\|^2 &\leq \frac{C}{h^{2\delta}} \int_{|\mathbf{y}| \leq 2h^\delta R} \mathrm{d}\mathbf{y} \int_{\Upsilon \cap \{|\mathbf{z}| \leq 2h^\delta R\}} \left| \Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big) \right|^2 \mathrm{d}\mathbf{z} \\ &\leq C h^{-2\delta}\, h^{(3-k)\delta}\, h^{\frac{k}{2}} \|\Phi\|^2_{L^2(\Upsilon)}, \end{aligned}$$

else, if $k = 3$ (here we use $\delta < \frac{1}{2}$)

$$\begin{aligned} \|\nabla \chi_h \Psi_h\|^2 &\leq \frac{C}{h^{2\delta}} \int_{\Upsilon \cap \{h^\delta R \leq |\mathbf{z}| \leq 2h^\delta R\}} \left| \Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big) \right|^2 \mathrm{d}\mathbf{z} \\ &\leq C h^{-2\delta}\, h^{\frac{k}{2}} \int_{\Upsilon \cap \{h^{\delta - \frac{1}{2}} R \leq |\mathbf{z}| \leq 2h^{\delta - \frac{1}{2}} R\}} \mathrm{e}^{-cRh^{\delta - \frac{1}{2}}}\, \mathrm{e}^{c|\mathbf{z}|}\, |\Phi(\mathbf{z})|^2 \, \mathrm{d}\mathbf{z} \\ &\leq C h^{-2\delta}\, h^{\frac{k}{2}}\, \mathrm{e}^{-cRh^{\delta - \frac{1}{2}}}\, \|\Phi\|^2_{L^2(\Upsilon)}. \end{aligned}$$

Let us now consider $\|\chi_h \Psi_h\|$ (we use that $2|\mathbf{y}| < R$ and $2|\mathbf{z}| < R$ implies $|\mathbf{x}| < R$):

$$\begin{aligned} \|\chi_h \Psi_h\|^2 &\geq \int_{2|\mathbf{y}| \leq h^\delta R} \mathrm{d}\mathbf{y} \int_{\Upsilon \cap \{2|\mathbf{z}| \leq h^\delta R\}} \left| \Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big) \right|^2 \mathrm{d}\mathbf{z} \\ &\geq C h^{(3-k)\delta} h^{\frac{k}{2}} \left(1 - \mathrm{e}^{-c\frac{R}{2} h^{\delta - \frac{1}{2}}}\right) \|\Phi\|^2_{L^2(\Upsilon)}. \end{aligned} \tag{6.20}$$

Consequently

$$\rho_h \leq \begin{cases} C h^{(3-k)\delta} h^{-2\delta} h^{\frac{k}{2}} h^{-(3-k)\delta} h^{-\frac{k}{2}} \left(1 - \mathrm{e}^{-c\frac{R}{2} h^{\delta - \frac{1}{2}}}\right)^{-1} = \mathcal{O}(h^{-2\delta}) & \text{if } k < 3 \\ C h^{-2\delta} h^{\frac{k}{2}} \mathrm{e}^{-cRh^{\delta - \frac{1}{2}}} h^{-(3-k)\delta} h^{-\frac{k}{2}} \left(1 - \mathrm{e}^{-c\frac{R}{2} h^{\delta - \frac{1}{2}}}\right)^{-1} = \mathcal{O}(\mathrm{e}^{-cRh^{\delta - \frac{1}{2}}}) & \text{if } k = 3. \end{cases}$$

We notice that all the constants in the above estimation depend only on $\underline{\chi}_R$ and on the model problem associated with $\mathbf{x}_0$. Lemma 6.5 is proved. □

6.3.2. *Linearization.* Note that $\psi_h^\tau \in \mathrm{Dom}(q_h[\widetilde{\mathbf{A}}, \Pi_{\mathbf{x}_0}])$. We can compare the quadratic form for the magnetic potential and its linear part by using (A.3) with $\mathbf{A} = \widetilde{\mathbf{A}}$, $\mathbf{A}' = \mathbf{A}_0$, $\mathcal{U} = \Pi_{\mathbf{x}_0}$, and $\psi = \psi_h^\tau$:

$$q_h[\widetilde{\mathbf{A}}, \Pi_{\mathbf{x}_0}](\psi_h^\tau) = q_h[\mathbf{A}_0, \Pi_{\mathbf{x}_0}](\psi_h^\tau) + 2\,\mathrm{Re}\left\langle (-ih\nabla + \mathbf{A}_0)\psi_h^\tau, (\widetilde{\mathbf{A}} - \mathbf{A}_0)\psi_h^\tau \right\rangle + \|(\widetilde{\mathbf{A}} - \mathbf{A}_0)\psi_h^\tau\|^2. \tag{6.21}$$

Combining with (6.19) we get

$$\frac{q_h[\widetilde{\mathbf{A}}, \Pi_{\mathbf{x}_0}](\psi_h^\tau)}{\|\psi_h^\tau\|^2} = h\lambda + h^2\rho_h + \frac{2\,\mathrm{Re}\left\langle (-ih\nabla + \mathbf{A}_0)\psi_h^\tau, (\widetilde{\mathbf{A}} - \mathbf{A}_0)\psi_h^\tau \right\rangle}{\|\psi_h^\tau\|^2} + \frac{\|(\widetilde{\mathbf{A}} - \mathbf{A}_0)\psi_h^\tau\|^2}{\|\psi_h^\tau\|^2}. \tag{6.22}$$

By Cauchy-Schwarz inequality, we obtain easily

$$\frac{q_h[\widetilde{\mathbf{A}}, \Pi_{\mathbf{x}_0}](\psi_h^\tau)}{\|\psi_h^\tau\|^2} \leq h\lambda + h^2\rho_h + 2\sqrt{h}\sqrt{\lambda + h\rho_h}\; a_h + a_h^2 \tag{6.23}$$

where we have set

$$a_h = \frac{\|(\widetilde{\mathbf{A}} - \mathbf{A}_0)\psi_h^\tau\|}{\|\psi_h^\tau\|}. \tag{6.24}$$

We now estimate the remainder due to the linearization of $\tilde{\mathbf{A}}$. Since $\mathbf{A}_0$ is the linear part of $\widetilde{\mathbf{A}}$, then for any $r \in (0, r_0]$ (cf. Lemma 6.3)

$$\forall \mathbf{x} \in \mathcal{B}(0, r), \quad |\widetilde{\mathbf{A}}(\mathbf{x}) - \mathbf{A}_0(\mathbf{x})| \leq \|\widetilde{\mathbf{A}}\|_{W^{2,\infty}(\mathcal{B}(0,r))} \frac{|\mathbf{x}|^2}{2}. \tag{6.25}$$

By construction, there exists $C(\Omega) > 0$ such that the support of ψ_h^τ is included in $\mathcal{B}(0, C(\Omega)h^\delta)$. Consequently, we obtain immediately

$$a_h \leq C(\Omega)\|\widetilde{\mathbf{A}}\|_{W^{2,\infty}(\mathrm{supp}(\psi_h^\tau))} h^{2\delta}. \tag{6.26}$$

Moreover using the definition of $\widetilde{\mathbf{A}}$ (see (6.5)) we get

$$\begin{aligned} \|\widetilde{\mathbf{A}}\|_{W^{2,\infty}(\mathrm{supp}(\psi_h^\tau))} &\leq \left(1 + \|\,\mathbb{I}_3 - J_{\mathbf{x}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{x}_0})} h^\delta\right) \|\mathbf{A}\|_{W^{2,\infty}(\mathcal{U}_{\mathbf{x}_0})} \\ &\leq \left(1 + C(\Omega)h^\delta\right) \|\mathbf{A}\|_{W^{2,\infty}(\Omega)} \end{aligned}$$

Thus, putting this last inequality in (6.26), we deduce

$$a_h \leq C(\Omega)\|\mathbf{A}\|_{W^{2,\infty}(\Omega)} h^{2\delta}. \tag{6.27}$$

Combining (6.23), (6.27) and Lemma 6.5 we get

$$\frac{q_h[\widetilde{\mathbf{A}}, \Pi_{\mathbf{x}_0}](\psi_h^\tau)}{\|\psi_h^\tau\|^2} \leq h\lambda + C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2-2\delta} + h^{2\delta+1/2} + h^{4\delta}). \tag{6.28}$$

Note that we have also used $\lambda \leq \|\mathbf{B}\|_{L^\infty(\Omega)} \leq \|\mathbf{A}\|_{W^{1,\infty}(\Omega)}$ (since $\mathbf{B} = \operatorname{curl}\mathbf{A}$) in order to control the crossed term $\sqrt{h}\sqrt{\lambda + h\rho_h}a_h$ in the right hand side of (6.23).

6.3.3. *Quasimode on Ω and estimation of the remainders.* We now define a quasimode for $q_h[\mathbf{A},\Omega]$. Let us note that for h small enough, ψ_h^τ is compactly supported in $\mathcal{V}_{\mathbf{x}_0}$. Therefore we can define $f_h \in H^1(\Omega)$ by

$$f_h := \psi_h^\tau \circ \mathsf{U}_{\mathbf{x}_0} \ .$$

Combining (6.28) with Lemma 6.3 for $r = h^\delta$ we get

$$\frac{q_h[\mathbf{A},\Omega](f_h)}{\|f_h\|^2} \leq \Big(h\lambda + C(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2-2\delta} + h^{2\delta+1/2} + h^{4\delta})\Big)\,(1 + C(\Omega)h^\delta) \ .$$

Therefore there exists a constant $C(\Omega) > 0$ such that

$$\frac{q_h[\mathbf{A},\Omega](f_h)}{\|f_h\|^2} \leq h\lambda + C(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2-2\delta} + h^{2\delta+1/2} + h^{4\delta} + h^{\delta+1}) \,. \tag{6.29}$$

We optimize this upper bound by taking $\delta = \frac{3}{8}$, which provides immediately estimate (6.10). The min-max principle then yields Theorem 6.1.

6.4. **Improvement for a straight polyhedron with constant magnetic field.** In this subsection we improve Theorem 6.1 for a straight polyhedral domain with constant magnetic field. Since there is no curvature, we expect smaller remainders in the asymptotics of $\lambda_h(\mathbf{B},\Omega)$. Moreover, in that case, we will see that the function $\mathbf{x} \mapsto E(\mathbf{B},\Pi_\mathbf{x})$ attains its minimum at a vertex of Ω.

Theorem 6.6. *Let Ω be a straight polyhedron and $\mathbf{B}$ be a constant magnetic field with $|\mathbf{B}| = 1$. Then*

$$\mathscr{E}(\mathbf{B},\Omega) = \min_{\mathbf{v}\in\mathfrak{V}} E(\mathbf{B},\Pi_\mathbf{v})$$

where $\mathfrak{V}$ denotes the set of the vertices of Ω. We have

$$\lambda_h(\mathbf{B},\Omega) \leq h\mathscr{E}(\mathbf{B},\Omega) + Ch^2 \ .$$

If there exists $\mathbf{v} \in \mathfrak{V}$ such that $E(\mathbf{B},\Pi_\mathbf{v}) = \mathscr{E}(\mathbf{B},\Omega) < \mathscr{E}^*(\mathbf{B},\Pi_\mathbf{v})$, then there exist positive constants C, c such that

$$\lambda_h(\mathbf{B},\Omega) \leq h\mathscr{E}(\mathbf{B},\Omega) + C\mathrm{e}^{-ch^{-1/2}}.$$

Proof. Since the polyhedral domain is assumed to have straight faces and edges and the magnetic field is constant, the function $\mathbf{x} \mapsto E(\mathbf{B},\Pi_\mathbf{x})$ is constant on each stratum of $\overline{\Omega}$. Let $\mathbf{v} \in \mathfrak{V}$. We apply Theorem 4.4 and relations (4.8) and (4.18) with $\Pi = \Pi_\mathbf{v}$:

$$E(\mathbf{B},\Pi_\mathbf{v}) \leq \mathscr{E}^*(\mathbf{B},\Pi_\mathbf{v}) = \min_{\mathbf{e}\in\mathfrak{E}_\mathbf{v}} E(\mathbf{B},\Pi_\mathbf{e}),$$

with $\mathfrak{E}_{\mathbf{v}}$ the subset of $\mathfrak{E}$ such that for any $\mathbf{e} \in \mathfrak{E}_{\mathbf{v}}$, $\mathbf{v} \in \partial\mathbf{e}$ and $\Pi_{\mathbf{e}}$ the wedge associated with the edge $\mathbf{e}$. In the same way we prove for each edge $\mathbf{e} \in \mathfrak{E}$:

$$E(\mathbf{B}, \Pi_{\mathbf{e}}) \leq \min_{\mathsf{f} \in \mathfrak{F}_{\mathbf{e}}} E(\mathbf{B}, \Pi_{\mathsf{f}}) \leq 1$$

where $\mathfrak{F}_{\mathbf{e}}$ denotes the set of the faces adjacent to an edge $\mathbf{e}$. Therefore

$$\min_{\mathbf{v} \in \mathfrak{V}} E(\mathbf{B}, \Pi_{\mathbf{v}}) = \mathscr{E}(\mathbf{B}, \Omega).$$

Let $\mathbf{v}_0$ be a vertex minimizing $\mathbf{x} \mapsto E(\mathbf{B}, \Pi_{\mathbf{x}})$. Let $\Pi_{\mathbb{X}}$ be the tangent cone given by Theorem 4.4. If $E(\mathbf{B}, \Pi_{\mathbf{v}_0}) < \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{v}_0})$ then $\Pi_{\mathbb{X}} = \Pi_{\mathbf{v}_0}$ and we have an (admissible generalized) eigenfunction on $\Pi_{\mathbf{v}_0}$ associated with $E(\mathbf{B}, \Pi_{\mathbf{v}_0})$. If $E(\mathbf{B}, \Pi_{\mathbf{v}}) = \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{v}})$, then there exists a stratum $\mathbf{t}$ of Ω associated with $\Pi_{\mathbb{X}}$ such that $\Pi_{\mathbb{X}}$ is the tangent cone to any point of $\mathbf{t}$. Moreover for any point $\mathbf{x} \in \mathbf{t}$ we have $E(\mathbf{B}, \Pi_{\mathbf{x}}) < \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{x}})$ therefore there exists a generalized eigenfunction on $\Pi_{\mathbf{x}}$ associated to $E(\mathbf{B}, \Pi_{\mathbf{x}})$. In both cases we have found a point $\mathbf{x} \in \overline{\Omega}$ such that there exists a generalized eigenfunction on $\Pi_{\mathbf{x}}$ associated to $\mathscr{E}(\mathbf{B}, \Omega)$. There exists $R_{\mathbf{x}} > 0$ such that

$$\Omega \cap \mathcal{B}(0, 2R_{\mathbf{x}}) = \Pi_{\mathbf{x}} \cap \mathcal{B}(0, 2R_{\mathbf{x}}). \tag{6.30}$$

We define the quasimode ψ_h as in (6.17) with $\delta = 0$, $\boldsymbol{\tau} = 0$ and $R = R_{\mathbf{x}}$. We have $\psi_h \in H^1(\Omega)$ and $q_h[\mathbf{A}, \Pi_{\mathbf{x}}](\psi_h) = q_h[\mathbf{A}, \Omega](\psi_h)$. Using (6.22) and the fact that $\mathbf{A}$ equals its linear part, we have:

$$\frac{q_h[\mathbf{A}, \Omega](\psi_h)}{\|\psi_h\|^2} = hE(\mathbf{B}, \Pi_{\mathbf{x}}) + h^2\rho_h. \tag{6.31}$$

Applying Lemma 6.5 with χ_h as defined in (6.15), $\delta = 0$ and $R = R_{\mathbf{x}}$, we have

$$\rho_h = \begin{cases} \mathcal{O}(1) & \text{if } k < 3, \\ \mathcal{O}(\mathrm{e}^{-ch^{-1/2}}) & \text{if } k = 3. \end{cases} \tag{6.32}$$

Then, by the min-max principle and (6.31), we deduce when $k < 3$:

$$\lambda_h(\mathbf{B}, \Omega) \leq h \inf_{\mathbf{x} \in \overline{\Omega}} E(\mathbf{B}, \Pi_{\mathbf{x}}) + \mathcal{O}(h^2) = h\mathscr{E}(\mathbf{B}, \Omega) + Ch^2.$$

If there exists $\mathbf{v} \in \mathfrak{V}$ such that $E(\mathbf{B}, \Pi_{\mathbf{v}}) = \mathscr{E}(\mathbf{B}, \Omega) < \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{v}})$, we use Theorem 4.4, Proposition 4.10 and there exists an (admissible generalized) eigenfunction with $k = 3$ of $H(\mathbf{A}, \Pi_{\mathbf{v}})$ for $E(\mathbf{B}, \Pi_{\mathbf{v}})$. According to (6.31) and (6.32), we have:

$$\lambda_h(\mathbf{B}, \Omega) \leq \frac{q_h[\mathbf{A}, \Omega](\psi_h)}{\|\psi_h\|^2} \leq hE(\mathbf{B}, \Pi_{\mathbf{v}}) + C\mathrm{e}^{-ch^{-1/2}}. \tag{6.33}$$

□

7. **Lower bound for first eigenvalues**

In this section we give a general lower bound on the first eigenvalue, namely:

Theorem 7.1. *Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$ be a polyhedral domain, $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential such that the associated magnetic field $\mathbf{B}$ does not vanish on $\overline{\Omega}$. Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that*

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \geq h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{5/4} \,. \tag{7.1}$$

We recall that the quantity $\mathscr{E}(\mathbf{B}, \Omega)$ is the lowest local energy defined in (1.6).

• *Idea of the proof.* We first make a partition of the unity of $\overline{\Omega}$ such that on each element we are able to use the change of variable given in (3.1). The local energy of the associated tangent model problem with frozen magnetic field is then bounded from below by $h\mathscr{E}(\mathbf{B}, \Omega)$. As above we then estimate the remainders due to the cut-off effects, the change of variables and the linearization of the potential.

• *IMS localization.* Let $\delta \in (0, \frac{1}{2})$. For h small enough, let us denote by $(\mathbf{x}_{j,h}, r_{j,h})$ a finite set of pairs (center, radius) provided by Lemma A.6 for $\rho = h^\delta$. Relying on Lemma A.7, we choose a finite associate partition of the unity $(\chi_{j,h})_j$ with $\chi_{j,h} \in \mathscr{C}^\infty_0(\mathcal{B}(\mathbf{x}_{j,h}, 2r_{j,h}))$ satisfying

$$\sum_j \chi^2_{j,h} = 1 \quad \text{on} \quad \overline{\Omega}$$

and the uniform estimate of gradients

$$\exists C > 0, \quad \forall h \in (0, h_0), \ \forall j, \quad \|\nabla \chi_{j,h}\|^2_{L^\infty(\Omega)} \leq C h^{-2\delta} \,. \tag{7.2}$$

The IMS formula (see A.4) provides for all $f \in H^1(\Omega)$

$$q_h[\mathbf{A}, \Omega](f) = \sum_j q_h[\mathbf{A}, \Omega](\chi_{j,h} f) - h^2 \sum_j \|\nabla \chi_{j,h} f\|^2_{L^2(\Omega)}$$

and using (7.2) and the finite covering property of the covering we get $C(\Omega) > 0$ such that

$$q_h[\mathbf{A}, \Omega](f) \geq \sum_j q_h[\mathbf{A}, \Omega](f_{j,h}) - C(\Omega) h^{2-2\delta} \|f\|^2_{L^2(\Omega)} \,. \tag{7.3}$$

where $f_{j,h}$ denotes the localized function $\chi_{j,h} f$.

• *Local control of the energy.* We estimate each term $q_h[\mathbf{A}, \Omega](f_{j,h})$ appearing in (7.3). By construction $\mathrm{supp}(f_{j,h}) \subset \mathcal{U}_{\mathbf{x}_{j,h}}$. Let $\widetilde{\mathbf{A}}^{j,h}$ defined as in (6.5) with $\mathbf{x} = \mathbf{x}_{j,h}$. Lemma 6.3 applied with $r = K r_{j,h} \leq C h^\delta$ provides $C(\Omega) > 0$ such that

$$\frac{q_h[\mathbf{A}, \Omega](f_{j,h})}{\|f_{j,h}\|^2} \geq (1 - C(\Omega) h^\delta) \frac{q_h[\widetilde{\mathbf{A}}^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h})}{\|\psi_{j,h}\|^2} \tag{7.4}$$

where we have denoted $\psi_{j,h} = f_{j,h} \circ \mathrm{U}_{\mathbf{x}_{j,h}}^{-1}$. Let $\mathbf{A}_0^{j,h}$ be the linear part of $\widetilde{\mathbf{A}}^{j,h}$ at the origin of $\Pi_{\mathbf{x}_{j,h}}$. We use (A.3) with $\psi = \psi_{j,h}$ and $\mathcal{U} = \Pi_{\mathbf{x}_{j,h}}$:

$$q_h[\widetilde{\mathbf{A}}^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) = q_h[\mathbf{A}_0^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) + 2\,\mathrm{Re}\left\langle (-ih\nabla + \mathbf{A}_0^{j,h})\psi_{j,h}, (\widetilde{\mathbf{A}}^{j,h} - \mathbf{A}_0^{j,h})\psi_{j,h}\right\rangle + \|(\widetilde{\mathbf{A}}^{j,h} - \mathbf{A}_0^{j,h})\psi\|^2. \tag{7.5}$$

Therefore Using the Cauchy-Schwarz inequality:

$$q_h[\widetilde{\mathbf{A}}^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) \geq q_h[\mathbf{A}_0^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) - 2\left(q_h[\mathbf{A}_0^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h})\right)^{1/2} \|(\widetilde{\mathbf{A}}^{j,h} - \mathbf{A}_0^{j,h})\psi_{j,h}\|\,.$$

We cannot conclude like in (6.23) because we do not have any a priori upper bound on $q_h[\mathbf{A}_0^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h})$. That is why we use the parametric estimate

$$\forall \eta > 0, \quad q_h[\widetilde{\mathbf{A}}^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) \geq (1-\eta) q_h[\mathbf{A}_0^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) - \eta^{-1}\|(\widetilde{\mathbf{A}}^{j,h} - \mathbf{A}_0^{j,h})\psi_{j,h}\|^2$$

based on the simple inequality $2ab \leq \eta a^2 + \eta^{-1} b^2$. Since $\mathrm{curl}\, \mathbf{A}_0^{j,h} = \mathbf{B}_{\mathbf{x}_{j,h}}$ we have

$$q_h[\mathbf{A}_0^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) \geq hE(\mathbf{B}_{\mathbf{x}_{j,h}}, \Pi_{\mathbf{x}_{j,h}})\|\psi_{j,h}\|^2\,.$$

Moreover using (6.25) and the same arguments as in Section 6.3.2 we get

$$\|(\widetilde{\mathbf{A}}^{j,h} - \mathbf{A}_0^{j,h})\psi_{j,h}\|^2 \leq C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{4\delta}\|\psi_{j,h}\|^2\,.$$

We deduce for all $\eta > 0$:

$$q_h[\widetilde{\mathbf{A}}^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) \geq (1-\eta)hE(\mathbf{B}_{\mathbf{x}_{j,h}}, \Pi_{\mathbf{x}_{j,h}})\|\psi_{j,h}\|^2 - \eta^{-1}C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{4\delta}\|\psi_{j,h}\|^2.$$

Choosing $\eta = h^{2\delta - 1/2}$ we get

$$\begin{aligned} q_h[\widetilde{\mathbf{A}}^{j,h}, \Pi_{\mathbf{x}_{j,h}}](\psi_{j,h}) &\geq \left(hE(\mathbf{B}_{\mathbf{x}_{j,h}}, \Pi_{\mathbf{x}_{j,h}}) - C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{2\delta+1/2}\right)\|\psi_{j,h}\|^2 \\ &\geq \left(h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^{2\delta+1/2}\right)\|\psi_{j,h}\|^2. \end{aligned} \tag{7.6}$$

- *Conclusion.* Combining the previous localized estimate (7.6) with (7.4) we deduce:

$$q_h[\mathbf{A}, \Omega](f_{j,h}) \geq \left(h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta+1/2} + h^{1+\delta})\right)\|f_{j,h}\|^2.$$

Summing up in j and using that $\sum_j f_{j,h}^2 = f^2$ we obtain

$$\sum_j q_h[\mathbf{A}, \Omega](f_{j,h}) \geq \left(h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta+1/2} + h^{1+\delta})\right)\|f\|^2, \tag{7.7}$$

and combining (7.7) with (7.3) we get $C(\Omega) > 0$ such that

$$\forall f \in H^1(\Omega), \quad \frac{q_h[\mathbf{A}, \Omega](f)}{\|f\|^2} \geq h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})\left(h^{2\delta+1/2} + h^{1+\delta} + h^{2-2\delta}\right). \tag{7.8}$$

We optimize this by taking $\delta = \frac{3}{8}$ and we deduce Theorem 6.1 from the min-max principle.

Like in the last section, we have a result using only a Hölder norm of the magnetic field:

Corollary 7.2. *Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$ be a polyhedral domain, $\mathbf{B} \in W^{1+\alpha,\infty}(\overline{\Omega})$ be a non-vanishing Hölder continuous magnetic field of order $1+\alpha$ with some $\alpha \in (0,1)$. Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that*

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \geq h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)(1 + \|\mathbf{B}\|^2_{W^{1+\alpha,\infty}(\Omega)})h^{5/4} \ . \tag{7.9}$$

Appendix A. Technical lemmas

A.1. Gauge transform.

Lemma A.1. *Let φ be a regular function in Ω.*
A function ψ_h is an eigenfunction for the operator $H_h(\mathbf{A}, \Omega)$ if an only if $\mathrm{e}^{i\varphi/h}\psi_h$ is an eigenfunction for $H_h(\mathbf{A} + \nabla\varphi, \Omega)$ associated with the same eigenvalue.

Proof. It is enough to notice

$$H_h(\mathbf{A} + \nabla\varphi, \Omega)\left(\mathrm{e}^{i\frac{\varphi}{h}}\psi_h\right) \ = \ \mathrm{e}^{i\frac{\varphi}{h}}H_h(\mathbf{A}, \Omega)\psi_h.$$

□

Lemma A.2. *Let $\mathbf{B}$ be a smooth magnetic field. There exists an associated magnetic potential $\mathbf{A}$ such that* curl $\mathbf{A} = \mathbf{B}$ *and one component of $\mathbf{A} - \mathbf{A}_0$ cancels, where $\mathbf{A}_0$ denotes the linear part of $\mathbf{A}$ at $\mathbf{0}$.*

Proof. Let us pros that we can cancel the first component of $\mathbf{A} - \mathbf{A}_0$.
Let $\tilde{\mathbf{A}}$ be a magnetic potential such that curl $\tilde{\mathbf{A}} = \mathbf{B}$ and $\mathbf{A}_0$ its linear part. We can write $\tilde{\mathbf{A}} - \mathbf{A}_0 = (P_1, P_2, P_3)$ where P_i satisfy $P_i(\mathbf{0}) = \mathbf{0}$ and $(\nabla P_i)(\mathbf{0}) = \mathbf{0}$. In particular, this means the the Taylor expansion of P_i starts with coefficients of degree at least 2. Let

$$\varphi(x_1, x_2, x_3) := \int_0^{x_1} P_1(x, x_2, x_3)dx.$$

We define $\mathbf{A} := \tilde{\mathbf{A}} - \nabla\varphi$. Then $\mathbf{A}_0$ is still the linear part of $\mathbf{A}$, and the first component of $\mathbf{A} - \mathbf{A}_0$ equals zero. □

A.2. Translation.

Let us remark that if $\mathbf{B}$ is a constant magnetic field, an associate magnetic potential is given by

$$\mathbf{A}^{\mathrm{S}}(\mathbf{x}) := \frac{1}{3}\mathbf{B} \wedge \mathbf{x} \ . \tag{A.1}$$

Indeed we have

$$\mathrm{curl}\, \mathbf{A}^{\mathrm{S}} = \frac{1}{3}\nabla \wedge (\mathbf{B} \wedge \mathbf{x}) = \frac{1}{3}\left((\nabla \cdot \mathbf{x})\mathbf{B} - (\nabla \cdot \mathbf{B})\mathbf{x}\right) = \mathbf{B} \ .$$

Lemma A.3 (Translation)**.** *Let Ω be a domain and $\mathbf{B}$ be a constant magnetic field. Let $\mathbf{A}$ be a linear magnetic potential such that* curl $\mathbf{A} = \mathbf{B}$*. Let $\mathsf{d} \in \mathbb{R}^3$ be a vector and $\Omega_\mathsf{d} := \Omega + \mathsf{d}$ be the translated domain. Then $H_h(\mathbf{A}, \Omega_\mathsf{d})$ and $H_h(\mathbf{A}, \Omega)$ are unitary equivalent. For $\psi \in \mathrm{Dom}(H_h(\mathbf{A}, \Omega))$ we denote by $\psi_\mathsf{d} : \mathbf{y} \mapsto \psi(\mathbf{y} - \mathsf{d})$ the translated function. Let $\varphi(\mathbf{y}) := \mathbf{y} \cdot \mathbf{A}(\mathsf{d})$. Then ψ is an eigenfunction of $H_h(\mathbf{A}, \Omega)$ if and only if $\mathrm{e}^{i\varphi/h}\psi_\mathsf{d}$ is an eigenfunction of $H_h(\mathbf{A}, \Omega_\mathsf{d})$.*
Moreover if the magnetic potential is $\mathbf{A} = \mathbf{A}^\mathsf{S}$ (given by (A.1)), we have

$$\varphi(\mathbf{y}) = \frac{1}{3}\mathbf{y} \cdot (\mathbf{B} \wedge \mathsf{d}) = \frac{1}{3}\mathsf{d} \cdot (\mathbf{y} \wedge \mathbf{B}) = \frac{1}{3}\mathbf{B} \cdot (\mathsf{d} \wedge \mathbf{y}) \ . \tag{A.2}$$

Proof. Let $\mathbf{A}_\mathsf{d}(\mathbf{y}) := \mathbf{A}(\mathbf{y} - \mathsf{d})$ be the translated magnetic potential. We have:

$$\begin{aligned} & \psi \quad \text{is an eigenfunction of} \quad (ih\nabla + \mathbf{A})^2 \quad \text{on} \quad \Omega \\ \iff & \psi_d \quad \text{is an eigenfunction of} \quad (ih\nabla + \mathbf{A}_\mathsf{d})^2 \quad \text{on} \quad \Omega_\mathsf{d} \\ \iff & \mathrm{e}^{i\varphi/h}\psi_d \quad \text{is an eigenfunction of} \quad (ih\nabla + \mathbf{A}_\mathsf{d} + \nabla\varphi)^2 \quad \text{on} \quad \Omega_\mathsf{d} \ . \end{aligned}$$

Therefore we are led to choose $\varphi(\mathbf{y})$ such that $\nabla\varphi = \mathbf{A} - \mathbf{A}_\mathsf{d}$. Since $\mathbf{A}$ is linear we get $\nabla\varphi(\mathbf{y}) = \mathbf{A}(\mathsf{d})$ and we take $\varphi(\mathbf{y}) := \mathbf{y} \cdot \mathbf{A}(\mathsf{d})$. Moreover if $\mathbf{A}$ is given by (A.1), we get (A.2) using basic vectorial identities. □

A.3. **Comparison between two potentials.** Let $\mathbf{A}$ and $\mathbf{A}'$ be two magnetic potentials. Then, for any function ψ of $\mathrm{Dom}(q_h[\mathbf{A}, \mathcal{U}]) \cap \mathrm{Dom}(q_h[\mathbf{A}', \mathcal{U}])$, we have:

$$q_h[\mathbf{A}, \mathcal{U}](\psi) = q_h[\mathbf{A}', \mathcal{U}](\psi) + 2\,\mathrm{Re}\left\langle(-ih\nabla + \mathbf{A}')\psi, (\mathbf{A} - \mathbf{A}')\psi\right\rangle + \|(\mathbf{A} - \mathbf{A}')\psi\|^2. \tag{A.3}$$

A.4. **Cut-off effect.**

Lemma A.4 (IMS formula)**.** *Assume that $\chi_1, \dots, \chi_L \in \mathscr{C}^\infty(\overline{\Omega})$ are such that*

$$\sum_{\ell=1}^{L} \chi_\ell^2 \equiv 1 \quad \text{on} \quad \Omega.$$

Then, for any $\psi \in H^1(\Omega)$

$$q[\mathbf{A}; \Omega, h](\psi) = \sum_{\ell=1}^{L} q[\mathbf{A}; \Omega, h](\chi_\ell\psi) - h^2 \sum_{\ell=1}^{L} \|\psi\nabla\chi_\ell\|^2_{L^2(\Omega)}$$

Lemma A.5. *Let $\psi \in \mathrm{Dom}_{\mathrm{loc}}(H_h(\mathbf{A}, \Omega, \mathrm{G}))$ and $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ a real smooth function. Then*

$$q_h[\mathbf{A}, \Omega, \mathrm{G}](\chi\psi) = \mathrm{Re}\left\langle\chi H_h(\mathbf{A}, \Omega, \mathrm{G})\psi, \chi\psi\right\rangle_\Omega + h^2 \int_\Omega \nabla\chi^\perp \mathrm{G} \nabla\chi\, |\psi|^2\, |\mathrm{G}|^{-1/2} \mathrm{d}x. \tag{A.4}$$

Compare with [13, (6.11)] and [11, Proposition 7.2.1].

Proof. Let

- $|\mathsf{G}|^{-1/2}\mathsf{G} = \big(g_{jk}\big)_{jk}$ (symmetric matrix)
- $\mathsf{X}_j u = (h\mathsf{D}_{x_j} + A_j)u$
- $\mathsf{Y}_j\chi = h\mathsf{D}_{x_j}\chi$

We have the commutation relation

$$\mathsf{X}_j(\chi\psi) = \chi\,\mathsf{X}_j\psi + \psi\,\mathsf{Y}_j\chi.$$

Then

$$\begin{aligned} q_h[\mathbf{A},\Omega,\mathsf{G}](\chi\psi) &= \sum_{jk}\int_\Omega g_{jk}\ \mathsf{X}_j(\chi\psi)\ \overline{\mathsf{X}_k(\chi\psi)}\ \mathrm{d}x \\ &= \sum_{jk}\int_\Omega g_{jk}\ (\chi\,\mathsf{X}_j\psi + \psi\,\mathsf{Y}_j\chi)\ (\overline{\chi\,\mathsf{X}_k\psi + \psi\,\mathsf{Y}_k\chi})\ \mathrm{d}x \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{A.5}$$

Let us analyze each part.

For I_1, we make an integration by part

$$\begin{aligned} I_1 &= \sum_{jk}\int_\Omega g_{jk}\ \chi\,\mathsf{X}_j\psi\ \overline{\chi\,\mathsf{X}_k\psi}\ \mathrm{d}x \\ &= \sum_{jk}\int_\Omega g_{jk}\ \chi^2\,\mathsf{X}_j\psi\ \overline{\mathsf{X}_k\psi}\ \mathrm{d}x \\ &\sum_{jk}\int_\Omega \mathsf{X}_k\big(g_{jk}\ \chi^2\,\mathsf{X}_j\psi\big)\ \overline{\psi}\ \mathrm{d}x + ih\sum_{jk}\int_{\partial\Omega} n_k\ g_{jk}\ \chi^2\,\mathsf{X}_j\psi\ \overline{\psi}\ \mathrm{d}x \end{aligned}$$

Using the boundary conditions of ψ, the sum of the contributions of the boundary terms cancels. In the first term, we have to commute χ^2 with X_k. Then

$$\begin{aligned} I_1 &= \sum_{jk}\int_\Omega \chi^2\,\mathsf{X}_k\big(g_{jk}\ \mathsf{X}_j\psi\big)\ \overline{\psi}\ \mathrm{d}x + \sum_{jk}\int_{\mathscr{O}} 2\chi\ \mathsf{Y}_k\chi\ g_{jk}\ \mathsf{X}_j\psi\ \overline{\psi}\ \mathrm{d}x \\ &= I_{1,1} + I_{1,2}. \end{aligned} \tag{A.6}$$

We write I_2

$$I_2 = \sum_{jk}\int_\Omega g_{jk}\ (\chi\,\mathsf{X}_j\psi)\ (\overline{\psi\,\mathsf{Y}_k\chi}) + g_{jk}\ (\psi\,\mathsf{Y}_j\chi)\ (\overline{\chi\,\mathsf{X}_k\psi})\ \mathrm{d}x$$

We combine this with the second member $I_{1,2}$ of I_1 and we use that χ is real and that $\mathsf{Y}_k\chi = -ih\partial_k\chi$:

$$\begin{aligned} I_{1,2}+I_2 &= -ih\sum_{jk}\int_\Omega 2\, g_{jk}\,\chi\,\partial_k\chi\,\mathsf{X}_j\psi\,\overline{\psi} - g_{jk}\,\chi\,\partial_k\chi\,\mathsf{X}_j\psi\,\overline{\psi} + g_{jk}\,\chi\,\partial_j\chi\,\psi\,\overline{\mathsf{X}_k\psi}\,\mathrm{d}x \\ &= -ih\sum_{jk}\int_\Omega g_{jk}\,\chi\,\partial_k\chi\,\mathsf{X}_j\psi\,\overline{\psi} + g_{jk}\,\chi\,\partial_j\chi\,\psi\,\overline{\mathsf{X}_k\psi}\,\mathrm{d}x \end{aligned}$$

Since $g_{jk} = g_{kj}$,

$$\sum_{jk} g_{jk}\,\chi\,\partial_k\chi\,\mathsf{X}_j\psi\,\overline{\psi} + g_{jk}\,\chi\,\partial_j\chi\,\psi\,\overline{\mathsf{X}_k\psi}$$

is a real function. We deduce

$$\mathrm{Re}(I_{1,2}+I_2) = 0. \tag{A.7}$$

Therefore

$$\begin{aligned} I_3 &= \sum_{jk}\int_\Omega g_{jk}\,\psi\,\mathsf{Y}_j\chi\,\overline{\psi\,\mathsf{Y}_k\chi}\,\mathrm{d}x \\ &= h^2\sum_{jk}\int_\Omega g_{jk}\,\partial_j\chi\,\partial_k\chi\,\psi\,\overline{\psi}\,\mathrm{d}x \end{aligned} \tag{A.8}$$

We deduce (A.4) from (A.5)-(A.8). □

A.5. **IMS partition.**

Lemma A.6. *Let $n \geq 1$ be the space dimension. M denotes $\mathbb{R}^n$ or $\mathbb{S}^n$. Let $\Omega \in \mathfrak{D}(M)$ and $K > 1$. There exist a positive integer L_Ω and two positive constants $\rho_\Omega \leq 1$, and $\kappa_\Omega \leq 1$ (depending also on K) such that for all $\rho \in (0, \rho_\Omega]$, there exists a (finite) set $\mathscr{X} \subset \overline{\Omega} \times [\kappa_\Omega\rho, \rho]$ satisfying the following four properties*

1. *We have the inclusion $\overline{\Omega} \subset \cup_{(\mathbf{x},r)\in\mathscr{X}} \mathcal{B}(\mathbf{x}, r)$*
2. *For any $(\mathbf{x}, r) \in \mathscr{X}$, the ball $\mathcal{B}(\mathbf{x}, Kr)$ is included in a map-neighborhood $\mathcal{U}_\mathbf{x}$ associated with $\mathbf{x}$, cf. (3.1)*
3. *Each point $\mathbf{x}_0$ of $\overline{\Omega}$ belongs to at most L_Ω different balls $\mathcal{B}(\mathbf{x}, Kr)$.*

Proof. The principle of the proof is a recursion on the dimension n.

The first step is an explicit construction when $d = 1$: The domain Ω is then an interval.

The second step consists in deducing a "local" result on a cone $\Gamma \in \mathfrak{P}_d$ or a wedge $\mathbb{R}^{n-d} \times \Gamma$ from the global result on the section $\Omega_0 \in \mathbb{S}^{d-1}$ of Γ.

The third step consists in applying a local diffeomorphism.

The fourth step consists in gluing together several local partitions to solve the case of Ω. This latter step simply amounts to merge the local coverings. □

Lemma A.7. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and, with the notations of Lemma A.6, $\rho \in (0, \rho_\Omega]$. Let $\mathscr{Z} \subset \overline{\Omega} \times [\kappa_\Omega \rho, \rho]$ be the associate set of pairs (center, radius). Then there exists a collection of smooth functions $(\chi_{(\mathbf{x},r)})_{(\mathbf{x},r)\in\mathscr{Z}}$ with $\chi_{(\mathbf{x},r)} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{x}, 2r))$ satisfying the identity (partition of unity)*

$$\sum_{(\mathbf{x},r)\in\mathscr{Z}} \chi^2_{(\mathbf{x},r)} = 1 \quad \text{on} \quad \overline{\Omega}$$

and the uniform estimate of gradients

$$\exists C > 0, \quad \forall (\mathbf{x}, r) \in \mathscr{Z}, \quad \|\nabla \chi_{(\mathbf{x},r)}\|_{L^\infty(\Omega)} \leq C\rho^{-1}\,,$$

where C only depends on Ω.

Proof. Let $\xi_{(\mathbf{x},r)} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{x}, 2r))$, with the property that $\xi_{(\mathbf{x},r)} \equiv 1$ in $\mathcal{B}(\mathbf{x}, r)$, and satisfying

$$\|\nabla \xi_{(\mathbf{x},r)}\|_{L^\infty(\mathbb{R}^3)} \leq C r^{-1}$$

where C is a universal constant. Then we set for each $(\mathbf{x}_0, r_0) \in \mathscr{Z}$

$$\chi_{(\mathbf{x}_0,r_0)} = \frac{\xi_{(\mathbf{x}_0,r_0)}}{(\sum_{(\mathbf{x},r)\in\mathscr{Z}} \xi^2_{(\mathbf{x},r)})^{1/2}}\,.$$

Due to property (1) in Lemma A.6, $\sum_{(\mathbf{x},r)\in\mathscr{Z}} \xi^2_{(\mathbf{x},r)} \geq 1$ and due to property (3),

$$\| \sum_{(\mathbf{x},r)\in\mathscr{Z}} \nabla \xi^2_{(\mathbf{x},r)}\|_{L^\infty(\mathbb{R}^3)} \leq C L_\Omega\,.$$

We deduce the lemma. □

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