# A pre-Lie algebra associated to a linear endomorphism and related algebraic structures 

Loïc Foissy<br>Fédération de Recherche Mathématique du Nord Pas de Calais FR 2956<br>Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville<br>Université du Littoral Côte dOpale-Centre Universitaire de la Mi-Voix 50, rue Ferdinand Buisson, CS 80699, 62228 Calais Cedex, France<br>email: foissy@lmpa.univ-littoral.fr


#### Abstract

We construct a functor $T$ from the category of endomorphisms of vector spaces to the category of Com-Pre-Lie algebras. For any endomorphism $f$ of a vector space $V$, we describe the enveloping algebra of the pre-Lie algebra $T(V, f)$, the dual Hopf algebra and the associated group of characters. For $f=I d_{V}$, we find the algebra of formal diffeomorphisms, seen as a subalgebra of the Connes-Kreimer Hopf algebra of rooted trees in the context of QFT; for other well-chosen nilpotent $f$, we obtain the groups of Fliess operators in Control Theory.

An algebraic study of these Com-Pre-Lie Hopf algebras is carried out: gradations, generation, subobject generated by $V$, etc.


AMS CLASSIFICATION. 16T05, 17B60, 93B25, 05C05.
KEYWORDS. Prelie algebras; Com-Pre-Lie Hopf algebras; Fliess operators; Faà di Bruno Hopf algebra.

## Contents

1 Algebraic structures associated to a linear endomorphism ..... 3
1.1 Construction of $T(V, f)$ ..... 4
1.2 Functoriality ..... 9
1.3 Gradations of $T(V, f)$ ..... 10
1.4 Enveloping algebra of $T(V, f)$ ..... 11
1.5 Dual Hopf algebra ..... 13
1.6 Group of characters ..... 15
2 From free Com-Pre-Lie algebras to $T(V, f)$ ..... 16
2.1 Free Com-Pre-Lie algebras ..... 16
2.2 A generic case ..... 17
2.3 Generation of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ ..... 17
2.4 A morphism from partitioned trees to words in biletters ..... 19
2.5 Freeness of $\mathfrak{g}_{\mathcal{D}}$ ..... 21
2.6 Prelie subalgebra generated by $V$ ..... 25
3 Examples ..... 27
3.1 The diagonalizable case ..... 27
3.2 Examples from Control theory ..... 31
4 Appendix ..... 34
4.1 Admissible words and Dyck paths ..... 34
4.2 Image of the morphism $\Phi_{C P L}$ ..... 36

## Introduction

The Hopf algebra $H$ of coordinates of the group of Fliess operators $G$, used to study the feedback in Control Theory, is described in [11]. This Hopf algebra is studied in the one-dimensional case in [10], and it is shown it is a right-sided Hopf algebra in the sense of [15]. Consequently, its dual Lie algebra $\mathfrak{g}$, which is the Lie algebra of $G$, inherits a right pre-Lie algebra structure. It is proved that $\mathfrak{g}$ is in fact a Hopf Com-Pre-Lie algebra, that is to say:

1. $\mathfrak{g}$ has a commutative product $\amalg$ and a coproduct $\Delta$, making it a commutative Hopf algebra.
2. $\mathfrak{g}$ has a nonassociative product $\bullet$ satisfying the (right) pre-Lie axiom: for all $a, b, c \in \mathfrak{g}$,

$$
a \bullet(b \bullet c)-(a \bullet b) \bullet c=a \bullet(c \bullet b)-(a \bullet c) \bullet b
$$

The Lie bracket of $\mathfrak{g}$ is the antisymmetrization of $\bullet$.
3. For all $a, b, c \in \mathfrak{g},(a \bullet b) \bullet c-a \bullet(b \bullet c)=(a \bullet c) \bullet b-a \bullet(c \bullet b)$.
4. For all $a, b \in T(V), \Delta(a \bullet b)=a^{(1)} \otimes a^{(2)} \bullet b+a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \boldsymbol{m} b^{(2)}$, with Sweedler's notations.

Our aim in this text is to give a generalization of the construction of $\mathfrak{g}$ and its relative $H$ and $G$, and to study some general properties of this construction. Let us take any linear endomorphism $f$ of a vector space $V$. We inductively define a pre-Lie product • on the shuffle Hopf algebra $(T(V), \boldsymbol{\omega}, \Delta)$, making it a Com-Pre-Lie Hopf algebra denoted by $T(V, f)$ (definition 1 and theorem 2). For example, if $x_{1}, x_{2}, x_{3} \in V$ and $w \in T(V)$ :

$$
\begin{aligned}
x_{1} \bullet w & =f\left(x_{1}\right) w, \\
x_{1} x_{2} \bullet w & =x_{1} f\left(x_{2}\right) w+f\left(x_{1}\right)\left(x_{2} Ш w\right), \\
x_{1} x_{2} x_{3} \bullet w & =x_{1} x_{2} f\left(x_{3}\right) w+x_{1} f\left(x_{2}\right)\left(x_{3} Ш w\right)+f\left(x_{1}\right)\left(x_{2} x_{3} Ш w\right) .
\end{aligned}
$$

Here are some examples:

1. Take $V$ be one-dimensional and $f=I d$. Taking a basis $(x)$ of $V$, we obtain a pre-Lie product on $\mathbb{K}[x]$ given by:

$$
x^{i} \bullet x^{j}=\binom{i+j}{i-1} x^{i+j} .
$$

We shall prove that $\mathbb{K}[x]$ admits a basis $\left(y_{n}\right)_{n \geq 1}$ such that for all $i, j \geq 0$ :

$$
\left[y_{i}, y_{j}\right]=(i-j) y_{i+j} .
$$

Hence, $\mathbb{K}[x]_{+}=\operatorname{Vect}\left(y_{i} \mid i \geq 1\right)$ is isomorphic, as a Lie algebra, to the Faà di Bruno Lie algebra $[7,8,9]$. This case is studied in section 3.1.
2. Take $V=\operatorname{Vect}\left(x_{0}, x_{1}\right)$ and $f: V \longrightarrow V$, with matrix in the basis $\left(x_{0}, x_{1}\right)$ given by:

$$
\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) .
$$

The Com-Pre-Lie Hopf algebra $T(V, f)$ is precisely the one obtained from the group of Fliess operators in dimension 1. In dimension $n \geq 2$, the group of Fliess operators admits a decomposition into a direct product of $n$ subgroups $G_{1}, \ldots, G_{n}$. The group $G_{i}$ is obtained from the Com-Pre-Lie Hopf algebra $T\left(V_{i}, f_{i}\right)$, where $V_{i}=V e c t\left(x_{0}, \ldots, x_{n}\right)$, and $f_{i}$ is defined by $f_{i}\left(v_{j}\right)=\delta_{i, j} v_{0}$. These cases are studied in section 3.2.

For any linear endomorphism $f$, the Com-Pre-Lie Hopf algebra $T(V, f)$ is graded by the length of words. We study the existence of other gradations of $T(V)$ as a pre-Lie algebra in proposition 10. This holds for the Prelie algebra of Fliess operators. In this case, the dimension of the homogeneous parts of the gradation are given by the Fibonacci polynomials (theorem 40), and consequently $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$ is a minimal space of generators of $T\left(V_{i}, f_{i}\right)$. In the one-dimensional case, we get in this way a result which was already proved in [10], where a presentation of the pre-Lie algebra of Fliess operators in dimension one is described.

According to [18, 19], we can give a description of the enveloping algebra $\mathcal{U}(T(V, f))$ in theorems 12 and 13. We describe the dual Hopf algebra in theorem 14 and its group of characters in theorem 15; this group is in a certain sense the exponentiation of the lie algebra $T(V, f)$. Because of the usual problems of duality of a infinite-dimensional Hopf algebra, we have to restrict here to the case where $f$ is locally nilpotent. This holds in the Fliess operators case, and we indeed recover the group of Fliess operators in this way.

Using a generic case, we give systems of generators of $T(V, f)$ as a pre-Lie algebra and as a Com-Pre-Lie algebra. We also study the pre-Lie and Com-Pre-Lie subalgebra of $T(V, f)$ generated by a subspace $W$ in the next section. We use for this a description of free Com-Pre-Lie algebras in terms of partitioned trees [10]. First, the generic case proves that the pre-Lie subalgebra generated by $V$ can be free (theorem 30 and its corollary 31); we give in theorem 32 and corollary 33 equivalent conditions for $T_{+}(V, f)$ to be generated as a pre-Lie or as a Com-Pre-Lie algebra by a subspace of $V$ in terms of the codimension of $\operatorname{Im}(f)$. Applied to the Faà di Bruno case, this gives by duality an injection of the Faà di Bruno Hopf algebra into the Connes-Kreimer Hopf algebra of rooted trees, and we get in this way the subalgebra of formal diffeomorphisms defined in [3, 4].

This paper is organized in the following way. The first sections groups all the algebraic structure associated to the linear endomorphism $V$ : the Com-Pre-Lie algebra $T(V, f)$, its enveloping algebra, the dual Hopf algebra and the associated group of characters. We also study the gradations of $T(V, f)$ here. Generation by $V$ and morphisms from free Com-Pre-Lie algebras to $T(V, f)$ are studied in the second section; it uses the notion of admissible words, whose study is reported to the appendix. The last section deals with the two main examples, the diagonalizable case, which leads to the Faà di Bruno Lie algebra, and the Fliess operators cases.

Notation. We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (algebra, coalgebras, pre-Lie algebras...) in this text will be taken over $\mathbb{K}$.

Aknowledgment. The research leading these results was partially supported by the French National Research Agency under the reference ANR-12-BS01-0017.

## 1 Algebraic structures associated to a linear endomorphism

## Notations.

1. Let $V$ be a $\mathbb{K}$-vector field. We denote by $T(V)$ the free associative, unitary algebra gener-
ated over $\mathbb{K}$ by $V$ :

$$
T(V)=\bigoplus_{n \geq 0} V^{\otimes n}
$$

As a vector space, it is generated by the set of words with letters in $V$; the unit is the empty word $\emptyset$ and the product is the concatenation of words. The augmentation ideal of $T(V)$ is:

$$
T_{+}(V)=\bigoplus_{n \geq 1} V^{\otimes n}
$$

2. We denote by $S h(k, l)$ the set of $(k, l)$-shuffles, that is to say permutations $\sigma \in \mathfrak{S}_{k+l}$ such that $\sigma(1)<\ldots<\sigma(k)$ and $\sigma(k+1)<\ldots<\sigma(k+l)$.
3. $T(V)$ is a commutative dendriform algebra (or Zinbiel algebra $[6,16,13,14,5]$ ), with the half-shuffle product $\prec$ : if $k, l \geq 1$ and if $x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l} \in V$,

$$
x_{1} \ldots x_{k} \prec y_{1} \ldots y_{l}=\sum_{\sigma \in \operatorname{Sh}(k, l), \sigma^{-1}(1)=1} \sigma \cdot x_{1} \ldots x_{k} y_{1} \ldots y_{l}
$$

the $n$-th symmetric group acts on words of length $n$ is by permutation of the letters. By convention, if $w$ is a nonempty word, $\emptyset \prec w=0$ and $w \prec \emptyset=w$; note that $\emptyset \prec \emptyset$ is not defined.
4. The induced commutative, associative product is the usual shuffle product $\boldsymbol{\amalg}$ [21]:
$x_{1} \ldots x_{k} Ш y_{1} \ldots y_{l}=x_{1} \ldots x_{k} \prec y_{1} \ldots y_{l}+y_{1} \ldots y_{l} \prec x_{1} \ldots x_{k}=\sum_{\sigma \in \operatorname{Sh}(k, l)} \sigma . x_{1} \ldots x_{k} y_{1} \ldots y_{l}$.
For all $x, y \in V, u, v \in T(V)$ :

$$
x u \prec v=x(u Ш v), \quad x u Ш y v=x(u Ш y v)+y(x u Ш v) .
$$

Examples. If $a, b, c, d \in V$ :

$$
\begin{array}{ll}
a \prec b c d=a b c d, & a Ш b c d=a b c d+b a c d+b c a d+b c d a, \\
a b \prec c d=a b c d+a c b d+a c d b, & \\
a b Ш c d=a b c d+a c b d+a c d b+c a b d+c a d b+c d a b, \\
a b c \prec d=a b c d+a b d c+a d b c, & \\
a b c Ш d=a b c d+a b d c+a d b c+d a b c .
\end{array}
$$

### 1.1 Construction of $T(V, f)$

We now fix a vector space $V$ and a linear endomorphism $f: V \longrightarrow V$.
Definition 1 We define a bilinear product • on $T(V)$ by induction on the length of words in the following way: for all $x \in V, w, w^{\prime} \in T(V)$,

$$
\left\{\begin{array}{l}
\emptyset \bullet w^{\prime}=0 \\
x w \bullet w^{\prime}=x\left(w \bullet w^{\prime}\right)+f(x)\left(w 山 w^{\prime}\right) .
\end{array}\right.
$$

Examples. If $x_{1}, x_{2}, x_{3} \in V, w \in T(V)$ :

$$
\begin{aligned}
x_{1} \bullet w & =f\left(x_{1}\right) w, \\
x_{1} x_{2} \bullet w & =x_{1} f\left(x_{2}\right) w+f\left(x_{1}\right)\left(x_{2} Ш w\right), \\
x_{1} x_{2} x_{3} \bullet w & =x_{1} x_{2} f\left(x_{3}\right) w+x_{1} f\left(x_{2}\right)\left(x_{3} Ш w\right)+f\left(x_{1}\right)\left(x_{2} x_{3} Ш w\right) .
\end{aligned}
$$

Theorem 2 1. For all $a, b, c \in T(V),(a Ш b) \bullet c=(a \bullet c) Ш b+a Ш(b \bullet c)$.
2. For all words $a, b, c \in A$ such that $(a, b) \neq(\emptyset, \emptyset),(a \prec b) \bullet c=(a \bullet c) \prec b+a \prec(b \bullet c)$.
3. For all $a, b, c \in T(V),(a \bullet b) \bullet c-a \bullet(b \bullet c)=(a \bullet c) \bullet b-a \bullet(c \bullet b)$.
4. For all $a, b \in T(V), \Delta(a \bullet b)=a^{(1)} \otimes a^{(2)} \bullet b+a^{(1)} \bullet b^{(1)} \otimes a^{(2)} Ш b^{(2)}$.

By points 1 and 3, $(T(V), Ш, \bullet)$ is a Com-Pre-Lie algebra [17]. This structure is denoted by $T(V, f)$.

Proof. 1. We assume that $a, b, c$ are words. If $a=\emptyset$, then:

$$
(\emptyset Ш b) \bullet c=b \bullet c=(\emptyset \bullet c) Ш b+\emptyset Ш(b \bullet c) .
$$

If $b=\emptyset$, then $(a \amalg \emptyset) \bullet c=a \bullet c=(a \bullet c) Ш \emptyset+a Ш(\emptyset \bullet c)$. We now assume that $a \neq \emptyset$ and $b \neq \emptyset$. We prove the result by induction on $n=\lg (a)+\lg (b)$. This is obvious if $n \leq 1$. Let us assume the result at rank $n-1$. We put $a=x u$ and $b=y v$. Then:

$$
\begin{aligned}
(a Ш b) \bullet c & =(x(u Ш y v)) \bullet c+(y(x u Ш v)) \bullet c \\
& =x((u Ш y v) \bullet c)+f(x)(u Ш y v Ш c)+y((x u Ш v) \bullet c)+f(y)(x u Ш v Ш c) \\
& =x((u \bullet c) Ш y v)+x(u Ш(y v \bullet c))+y((x u \bullet c) Ш b)+y(x u Ш(v \bullet c)) \\
& +f(x)(u Ш y v Ш c)+f(y)(x u Ш v Ш c) \\
& =x((u \bullet c) Ш y v)+x(u Ш y(v \bullet c))+x(u Ш f(y)(v Ш c))+y(x(u \bullet c) Ш b) \\
& +y(f(x)(u Ш c) Ш b)+y(x u Ш(v \bullet c))+f(x)(u Ш y v Ш c)+f(y)(x u Ш v Ш c) \\
& =x((u \bullet c) Ш y v)+y(x(u \bullet c) Ш b)+x(u Ш y(v \bullet c))+y(x u Ш(v \bullet c)) \\
& +x(u Ш f(y)(v Ш c))+f(y)(x u Ш v Ш c)+y(f(x)(u Ш c) Ш b)+f(x)(u Ш y v Ш c) \\
& =(x(u \bullet c)) Ш y v+x u Ш y(v \bullet c)+x u Ш f(y)(v Ш c)+f(x)(u Ш c) Ш y v \\
& =((x u) \bullet c) Ш y v+x u Ш(y v \bullet c) \\
& =(a \bullet c) Ш b+a Ш(b \bullet c) .
\end{aligned}
$$

2. If $a=\emptyset,(\emptyset \prec b) \bullet c=0=(\emptyset \bullet b) \prec c+\emptyset \prec(b \bullet c)$. If $a=x u$, then:

$$
\begin{aligned}
(a \prec b) \bullet c & =(x(u Ш b)) \bullet c \\
& =x((u Ш b) \bullet c)+f(x)(u Ш b Ш c) \\
& =x((u \bullet c) Ш b)+x(u Ш(b \bullet c))+f(x)(u Ш b Ш c) \\
& =x((u \bullet c) Ш b)+f(x)(u Ш c Ш b)+x(u Ш(b \bullet c)) \\
& =(x(u \bullet c)+f(x)(u Ш c)) \prec b+x(u Ш(b \bullet c)) \\
& =(x u \bullet c) \prec b+x u \prec(b \bullet c) \\
& =(a \bullet c) \prec b+a \prec(b \bullet c) .
\end{aligned}
$$

3. We assume that $a, b, c$ are words and we proceed by induction on $\lg (a)$. If $a=\emptyset$ :

$$
(\emptyset \bullet b) \bullet c-\emptyset \bullet(b \bullet c)=0
$$

If $a=x u$, then:

$$
\begin{aligned}
(a \bullet b) \bullet c-a \bullet(b \bullet c) & =(x(u \bullet b)) \bullet c+(f(x)(u Ш b)) \bullet c-x(u \bullet(b \bullet c))-f(x)(u Ш(b \bullet c)) \\
& =x((u \bullet b) \bullet c)+f(x)((u \bullet b) Ш c)+f(x)((u Ш b) \bullet c)+f^{2}(x)(u Ш b Ш c) \\
& -x(u \bullet(b \bullet c))-f(x)(u Ш(b \bullet c)) \\
& =x((u \bullet b) \bullet c-u \bullet(b \bullet c)) \\
& +f(x)((u \bullet b) Ш c+(u \bullet c) Ш b)+f^{2}(x)(u Ш b Ш c) .
\end{aligned}
$$

The second and third terms are obviously symmetric in $b, c$. By the induction hypothesis applied to $u$, the first term is also symmetric in $b, c$, so the pre-Lie relation is satisfied for $a, b, c$.
4. Let us assume that $a$ and $b$ are words. We proceed by induction on the length of $a$. If $a=\emptyset:$

$$
\Delta(\emptyset \bullet b)=\emptyset \otimes \emptyset \bullet b+\emptyset \bullet b^{(1)} \otimes \emptyset \amalg b^{(2)}=0 .
$$

If the length of $a$ is $\geq 1$, we put $a=x u$. Applying the induction hypothesis to $u$ :

$$
\begin{aligned}
\Delta(a \bullet b) & =\Delta(x(u \bullet b)+f(x)(u Ш b)) \\
& =x u^{(1)} \otimes u^{(2)} \bullet b+x\left(u^{(1)} \bullet b\right) \otimes u^{(2)} Ш b^{(2)}+\emptyset \otimes x(u \bullet b) \\
& +f(x)\left(u^{(1)} Ш b^{(2)}\right) \otimes u^{(2)} Ш b^{(2)}+\emptyset \otimes f(x)(u Ш b) \\
& =x u^{(1)} \otimes u^{(2)} \bullet b+\left(x u^{(1)}\right) \bullet b^{(1)} \otimes u^{(2)} Ш b^{(2)}+\emptyset \otimes(x u) \bullet b ; \\
a^{(1)} \otimes a^{(2)} \bullet b & =x u^{(1)} \otimes u^{(2)} \bullet b+\emptyset \otimes(x u) \bullet b, \\
a^{(1)} \bullet b^{(1)} \otimes a^{(2)} Ш b^{(2)} & =\left(x u^{(1)}\right) \bullet b^{(1)} \otimes u^{(2)} Ш b^{(2)}+\emptyset \bullet b^{(1)} \otimes(x u) Ш b \\
& =\left(x u^{(1)}\right) \bullet b^{(1)} \otimes u^{(2)} Ш b^{(2)} .
\end{aligned}
$$

Hence, the result holds for all words $a, b$.

Let us now give a closed formula for the pre-Lie product of two words.

Definition 3 Let $k, l \in \mathbb{N}$. For all $\sigma \in \mathfrak{S}_{k+l}$, we put:

$$
m_{k}(\sigma)=\max \{i \in\{1, \ldots, k\} \mid \sigma(1)=1, \ldots, \sigma(i)=i\},
$$

with the convention $m_{k}(\sigma)=0$ if there is no $i \in\{1, \ldots, k\}$ such that $\sigma(i)=i$.

Proposition 4 Let $k, l \in \mathbb{N}$ and let $x_{1}, \ldots, x_{k+l} \in V$. Then:

$$
x_{1} \ldots x_{k} \bullet x_{k+1} \ldots x_{k+l}=\sum_{\sigma \in S h(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(k+l-i)}\right)\left(\sigma . x_{1} \ldots x_{k+l}\right) .
$$

Proof. By induction on $k$. If $k=0$, then $x_{1} \ldots x_{k}=\emptyset$. For all $\sigma \in S h(0, l), m_{0}(\sigma)=0$ by convention, so:

$$
\sum_{\sigma \in S h(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(k+l-i)}\right)\left(\sigma . x_{1} \ldots x_{k+l}\right)=0=\emptyset \bullet x_{k+1} \ldots x_{k+l} .
$$

Let us assume the result at rank $k-1$.

$$
\begin{aligned}
& x_{1} \ldots x_{k} \bullet x_{k+1} \ldots x_{k+l} \\
& =x_{1}\left(x_{2} \ldots x_{k} \bullet x_{k+1} \ldots x_{k+l}\right)+f\left(x_{1}\right)\left(x_{2} \ldots x_{k} Ш x_{k+1} \ldots x_{k+l}\right) \\
& =\sum_{\tau \in S h(k-1, l)}\left(\sum_{i=1}^{m_{k-1}(\tau)} I d \otimes I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(k+l)}\right)\left(x_{1} \tau .\left(x_{2} \ldots x_{k+l}\right)\right) \\
& +\sum_{\tau \in S h(k-1, l)} f \otimes I d^{k+l-1}\left(x_{1} \tau \cdot\left(x_{2} \ldots x_{k+l}\right)\right) \\
& =\sum_{\sigma \in S h(k, l), \sigma(1)=1}\left(\sum_{i=2}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(k+l)}\right)\left(\sigma .\left(x_{1} \ldots x_{k+l}\right)\right) \\
& +\sum_{\sigma \in S h(k, l), \sigma(1)=1} f \otimes I d^{k+l-1}\left(\sigma \cdot\left(x_{1} \ldots x_{k+l}\right)\right) \\
& =\sum_{\sigma \in S h(k, l), \sigma(1)=1}\left(\sum_{i=1}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(k+l)}\right)\left(\sigma \cdot\left(x_{1} \ldots x_{k+l}\right)\right) \\
& =\sum_{\sigma \in S h(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(k+l)}\right)\left(\sigma \cdot\left(x_{1} \ldots x_{k+l}\right)\right),
\end{aligned}
$$

so the result holds for all $k$.

Remark. In particular, if $x_{1}, \ldots, x_{k} \in V$ :

$$
x_{1} \ldots x_{k} \bullet \emptyset=\sum_{i=1}^{k} x_{1} \ldots x_{i-1} f\left(x_{i}\right) x_{i+1} \ldots x_{k}
$$

The pre-Lie product • is generally non associative, as proved in the following proposition:
Proposition 5 The following conditions are equivalent:

1.     - is trivial.
2.     - is associative.
3. $f=0$.

Proof. 1. $\Longrightarrow 2$. Obvious.
2. $\Longrightarrow 3$. Let $x \in V$. Then:

$$
(x \bullet \emptyset) \bullet \emptyset-x \bullet(\emptyset \bullet \emptyset)=f(x) \bullet \emptyset-0=f^{2}(x)
$$

so $f^{2}=0$. Moreover:

$$
\begin{aligned}
(x x \bullet x) \bullet \emptyset-x x \bullet(x \bullet \emptyset) & =(2 f(x) x x+x f(x) x) \bullet \emptyset-x x \bullet f(x) \\
& =2 f^{2}(x) x x+2 f(x) f(x) x+2 f(x) x f(x)+f(x) f(x) x \\
& +x f^{2}(x) x+x f(x) f(x)-f(x) f(x) x-f(x) x f(x)-x f(x) f(x) \\
& =2 f(x) f(x) x+f(x) x f(x) .
\end{aligned}
$$

Let us assume that $f(x) \neq 0$. There exists $g \in V^{*}$, such that $g(f(x))=1$.

$$
(g \otimes I d \otimes g)((x x \bullet x) \bullet \emptyset-x x \bullet(x \bullet \emptyset))=2 f(x) g(x)+x=0
$$

so there exists $\lambda \in \mathbb{K}$ such that $x=\lambda f(x)$. Then $f(x)=\lambda f^{2}(x)=0$ : this is a contradiction. Finally, $f(x)=0$.
$3 . \Longrightarrow 1$. This is a direct consequence of proposition 4 .
Corollary 6 We have:

$$
T(V, f) \bullet T(V, f)=\bigoplus_{n \geq 1}\left(\sum_{i=1}^{n} T(V, f)^{\otimes(i-1)} \otimes \operatorname{Im}(f) \otimes T(V, f)^{\otimes(n-i)}\right) .
$$

Consequently, $T(V, f) \bullet T(V, f) \subseteq T_{+}(V, f)$.
Proof. We put:

$$
W=\bigoplus_{n \geq 1}\left(\sum_{i=1}^{n} T(V, f)^{\otimes(i-1)} \otimes \operatorname{Im}(f) \otimes T(V, f)^{\otimes(n-i)}\right) .
$$

Proposition 4 directly implies that $T(V, f) \bullet T(V, f) \subseteq W$. Let $w=x_{1} \ldots x_{i-1} f\left(x_{i}\right) x_{i+1} \ldots x_{n} \in$ $T(V, f)^{\otimes(i-1)} \otimes \operatorname{Im}(f) \otimes T(V, f)^{\otimes(n-i)}$, with $1 \leq i \leq n$. Let us prove that $w \in T(V, f) \bullet T(V, f)$ by induction on $i$. If $i=1$, then:

$$
x_{1} \bullet x_{2} \ldots x_{n}=f\left(x_{1}\right) x_{2} \ldots x_{n}+0 \in T(V, f) \bullet T(V, f) .
$$

Let us assume the result at all rank $j<i$. So:

$$
\begin{aligned}
& x_{1} \ldots x_{i} \bullet x_{i+1} \ldots x_{n}-x_{1} \ldots x_{i-1} f\left(x_{i}\right) x_{i+1} \ldots x_{n} \\
& =\sum_{\sigma \in S h(i, n-i), \sigma \neq I d}\left(\sum_{j=1}^{m_{i}(\sigma)} I d^{\otimes(j-1)} \otimes f \otimes I d^{\otimes(n-j)}\right)\left(\sigma . x_{1} \ldots x_{n}\right) \\
& +\left(\sum_{j=1}^{i} I d^{\otimes(j-1)} \otimes f \otimes I d^{\otimes(n-j)}\right)\left(x_{1} \ldots x_{n}\right)-x_{1} \ldots x_{i-1} f\left(x_{i}\right) x_{i+1} \ldots x_{n} \\
& =\sum_{\sigma \in S h(i, n-i), \sigma \neq I d}\left(\sum_{j=1}^{m_{i}(\sigma)} I d^{\otimes(j-1)} \otimes f \otimes I d^{\otimes(n-j)}\right)\left(\sigma . x_{1} \ldots x_{n}\right) \\
& +\left(\sum_{j=1}^{i-1} I d^{\otimes(j-1)} \otimes f \otimes I d^{\otimes(n-j)}\right)\left(\sigma . x_{1} \ldots x_{n}\right)
\end{aligned}
$$

If $\sigma \in \operatorname{Sh}(i, n-i)$ is different from the identity of $\mathfrak{S}_{n}$, then $m_{i}(\sigma)<i$. By the induction hypothesis, $x_{1} \ldots x_{i} \bullet x_{i+1} \ldots x_{n}-x_{1} \ldots x_{i-1} f\left(x_{i}\right) x_{i+1} \ldots x_{n} \in T(V, f) \bullet T(V, f)$, so $x_{1} \ldots x_{i-1} f\left(x_{i}\right) x_{i+1} \ldots x_{n} \in$ $T(V, f) \bullet T(V, f)$. Finally, $W=T(V, f) \bullet T(V, f)$.

Notice that $\mathbb{K} \emptyset$ is a trivial pre-Lie subalgebra of $T(V, f)$, and $T_{+}(V, f)$ is a pre-Lie ideal of $T(V, f)$. Hence:

Proposition 7 As a Lie algebra, $T(V, f)=T_{+}(V, f) \rtimes \mathbb{K} \emptyset$. The right action of $\emptyset$ on $T(V, f)$ is given by:

$$
x_{1} \ldots x_{k} \bullet \emptyset=\sum_{i=1}^{k} x_{1} \ldots f\left(x_{i}\right) \ldots x_{k}
$$

Proof. It remains to compute the bracket $\left[x_{1} \ldots x_{k}, \emptyset\right]$. First:

$$
\left[x_{1} \ldots x_{k}, \emptyset\right]=x_{1} \ldots x_{k} \bullet \emptyset-\emptyset \bullet x_{1} \ldots x_{k}=x_{1} \ldots x_{k} \bullet \emptyset .
$$

The result comes then from proposition 4 .

### 1.2 Functoriality

Lemma 8 Let $W$ be a subspace of $V$. We put:

$$
I=\bigoplus_{n=1}^{\infty}\left(\sum_{i=1}^{n} V_{1}^{\otimes(i-1)} \otimes W \otimes V_{1}^{\otimes(n-i)}\right)
$$

Then I is a Com-Pre-Lie ideal of $T(V, f)$ if, and only if, $f(W) \subseteq W$. Moreover, if it is the case and if $f$ is surjective, then $I$ is generated by $W$ as a Com-Pre-Lie ideal.

Proof. $I$ is clearly an ideal for the shuffle product. Let us assume that it is an ideal for -. For all $w \in W, w \in I$, so $w \bullet \emptyset=f(w) \in I \cap V=W$ : $W$ is stable under $f$. Conversely, if $f(W) \subseteq W$, by proposition $4, I$ is an ideal for $\bullet$.

Let us assume that $f$ is surjective. We denote by $J$ the Com-Pre-Lie ideal of $T(V, f)$ generated by $W$. As $I$ is a Com-Pre-Lie ideal of $T(V, f)$ which contains $W, J \subseteq I$. Let $w=x_{1} \ldots x_{n} \in$ $V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)}$, and let us prove that $w \in J$ by induction on $n$. If $n=1$, then $w \in W \subseteq J$. Let us assume the result at rank $n-1$. If $i \geq 2$, then $x_{2} \ldots x_{n} \in J$. As $f$ is surjective, there exists $y_{1} \in V$, such that $f_{1}\left(y_{1}\right)=x_{1}$. Then:

$$
y_{1} \bullet x_{2} \ldots x_{n}=f\left(y_{1}\right) x_{2} \ldots x_{n}=w \in J .
$$

If $i=1$, then $x_{1} \in J$ and:

$$
x_{1} Ш x_{2} \ldots x_{n}=x_{1} \ldots x_{n}+\sum_{i=2}^{n} x_{2} \ldots x_{i} x_{1} x_{i+1} \ldots x_{n} \in J .
$$

We already proved that $x_{2} \ldots x_{i} x_{1} x_{i+1} \ldots x_{n} \in J$ for all $2 \leq i \leq n$, so $w \in J$. As a conclusion, $I \subseteq J$.

Proposition 9 Let $V_{1}, V_{2}$ be two vector spaces and $f_{1}: V_{1} \longrightarrow V_{1}, f_{2}: V_{2} \longrightarrow V_{2}$ be two endomorphisms. If $f: V_{1} \longrightarrow V_{2}$ satisfies $f \circ f_{1}=f_{2} \circ f$, then the following map is a morphism of Com-Pre-Lie algebras:

$$
F:\left\{\begin{aligned}
T\left(V_{1}\right) & \longrightarrow T\left(V_{2}\right) \\
x_{1} \ldots x_{n} & \longrightarrow f\left(x_{1}\right) \ldots f\left(x_{n}\right) .
\end{aligned}\right.
$$

Moreover, the image of this morphism is $T\left(\operatorname{Im}(f),\left(f_{2}\right)_{\mid \operatorname{Im}(f)}\right)$ and its kernel is:

$$
\operatorname{Ker}(F)=\bigoplus_{n=1}^{\infty}\left(\sum_{i=1}^{n} V_{1}^{\otimes(i-1)} \otimes \operatorname{Ker}(f) \otimes V_{1}^{\otimes(n-i)}\right) .
$$

If $f_{1}$ its surjective, $\operatorname{Ker}(F)$ is generated, as a Com-Pre-Lie ideal, by $\operatorname{Ker}(f)$.
Proof. The map $F$ is clearly an algebra morphism from $\left(T\left(V_{1}\right), \boldsymbol{\omega}\right)$ to $\left(T\left(V_{2}\right), \boldsymbol{w}\right)$. Let $x_{1}, \ldots, x_{k+l} \in V_{1}$. By proposition 4:

$$
\begin{aligned}
F\left(x_{1} \ldots x_{k} \bullet x_{k+1} \ldots x_{k+l}\right) & =\sum_{\sigma \in S h(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} f^{\otimes(i-1)} \otimes\left(f \circ f_{1}\right) \otimes f^{\otimes(k+l-i)}\right)\left(\sigma . x_{1} \ldots x_{k+l}\right) \\
& =\sum_{\sigma \in S h(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} f^{\otimes(i-1)} \otimes\left(f_{2} \circ f\right) \otimes f^{\otimes(k+l-i)}\right)\left(\sigma . x_{1} \ldots x_{k+l}\right) \\
& =\sum_{\sigma \in S h(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f_{2} \otimes I d^{\otimes(k+l-i)}\right)\left(\sigma . f\left(x_{1}\right) \ldots f\left(x_{k+l}\right)\right) \\
& =F\left(x_{1} \ldots x_{k}\right) \bullet F\left(x_{k+1} \ldots x_{k+l}\right) .
\end{aligned}
$$

So $F$ is a pre-Lie algebra morphism. By lemma 8 , if $f_{1}$ is surjective, then $\operatorname{Ker}(F)$ is generated by $\operatorname{Ker}(f)$.

### 1.3 Gradations of $T(V, f)$

By proposition 4, the Com-Pre-Lie algebra $T(V, f)$ is graded by the the length of words. This is not a connected gradation of the pre-Lie algebra $T(V, f)$, as $\emptyset$ is homogeneous of degree 0 . Let us now define other gradations, when $V$ itself is graded.

Proposition 10 Let us assume that $V$ admits a gradation $\left(V_{n}\right)_{n \geq 0}$ such that $f$ is homogeneous of degree $N \in \mathbb{N}$. For all $n \geq 0$, we put:

$$
T(V, f)_{n}=\bigoplus_{k=1}^{n-N} \bigoplus_{i_{1}+\ldots+i_{k}=n-N}\left(V_{i_{1}} \otimes \ldots \otimes V_{i_{k}}\right) .
$$

Then $\left(T(V, f)_{n}\right)_{n \geq 0}$ is a gradation of the pre-Lie algebra $T(V, f)$.
Proof. Let $u=x_{1} \ldots x_{k}$ and $v=y_{1} \ldots y_{l}$ be two words of $T(V, f)$ with homogeneous letters. Then $u$ is homogeneous of degree $\operatorname{deg}(u)=\operatorname{deg}\left(x_{1}\right)+\ldots+\operatorname{deg}\left(x_{k}\right)+N$ and $v$ is homogeneous of degree $\operatorname{deg}(v)=\operatorname{deg}\left(y_{1}\right)+\ldots+\operatorname{deg}\left(y_{l}\right)+N$. Let us prove that $u \bullet v$ is homogeneous of degree $\operatorname{deg}(u)+\operatorname{deg}(v)$. We proceed by induction on $k$. If $k=0$, then $u=\emptyset, u \bullet v=0$ and the result is obvious. If $k \geq 1$, then:

$$
u \bullet v=x_{1}\left(x_{2} \ldots x_{k} \bullet v\right)+f\left(x_{1}\right)\left(x_{2} \ldots x_{k} Ш v\right) .
$$

By the induction hypothesis, $x_{1} \ldots x_{k} \bullet y$ is homogeneous of degree $\operatorname{deg}\left(x_{2}\right)+\ldots+\operatorname{deg}\left(x_{k}\right)+$ $N+\operatorname{deg}(v)$, so $x_{1}\left(x_{2} \ldots x_{k} \bullet y\right)$ is homogeneous of degree $\operatorname{deg}\left(x_{1}\right)+\ldots+\operatorname{deg}\left(x_{k}\right)+N+\operatorname{deg}(v)=$ $\operatorname{deg}(u)+\operatorname{deg}(v)$. Moreover, $f\left(x_{1}\right)$ is homogeneous of degree $\operatorname{deg}\left(x_{1}\right)+N$ and $x_{2} \ldots x_{k} Ш v$ is homogeneous of degree $\operatorname{deg}\left(x_{2}\right)+\ldots+\operatorname{deg}\left(x_{k}\right)+\operatorname{deg}\left(y_{1}\right)+\ldots+\operatorname{deg}\left(y_{l}\right)+N$, so $f\left(x_{1}\right)\left(x_{2} \ldots x_{k} Ш v\right)$ is homogeneous of degree $\operatorname{deg}\left(x_{1}\right)+N+\operatorname{deg}\left(x_{2}\right)+\ldots+\operatorname{deg}\left(x_{k}\right)+\operatorname{deg}\left(y_{1}\right)+\ldots+\operatorname{deg}\left(y_{l}\right)+N=$ $\operatorname{deg}(u)+\operatorname{deg}(v)$. Finally, $u \bullet v$ is homogeneous of degree $\operatorname{deg}(u)+\operatorname{deg}(v)$.

Under the hypothesis of proposition $10, T(V, f)$ is a bigraded pre-Lie algebra, with, for all $k, n \geq 0$ :

$$
T(V, f)_{n, k}=\bigoplus_{i_{1}+\ldots+i_{k}=n-N}\left(V_{i_{1}} \otimes \ldots \otimes V_{i_{k}}\right) .
$$

If the gradation of $V$ is finite-dimensional, then the bigradation of $T(V, f)$ is finite-dimensional. We put:

$$
F_{V}(X)=\sum_{n=0}^{\infty} \operatorname{dim}\left(V_{n}\right) X^{n}, \quad F_{T(V, f)}(X, Y)=\sum_{n, k \geq 0} \operatorname{dim}\left(T(V, f)_{n, k}\right) X^{n} Y^{k}
$$

We obtain:
Corollary 11 1. Under the hypothesis of proposition 10:

$$
F_{T(V, f)}(X, Y)=\frac{X^{N}}{1-F_{V}(X) Y}
$$

2. Under the hypothesis of proposition 10, if for all $i \geq 0, V_{i}$ is finite-dimensional and $V_{0}=(0)$, then for all $n \geq 0, T(V, f)_{n}$ is finite-dimensional and:

$$
\sum_{n \geq 0} \operatorname{dim}\left(T(V, f)_{n}\right) X^{n}=\frac{X^{N}}{1-F_{V}(X)}
$$

Moreover, if $N \geq 1$, then $T(V, f)_{0}=(0)$.

Proof. 1. Indeed, putting $p_{i}=\operatorname{dim}\left(V_{i}\right)$ for all $i$ :

$$
\begin{aligned}
F_{T(V, f)}(X, Y) & =\sum_{k, n \geq 0} \sum_{i_{1}+\ldots+i_{k}=n-N} p_{i_{1}} \ldots p_{i_{k}} X^{n} Y^{k} \\
& =\sum_{k \geq 0} \sum_{i_{1}, \ldots, i_{k}} p_{i_{1}} \ldots p_{i_{k}} X^{i_{1}+\ldots+i_{k}+N} Y^{k} \\
& =X^{N} \sum_{k \geq 0} \sum_{i_{1}, \ldots, i_{k}} p_{i_{1}} X^{i_{1}} Y \ldots p_{i_{k}} X^{i_{k}} Y \\
& =\frac{X^{N}}{1-F_{V}(X) Y}
\end{aligned}
$$

2. Take $Y=1$ in the first point.

### 1.4 Enveloping algebra of $T(V, f)$

We use here the Oudom-Guin construction of the enveloping algebra on $S(T(V, f)$ ) [18, 19]. In order to avoid confusions between the product of $S(V)$ and the concatenation of words, we shall denote by $\times$ the product of $S(T(V, f))$ and by 1 its unit. We extend the pre-Lie product into a product from $S(T(V, f)) \otimes S(T(V, f))$ into $S(T(V, f))$ in the following way:

1. If $u_{1}, \ldots, u_{k} \in T(V, f),\left(u_{1} \times \ldots \times u_{k}\right) \bullet 1=u_{1} \times \ldots \times u_{k}$.
2. If $u, u_{1}, \ldots, u_{k} \in T(V, f)$ :

$$
u \bullet\left(u_{1} \times \ldots \times u_{k}\right)=\left(u \bullet\left(u_{1} \times \ldots \times u_{k-1}\right)\right) \bullet u_{k}-u \bullet\left(\left(u_{1} \times \ldots \times u_{k-1}\right) \bullet u_{k}\right)
$$

3. If $x, y, z \in S(T(V, f)),(x \times y) \bullet z=\left(x \bullet z^{(1)}\right) \times\left(y \bullet z^{(2)}\right)$, where $\Delta(z)=z^{(1)} \otimes z^{(2)}$ is the usual coproduct of $S(T(V, f))$, defined by:

$$
\Delta\left(u_{1} \times \ldots \times u_{k}\right)=\left(u_{1} \otimes 1+1 \otimes u_{1}\right) \times \ldots \times\left(u_{k} \otimes 1+1 \otimes u_{k}\right)
$$

for all $u_{1}, \ldots, u_{k} \in T(V, f)$.
In particular, if $u_{1}, \ldots, u_{k}, u \in T(V, f)$ :

$$
\left(u_{1} \times \ldots \times u_{k}\right) \bullet u=\sum_{i=1}^{k} u_{1} \times \ldots \times\left(u_{i} \bullet u\right) \times \ldots \times u_{k}
$$

Denoting by $\star$ the associative product induced on $S(T(V, f))$ by $\bullet$, for all $x, y \in S(T(V, f))$ :

$$
x \star y=\left(x \bullet y^{(1)}\right) \times y^{(2)}
$$

Notations. Let $w_{1}, \ldots, w_{k} \in T(V, f)$. For all $I=\left\{i_{1}, \ldots, i_{p}\right\} \subseteq\{1, \ldots, k\}$, we put:

$$
w_{I}=w_{i_{1}} \times \ldots \times w_{i_{p}}, \quad w_{I}^{Ш}=w_{i_{1}} Ш \ldots Ш w_{i_{p}}
$$

With these notations, the coproduct of $S(T(V, f))$ is given by:

$$
\Delta\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I \subseteq\{1, \ldots, k\}} w_{I} \otimes w_{\bar{I}}
$$

Theorem 12 Let $w_{1}, \ldots, w_{k} \in T(V, f), k \geq 0, x \in V, w \in T(V, f)$.

1. If $k \geq 1, \emptyset \bullet\left(w_{1} \times \ldots \times w_{k}\right)=0$.
2. $x w \bullet\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I \subseteq\{1, \ldots, k\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{\bar{I}}^{\omega}\right)$.

3．$\emptyset \star\left(w_{1} \times \ldots \times w_{k}\right)=\emptyset \times w_{1} \times \ldots \times w_{k}$ ．
4．$x w \star\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I \sqcup J \sqcup K=\{1, \ldots, k\}}\left(f^{|J|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{J}^{\amalg}\right)\right) \times w_{K}$ ．
Proof．1．We proceed by induction on $k$ ．If $k=1, \emptyset \bullet w_{1}=0$ ．Let us assume the result at rank $k_{1}$ ．Then：

$$
\emptyset \bullet w_{1} \times \ldots \times w_{k}=\left(\emptyset \bullet w_{1} \times \ldots \times w_{k-1}\right) \bullet w_{k}-\sum_{i=1}^{k-1} \emptyset \bullet\left(w_{1} \times \ldots \times\left(w_{i} \bullet w_{k}\right) \times \ldots \times w_{k-1}\right) .
$$

By the induction hypothesis，both terms of this sum are equal to zero．
2．If $k=0$ ，then：

$$
\sum_{I \subseteq \emptyset} f^{0-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{\bar{I}}^{\amalg}\right)=x((w \bullet 1) Ш \emptyset)=x w=x w \bullet 1 .
$$

If $k=1$ ，then：

$$
\begin{aligned}
\sum_{I \subseteq\{1\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{\bar{I}} \frac{\mathrm{~L}}{}\right) & =f(x)\left((w \bullet 1) Ш w_{1}\right)+x\left(\left(w \bullet w_{1}\right) Ш \emptyset\right) \\
& =f(x)\left(w 山 w_{1}\right)+x\left(w \bullet w_{1}\right) \\
& =x w \bullet w_{1} .
\end{aligned}
$$

Let us assume the result at rank $k-1$ ．For all $I \subseteq\{1, \ldots, k-1\}$ ，we denote by $\tilde{I}$ its complement in $\{1, \ldots, k-1\}$ and by $\bar{I}=\tilde{I} \cup\{k\}$ its complement in $\{1, \ldots, k\}$ ．Then：

$$
\begin{aligned}
& x w \bullet\left(w_{1} \times \ldots \times w_{k}\right) \\
& =\left(x w \bullet\left(w_{1} \times \ldots \times w_{k-1}\right)\right) \bullet w_{k}-\sum_{i=1}^{k-1} x w\left(w_{1} \times \ldots \times\left(w_{i} \bullet w_{k}\right) \times \ldots \times w_{k-1}\right) \\
& =\sum_{I \subseteq\{1, \ldots, k-1\}}\left(f^{k-1-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{\tilde{I}}^{\amalg}\right)\right) \bullet w_{k} \\
& -\sum_{i=1}^{k-1} \sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-1-|I|}(x)\left(\left(w \bullet w_{I}^{\prime}\right) Ш w^{\prime} \underset{I}{\tilde{I}}\right) \\
& \left.=\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-1-|I|}(x)\left(\left(\left(w \bullet w_{I}\right) \bullet w_{k}\right) Ш w_{\tilde{I}}^{山}\right)+\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-1-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш\left(w_{\tilde{I}}^{山} \bullet w_{k}\right)\right)\right) \\
& \left.+\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш\left(w_{\tilde{I}}^{\amalg} Ш w_{k}\right)\right)\right) \\
& -\sum_{I \subseteq\{1, \ldots, k-1\}} \sum_{i \in I} f^{k-1-|I|}(x)\left(\left(w \bullet w_{I}^{\prime}\right) Ш w_{\tilde{I}}^{Ш}\right)-\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-1-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш\left(w_{\tilde{I}}^{山} \bullet w_{k}\right)\right) \\
& =\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-1-|I|}(x)\left(\left(\left(w \bullet w_{I}\right) \bullet w_{k}\right) Ш w_{\tilde{I}}^{\amalg}\right)-\sum_{I \subseteq\{1, \ldots, k-1\}} \sum_{i \in I} f^{k-1-|I|}(x)\left(\left(w \bullet w_{I}^{\prime}\right) Ш w_{\tilde{I}}^{山}\right) \\
& \left.+\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш\left(w_{\tilde{I}}^{\amalg} Ш w_{k}\right)\right)\right) \\
& =\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-1-|I|}(x)\left(\left(\left(w \bullet w_{I \cup\{k\}}\right) Ш w_{\tilde{I}}^{\amalg}\right)+\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{\bar{I}}^{山}\right)\right. \\
& =\sum_{J \subseteq\{1, \ldots, k\}, k \in I} f^{k-|J|}(x)\left(\left(\left(w \bullet w_{J}\right) Ш w_{\underset{J}{\amalg}}^{\stackrel{\omega}{x}}\right)+\sum_{I \subseteq\{1, \ldots, k-1\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{I}^{\amalg}\right)\right. \\
& =\sum_{I \subseteq\{1, \ldots, k\}} f^{k-|I|}(x)\left(\left(w \bullet w_{I}\right) Ш w_{\bar{I}}^{\amalg}\right)
\end{aligned}
$$

where $w_{j}^{\prime}=w_{j}$ if $j \neq i$ and $w_{i}^{\prime}=w_{i} \bullet w_{k}$.
3 and 4．For all $u \in T(V, f)$ ：

$$
u \star\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I \subseteq\{1, \ldots, k\}}\left(u \bullet w_{I}\right) \times w_{\bar{I}} .
$$

The results come then directly from points 1 and 2 ．

Theorem 13 Let $x_{1}, \ldots, x_{i} \in V, w_{1}, \ldots, w_{k} \in T(V, f), i \geq 1, k \geq 0$ ．If $w=x_{1} \ldots x_{i}$ ：

$$
\begin{aligned}
& \left.w \bullet\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I_{1} \sqcup \ldots \sqcup I_{i}=\{1, \ldots, k\}} f^{\left|I_{1}\right|}\left(x_{1}\right)\left(f^{\left|I_{2}\right|}\left(x_{2}\right)\left(\ldots\left(f^{\left|I_{i}\right|}\left(x_{i}\right) w_{I_{i}}^{\amalg}\right) Ш w_{I_{i-1}}^{\amalg}\right) \ldots\right) Ш w_{I_{1}}^{\amalg}\right), \\
& \left.w \star\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I_{1} \sqcup \ldots \sqcup I_{i+1}=\{1, \ldots, k\}} f^{\left|I_{1}\right|}\left(x_{1}\right)\left(f^{\left|I_{2}\right|}\left(x_{2}\right)\left(\ldots\left(f^{\left|I_{i}\right|}\left(x_{i}\right) w_{I_{i}}^{\amalg}\right) Ш w_{I_{i-1}}^{\amalg}\right) \ldots\right) Ш w_{I_{1}}^{\amalg}\right) \times w_{I_{i+1}} .
\end{aligned}
$$

Proof．By induction on $i$ ．If $i=1$ ，then：

$$
\begin{aligned}
x_{1} \bullet\left(w_{1} \times \ldots \times w_{k}\right) & =\sum_{I \subseteq\{1, \ldots, k\}} f^{k-|I|}\left(x_{1}\right)\left(\left(\emptyset \bullet w_{I}\right) Ш w_{I}^{\amalg}\right) \\
& =f^{k}\left(x_{1}\right)\left((\emptyset \bullet 1) Ш w_{1} Ш \ldots Ш w_{k}\right)+0,
\end{aligned}
$$

so the result holds if $k=1$ ．Let us assume the result at rank $i-1$ ．Then：

$$
x_{1} \ldots x_{i} \bullet\left(w_{1} \times \ldots \times w_{k}\right)=\sum_{I_{1} \sqcup J=\{1, \ldots, k\}} f^{\left|I_{1}\right|}\left(x_{1}\right)\left(\left(x_{2} \ldots x_{i} \bullet w_{J}\right) Ш w_{I_{1}}^{\amalg}\right) .
$$

The induction hypothesis applied to $x_{2} \ldots x_{i} \bullet w_{J}$ gives the result．
Remark．In particular，$x_{1} \bullet\left(w_{1} \times \ldots \times w_{k}\right)=f^{k}\left(x_{1}\right)\left(w_{1} \amalg \ldots 山 w_{k}\right)$ ．

## 1．5 Dual Hopf algebra

By duality，if $f$ is locally nilpotent，the symmetric algebra $S(T(V, f)$ ），with its usual product $\times$ ，inherits a Hopf algebra structure．Let us describe its coproduct $\Delta$ ，dual to the product $\star$ on $S\left(T\left(V^{*}, f^{*}\right)\right)$ ．

Theorem 14 Let us assume that $f$ is locally nilpotent．We define a coproduct $\tilde{\Delta}$ on $S(T(V, f))$ in the following way：

1．$\tilde{\Delta}(\emptyset)=\emptyset \otimes 1$ ．
2．If $x \in V$ and $u \in T(V, f)$ ：

$$
\tilde{\Delta}(x u)=\sum_{i \geq 0}\left(\theta_{f^{i}(x)} \otimes I d\right) \circ\left(I d \otimes m^{(i+1)}\right) \circ\left(\tilde{\Delta} \otimes I d^{\otimes i}\right) \circ \Delta_{山}^{(i+1)}(u) .
$$

We used the iterated products $m^{(j)}: S(T(V, f))^{\otimes j} \longrightarrow S(T(V, f))$ and the iterated coprod－ ucts $\Delta_{山}^{(j)}: T(V, f) \longrightarrow T(V, f)^{\otimes j}$ ．

For all $w \in T(V, f)$ ，we put $\Delta(w)=\tilde{\Delta}(w)+1 \otimes w$ ．With this coproduct，$S(T(V, f))$ is a Hopf algebra．

Proof. We denote by $g$ the transpose of $f$. As $S(T(V, f))$ is the dual of the enveloping algebra of a right pre-Lie algebra, for all $w \in T(V, f)$ :

$$
\Delta(w)-1 \otimes w \in T(V, f) \otimes S(T(V, f))
$$

We let $\tilde{\Delta}(w)=\Delta(w)-1 \otimes w$ for all $w \in T(V, f)$. Let $v, v_{1}, \ldots, v_{k}$ be words in $T\left(V^{*}, f^{*}\right)$. If $v \neq \emptyset$, then $v \bullet v_{1} \times \ldots \times v_{k} \in T_{+}\left(V^{*}, f^{*}\right)$, so:

$$
\left\langle v \otimes v_{1} \times \ldots \times v_{k}, \tilde{\Delta}(\emptyset)\right\rangle=\left\langle v \bullet\left(v_{1} \times \ldots \times v_{k}\right), \emptyset\right\rangle=0=\left\langle v \otimes v_{1} \times \ldots \times v_{k}, \emptyset \otimes 1\right\rangle .
$$

If $v=\emptyset$ :

$$
\left\langle\emptyset \otimes v_{1} \times \ldots \times v_{k}, \tilde{\Delta}(\emptyset)\right\rangle=\left\langle\emptyset \bullet\left(v_{1} \times \ldots \times v_{k}\right), \emptyset\right\rangle=\delta_{k, 0}=\left\langle\emptyset \otimes v_{1} \times \ldots \times v_{k}, \emptyset \otimes 1\right\rangle
$$

So $\tilde{\Delta}(\emptyset)=\emptyset \otimes 1$.

Let $v_{1}, \ldots, v_{k} \in T\left(V^{*}, f^{*}\right)$. Then:

$$
\begin{aligned}
& \left\langle\emptyset \otimes v_{1} \times \ldots \times v_{k}, \tilde{\Delta}(x u)\right\rangle \\
& =\left\langle\emptyset \bullet\left(v_{1} \times \ldots \times v_{k}\right), x u\right\rangle \\
& =\delta_{k, 0}\langle\emptyset, x u\rangle \\
& =0 \\
& =\left\langle\emptyset \otimes v_{1} \times \ldots \times v_{k}, \sum_{i \geq 0}\left(\theta_{f^{i}(x)} \otimes I d\right) \circ\left(I d \otimes m^{(i+1)}\right) \circ\left(\tilde{\Delta} \otimes I d^{\otimes i}\right) \circ \Delta_{\amalg}^{(i+1)}(u)\right\rangle .
\end{aligned}
$$

Let $y \in V^{*}, v, v_{1}, \ldots, v_{k} \in T\left(V^{*}, f^{*}\right)$. Then:

$$
\begin{aligned}
& \left\langle y v \otimes v_{1} \times \ldots \times v_{k}, \tilde{\Delta}(x u)\right\rangle \\
& =\left\langle y v \bullet\left(v_{1} \times \ldots \times v_{k}\right), x u\right\rangle \\
& =\sum_{I \subseteq\{1, \ldots, k\}}\left\langle g^{k-|I|}(y), x\right\rangle\left\langle\left(v \bullet v_{I}\right) Ш v_{\bar{I}}^{\omega}, u\right\rangle \\
& =\sum_{I \subseteq\{1, \ldots, k\}}\left\langle y, f^{k-|I|}(x)\right\rangle\left\langle v \otimes v_{I} \otimes \Delta^{(|\bar{I}|)}\left(w_{\bar{I}}\right),\left(\tilde{\Delta} \otimes I d^{\otimes|I|}\right) \circ \Delta_{\amalg}^{(|\bar{I}|+1)}(u)\right\rangle \\
& =\sum_{i=0}^{k}\left\langle y, f^{k-|I|}(x)\right\rangle\left\langle v \otimes v_{1} \times \ldots \times v_{k},\left(I d \otimes m^{(i+1)}\right) \circ\left(\tilde{\Delta} \otimes I d^{\otimes i}\right) \circ \Delta_{\amalg}^{(i+1)}(u)\right\rangle \\
& =\sum_{i \geq 0}\left\langle u v \otimes v_{1} \times \ldots \times v_{k},\left(I d \otimes m^{(i+1)}\right) \circ\left(\tilde{\Delta} \otimes I d^{\otimes i}\right) \circ \Delta_{\amalg}^{(i+1)}(u)\right\rangle .
\end{aligned}
$$

This implies the announced result.

Examples. Let $x_{1}, x_{2} \in V$.

$$
\begin{aligned}
\tilde{\Delta}(\emptyset) & =\emptyset \otimes 1 \\
\tilde{\Delta}\left(x_{1}\right) & =\sum_{i \geq 0} f^{i}\left(x_{1}\right) \otimes \emptyset^{\times i} \\
\tilde{\Delta}\left(x_{1} x_{2}\right) & =\sum_{i, j \geq 0} f^{i}\left(x_{1}\right) f^{j}\left(x_{2}\right) \otimes \emptyset^{\times(i+j)}+\sum_{i \geq 0} i f^{i}\left(x_{1}\right) \otimes x_{2} \times \emptyset^{\times(i-1)} .
\end{aligned}
$$

In particular, the second line shows that the condition of local nilpotence of $f$ is necessary in order to obtain a coproduct with values in $S(T(V, f)) \otimes S(T(V, f))$.

### 1.6 Group of characters

Let us now describe the character group $G$ associated to $f$. Any character of $S\left(T\left(V^{*}, f^{*}\right)\right)$ is the extension of a linear map from $T\left(V^{*}, f^{*}\right)$ to $\mathbb{K}$, so, as a set, we identify $G$ and $\prod_{n=0}^{\infty}\left(V^{* *}\right)^{\otimes n}$. The composition of $G$ is denoted by $\diamond$. We shall consider the following subset of $G^{\prime}$ :

$$
\bar{T}(V)=\prod_{n=0}^{\infty} V^{\otimes n}
$$

We give $G^{\prime}$ and $\bar{T}(V)$ their usual ultrametric topology.
We shall prove that this is a subgroup of $G$. Note that $G=\bar{T}(V)$ if, and only if, $V$ is finite-dimensional.

Theorem 15 We inductively define a continuous composition $\tilde{\diamond}$ on $\bar{T}(V)$ : for all $v \in \bar{T}(V)$,

1. $\emptyset \tilde{v} v=\emptyset$.
2. if $x \in V$ and $u \in G$, $x u \tilde{\Delta} v=\sum_{i \geq 0} f^{(i)}(x)\left((u \tilde{\Delta} v) Ш v^{Ш i}\right)$.

Then for all $u, v \in G, u \diamond v=u \tilde{v} v+v$.
Proof. We define a composition $\tilde{\triangleleft}$ on $\bar{T}(V)$ by $u \tilde{\triangleleft} v(w)=(u \otimes v) \circ \tilde{\Delta}(w)$ for all $w \in T(V, f)$. Then, for all $w \in T(V, f)$ :

$$
(u \diamond v)(w)=(u \otimes v) \circ \Delta(w)=(u \otimes v) \circ \tilde{\Delta}(w)+u(1) v(w)=(u \tilde{\triangleleft} v)(w)+v(w) .
$$

So $u \diamond v=u \tilde{\triangleleft} v+v$.
Let $w \in T(V, f)$. Then $(\emptyset \tilde{\triangleleft} v)(w)=(\emptyset \otimes v) \circ \tilde{\Delta}(w)$. If $w \neq \emptyset$, we observed that $\tilde{\Delta}(w) \in$ $T_{+}(V, f) \otimes T(V, f)$, so $(\emptyset \otimes v) \circ \tilde{\Delta}(w)=0$. Hence, $\emptyset \tilde{\sim} v=\lambda \emptyset$ for a particular $\lambda$. If $w=\emptyset$, we obtain $(\emptyset \sim v)(w)=\emptyset(\emptyset) v(1)=1$, so $\lambda=1$.

Let us compute $x u \tilde{\triangleleft} v$. First, $x u \tilde{\triangleleft} v(\emptyset)=(x u \otimes v)(\emptyset \otimes 1)=0$. Let us take $y \in V^{*}$ and $w \in T\left(V^{*}, f^{*}\right)$. Then:

$$
\begin{aligned}
x u \tilde{\triangleleft} v(y w) & =\sum_{i \geq 0}(x u \otimes v)\left(\left(\theta_{g^{i}(y)} \otimes I d\right) \circ\left(I d \otimes m^{(i+1)}\right) \circ(\tilde{\Delta} \otimes I d) \circ \Delta_{\amalg}^{(i+1)}(w)\right) \\
& =\sum_{i \geq 0} g^{i}(y)(x)(u \otimes v)\left(\left(I d \otimes m^{(i+1)}\right) \circ(\tilde{\Delta} \otimes I d) \circ \Delta_{\uplus}^{(i+1)}(w)\right) \\
& =\sum_{i \geq 0} y\left(f^{i}(x)\right)\left(u \otimes v^{\otimes(i+1)}\right)\left((\tilde{\Delta} \otimes I d) \circ \Delta_{\amalg}^{(i+1)}(w)\right) \\
& =\sum_{i \geq 0} y\left(f^{i}(x)\right)\left(u \tilde{\triangleleft} v \otimes v^{\otimes i}\right)\left(\Delta_{\amalg}^{(i+1)}(w)\right) \\
& =\sum_{i \geq 0} y\left(f^{i}(x)\right)\left((u \tilde{\triangleleft} v) Ш v^{\amalg i}\right)(w) \\
& =\sum_{i \geq 0} f^{i}(x)\left((u \tilde{\triangleleft} v) Ш v^{Ш i}\right)(y w) .
\end{aligned}
$$

So the composition $\tilde{\triangleleft}$ defined in this proof is the composition $\tilde{\triangleleft}$ defined in theorem 15. Moreover, an easy induction on the length proves that for all word $u$ and all $w \in \bar{T}(V), u \tilde{\sim} w \in \bar{T}(V)$. By linearity and continuity, $\bar{T}(V) \tilde{\diamond} \bar{T}(V) \subseteq \bar{T}(V)$.

Corollary 16 For all $x_{1}, \ldots, x_{k} \in V, v \in G$ :

$$
\left.x_{1} \ldots x_{k} \diamond v=\sum_{i_{1}, \ldots, i_{k} \geq 0} f^{i}\left(x_{1}\right)\left(f^{i_{2}}\left(x_{2}\right)\left(\ldots\left(f^{i_{k}}\left(x_{k}\right) v^{\amalg i_{k}}\right) Ш v^{\amalg i_{k-1}}\right) \ldots\right) Ш v^{\amalg i_{1}}\right)+v .
$$

Proof. By induction on $k$.

## 2 From free Com-Pre-Lie algebras to $T(V, f)$

### 2.1 Free Com-Pre-Lie algebras

Let us recall the construction of free Com-Pre-Lie algebras [10].
Definition 17 1. A partitioned forest is a pair $(F, I)$ such that:
(a) $F$ is a rooted forest (the edges of $F$ being oriented from the roots to the leaves).
(b) I is a partition of the vertices of $F$ with the following condition: if $x, y$ are two vertices of $F$ which are in the same part of $I$, then either they are both roots, or they have the same direct ascendant.

The parts of the partition are called blocks.
2. We shall say that a partitioned forest is a partitioned tree if all the roots are in the same block.
3. Let $\mathcal{D}$ be a set. A partitioned tree decorated by $\mathcal{D}$ is a pair $(t, d)$, where $t$ is a partitioned tree and $d$ is a map from the set of vertices of tinto $\mathcal{D}$. For any vertex $x$ of $t, d(x)$ is called the decoration of $x$.
4. The set of isoclasses of partitioned trees will be denoted by $\mathcal{P} \mathcal{T}$. For any set $\mathcal{D}$, the set of isoclasses of partitioned trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{P} \mathcal{T}(\mathcal{D})$.

Examples. We represent partitioned trees by the Hasse graph of the underlying rooted forest, the blocks of cardinality $\geq 2$ being represented by horizontal edges of different colors. Here are the partitioned trees with $\leq 4$ vertices:

$$
\begin{aligned}
& \cdot ; \mathfrak{\sharp} ; \boldsymbol{V}, \nabla, \downarrow, \perp=\lrcorner, \ldots ; \boldsymbol{V}, \nabla=\nabla, \nabla, \forall=\vee, \nabla=\nabla, Y, \nabla, \ddagger, \\
& K=\Sigma, L=\downarrow, Z=\nabla, L, L=\perp=\omega, \ldots .
\end{aligned}
$$

Definition 18 Let $t=(t, I)$ and $t^{\prime}=\left(t^{\prime}, J\right) \in \mathcal{P T}$.

1. Let $s$ be a vertex of $t^{\prime}$. The partitioned tree $t \bullet_{s} t^{\prime}$ is defined as follows:
(a) As a rooted forest, $t \bullet{ }^{\prime} t^{\prime}$ is obtained by grafting all the roots of $t^{\prime}$ on the vertex $s$ of $t$.
(b) We put $I=\left\{I_{1}, \ldots, I_{k}\right\}$ and $J=\left\{J_{1}, \ldots, J_{l}\right\}$. The partition of the vertices of this rooted forest is $I \sqcup J=\left\{I_{1}, \ldots, I_{k}, J_{1}, \ldots, J_{l}\right\}$.
2. The partitioned tree $t \boldsymbol{\omega} t^{\prime}$ is defined as follows:
(a) As a rooted forest, $t \boldsymbol{\omega} t^{\prime}$ is $t t^{\prime}$.
(b) We put $I=\left\{I_{1}, \ldots, I_{k}\right\}$ and $J=\left\{J_{1}, \ldots, J_{l}\right\}$ and we assume that the set of roots of $t$ is $I_{1}$ and the set of roots of $t^{\prime}$ is $J_{1}$. The partition of the vertices of $t 山 t^{\prime}$ is $\left\{I_{1} \sqcup J_{1}, I_{2}, \ldots, I_{k}, J_{1}, \ldots, J_{l}\right\}$.

## Examples.

1. Here are the three possible graftings $\nabla \bullet_{s}: \vee, \forall$ and $\nabla$.
2. Here are the two possible graftings $\boldsymbol{1} \bullet_{s} \ldots: \boldsymbol{V}$ and $\boldsymbol{\gamma}$.
3. $\vee \boldsymbol{V} \cdot=\boldsymbol{K}$.

These operations are similarly defined for decorated partitioned trees.
The free non unitary Com-Pre-Lie algebra is described in [10] :
Proposition 19 Let $\mathcal{D}$ be a set. The free non unitary Com-Pre-Lie algebra ComPrelie(D) generated by $\mathcal{D}$ is the vector space generated by $\mathcal{P} \mathcal{T}(\mathcal{D})$. We extend $\boldsymbol{\omega}$ by bilinearity and the pre-Lie product • is defined on in the following way: if $t, t^{\prime} \in \mathcal{P} \mathcal{T}(\mathcal{D})$,

$$
t \bullet t^{\prime}=\sum_{s \in V(t)} t \bullet \bullet_{s} t^{\prime}
$$

It satisfies the following universal property: if $A$ is a Com-Pre-Lie algebra and $a_{d} \in \mathcal{D}$ for all $d \in \mathcal{D}$, there exists a unique Com-Pre-Lie algebra morphism $\phi: \operatorname{ComPrelie}(\mathcal{D}) \longrightarrow A$, such that $\phi(\cdot d)=a_{d}$ for all $d \in \mathcal{D}$.

Rooted trees are identified with partitioned rooted trees such that any block has cardinality one. The free pre-Lie algebra $\operatorname{Prelie}(\mathcal{D})$, which is based on decorated rooted trees [2], is seen as a pre-Lie subalgebra of $\operatorname{ComPrelie}(\mathcal{D})$.

### 2.2 A generic case

Definition 20 1. Let $\mathcal{D}$ be a set. The set of biletters in $\mathcal{D}$ is $\mathbb{N} \times \mathcal{D}$. A biletter will be denoted by $\binom{k}{d}$, with $k \in \mathbb{N}$ and $d \in \mathcal{D}$. A biword in $\mathcal{D}$ is a word in biletters in $\mathcal{D}$.
2. Let $V_{\mathcal{D}}$ be the space generated by the biletters in $\mathcal{D}$. We define a map $f_{\mathcal{D}}: V_{\mathcal{D}} \longrightarrow V_{\mathcal{D}}$ by $f_{\mathcal{D}}\left(\binom{k}{i}\right)=\binom{k+1}{i}$ for all $k \geq 0, d \in \mathcal{D}$.
3. The pre-Lie subalgebra $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ generated by the billeters $\binom{0}{d}, d \in \mathcal{D}$, is denoted by $\mathfrak{g}_{\mathcal{D}}$.

The pair $\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ satisfies a universal property:
Proposition 21 Let $V$ be a vector space, $g: V \longrightarrow V$ a linear map, and for all $d \in \mathcal{D}$ $x_{d} \in V$. There exists a unique linear map $F: V_{\mathcal{D}} \longrightarrow V$ such that $F \circ f_{\mathcal{D}}=g \circ F$ and $F\binom{0}{d}=x_{d}$ for all $d \in \mathcal{D}$.

Proof. The map $F$ is defined by $F\binom{k}{d}=g^{k}\left(x_{d}\right)$.

### 2.3 Generation of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$

We now assume that $\mathcal{D}$ is a totally ordered set. Consequently, the set of biletters $\binom{0}{d}, d \in \mathcal{D}$, is also totally ordered.

Proposition 22 1. The pre-Lie algebra $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is generated by the set of words in biletters $\binom{0}{d}, d \in \mathcal{D}$. Moreover, this set is a minimal set of generators.
2. The Com-Pre-Lie algebra $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is generated by $\emptyset$ and the set of Lyndon words in biletters $\binom{0}{d}, d \in \mathcal{D}$. Moreover, this set is a minimal set of generators.

Proof. 1. We denote by $W$ the subspace of $V_{\mathcal{D}}$ generated by the biletters $\binom{0}{d}, d \in \mathcal{D}$. Let $A$ be the pre-Lie subalgebra of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ generated by $T(W)$. Let $w=x_{1} \ldots x_{n}$ be a words in biletters $\binom{i}{d}, i \geq 0, d \in \mathcal{D}$, If all its biletters are of the form $\binom{0}{d}$, then it belongs to $T(W) \subseteq A$. Let us now assume that it contains at least one biletter $\binom{i}{d}$, with $i \geq 1$. We prove that $w \in A$ by induction on $n$. If $n=1$, we put $x_{1}=\binom{i}{d}$, and:

$$
\binom{0}{d} \bullet\left(\emptyset^{\times i}\right)=\binom{i}{d} \in A
$$

So the result is true at rank 1 . Let us assume it at all rank $<n, n \geq 2$. Let $k$ be the smallest integer such that $x_{k}=\binom{i}{d}$, with $i \geq 1$. We proceed by induction on $k$. If $k=1$, then by the induction hypothesis on $n, x_{2} \ldots x_{n} \in A$, and by theorem 13:

$$
\binom{0}{d} \bullet\left(\emptyset^{\times(i-1)} \times x_{2} \ldots x_{n}\right)=\binom{i}{d} x_{2} \ldots x_{n}=x_{1} \ldots x_{n} \in A
$$

Let us assume the result at all rank $<k, k \geq 2$. We already know that $x_{1} \ldots x_{k-1}\binom{0}{d}$, and $x_{k+1} \ldots x_{n} \in A$. By proposition 13 :

$$
\begin{aligned}
& x_{1} \ldots x_{k-1}\binom{0}{d} \bullet\left(\emptyset^{\times(i-1)} \times x_{k+1} \ldots x_{n}\right) \\
& =x_{1} \ldots x_{n}+\text { words of length } n \text { satisfying the induction hypothesis on } k \in A
\end{aligned}
$$

So $x_{1} \ldots x_{n} \in A$. Finally, $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is generated by $T(W)$ as a pre-Lie algebra.

As $\operatorname{Im}\left(f_{\mathcal{D}}\right) \cap W=(0)$, by corollary 6 :

$$
T(W) \oplus\left(T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \bullet T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)\right)=T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)
$$

So $T(W)$ is a minimal subspace of generators of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$.
2. Let $W^{\prime}$ be the subspace of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ generated by $\emptyset$ and the Lyndon words in biletters $\binom{0}{d}$. We denote by $B$ the Com-Pre-Lie subalgebra of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ generated by $W$. Then $B$ contains the subalgebra generated by $W^{\prime}$ (for the product $山$ ), that is to say the shuffle algebra $T(W)$. By the first point, $B=T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$.

Let us prove now the minimality of $W^{\prime}$. First, Lyndon words are a minimal system of generators of the shuffle algebra $T(W)$. Moreover, $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \bullet T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is contained in the set of words in biletters $\binom{i}{d}$, so:

$$
W^{\prime} \cap\left(\left(T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) Ш T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)\right)+\left(T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \bullet T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)\right)\right)=(0) .
$$

Consequently, $W^{\prime}$ is a minimal subspace of generators of $\left(T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right), Ш, \bullet\right)$.

Corollary 23 Let $V$ be a vector space and let $g: V \longrightarrow V$. Let $W$ be a subspace of $V$ such that:

$$
V=\sum_{n=0}^{\infty} g^{n}(W)
$$

Then $T(V, g)$ is generated, as a pre-Lie algebra, by $T(W)$. We give $W$ a totally ordered basis $\left(w_{d}\right)_{d \in D}$. Then $T(V, g)$ is generated, as a Com-Pre-Lie algebra, by $\emptyset$ and the set of Lyndon words in letters $w_{d}, d \in \mathcal{D}$.

Proof. Let $f: V_{\mathcal{D}} \longrightarrow V$ be the unique map such that $f \circ f_{\mathcal{D}}=g \circ f$, and let $F$ : $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \longrightarrow T(V, g)$ be obtained by functoriality. We obtain:

$$
\operatorname{Im}(f)=\sum_{n=0}^{\infty} g^{n}(W)=V
$$

so $\operatorname{Im}(F)=T(\operatorname{Im}(f))=T(V, g)$. As $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is generated, as a pre-Lie algebra, by the words in biletters $\binom{0}{d}, T(V, g)$ is generated by their images, that is to say words in $W$.

As $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is generated, as a Com-Pre-Lie algebra, by $\emptyset$ and the Lyndon words in biletters $\binom{0}{d}, d \in \mathcal{D}, T(V, g)$ is generated by their images, that is to say $\emptyset$ and the Lyndon words in letters $w_{d}, d \in \mathcal{D}$.

## Remarks.

1. If $W$ is 1-dimensional, that is to say if $(V, g)$ is cyclic, then $T(V, g)$ is generated, as a Com-Pre-Lie algebra, by $\emptyset$ and any nonzero element of $W$.
2. These sets of generators are not necessarily minimal. For example, if $g$ is surjective, we shall prove in corollary 38 that $\emptyset$ and $W$ generate $T(V, g)$, even if $V$ is not one-dimensional.

### 2.4 A morphism from partitioned trees to words in biletters

Our aim is now to prove that $\mathfrak{g}_{\mathcal{D}}$ the pre-Lie subalgebra of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ generated by the biletters $\binom{0}{d}, d \in \mathcal{D}$, is free. We shall use the following morphism:

Definition 24 We denote by $\Phi_{C P L}$ the Com-Pre-Lie algebra morphism defined by:

$$
\Phi_{C P L}:\left\{\begin{aligned}
\operatorname{ComPrelie}(\mathcal{D}) & \longrightarrow T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \\
\cdot d & \longrightarrow\binom{0}{d}
\end{aligned}\right.
$$

Its restriction to $\operatorname{Prelie}(\mathcal{D})$ is denoted by $\Phi_{P L}$. Note that $\mathfrak{g}_{\mathcal{D}}=\Phi_{P L}(\operatorname{ComPrelie}(\mathcal{D}))$.
Examples. Let $a, b, c, d \in \mathcal{D}$.

$$
\begin{aligned}
& \Phi_{C P L}\left(\mathfrak{t}_{a}^{b}\right)=\binom{10}{a b}, \\
& \Phi_{C P L}\left({ }_{a}^{b} \mathrm{~L}_{\mathrm{c}}\right)=\binom{100}{a b c}+\binom{100}{a c b}+\binom{010}{c a b}, \\
& \Phi_{C P L}\left({ }^{b} \boldsymbol{V}_{a}{ }^{c}\right)=\binom{200}{a b c}+\binom{200}{a c b}, \\
& \Phi_{C P L}\left(\mathfrak{f}_{a}^{c}\right)=\binom{110}{a b c}, \\
& \Phi_{C P L}\left({ }^{b} \stackrel{\rightharpoonup}{⿶}_{a}^{d}{ }^{d}\right)=\binom{3000}{a b c d}+\binom{3000}{a b d c}+\binom{3000}{a c b d}+\binom{3000}{a c d b}+\binom{3000}{a d b c}+\binom{3000}{a d c b}, \\
& \Phi_{C P L}\left({ }^{\left.{ }^{( }{ }^{g} \bigvee_{a}{ }^{d}\right)}=\binom{2100}{a b c d}+\binom{2100}{a b d c}+\binom{2010}{a d b c}\right. \text {, } \\
& \Phi_{C P L}\left(\bigcup_{a}^{d}{ }_{a}^{c}\right)=\binom{1200}{a b c d}+\binom{1200}{a b d c}, \\
& \Phi_{C P L}\left({ }_{a}\left\llcorner_{b}^{d}\right)=\binom{1010}{a c b d}+\binom{1100}{a b c d}+\binom{1100}{a b d c}+\binom{1100}{b a c d}+\binom{1100}{b a d c}+\binom{1010}{b d a c},\right. \\
& \Phi_{C P L}\left(\begin{array}{c}
\stackrel{q}{b} \nabla_{a}^{d}
\end{array}\right)=\binom{1100}{a b c d}+\binom{1100}{a b d c}+\binom{1010}{a d b c}, \\
& \Phi_{C P L}\left(\begin{array}{l}
\mathfrak{i}_{a}^{d} \\
a \\
a
\end{array}\right)=\binom{1110}{a b c d} .
\end{aligned}
$$

Remark. $\Phi_{C P L}$ is not injective: for any $d \in \mathcal{D}, \Phi_{C P L}\left({ }_{d}^{d} \boldsymbol{L}_{d}^{d}-2^{d}{ }_{d}^{d}{ }_{d}^{d}\right)=0$. However, we will prove that the restriction $\Phi_{P L}$ is injective (theorem 30).

By the universal property of $V_{\mathcal{D}}$ and the factoriality of the construction of $T(V, f)$ :
Theorem 25 Let $V$ be a vector space, $f: V \longrightarrow V$ a linear map and $x_{d} \in V$ for all $d \in \mathcal{D}$. The following map is a Com-Pre-Lie algebra morphism:

$$
\varphi_{(V, f)}:\left\{\begin{aligned}
T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) & \longrightarrow T(V, f) \\
\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}} & \longrightarrow f^{a_{1}}\left(x_{d_{1}}\right) \ldots f^{a_{n}}\left(x_{d_{n}}\right) .
\end{aligned}\right.
$$

Moreover, denoting by $W$ the smallest subspace of $V$ stable under $f$ and containing all the $x_{d}$ 's, the image of $\varphi_{(V, f)}$ is $T(W)$.

Definition 26 Let $t$ be a partitioned tree decorated by $\mathcal{D}$.

1. The number of vertices of $t$ is denoted by $|t|$.
2. Let $s$ be a vertex of $t$.
(a) The fertility of $s$ is the number of blocks $B$ such that there is an edge from $s$ to any vertex of $B$. It is denoted by fert $(t)$.
(b) The decoration of $s$ is denoted by $d(s)$.
3. A linear extension of $t$ is a bijection $\sigma:\{1, \ldots,|t|\} \longrightarrow \operatorname{Vert}(t)$, such that if $x$ is a child of $y$, then $\sigma^{-1}(x)>\sigma^{-1}(y)$ (the edges of $t$ being oriented from the roots to the leaves). The set of linear extensions of $t$ is denoted by $L(t)$.

Let us now give a direct description of $\Phi_{C P L}$.
Proposition 27 For all partitioned tree decorated by $\mathcal{D}$ :

$$
\Phi_{C P L}(t)=\sum_{\sigma \in L(t)}\left(\begin{array}{ccc}
f \operatorname{ert}(\sigma(1)) & \ldots & f \operatorname{ert}(\sigma(|t|)) \\
d(\sigma(1)) & \ldots & d(\sigma(|t|))
\end{array}\right)
$$

Proof. By induction on $n=|t|$. It is obvious if $n=0$ or $n=1$. Let us assume the result at all ranks $<n$. Two cases can occur.

First case. Let us assume that $t$ has several roots. We can write $t=t_{1} \amalg t_{2}$, with $t_{1}, t_{2}$ two partitioned trees with $<n$ vertices. Let us denote by $k$ and $l$ the number of vertices of $t_{1}$ and $t_{2}$. Then:

$$
L(t)=\bigsqcup_{\sigma \in \operatorname{Sh}(k, l)}\left(L\left(t_{1}\right) \otimes L\left(t_{2}\right)\right) \circ \sigma^{-1}
$$

Hence, by the induction hypothesis applied to $t_{1}$ and $t_{2}$ :

$$
\begin{aligned}
& \Phi_{C P L}(t)=\Phi_{C P L}\left(t_{1}\right) Ш \Phi_{C P L}\left(t_{2}\right) \\
& =\sum_{\sigma_{1} \in L\left(t_{1}\right), \sigma_{2} \in L\left(t_{2}\right)}\left(\begin{array}{ccc}
f \operatorname{ert}\left(\sigma_{1}(1)\right) & \ldots & f \operatorname{ert}\left(\sigma_{1}(k)\right) \\
d\left(\sigma_{1}(1)\right) & \ldots & d\left(\sigma_{1}(k)\right)
\end{array}\right) 山\left(\begin{array}{ccc}
f \operatorname{ert}\left(\sigma_{2}(1)\right) & \ldots & f \operatorname{ert}\left(\sigma_{2}(l)\right) \\
d\left(\sigma_{2}(1)\right) & \ldots & d\left(\sigma_{2}(n)\right)
\end{array}\right) \\
& =\sum_{\sigma \in \operatorname{Sh}(k, l)} \sum_{\sigma_{1} \in L\left(t_{1}\right), \sigma_{2} \in L\left(t_{2}\right)}\left(\begin{array}{ccc}
f \operatorname{ert}\left(\left(\sigma_{1} \otimes \sigma_{2}\right) \circ \sigma^{-1}(1)\right) & \ldots & f \operatorname{ert}\left(\left(\sigma_{1} \otimes \sigma_{2}\right) \circ \sigma^{-1}(k+l)\right) \\
d\left(\left(\sigma_{1} \otimes \sigma_{2}\right) \circ \sigma^{-1}(1)\right) & \ldots & d\left(\left(\sigma_{1} \otimes \sigma_{2}\right) \circ \sigma^{-1}(k+l)\right)
\end{array}\right) \\
& =\sum_{\sigma \in L(\sigma)}\left(\begin{array}{ccc}
f \operatorname{ert}(\sigma(1)) & \ldots & f \operatorname{ert}(\sigma(n)) \\
d(\sigma(1)) & \ldots & d(\sigma(n))
\end{array}\right) .
\end{aligned}
$$

Second case. Let us assume that $t$ has a single root. Let $t^{\prime}=t_{1} Ш \ldots \amalg t_{k}$ be the partitioned tree obtained by deleting the root of $t$, where $t_{1}, \ldots, t_{k}$ are partitioned trees with a single root. If $d$ is the decoration of the root of $t$, then $t=\bullet \bullet \bullet t_{1} \times \ldots \times t_{k}$. We put $w_{i}=\Phi_{C P L}\left(t_{i}\right)$ for all $i$. Then, by theorem 12:

$$
\begin{aligned}
\Phi_{C P L}(t) & =\Phi_{C P L}(\cdot d) \bullet w_{1} \times \ldots \times w_{k} \\
& =\binom{0}{d} \bullet w_{1} \times \ldots \times w_{k} \\
& =\sum_{I \subseteq\{1, \ldots, k\}}\binom{k-|I|}{d}\left(\left(\emptyset \bullet w_{I}\right) Ш w_{\bar{I}}^{\omega}\right) \\
& =0+\binom{k}{d}\left(\emptyset Ш w_{\bar{\emptyset}}^{\omega}\right) \\
& =\binom{k}{d}\left(w_{1} Ш \ldots Ш w_{k}\right) \\
& =\binom{k}{d}\left(\Phi_{C P L}(G)\right) \\
& =\sum_{\tau \in L\left(t^{\prime}\right)}\left(\begin{array}{cccc}
k & f e r t(\tau(1)) & \ldots & \operatorname{fert}(\tau(n-1)) \\
d & d(\tau(1)) & \ldots & d(\tau(n-1))
\end{array}\right) .
\end{aligned}
$$

Moreover, for all $\sigma \in L(t), \sigma(1)$ is the root of $t$. Hence:

$$
L(t)=(1) \otimes L\left(t^{\prime}\right)
$$

Finally:

$$
\Phi_{C P L}(t)=\sum_{\tau \in L(t)}\left(\begin{array}{cccc}
k & \operatorname{fert}((1) \otimes \tau(1)) & \ldots & \operatorname{fert}((1) \otimes \tau(n)) \\
d & d((1) \otimes \tau(1)) & \ldots & d((1) \otimes \tau(n))
\end{array}\right)
$$

So the result holds for all partitioned tree $t$.
Remark. If $t$ is a partitioned tree with $k$ blocks:

$$
\sum_{i=1}^{|t|} f \operatorname{ert}(\sigma(i))=\sum_{x \in V(t)} \sharp\{\text { blocks of direct descendants of } x\}=k-1 \leq|t|-1 \text {. }
$$

Moreover, this is an equality is an equality if, and only if, $t$ is a rooted tree. Hence, for all partitioned tree of degree $n, \Phi_{C P L}(t)$ is a sum of biwords of length $n$, such that the sum of its upper letters is $\leq n-1$. Consequently, $\Phi_{C P L}$ is not surjective.

### 2.5 Freeness of $\mathfrak{g}_{\mathcal{D}}$

We now prove that the restriction of $\Phi_{C P L}$ to the the free pre-Lie algebra $\operatorname{Prelie}(\mathcal{D})$ is injective. We shall use certain families of biwords:

Definition 28 Let $w=\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}}$ be a biword.

1. We shall say that $w$ is admissible if:

- For all $1 \leq i \leq n, a_{i}+\ldots+a_{n} \leq n-i$.
- $a_{1}+\ldots+a_{n}=n-1$.

2. We shall say that $w$ is $\sigma$-admissible if it can be written as $w=w_{1} \ldots w_{k}$, with $w_{1}, \ldots, w_{k}$ admissible.

## Examples.

1. Here are the possible upper words for admissible words of length $\leq 4$ :

$$
0 ; 10 ; 200,110 ; 3000,2100,1200,2010,1110 .
$$

2. Here are the possible upper words for $\sigma$-admissible words of length $\leq 4$ :

$$
0 ; 00,10 ; 000,100,200,010,110 ;
$$

$$
0000,1000,2000,3000,0100,1100,2100,0200,1200,0010,1010,2010,0110,1110 .
$$

Remark. If $w=\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}}$ is admissible and $n \geq 1$, then $a_{n} \leq n-n=0$, so $a_{n}=0$.
Lemma 29 1. Let $w=\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}}$ be a $\sigma$-admissible biword. Then it can be uniquely written as $w=w_{1} \ldots w_{k}$, with $w_{1}, \ldots, w_{k}$ admissible. Moreover, $a_{1}+\ldots+a_{n}=n-k$.
2. Let $w=\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}}$ be a $\sigma$-admissible word. Then for all $1 \leq i \leq n, a_{i}+\ldots+a_{n} \leq n-i$.
3. Let $w=\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}}$ be a word such that for all $1 \leq i \leq n$, $a_{i}+\ldots+a_{n} \leq n-i$. We put $k=n-\left(a_{1}+\ldots+a_{n}\right)$. Then $k \geq 1$ and for $d \in \mathcal{D},\binom{k a_{1} \ldots a_{n}}{d d_{1} \ldots d_{n}}$ is admissible.

Proof. 1. Let $w=w_{1} \ldots w_{k}=w_{1}^{\prime} \ldots w_{l}^{\prime}$ be two decompositions of $w$ in admissible words.

$$
a_{1}+\ldots+a_{n}=\sum_{i=1}^{k}\left(\text { sum of the letters of } w_{i}\right)=\sum_{i=1}^{k}\left(l g\left(w_{i}\right)-1\right)=l g(w)-k=n-k .
$$

Similarly, $a_{1}+\ldots+a_{n}=n-l$, so $k=l$. Let us assume that $w_{1}$ is formed by the first $i$ biletters of $w, w_{1}^{\prime}$ is formed by the first $j$ biletters of $w$, with $i<j$. As $w_{1}$ and $w_{1}^{\prime}$ are admissible, $a_{1}+\ldots+a_{j}=j-1$ and $a_{1}+\ldots+a_{i}=i-1$, so $a_{i+1}+\ldots+a_{j}=(j-1)-(i-1)=j-i$. As $w_{1}^{\prime}$ is admissible, $a_{i+1}+\ldots+a_{j} \leq j-i-1$ : this is a contradiction. So $i=j$, and $w_{1}=w_{1}^{\prime}$; hence, $w_{2} \ldots w_{k}=w_{2}^{\prime} \ldots w_{k}^{\prime}$. Iterating the process, $w_{2}=w_{2}^{\prime}, \ldots, w_{k}=w_{k}^{\prime}$.
2. We put $w=w_{1} \ldots w_{k}$, where $w_{1}, \ldots, w_{k}$ are admissible. We put, for all $i$, $w_{i}=\binom{a_{i, 1} \ldots a_{i, l_{i}}}{d_{i, 1} \ldots d_{i, l_{i}}}$. For all $i, j$ :
$a_{i, j}+\ldots+a_{i, l_{i}}+a_{i+1,1}+\ldots+a_{k, l_{k}} \leq l_{i}-j+l_{I+1}-1+\ldots+l_{k}-1 \leq l_{i}-j+l_{i+1}+\ldots+l_{l}$.
Hence, if $w=\binom{a_{1} \ldots a_{n}}{d_{1} \ldots d_{n}}$, for all $1 \leq i \leq n, a_{i}+\ldots+a_{n} \leq n-i$.
3. We put $b_{1} \ldots b_{n+1}=k a_{1} \ldots a_{n}$. If $2 \leq i \leq n+1$ :

$$
b_{i}+\ldots+b_{n+1}=a_{i-1}+\ldots+a_{n} \leq n-(i-1) \leq n+1-i .
$$

Moreover, $b_{1}+\ldots+b_{n+1}=k+a_{1}+\ldots+a_{n}=n=(n+1)-1$. So $\binom{b_{1} \ldots b_{n+1}}{d_{1} \ldots d_{n+1}}$ is admissible.
Theorem 30 The map $\Phi_{P L}: \operatorname{Prelie}(\mathcal{D}) \longrightarrow T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is injective.
Proof. We shall consider the subalgebra (for the product ш) generated by $\operatorname{Prelie}(\mathcal{D})$ in ComPrelie $(\mathcal{D})$. We denote it by $\mathcal{H}_{\mathcal{D}}$. A basis of $\mathcal{H}_{\mathcal{D}}$ is given by the set of partitioned trees decorated by $\mathcal{D}$ such that any block has cardinality 1 , except the block containing the roots. Forgetting the blocks, these special partitioned trees are identified with rooted forests. Here are for example rooted forests with $\leq 4$ vertices:

$$
., 1, \ldots, V, \sharp, \downarrow,\lrcorner, \ldots, V, \forall, Y, \ddagger, \mathcal{L}, \downarrow, \mu, \ldots, \ldots .
$$

Let $\mathcal{W}_{n}$ be the set of $\sigma$-admissible words of length $n$ and let $\mathcal{W}$ be the union of the $\mathcal{W}_{n}$. We choose a total order on $\mathcal{W}$ such that:

1. If $u \in \mathcal{W}_{m}$ and $v \in \mathcal{W}_{n}$ with $m>n$, then $u>v$.
2. For all $n \geq 0$, for all $d \in \mathcal{D}$ the following map is increasing:

$$
\begin{cases}\mathcal{W}_{n} & \longrightarrow \mathcal{W}_{n+1} \\
\left(\begin{array}{ccccc}
a_{1} & \ldots a_{n} \\
d_{1} & \ldots d_{n}
\end{array}\right) & \longrightarrow\left(\begin{array}{cccc}
n-a_{1} \ldots-a_{n} & a_{1} & \ldots & a_{n} \\
d & d_{1} & \ldots & d_{n}
\end{array}\right)\end{cases}
$$

Note that this maps takes its values in the set of admissible words, by lemma 29.
3. If $u=u_{1} \ldots u_{k}$ and $v=v_{1} \ldots v_{l}$ are two elements of $\mathcal{W}_{n}$ decomposed into admissible words, if for a particular index $i, u_{k}=v_{l}, u_{k-1}=v_{l-1}, \ldots, u_{k-i+1}=v_{k-i+1}$ and $u_{k-i}>v_{l-i}$, then $u>v$.

For any decorated rooted forest $F$ (still identified with a partitioned tree by putting all the roots in the same block), we inductively define a biword $w_{F}$ in the following way:

- $w_{1}=\emptyset$.
- If $F$ is not a tree, we put $F=t_{1} \ldots t_{k}$, indexed such that $w_{t_{1}} \leq \ldots \leq w_{t_{k}}$. Then $w_{F}=w_{t_{1}} \ldots w_{t_{k}}$.
- If $F$ is a tree, let $d$ be the decoration of its root and let $t_{1} \ldots t_{k}$ be the forest obtained by deleting the root of $F$, its trees being indexed such that $w_{t_{1}} \leq \ldots \leq w_{t_{k}}$. Then $w_{F}=\binom{k}{d} w_{t_{1}} \ldots w_{t_{k}}$.

First step. Let us prove that for all rooted forest $F$ :

1. the coefficient of $w_{F}$ in $\Phi_{C P L}(F)$ is not zero.
2. $w_{F}$ is $\sigma$-admissible. Moreover, if $F$ is a tree, $w_{F}$ is admissible.

We proceed by induction on the number $n$ of vertices of $F$. If $n=0$, this is obvious. Let us assume the result at all rank $<k$. If $F$ is not a tree, we put $F=t_{1} \ldots t_{k}$. By the induction hypothesis, $w_{t_{1}}, \ldots, w_{t_{k}}$ are admissible, so $w_{t_{1}} \ldots w_{t_{k}}$ is $\sigma$-admissible. For all $1 \leq i \leq k$, there exists a linear extension $\sigma_{i}$ of $t_{i}$ such that:

$$
w_{t_{i}}=\left(\begin{array}{ccc}
f \operatorname{ert}\left(\sigma_{i}(1)\right) & \ldots & f \operatorname{ert}\left(\sigma_{i}\left(n_{i}\right)\right) \\
d\left(\sigma_{i}(1)\right) & \ldots & d\left(\sigma_{i}\left(n_{i}\right)\right)
\end{array}\right)
$$

Then $\sigma=\sigma_{1} \otimes \ldots \otimes \sigma_{k}$ is a linear extension of $F$ and:

$$
\left(\begin{array}{ccc}
f \operatorname{ert}(\sigma(1)) & \ldots & f \operatorname{ert}(\sigma(n)) \\
d(\sigma(1)) & \ldots & d(\sigma(n))
\end{array}\right)=w_{t_{1}} \ldots w_{t_{n}},
$$

so this biword appears in $\Phi_{C P L}(F)$. If $F$ is a tree, keeping the notations of the the definition of $w_{F}$, by the induction hypothesis, $w_{t_{1}}, \ldots, w_{t_{k}}$ are admissible. By lemma $29,\binom{k}{d} w_{t_{1}} \ldots w_{t_{k}}$ is admissible. For all $1 \leq i \leq k$, there exists a linear extension $\sigma_{i}$ of $t_{i}$ such that:

$$
w_{t_{i}}=\left(\begin{array}{ccc}
f \operatorname{ert}\left(\sigma_{i}(1)\right) & \ldots & f \operatorname{ert}\left(\sigma_{i}\left(n_{i}\right)\right) \\
d\left(\sigma_{i}(1)\right) & \ldots & d\left(\sigma_{i}\left(n_{i}\right)\right)
\end{array}\right) .
$$

Then $\sigma=(1) \otimes \sigma_{1} \otimes \ldots \otimes \sigma_{k}$ is a linear extension of $F$ and:

$$
\left(\begin{array}{ccc}
f \operatorname{ert}(\sigma(1)) & \ldots & f \operatorname{ert}(\sigma(n)) \\
d(\sigma(1)) & \ldots & d(\sigma(n))
\end{array}\right)=\binom{k}{d} w_{t_{1}} \ldots w_{t_{n}},
$$

so this biword appears in $\Phi_{C P L}(F)$.

Second step. Let $F, G$ be two forests, such that $w_{F}=w_{G}$. As $w=w_{F}=w_{G}$ appears in $\Phi_{C P L}(F)$ and $\Phi_{C P L}(G)$, then the number of vertices of $F$ and $G$ is the length $n$ of $w$. Let us prove that $F=G$ by induction on $n$. It is obvious if $n=1$. Let us assume the result at all ranks $<n$. Let us put $F=t_{1} \ldots t_{k}$, and $G=t_{1}^{\prime} \ldots t_{l}^{\prime}$. As the biwords $w_{t_{i}}$ are admissible, by lemma 29 , the sum of letters of $w$ is $n-k$; similarly, this sum is $n-l$, so $k=l$. If $k \geq 2$, by the unicity of the decomposition of $w$ into admissible biwords, $w_{t_{i}}=w_{t_{i}^{\prime}}$ for all $i$. By the induction hypothesis, $t_{i}=t_{i}^{\prime}$ for all $i$, so $F=G$. If $k=1$, considering the first biletter of $w$, the roots of $F$ and $G$ have the same decoration and the same fertility. Considering the biword obtained by deleting the first biletter of $w$, if $F^{\prime}$ and $G^{\prime}$ are the forests obtained by deleting the roots of $F$ and $G$, the induction hypothesis implies that $F^{\prime}=G^{\prime}$; hence, $F=G$.

Third step. For all forest $F$, we put:

$$
w_{F}^{\prime}=\max \left\{w \mid \text { the coefficient of } w \text { in } \Phi_{C P L}(F) \text { is not } 0 \text { and } w \text { is admissible }\right\} .
$$

By the preceding observations, $w_{F}^{\prime}$ exists and $w_{F}^{\prime} \geq w_{F}$. Let us prove that $w_{F}^{\prime}=w_{F}$ for any forest $F$. We proceed by induction on the number $n$ of vertices of $F$. It is obvious if $F=1$. Let us assume the result at all rank $<n$.

We put $w_{F}^{\prime}=w_{1} \ldots w_{k}$, with $w_{1} \ldots w_{k}$ admissible, and $f$ the linear extension of $F$ corresponding th $w_{F}^{\prime}$. For all $1 \leq i \leq k$, let $X_{i}$ be the subset of the vertices of $F$ corresponding to the biletters of $w_{i}$. If $x \in X_{i}$ and $y \in X_{j}$ is a child of $x$ in $F$, then as $f$ is a linear extension of $F$, $j \geq i$. So for all $1 \leq i \leq k, F_{i}=X_{i} \sqcup \ldots \sqcup X_{k}$ is an ideal of $F$, and $w_{i} \ldots w_{k}$ appears in $\Phi_{C P L}\left(F_{i}\right)$. By lemma 29, the sum of the first letters of $w_{i} \ldots w_{k}$ is $\left|F_{i}\right|-(k-i+1)=\left|F_{i}\right|-\lg \left(F_{i}\right)$, so $F_{i}$ is a forest of $k-i+1$ trees.

Let us assume there exists an index $i$ such that $F_{i}$ is not the disjoint union of a tree formed by the vertices in $X_{i}$ and $F_{i+1}$. We choose $i$ maximal; for all $j>i$, let $t_{j}$ be the tree formed by the vertices of $X_{j}$, such that $F_{i+1}=t_{i+1} \ldots t_{k}$. As $F_{i}$ is a forest of length $k-i+1$, and $X_{i}$ does not form a subtree of $F_{i}, X_{i}$ forms a forest $s_{1} \ldots s_{l}$ with $l>1$, and at least one of the $s_{p}$, say $s_{l}$ for example, has for child a root of one of the $t_{j}$. Let $t$ be the subtree of $F$ containing $s_{l}$ and $t_{j}$. Then a word of the form $\ldots w_{t} w_{t_{j+1}} \ldots w_{t_{k}}$ appears in $\Phi_{C P L}(F)$. As $|t|>\left|t_{j}\right|$, by the first condition on the order on admissible words, $w_{t_{j+1}}>w_{t_{j}}$. By the third condition on the order, $\ldots w_{t} w_{t_{j+1}} \ldots w_{t_{k}}>w_{t_{1}} \ldots w_{t_{k}}=w_{F}^{\prime}$ : contradicts the maximality of $w_{F}^{\prime}$. We obtain that $F=t_{1} \ldots t_{k}$, the vertices of $t_{i}$ corresponding via $f$ to the billetters of $w_{i}$ for all $i$. Consequently, $w_{i} \leq w_{t_{i}}^{\prime}=w_{t_{i}}$ for all $i$.

Let us assume that $i<j$ and $w_{i}>w_{j}$. Shuffling the biletters corresponding to the vertices of $t_{i}$ and $t_{j}, w_{1} \ldots w_{j} \ldots w_{i} \ldots w_{k}$ also appears in $\Phi_{C P L}(F)$ and is strictly greater than $w_{F}^{\prime}$, by the third condition on the order: contradiction. So $w_{1} \leq \ldots \leq w_{k}$.

Let us assume that $k \geq 2$. We put $F=t_{1}^{\prime} \ldots t_{k}^{\prime}$, with $w_{t_{1}^{\prime}} \leq \ldots \leq w_{t_{k}^{\prime}}$, such that $w_{F}=$ $w_{t_{1}^{\prime}} \ldots w_{t_{k}^{\prime}}$. By the induction hypothesis, for all $i, w_{t_{i}^{\prime}}=w_{t_{i}^{\prime}}^{\prime}$. In particular, $w_{k} \leq w_{t_{k}}^{\prime} \leq w_{t_{k}^{\prime}}$, as $t_{k}$ is one of the $t_{j}^{\prime}$. If $w_{k}<w_{t_{k}^{\prime}}$, then $w_{1} \ldots w_{k}<w_{F}$ by the third condition of the total order, and this contradicts the maximality of $w_{F}^{\prime}$. So $w_{k}=w_{t_{k}^{\prime}}^{\prime}$. Moreover, $w_{1} \ldots w_{t_{k}}^{\prime}$ also appears in $\Phi_{C P L}(F)$; by maximality of $w_{F}^{\prime}$, necessarily $w_{t_{k}}^{\prime} \leq w_{k}$, so $w_{t_{k}}^{\prime}=w_{k}$. By the second step, $t_{k}=t_{k}^{\prime}$. We then obtain that $t_{1} \ldots t_{k-1}=t_{1}^{\prime} \ldots t_{k-1}^{\prime}$ and that $w_{t_{1} \ldots t_{k-1}}^{\prime}=w_{1} \ldots w_{k-1}$. By the induction hypothesis, $w_{1} \ldots w_{k-1}=w_{t_{1}^{\prime}} \ldots w_{t_{k-1}^{\prime}}$, so $w_{F}^{\prime}=w_{F}$.

Let us assume that $k=1$. Let $d$ the decoration of the root of $F, l$ its fertility and $G$ be the forest obtained by deleting the root of $F$. Then $\Phi_{C P L}(F)=\binom{l}{d} \Phi_{C P L}(G)$. By the second condition on the total order on $\sigma$-admissible biwords, $w_{F}^{\prime}=\binom{l}{d} w_{G}^{\prime}$. By the induction hypothesis, $w_{G}^{\prime}=w_{G}$, so $w_{F}^{\prime}=w_{F}$.

Last step. Let $x=\sum a_{F} F$ be a nonzero element of $\mathcal{H}_{\mathcal{D}}$. As $F \longrightarrow w_{F}$ is injective by the second step, let us totally order the set of decorated rooted forests such that $F<G$ if, and only if, $w_{F}<w_{G}$. Let $F_{0}$ be the maximal forest such that $a_{F} \neq 0$. If $F<F_{0}$, then $w_{F}<w_{F_{0}}$; as
$w_{F}=w_{F}^{\prime}, w_{F_{0}}$ does not appear in $\Phi_{C P L}(F)$. If $F>F_{0}, a_{F}=0$. So the coefficient of $w_{F}$ in $\Phi_{C P L}(F)$ is $a_{F_{0}}$ times the coefficient of $w_{F_{0}}$ in $\Phi_{C P L}\left(F_{0}\right)$, which is not zero. So $\Phi_{C P L}(x) \neq 0$ : $\Phi_{C P L}$ is injective.

## Remarks.

1. We even prove that the restriction of $\Phi_{C P L}$ to $\mathcal{H}_{\mathcal{D}}$ is injective.
2. Although it contains a free pre-Lie subalgebra, $T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is not free. Let us assume it is free. As the length of words gives a connected gradation of $T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$, it is freely generated by any graded complement of $T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \bullet T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$. Let us choose $d \in \mathcal{D}$. Then it is not difficult to prove that we can choose a complement of $T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right) \bullet T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ which contains $\binom{0}{d},\binom{1}{d}$ and $\binom{00}{d d}$, which implies that the pre-Lie subalgebra generated by these three elements is free. Hence, the following pre-Lie morphism is injective:

$$
\psi:\left\{\begin{aligned}
\operatorname{Prelie}(\{0,1,00\}) & \longrightarrow\left(T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)\right. \\
\cdot 0 & \longrightarrow\binom{0}{d} \\
\cdot 1 & \longrightarrow\binom{1}{d} \\
\cdot 00 & \longrightarrow\binom{00}{d d} .
\end{aligned}\right.
$$

But $\psi\left({ }^{0} \bigvee_{0}{ }^{0}-2 ป_{1}^{00}\right)=0$ : this is a contradiction. So $T_{+}\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is not free.
Corollary 31 The pre-Lie subalgebra $\mathfrak{g}_{\mathcal{D}}$ of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$ is freely generated by the biletters $\binom{0}{d}$, $d \in \mathcal{D}$.

We shall give more results on the morphism $\Phi_{C P L}$ and the admissible words in the appendix.

### 2.6 Prelie subalgebra generated by $V$

We here consider the pre-Lie subalgebra of $V$ generated by $V$. We denote by $\operatorname{Prelie}(V)$ the free pre-Lie algebra generated by $V$. This is the space of rooted trees decorated by $V$, each vertex being linear in its decoration. The pre-Lie product is also given by grafting. We shall also consider the free Com-Pre-Lie algebra generated by $V$. This is the space of partitioned trees decorated by $V$, each vertex being linear in its decoration. We consider the two morphisms:

$$
\phi_{P L}:\left\{\begin{array}{rll}
\operatorname{Prelie}(V) & \longrightarrow & T_{+}(V) \\
\cdot v & \longrightarrow & v,
\end{array} \phi_{C P L}:\left\{\begin{aligned}
& \operatorname{ComPrelie}(V) \longrightarrow \\
& \bullet_{v} \longrightarrow \\
& \hline
\end{aligned}\right.\right.
$$

Note that $\phi_{P L}$ is the restriction of $\phi_{C P L}$ on $\operatorname{Prelie}(V) \subseteq \operatorname{ComPrelie}(V)$.
Combining theorem 25 and proposition $27, \phi_{C P L}=\varphi_{(V, f)} \circ \Phi_{C P L}$. Hence, for all partitioned tree $t$ decorated by $V$ :

$$
\phi_{C P L}(t)=\sum_{\sigma \in L(t)} f^{f e r t(\sigma(1))}(d(\sigma(1))) \ldots f^{f e r t(\sigma(|t|))}(d(\sigma(|t|))) .
$$

Theorem 32 1. $\phi_{P L}$ is surjective if, and only if, $f(V)=V$.
2. $\phi_{C P L}$ is surjective if, and only if, the codimension of $f(V)$ in $V$ is $\leq 1$.

Proof. 1. $\Longrightarrow$. The length of words is a gradation of $T(V, f)$; moreover, $\phi_{C P L}$ is homogeneous for this gradation. Hence:

$$
\operatorname{Im}\left(\phi_{P L}\right) \cap V^{\otimes 2}=\operatorname{Vect}\left(\phi_{P L}\left(\mathbf{t}_{v}^{w}\right) \mid v, w \in V\right)=\operatorname{Vect}(f(v) w \mid v, w \in V)=f(V) \otimes V .
$$

If $\phi_{P L}$ is surjective, then $f(V) \otimes V=V \otimes V$, so $f(V)=V$.

1. $\Longleftarrow$. Let $v_{1} \ldots v_{n} \in V^{\otimes n}$. As $f(V)=V$, for all $1 \leq i \leq n-1$, there exists $w_{i} \in V$ such that $f\left(w_{i}\right)=v_{i}$. Let $t$ be the ladder with $n$ vertices, decorated from the root to the leaf by $w_{1}, \ldots, w_{n-1}, v_{n}$. Then $L(t)$ is reduced to a single element, and $\phi_{P L}(t)=f\left(w_{1}\right) \ldots f\left(w_{n-1}\right) v_{n}=$ $v_{1} \ldots v_{n}$. So $\phi_{P L}$ is surjective.
$2 . \Longrightarrow$. By homogeneity of $\phi_{C P L}$ :
$\operatorname{Im}\left(\phi_{C P L}\right) \cap V^{\otimes 2}=\operatorname{Vect}\left(\phi_{C P L}\left(\mathfrak{l}_{v}^{w}\right), \phi_{C P L}\left({ }_{v \varpi w}\right) \mid v, w \in V\right)=V \operatorname{ect}(f(v) w, v w+w v \mid v, w \in V)$.
Let us assume that the codimension of $f(V)$ in $V$ is $\geq 2$. Let $x, y \in V$, linearly independent, such that $f(V) \cap \operatorname{Vect}(x, y)=(0)$. There exists a linear map $g: V \longrightarrow V$, such that $g(f(V))=(0)$ and $g(x)=x, g(y)=y$. As $\phi_{C P L}$ is surjective, we can write:

$$
x y=\sum_{i \in I} f\left(v_{i}\right) w_{i}+\sum_{j \in J}\left(v_{j} w_{j}+w_{j} v_{j}\right)
$$

Hence:

$$
\begin{aligned}
x y & =g(x) g(y) \\
& =\sum_{i \in I} g\left(f\left(v_{i}\right)\right) g\left(w_{i}\right)+\sum_{j \in J}\left(g\left(v_{j}\right) g\left(w_{j}\right)+g\left(w_{j}\right) g\left(v_{j}\right)\right) \\
& =0+\sum_{j \in J}\left(g\left(v_{j}\right) g\left(w_{j}\right)+g\left(w_{j}\right) g\left(v_{j}\right)\right) \\
& =\sum_{j \in J}\left(g\left(w_{j}\right) g\left(v_{j}\right)+g\left(v_{j}\right) g\left(w_{j}\right)\right) \\
& =g(y) g(x) \\
& =y x .
\end{aligned}
$$

So $x y=y x$, and consequently $x$ and $y$ are linearly independent: this is a contradiction. So the codimension of $f(V)$ in $V$ is $\leq 1$.
$2 . \Longleftarrow$. If $f(V)=V$, then by the first point, $\phi_{P L}$, and consequently $\phi_{C P L}$, is surjective. Let us assume that the codimension of $f(V)$ in $V$ is 1 . Let us choose $x \in V$, such that $V=$ $f(V) \oplus \operatorname{Vect}(x)$. In order to prove that $\phi_{C P L}$ is surjective, it is enough to prove that $v_{1} \ldots v_{n} \in$ $\operatorname{Im}\left(\phi_{C P L}\right)$ for all $n \geq 1, v_{1}, \ldots, v_{n} \in\{x\} \cup f(V)$. We proceed by induction on $n$. If $n=1$, then $v_{1}=\phi_{C P L}\left(\cdot v_{1}\right)$. Let us assume the result at rank $n-1$. Let $k$ be the greatest integer such that $v_{1}=\ldots=v_{k}=x$, with the convention that $k=0$ if $v_{1} \neq x$. We proceed by induction on $k$. If $k=0$, let us put $v_{1}=f\left(w_{1}\right)$. By the induction on $n$, there exists $P \in \operatorname{ComPrelie}(V)$ such that $\phi_{C P L}(P)=v_{2} \ldots v_{n}$. Then:

$$
\phi_{C P L}\left(\bullet w_{1} \bullet X\right)=w_{1} \bullet v_{2} \ldots v_{n}=f\left(w_{1}\right) v_{2} \ldots v_{n}=v_{1} \ldots v_{n}
$$

Let us assume the result at rank $k-1$, with $k \geq 1$. By the induction hypothesis on $n$, there exists $X \in \operatorname{ComPrelie}(V)$ such that $\phi_{C P L}(X)=x^{k-1} v_{k+1} \ldots v_{n}$. Then:

$$
\begin{aligned}
\phi_{C P L}\left({ }_{x} Ш X\right) & =x Ш x^{k-1} v_{k+1} \ldots v_{n} \\
& =k x^{k} v_{k+1} \ldots v_{n}+\sum_{i=k+1}^{n} x^{k-1} v_{k+1} \ldots v_{i} x v_{i+1} \ldots v_{n}
\end{aligned}
$$

By the induction hypothesis on $k$, for all $k+1 \leq i \leq n, x^{k-1} v_{k+1} \ldots v_{i} x v_{i+1} \ldots v_{n} \in \operatorname{Im}\left(\phi_{C P L}\right)$. So $k x^{k} k v_{k+1} \ldots v_{n} \in \operatorname{Im}\left(\phi_{C P L}\right)$. As the characteristic of the base field is zero, $v_{1} \ldots v_{n}=$ $x^{k} k v_{k+1} \ldots v_{n} \in \operatorname{Im}\left(\phi_{C P L}\right)$. So $\phi_{C P L}$ is surjective.

Corollary 33 1. (a) The subspace $V$ generates the pre-Lie algebra $T_{+}(V)$ if, and only if, $f(V)=V$.
(b) The subspace $V$ generates the Com-Pre-Lie algebra $T_{+}(V)$ if, and only if, the codimension of $f(V)$ in $V$ is $\leq 1$.
2. Let $W$ be a subspace of $V$. We put:

$$
\bar{W}=\sum_{k=0}^{\infty} f^{k}(W)
$$

that is to say the smallest subspace of $V$ stable by $f$ which contains $W$.
(a) $K \emptyset \oplus W$ generates the pre-Lie algebra $T(V, f)$ if, and only if, $\bar{W}=V$ and $f(V)=V$.
(b) $K \emptyset \oplus W$ generates the Com-Pre-Lie algebra $T(V, f)$ if, and only if, $\bar{W}=V$ and the codimension of $f(V)$ in $V$ is $\leq 1$.

Proof. 1. It is a direct consequence of theorem 32.
2. (a) We denote by $A$ the pre-Lie subalgebra of $T(V, f)$ generated by $K \emptyset \oplus W$ and by $A^{\prime}$ the pre-Lie subalgebra of $T(V, f)$ generated by $\bar{W}$. Let us prove that $A=K \emptyset \oplus A^{\prime}$. First, for all $w_{1}, \ldots, w_{k} \in \bar{W}:$

$$
w_{1} \ldots w_{k} \bullet \emptyset=\sum_{i=1}^{k} w_{1} \ldots w_{i-1} f\left(w_{i}\right) w_{i+1} \ldots w_{k} \in A^{\prime}
$$

as $\bar{W}$ is stable under $f$. As $\emptyset \circ w=0$ for any $w \in A^{\prime}, K \emptyset \oplus A^{\prime}$ is a pre-Lie subalgebra which contains $W$, so it contains $A$. For any $x \in W$, for any $k \geq 0$ :

$$
f^{k}(x)=(\ldots((x \bullet \emptyset) \bullet \emptyset) \ldots) \bullet \emptyset \in A
$$

so $\bar{W} \subseteq A$, which implies $A^{\prime} \subseteq A$ and $K \emptyset \oplus A^{\prime} \subseteq A$.
$\Longrightarrow$. If $A=T(V, f)$, then $A^{\prime}=T(V, f)$. Comparing the homogeneous component of degree $1, \bar{W}=V$. By the first point, $f(V)=V$.
$\Longleftarrow$. In this case, $A^{\prime}=T_{+}(V)$, so $A=K \emptyset \oplus T_{+}(V)=T(V)$.
Point 2 is proved similarly.

## 3 Examples

### 3.1 The diagonalizable case

We here assume that the endomorphism $f$ is diagonalizable. Let $\left(x_{d}\right)_{d \in \mathcal{D}}$ be a basis of $V$, such that $f\left(x_{d}\right)=\lambda_{d} x_{d}$ for all $d \in \mathcal{D}$. We now construct the pre-Lie algebra $T_{+}(V, f)$ as the dual of a Hopf subalgebra of the Connes-Kreimer Hopf algebra $\mathcal{H}_{C K}^{\mathcal{D}}$ of rooted trees decorated by $\mathcal{D}[4,1]$. As an algebra, it is the free commutative, associative algebra generated by the set of rooted trees decorated by $\mathcal{D}$. Its coproduct is given by admissible cuts : for any tree $t$,

$$
\Delta(t)=\sum_{c \text { admissible cut of } t} R^{c}(t) \otimes P^{c}(t)
$$

For example:

$$
\Delta\left({ }^{b} \mathfrak{\bigvee}_{a}^{d}\right)={ }^{b} \mathfrak{V}_{a}^{d}{ }^{d} \otimes 1+1 \otimes{ }^{b} \mathfrak{\bigvee}_{a}^{d}{ }^{d}+\dot{!}_{a}^{d}{ }_{a}^{d} \otimes \cdot d+{ }^{b} \bigvee_{a}{ }^{c} \otimes \cdot{ }_{d}+\mathfrak{l}_{a}^{b} \otimes \mathfrak{l}_{c}^{d}+\mathfrak{l}_{a}^{c} \otimes \cdot{ }_{b} \cdot d+\cdot{ }_{a} \otimes \cdot{ }_{b} \mathfrak{l}_{c}^{d} .
$$

The enveloping algebra of $\operatorname{Prelie}(\mathcal{D})$ and $\mathcal{H}_{C K}^{\mathcal{D}}$ are in duality by the pairing [20, 12]:

$$
\langle-,-\rangle:\left\{\begin{array}{rll}
\mathcal{U}(\operatorname{Prelie}(\mathcal{D})) \times \mathcal{H}_{C K}^{\mathcal{D}} & \longrightarrow & \mathbb{K} \\
(F, G) & \longrightarrow & \delta_{F, G} s_{F}
\end{array}\right.
$$

where $s_{F}$ is the number of symmetries of the forest $F$. As the characteristic of $\mathbb{K}$ is zero, this pairing is non degenerate. Consequently, the subspace of $\mathcal{H}_{C K}^{\mathcal{D}}$ generated by the set of rooted trees inherits a pre-Lie cobracket defined by $\delta=(\pi \otimes \pi) \circ \Delta$, where $\pi$ is defined by:

$$
\pi(F)= \begin{cases}F & \text { if } F \text { is tree } \\ 0 & \text { if } F \text { is a forest which is not a tree. }\end{cases}
$$

Following [3, 4], for all $d \in \mathcal{D}$, we define a linear map $N_{d}: \mathcal{H}_{C K}^{\mathcal{D}} \longrightarrow \mathcal{H}_{C K}^{\mathcal{D}}$ by:

$$
N_{d}(F)=\sum_{s \in V(F)} \lambda_{d(s)} F \bullet_{s} \cdot d
$$

for any forest $F$. We shall also need the following map:

$$
\phi_{\lambda}:\left\{\begin{array}{rll}
\mathcal{H}_{C K}^{\mathcal{D}} & \longrightarrow & \mathcal{H}_{C K}^{\mathcal{D}} \\
F & \longrightarrow & \left(\sum_{s \in V(F)} \lambda_{d(s)}\right) F .
\end{array}\right.
$$

Proposition 34 For all $d \in \mathcal{D}$ :

1. For all $x, y \in \mathcal{H}_{C K}^{\mathcal{D}}, N_{d}(x y)=N_{d}(x) y+x N_{d}(y)$.
2. $N_{d} \circ \pi=\pi \circ N_{d}$.
3. For all $x \in \mathcal{H}_{C K}^{\mathcal{D}}, \Delta \circ N_{d}(x)=\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ \Delta(x)+(1 \otimes \cdot d) \cdot\left(\phi_{\lambda} \otimes I d\right) \circ \Delta(x)$.
4. If $x$ is a linear span of trees, $\delta \circ N_{d}(x)=\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ \delta(x)+\phi_{\lambda}(x) \otimes \cdot{ }_{d}$.

Proof. 1. This is obvious if $x$ and $y$ are forests, as $V(x y)=V(x) \sqcup V(y)$.
2. If $F$ is a tree, then $N_{d}(F)$ is a linear span of trees, so $N_{d} \circ \pi(F)=N_{d}(F)=\pi \circ N_{d}(F)$. If $F$ is a forest which is not a tree, then $N_{d}(F)$ is a linear span of forests which are not trees, so $N_{d} \circ \pi(F)=0=\pi \circ N_{d}(F)$.
3. Let $F$ be a forest, and $s \in V(F)$. We denote by $l$ the leaf decorated by $d$ grafted onto $F$ to obtain $G=F \circ_{s} \cdot d$. There are three types of admissible cuts of $G$ :

- Admissible cuts $c$ such that $s \in V\left(P^{c}(G)\right)$. Then $R^{c}(G)=R^{c \mid F}(F)$ and $P^{c}(G)=R^{c_{\mid F}}(F) \bullet_{s}$ $\cdot d$.
- Admissible cuts $c$ such that $s, \ell \in V\left(R^{c}(G)\right)$. Then $R^{c}(G)=R^{c \mid F}(F) \bullet{ }_{s} \cdot d$ and $P^{c}(G)=$ $R^{c_{\mid F}}(F)$.
- Admissible cuts $c$ such that $s \in V\left(R^{c}(G)\right)$ and $\ell \in V\left(P^{c}(G)\right)$. Then $R^{c}(G)=R^{c_{\mid F}}(F)$ and $P^{c}(G)={ }_{\cdot d} R^{c}{ }^{\mid F}(F)$

Summing all these possibilities, we obtain:

$$
\begin{aligned}
\Delta\left(N_{d}(F)\right) & =\sum_{c \text { admissible cut of } F} \sum_{s \in V\left(P^{c}(F)\right)} \lambda_{d(s)} R^{c}(F) \otimes P^{c}(F) \bullet_{s} \cdot d \\
& =\sum_{c \text { admissible cut of } F s \in V\left(R^{c}(F)\right)} \lambda_{d(s)} R^{c}(F) \bullet_{s} \cdot d \otimes P^{c}(F) \\
& =\sum_{c \text { admissible cut of } F s \in V\left(R^{c}(F)\right)} \lambda_{d(s)} R^{c}(F) \otimes \cdot{ }_{d} P^{c}(F) \\
& =\sum_{c \text { admissible cut of } F} N_{d}\left(R^{c}(F)\right) \otimes P^{c}(F)+R^{c}(F) \otimes N_{d}\left(P^{c}(F)\right)+\phi_{\lambda}\left(R^{c}(F)\right) \otimes \cdot{ }_{d} P^{c}(F) \\
& \left.=\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ \Delta(F)+1 \otimes \cdot{ }_{d}\right) \cdot\left(\phi_{\lambda} \circ I d\right) \circ \Delta(F) .
\end{aligned}
$$

4. Let $t$ be a tree. We put $\Delta(t)=t \otimes 1+\sum t^{\prime} \otimes t^{\prime \prime}$, where $\sum t^{\prime} \otimes t^{\prime \prime}$ is a sum of tensors such that the left part is a forest and the right part is a tree. Then:

$$
\begin{aligned}
(\pi \otimes \pi)\left((1 \otimes \cdot d) \cdot\left(\phi_{\lambda} \otimes I d\right) \circ \Delta(t)\right) & =(\pi \otimes \pi)\left(\phi_{\lambda}(t) \otimes \cdot{ }_{d}+\sum \phi_{\lambda}\left(t^{\prime}\right) \otimes t^{\prime \prime} \cdot{ }^{\prime}\right) \\
& =\phi_{\lambda}(t) \otimes \cdot{ }_{d} .
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\delta \circ N_{d}(t) & =(\pi \otimes \pi) \circ \Delta \circ N_{d}(x) \\
& =(\pi \otimes \pi) \circ\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ \Delta(t)+(\pi \otimes \pi)\left((1 \otimes \cdot d) \cdot\left(\phi_{\lambda} \otimes I d\right) \circ \Delta(t)\right) \\
& =\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ(\pi \otimes \pi) \circ \Delta(t)+\phi_{\lambda}(t) \otimes \cdot d \\
& =\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ \delta(t)+\phi_{\lambda}(t) \otimes \cdot d,
\end{aligned}
$$

which proves the last point.
Definition 35 Let $d_{1}, \ldots, d_{n} \in \mathcal{D}, n \geq 1$. We inductively defined $t_{d_{1} \ldots d_{n}}$ in the following way:

1. If $n=1, t_{d_{1}}=\cdot{ }_{d_{1}}$.
2. If $n \geq 2, t_{d_{1} \ldots d_{n}}=N_{d_{n}}\left(t_{d_{1} \ldots d_{n-1}}\right)$.

We denote by $\mathcal{H}_{\lambda}$ the subalgebra of $\mathcal{H}_{C K}^{\mathcal{D}}$ generated by the elements $t_{d_{1} \ldots d_{n}}$.
Examples. If $a, b, c, d \in \mathcal{D}$ :

$$
\begin{aligned}
& t_{a}=._{a}, \\
& t_{a b}=\lambda_{a} \mathbf{l}_{a}^{b}, \\
& \left.t_{a b c}=\lambda_{a}^{2 b} \bigvee_{a}^{c}+\lambda_{a} \lambda_{b}\right\}_{a}^{c},
\end{aligned}
$$

Proposition $36 \mathcal{H}_{\lambda}$ is stable under $N_{d}$ for all $d \in \mathcal{D}$ and under $\phi_{\lambda}$, and is a Hopf subalgebra of $\mathcal{H}_{C K}^{\mathcal{D}}$.

Proof. We consider

$$
A=\left\{x \in \mathcal{H}_{\lambda} \mid \phi_{\lambda}(x), N_{d}(x) \in \mathcal{H}_{\lambda} \text { for all } d \in \mathcal{D}\right\} /
$$

As $N_{d}$ and $\phi_{\lambda}$ are derivations, this is a subalgebra of $\mathcal{H}_{\lambda}$. For all nonempty word $w=d_{1} \ldots d_{n}$, $N_{d}\left(t_{w}\right)=t_{w d}$ and $\phi_{\lambda}\left(t_{w}\right)=\left(\lambda_{d_{1}}+\ldots+\lambda_{d_{n}}\right) t_{w}$, so $A$ contains the generators of $\mathcal{H}_{\lambda}: A=\mathcal{H}_{\lambda}$, which
proves the first point. Let us prove that $\Delta\left(t_{w}\right) \in \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}$ for all nonempty word $w$ by induction on the length $n$. If $n=1$, then $t_{w}={ }_{\cdot d}$ for a particular $d$, so $\Delta\left(t_{w}\right)=t_{w} \otimes 1+1 \otimes t_{w} \in \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}$. Let us assume the result at rank $n-1$. We put $w=w^{\prime} d$, with $d \in \mathcal{D}$. By the induction hypothesis, $\Delta\left(t_{w}^{\prime}\right) \in \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}$. Then:

$$
\begin{aligned}
\Delta\left(t_{w}\right) & =\Delta \circ N_{d}\left(t_{w^{\prime}}\right) \\
& =\left(N_{d} \otimes I d+I d \otimes N_{d}\right) \circ \Delta\left(t_{w^{\prime}}\right)+(1 \otimes \cdot d) \cdot\left(\phi_{\lambda} \otimes I d\right) \circ \Delta\left(\left(t_{w^{\prime}}\right)\right.
\end{aligned}
$$

As $\mathcal{H}_{\lambda}$ is stable under $N_{d}$ and $\phi_{\lambda}$ and contains ${ }^{d}$, this belongs to $\mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}$. So $\mathcal{H}_{\lambda}$ is a Hopf subalgebra of $\mathcal{H}_{C K}^{\mathcal{D}}$.

As its generators are linear spans of trees, $\mathcal{H}_{\lambda}$ is stable under $\pi$. As it is also stable under $\Delta$, the vector space generated by the elements $t_{w}$ is stable under $\delta$, so is a pre-Lie coalgebra. Let us give a formula for this pre-Lie cobracket.

Proposition 37 Let $d_{1}, \ldots, d_{n} \in \mathcal{D}$. For all $J=\left\{i_{1}, \ldots, i_{k}\right\} \subseteq\{1, \ldots, n\}$, with $i_{1}<\ldots<$ $i_{k}$, we put:

- $w_{J}=d_{i_{1}} \ldots d_{i_{k}}$.
- $m(J)=\max \{i \mid\{1, \ldots, i\} \subseteq J\}$, with the convention $m(J)=0$ if $1 \notin J$.

Then:

$$
\delta\left(t_{d_{1} \ldots d_{n}}\right)=\sum_{I \subsetneq\{1, \ldots, n\}}\left(\sum_{i=1}^{m(I)} \lambda_{i}\right) t_{w_{I}} \otimes t_{w_{I^{c}}}
$$

Proof. By induction on $n$. It is obvious if $n=1$, as $\delta\left(t_{d_{1}}\right)=0$. Let us assume the result at rank $n-1$.

$$
\begin{aligned}
\delta\left(t_{d_{1} \ldots d_{n}}\right)= & \delta \circ N_{d_{n}}\left(t_{d_{1} \ldots d_{n-1}}\right) \\
= & \left(N_{d_{n}} \otimes I d+I d \otimes N_{d_{n}}\right) \circ \delta\left(t_{d_{1} \ldots d_{n-1}}\right)+\phi_{\lambda}\left(t_{d_{1} \ldots d_{n-1}}\right) \otimes \sum_{d_{n}} \\
= & \sum_{I \subsetneq\{1, \ldots, n-1\}}^{m(I)}\left(\sum_{i=1} \lambda_{d_{i}}\right)\left(N_{d_{n}} \otimes I d+I d \otimes N_{d_{n}}\right)\left(t_{w_{I}} \otimes t_{\left.w_{\{1, \ldots, n-1\} \backslash I}\right)}\right) \\
& +\left(\sum_{i=1}^{n-1} \lambda_{d_{i}}\right) t_{d_{1} \ldots d_{n-1}} \otimes t_{d_{n}} \\
= & \sum_{I \subsetneq\{1, \ldots, n-1\}}\left(\sum_{i=1}^{m(I)} \lambda_{d_{i}}\right)\left(t_{w_{I \cup\{n\}}} \otimes t_{w_{\{1, \ldots, n-1\} \backslash I}}+t_{w_{I}} \otimes t_{\left.w_{\{1, \ldots, n\} \backslash I}\right)}\right) \\
+ & \left(\sum_{i=1}^{n-1} \lambda_{d_{i}}\right) t_{d_{1} \ldots d_{n-1}} \otimes t_{d_{n}} \\
= & \left.\sum_{J \subsetneq\{1, \ldots, n\}, n \in J}\left(\sum_{i=1}^{m(J)} \lambda_{d_{i}}\right) t_{w_{J}} \otimes t_{w_{J c}}+\sum_{i \subsetneq\{1, \ldots, n-1\}}^{m(J)} \lambda_{d_{i}}\right) t_{w_{J}} \otimes t_{w_{J^{c}}} \\
& +\left(\sum_{i=1}^{n-1} \lambda_{d_{i}}\right) t_{d_{1} \ldots d_{n-1}} \otimes t_{d_{n}} \\
= & \sum_{J \subsetneq\{1, \ldots, n\}}\left(\sum_{i=1}^{m(J)} \lambda_{d_{i}}\right) t_{w_{J}} \otimes t_{w_{J c} .}
\end{aligned}
$$

So the result holds for all $n$.
Let us assume that all the $\lambda_{d}$ are non zero. For all word $w=d_{1} \ldots d_{n}$, in $t_{w}$ appears only one ladder, that is to say a tree with no ramification: its vertices, from the root to the unique leaf, are decorated by $d_{1}, \ldots, d_{n}$, and its coefficient is $\lambda_{d_{1}} \ldots \lambda_{d_{n-1}}$. So the elements $t_{w}$ are linearly independent. Dually, identifying the dual basis $\left(t_{d_{1} \ldots d_{n}}^{*}\right)_{d_{1}, \ldots, d_{n} \in \mathcal{D}}$ of the graded dual of the preLie coalgebra of the generators of $\mathcal{H}_{\lambda}$ and the basis $\left(x_{d_{1}} \ldots x_{d_{n}}\right)_{d_{1}, \ldots, d_{n} \in \mathcal{D}}$, we obtain a pre-Lie algebra structure on the space $T(V)$. If $d_{1}, \ldots, d_{k+l} \in \mathcal{D}$ :

$$
\begin{aligned}
x_{d_{1}} \ldots x_{d_{k}} \bullet x_{d_{k+1}} \ldots x_{d_{k}} & =\sum_{\sigma \in \operatorname{Sh}(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} \lambda_{d_{i}}\right)\left(\sigma \cdot\left(x_{d_{1}} \ldots x_{d_{k+l}}\right)\right) \\
& =\sum_{\sigma \in \operatorname{Sh}(k, l)}\left(\sum_{i=1}^{m_{k}(\sigma)} I d^{\otimes(i-1)} \otimes f \otimes I d^{\otimes(n-i)}\right)\left(\sigma \cdot\left(x_{d_{1}} \ldots x_{d_{k+l}}\right)\right),
\end{aligned}
$$

as if $i \leq m_{k}(\sigma), \sigma^{-1}(i)=i$ and $f\left(x_{d_{\sigma-1}(i)}\right)=\lambda_{d_{i}} x_{d_{i}}$. Hence, we obtain the pre-Lie algebra ( $T(V, f), \circ$ ).

If $x$ is an eigenvector of $f$ of eigenvalue $\lambda$, the preceding formula gives:

$$
x^{k} \bullet x^{l}=\lambda\binom{k+l}{k} x^{k+l} .
$$

Let us assume that $\lambda \neq 0$. For all $k \geq 0$, we put $y_{k}=\frac{(k+1)!}{\lambda} x^{k}$. As the characteristic of $\mathbb{K}$ is zero, $\left(y_{k}\right)_{k \geq 0}$ is a basis of $\mathbb{K}[x]$. We obtain:

$$
y_{k} \bullet y_{l}=\frac{k(k+1)}{k+l+1} y_{k+l},
$$

so $\left[y_{k}, y_{l}\right]=(k-l) y_{k+l}$. Hence, if $\lambda \neq 0, \mathbb{K}[x]_{+} \subseteq T(V, f)$ is isomorphic, as a Lie algebra, to the Faà di Bruno Lie algebra $[7,8,9]$.

Remark. In particular, if $V$ is one-dimensional and $f=I d_{V}$, we can work with a single decoration and delete it everywhere. The dual of the enveloping algebra of $T(V, f)$ is the subalgebra of $\mathcal{H}_{C K}$ generated by the elements $t_{n}$ inductively defined by $t_{1}=\cdot$ and $t_{n+1}=N\left(t_{n}\right)$ :

$$
\begin{aligned}
& t_{1}=\boldsymbol{\bullet} \\
& t_{2}=1 \\
& t_{3}=\bigvee+\downarrow \\
& t_{4}=\boldsymbol{V}+3 \bigvee+Y+\downarrow \\
& t_{5}=\boldsymbol{V}+6 \bigvee+3 \bigvee+4 \bigvee+4 \bigvee+Y+3 \bigvee+\vdots+\ddagger
\end{aligned}
$$

This is the Connes-Moscovici subalgebra of $\mathcal{H}_{C K}[3,4]$.

### 3.2 Examples from Control theory

We now consider the group of Fliess operators [11]. We fix an integer $n \geq 1$. We denote by $X^{*}$ the set of words in letters $x_{0}, \ldots, x_{n}$. As a set, the group of Fliess operators is isomorphic
to $G_{\text {Fliess }}=\mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle^{n} \approx \mathbb{K}^{n}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$. Let us now describe the composition. Let $c, d=\left(d_{1}, \ldots, d_{n}\right) \in G_{\text {Fliess }}$. We put:

$$
c=\sum_{w \in X^{*}} c_{w} w,
$$

with $c_{w} \in \mathbb{K}^{n}$ for all $w \in X^{*}$.

- For all $0 \leq i \leq n$, we consider:

$$
\tilde{D}_{x_{i}}:\left\{\begin{aligned}
G_{\text {Fliess }} & \longrightarrow G_{\text {Fliess }} \\
e & \longrightarrow x_{i} e+x_{0}\left(d_{i} \amalg e\right)
\end{aligned}\right.
$$

with $d_{0}=0$. If $w=x_{i_{1}} \ldots x_{i_{k}} \in X^{*}$, we put $\tilde{D}_{w}=\tilde{D}_{x_{i_{1}}} \circ \ldots \circ \tilde{D}_{x_{i_{k}}}$. By convention, $\tilde{D}_{\emptyset}=I d_{G_{\text {Fliess }}}$.

- The composition cõ $d$ is defined by:

$$
c \tilde{\diamond} d=\sum_{w \in X^{*}} c_{w} \tilde{D}_{w}(\emptyset) .
$$

In particular, $\emptyset \tilde{\delta} d=\tilde{D}_{\emptyset}(\emptyset)=\emptyset$.

- The product of $G_{F l i e s s}$ is then given by $c \diamond d=c \tilde{\diamond} d+d$.

Let $0 \leq i \leq n, c, d \in G_{\text {Fliess }}$. We keep the same notation as before.

$$
\begin{aligned}
x_{i} c \tilde{\diamond} d & =\sum_{w \in X^{*}} c_{w} \tilde{D}_{x_{i} w}(\emptyset) \\
& =\sum_{w \in X^{*}} c_{w} \tilde{D}_{x_{i}} \tilde{D}_{w}(\emptyset) \\
& =\sum_{w \in X^{*}} c_{w}\left(x_{i} \tilde{D}_{w}(\emptyset)+x_{0}\left(d_{i} Ш \tilde{D}_{w}(\emptyset)\right)\right. \\
& =x_{i}\left(\sum_{w \in X^{*}} c_{w} \tilde{D}_{w}(\emptyset)\right)+x_{0}\left(d_{i} Ш\left(\sum_{w \in X^{*}} c_{w} \tilde{D}_{w}(\emptyset)\right)\right) \\
& =x_{i}(c \tilde{\diamond} d)+x_{0}\left(d_{i} Ш(c \tilde{\diamond} d)\right) .
\end{aligned}
$$

In other words:

- $\tilde{\delta}$ is $\mathbb{K}^{n}$-linear on the left.
- For all $d \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle^{n}, \emptyset \tilde{\sim} d=\emptyset$.
- For all $c, d \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle^{n}, x_{i} c \tilde{c} d=x_{i}(c \tilde{\imath} d)+x_{0}\left(d_{i} ш(c \tilde{\delta} d)\right)$, with the convention $d_{0}=0$.

Notations. Let $\epsilon_{i}$ be the $i$-th element of the canonical basis. We put $G_{i}=\epsilon_{i} \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle^{n}$.
Lemma 38 If $c, d \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$ :

$$
\left(\epsilon_{i} \emptyset\right) \tilde{\diamond} v=\epsilon_{i} \emptyset, \quad\left(\epsilon_{i} x_{j} d\right) \tilde{\diamond} d=\epsilon_{i} x_{j}(c \tilde{\diamond} d)+\epsilon_{i} \delta_{i, j} x_{0}((c \tilde{\diamond} d) Ш d) .
$$

Proof. Comes from the left $\mathbb{K}^{n}$-linearity and that $\epsilon_{i} \epsilon_{j}=\delta_{i, j} \epsilon_{i}$.
Theorem 39 1. For all $1 \leq i \leq n, G_{i}$ is a subgroup of $G$. Moreover, it is isomorphic to the group of characters associated to $\left(V, f_{i}\right)$, with $V=\operatorname{Vect}\left(x_{0}, \ldots, x_{n}\right)$ and:

$$
f_{i}:\left\{\begin{array}{rll}
V & \longrightarrow & V \\
x_{j} & \longrightarrow & \delta_{i, j} x_{0} .
\end{array}\right.
$$

2. The group $G$ is the direct product of the subgroups $G_{i}$ 's, $1 \leq i \leq n$.

Proof. 1. As a set, we identify $G_{i}$ with $\mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$, via the multiplication by $\epsilon_{i}$. In this group, If $u, v \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$, by lemma 38:

$$
\emptyset \tilde{\Delta} v=\emptyset, \quad x_{j} u \tilde{\nu} v=x_{j}(u \tilde{\nu} v)+\delta_{i, j} x_{0}((u \tilde{\nu} v) Ш v) .
$$

We consider the character group associated to $f_{i}$. As $f_{i}^{2}=0$, theorem $15-2$ becomes:

$$
\epsilon_{i} x_{j} u \tilde{\sim} v=x_{j}(u \tilde{\diamond} v)+\delta_{i, j} x_{0}((u \tilde{\Sigma} v) Ш v) .
$$

So $G_{i}$ is isomorphic to this group of characters, via the multiplication by $\epsilon_{i}$.
2. Let $1 \leq i, j \leq n$, with $i \neq j$. Then, by lemma 38, if $u, v \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$ :

$$
\begin{aligned}
\left(\epsilon_{i} \emptyset\right) \tilde{\diamond}\left(\epsilon_{j} v\right) & =\epsilon_{i} \emptyset \\
\left(\epsilon_{i} x_{k} u\right) \tilde{\diamond}\left(\epsilon_{j} v\right) & =\epsilon_{i} x_{k}\left(u \tilde{\diamond}\left(\epsilon_{j} v\right)\right)+\epsilon_{i} \delta_{i, k} x_{0}\left(\left(u \tilde{\diamond}\left(\epsilon_{j} v\right)\right) Ш\left(\epsilon_{j} v\right)\right) \\
& =\epsilon_{i} x_{k}\left(u \tilde{\diamond}\left(\epsilon_{j} v\right)\right)+\epsilon_{i} \epsilon_{j} \delta_{i, k} x_{0}\left(\left(u \tilde{\diamond}\left(\epsilon_{j} v\right)\right) Ш v\right) \\
& =x_{k}\left(\epsilon_{i} u \tilde{\diamond}\left(\epsilon_{j} v\right)\right)+0 .
\end{aligned}
$$

An easy induction on the length then proves that for all word $u$, for all $v \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$, $\left(\epsilon_{i} u\right) \tilde{\sim}\left(\epsilon_{j} v\right)=\epsilon_{i} u$. By continuity of $\tilde{\varnothing}$, for all $u, v \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle,\left(\epsilon_{i} u\right) \tilde{\sim}\left(\epsilon_{j} v\right)=\epsilon_{i} u$. Hence, for all $u, v \in \mathbb{K}\left\langle\left\langle x_{0}, \ldots, x_{n}\right\rangle\right\rangle$ :

$$
\left(\epsilon_{i} u\right) \diamond\left(\epsilon_{j} v\right)=\epsilon_{i} u+\epsilon_{j} v
$$

As a conclusion, $G$ is the direct sum of the subgroups $G_{i}$ 's.
The Lie algebra of $G_{i}$ is the pre-Lie algebra $T\left(V, f_{i}\right)=\mathbb{K}\left\langle x_{0}, \ldots, x_{n}\right\rangle$ with the composition associated to $f_{i}$. Let us give a few results on this pre-Lie algebra:

Theorem $40 T\left(V, f_{i}\right)$ inherits a gradation such that $x_{i}$ is homogeneous of degree 1 for all $1 \leq i \leq n$ and $x_{0}$ is homogeneous of degree 0 . The formal series of this gradation is:

$$
F_{T\left(V, f_{i}\right)}(X)=\frac{X}{1-n X-X^{2}}=\sum_{k \geq 0} \frac{1}{\sqrt{n^{2}+4}}\left(\left(\frac{n+\sqrt{n^{2}+4}}{2}\right)^{k}-\left(\frac{n-\sqrt{n^{2}+4}}{2}\right)^{k}\right) X^{k} .
$$

Proof. We put $V_{1}=\operatorname{Vect}\left(x_{1}, \ldots, x_{n}\right)$ and $V_{2}=\operatorname{Vect}\left(x_{0}\right)$. It defines a gradation of $V$, such that $f_{i}$ is homogeneous of degree 1 . We apply corollary 11 .

## Examples.

$$
\begin{aligned}
\operatorname{dim}\left(T\left(V, f_{i}\right)_{0}\right) & =0 \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{1}\right) & =1 \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{2}\right) & =n \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{3}\right) & =n^{2}+1, \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{4}\right) & =n\left(n^{2}+2\right), \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{5}\right) & =n^{4}+3 n^{2}+1 \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{7}\right) & =n^{6}+5 n^{4}+6 n^{2}+1 \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{8}\right) & =n\left(n^{2}+2\right)\left(n^{4}+4 n^{2}+2\right), \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{9}\right) & =\left(n^{2}+1\right)\left(n^{6}+6 n^{4}+9 n^{2}+1\right), \\
\operatorname{dim}\left(T\left(V, f_{i}\right)_{10}\right) & =n\left(n^{4}+3 n^{2}+1\right)\left(n^{4}+5 n^{2}+5\right) .
\end{aligned}
$$

These are the Fibonacci polynomials, see sequence A011973 in [22]. Here are several specializations of these sequences, which can be found in [22]:

| $n$ | ref. | $n$ | ref. | $n$ | ref. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $A 000045$ | 2 | $A 000129$ | 3 | $A 006190$ |
| 4 | $A 001076$ | 5 | $A 052918$ | 6 | $A 005668$ |
| 7 | $A 054413$ | 8 | $A 041025$ | 9 | $A 099371$ |
| 10 | $A 041041$ | 11 | $A 049666$ | 12 | $A 041061$ |
| 13 | $A 140455$ | 14 | $A 041085$ | 16 | $A 041113$ |
| 18 | $A 041145$ | 20 | $A 041181$ | 22 | $A 041221$ |

These sequences are generically called generalized Fibonacci sequences. If $n=1$, this is the Fibonacci sequence; if $n=2$, this is the Pell sequence.

Remarks. By corollary 23, $T\left(V, f_{i}\right)$ is generated, as a pre-Lie algebra, by $\mathbb{K}\left\langle x_{1}, \ldots, x_{n}\right\rangle$. Moreover, by corollary $33, T_{+}\left(V, f_{i}\right)$ is generated, as a Com-Pre-Lie algebra, by $\operatorname{Vect}\left(x_{0}, \ldots, x_{n}\right)$ if, and only if, $n=1$. We recover in this way a result of [10].

## 4 Appendix

### 4.1 Admissible words and Dyck paths

Recall that a Dyck path of length $2 n$ is a path from $(0,0)$ to $(n, n)$, made of steps $\rightarrow$ and $\uparrow$, always staying under the diagonal, but possibly touching it in other points that $(0,0)$ and $(n, n)$. The set of Dyck paths of length $2 n$ is denoted by $D(n)$.

## Examples.

$$
\begin{aligned}
& D(0)=\{.\} \\
& D(1)=\{\rightarrow \uparrow\} \\
& D(2)=\{\rightarrow \uparrow \rightarrow \uparrow, \rightarrow \rightarrow \uparrow \uparrow\}, \\
& D(3)=\{\rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow, \rightarrow \rightarrow \uparrow \uparrow \rightarrow \uparrow, \rightarrow \uparrow \rightarrow \rightarrow \uparrow \uparrow, \rightarrow \rightarrow \uparrow \rightarrow \uparrow \uparrow, \rightarrow \rightarrow \rightarrow \uparrow \uparrow \uparrow\} .
\end{aligned}
$$

Note that a nonempty Dyck path always starts by $\rightarrow$ and ends by $\uparrow$.

Lemma 41 Let $\rightarrow \uparrow^{a_{1}} \rightarrow \uparrow^{a_{2}} \ldots \rightarrow \uparrow a_{n}$ be a path starting from ( 0,0 ), with $a_{1}, \ldots, a_{n} \geq 0$.

1. It is a path from $(0,0)$ to $(n, n)$ if, and only if, $a_{1}+\ldots+a_{n}=n$.
2. This path is under the diagonal if, and only if, for all $1 \leq i \leq n, a_{1}+\ldots+a_{i} \leq i$.

Proof. 1. The path contains $n \rightarrow$ and $a_{1}+\ldots+a_{n} \uparrow$, so is a path from $(0,0)$ to $\left(n, a_{1}+\right.$ $\left.\ldots+a_{n}\right)$.
2. Immediate.

Notation. For all $n \geq 1$, we denote by $\mathcal{W}(n)$ the set of first lines of admissible biwords of length $n$, that is to say words $a_{1} \ldots a_{n}$ with letters in $\mathbb{N}$, such that:

- For all $1 \leq i \leq n, a_{i}+\ldots+a_{n} \leq n-i$.
- $a_{1}+\ldots+a_{n}=n-1$.

We denote by $\mathcal{W}_{\sigma}(n)$ the set of first lines of $\sigma$-admissible biwords of length $n$, that is to say words $w_{1} \ldots w_{k}$ of length $k$, such that $w_{1}, \ldots w_{k}$ are admissible words.

Proposition 42 For all $n \geq 0$, the following map is a bijection:

$$
\theta_{n}: \begin{cases}\mathcal{W}(n+1) & \longrightarrow D(n) \\ a_{1} \ldots a_{n+1} & \longrightarrow \rightarrow \uparrow^{a_{n}} \ldots \rightarrow \uparrow^{a_{1}}\end{cases}
$$

Proof. First, $\theta_{n}$ is well-defined: if $a_{1} \ldots a_{n+1} \in \mathcal{W}(n+1)$, by definition $a_{n+1}=0$. Moreover, $a_{1}+\ldots+a_{n}=a_{1}+\ldots+a_{n+1}=n+1-1=n$. By lemma 41, it is a path from $(0,0)$ to $(n, n)$. For all $1 \leq i \leq n, a_{i}+\ldots+a_{n}=a_{i}+\ldots+a_{n+1} \leq n+1-i$. By lemma 41, it is under the diagonal, so it is indeed in $D(n)$.
$\theta_{n}$ is injective: if $\theta_{n}\left(a_{1} \ldots a_{n+1}\right)=\theta_{n}\left(b_{1} \ldots b_{n+1}\right)$, then immediately $a_{1}=b_{1}, \ldots, a_{n}=b_{n}$; moreover, $a_{n+1}=b_{n+1}=0$.
$\theta_{n}$ is surjective: if $P$ is a Dyck path of length $n$, as it starts by $\rightarrow$ and contains exactly $n \rightarrow$ and $\uparrow$, it can be written under the form $\rightarrow \uparrow^{a_{n}} \ldots \rightarrow \uparrow^{a_{1}}$, with $a_{1}+\ldots+a_{n}=n$. As it is under the diagonal, by lemma 41 , for all $1 \leq i \leq n, a_{i}+\ldots+a_{n} \leq n+1-i$. So $a_{1} \ldots a_{n} 0 \in \mathcal{W}(n+1)$ and $\theta\left(a_{1} \ldots a_{n} 0\right)=P$.

Corollary 43 For all $n \geq 1,|\mathcal{W}(n)|$ is the $(n-1)$-th Catalan number $\frac{1}{n}\binom{n n-2}{n-1}$.
Corollary 44 1. For all $n \geq 0,\left|\mathcal{W}_{\sigma}(n)\right|$ is the $n$-th Catalan number $\frac{1}{n+1}\binom{2 n}{n}$.
2. Let $a_{1} \ldots a_{n}$ be a word with letters in $\mathbb{N}$. It is $\sigma$-admissible if, and only if, for all $1 \leq i \leq n$, $a_{i}+\ldots+a_{n} \leq n-i$.

Proof. 1. Let $C(X)$ be the formal series of Catalan numbers:

$$
C(X)=\sum_{k=0}^{\infty} \frac{1}{n+1}\binom{2 n}{n} X^{n}=\frac{1-\sqrt{1-4 X}}{2 X} .
$$

Then the formal series of admissible words is:

$$
f(X)=\sum_{n \geq 1}|\mathcal{W}(n)| X^{n}=X C(X)=\frac{1-\sqrt{1-4 X}}{2}
$$

so $f^{2}(X)-f(X)+X=0$ or, equivalently:

$$
\frac{1}{1-f(X)}=\frac{f(X)}{X}
$$

By the unicity of decomposition of $\sigma$-admissible words in lemma 29 , the formal series of $\sigma$ admissible words is:

$$
g(X)=\sum_{n \geq 0}\left|\mathcal{W}_{\sigma}(n)\right| X^{n}=\frac{1}{1-f(X)}=\frac{f(X)}{X}=C(X)
$$

so $\left|\mathcal{W}_{\sigma}(n)\right|$ is the $n$-th Catalan number.
2. We denote by $\mathcal{W}_{\sigma}^{\prime}(n)$ the set of words $a_{1} \ldots a_{n}$ such that for all $i, a_{i}+\ldots+a_{n} \leq n-i$. By lemma $29, \mathcal{W}_{\sigma}(n) \subseteq \mathcal{W}_{\sigma}^{\prime}(n)$. By lemma 29 , the following map is well-defined:

$$
\alpha_{n}: \begin{cases}\mathcal{W}_{\sigma}^{\prime}(n) & \longrightarrow \mathcal{W}(n+1) \\ a_{1} \ldots a_{n} & \longrightarrow\left(n-a_{1}-\ldots-a_{n}\right) a_{1} \ldots a_{n}\end{cases}
$$

It is clearly injective, so $\left|\mathcal{W}_{\sigma}^{\prime}(n)\right| \leq|\mathcal{W}(n+1)|=\left|\mathcal{W}_{\sigma}(n)\right|$. So $\mathcal{W}_{\sigma}(n)=\mathcal{W}_{\sigma}^{\prime}(n)$.

### 4.2 Image of the morphism $\Phi_{C P L}$

Proposition $45 \mathfrak{g}_{\mathcal{D}}=\Phi_{P L}(\operatorname{Prelie}(\mathcal{D}))$ is included in the space generated by admissible words and $\Phi_{C P L}(\operatorname{ComPrelie}(\mathcal{D}))$ is included in the space generated by $\sigma$-admissible words.

Proof. Let $t$ be a partitioned tree with $n$ vertices and $k$ blocks, and let $f$ be a linear extension of $t$. We put $f \operatorname{ert}(f(i))=a_{i}$ for all $i$. It is not difficult to prove that $a_{1}+\ldots+a_{n}=k-1$. As $k \leq n, a_{1}+\ldots+a_{n} \leq n-1$; if $t$ is a rooted tree, $k=n$ and $a_{1}+\ldots+a_{n}=n-1$. Let us prove that $a_{i}+\ldots+a_{n} \leq n-i$ for all $1 \leq i \leq n$ by induction on $n$. It is obvious if $n=0$. If $n \geq 1$, we already proved the result if $i=1$. Let us assume that $i \geq 2$. We put $t^{\prime}=f^{-1}(\{2, \ldots, n\})$. As $f$ is a linear extension of $t, t^{\prime}$ is an ideal of $t$, and $b_{1} \ldots b_{n-1}=a_{2} \ldots a_{n}$ appears in $\Phi_{C P L}\left(t^{\prime}\right)$. By the induction hypothesis:

$$
a_{i}+\ldots+a_{n}=b_{i-1}+\ldots+b_{n-1} \leq(n-1)-(i-1)=n-i .
$$

Finally, if $t$ is a partitioned tree, any biword appearing in $\Phi_{C P L}(t)$ is $\sigma$-admissible; if $t$ is a rooted tree, any biword appearing in $\Phi_{C P L}(t)$ is admissible.

Remark. It is not difficult to see that $\mathfrak{g}_{\mathcal{D}}$ is strictly included in the space of admissible words and that $\Phi_{C P L}(\operatorname{ComPrelie}(\mathcal{D}))$ is strictly included in the space of $\sigma$-admissible words.

Corollary 46 The space of $\sigma$-admissible words, with the shuffle product $\amalg$ and the pre-Lie product • of $T\left(V_{\mathcal{D}}, f_{\mathcal{D}}\right)$, is a Com-Pre-Lie algebra.

Proof. Let $w$ be a $\sigma$-admissible biwords. Let us first prove that there exists a partitioned tree $t$ such that $w$ appears in $\Phi_{C P L}(t)$. We proceed by induction on the length $n$ of $w$. If $n=0$, then we take $t=1$. Let us assume the result at all rank $<n$. If $w=a_{1} \ldots a_{n}$ is admissible, then the second point implies that $a_{2} \ldots a_{n}$ is $\sigma$-admissible, and $a=1\binom{n-a_{1}-\ldots-a_{n}}{d}$ for a certain $d \in \mathcal{D}$. By the induction hypothesis, there exists a forest $t^{\prime}$ such that $a_{2} \ldots a_{n}$ appears in $\Phi_{C P L}\left(t^{\prime}\right)$. Then $a_{1} \ldots a_{n}$ appears in $\Phi_{C P L}\left(\cdot{ }_{d} \bullet t^{\prime}\right)$. If $w=w_{1} \ldots w_{k}$, with $w_{1}, \ldots, w_{k}$ admissible and $k \geq 2$, then for all $i$ there exists $t_{i}$ such that $w_{i}$ appears in $\Phi_{C P L}\left(t_{i}\right)$. Hence, $w_{1} \ldots w_{k}$, being a particular shuffling of $w_{1}, \ldots, w_{k}$, appears in $\Phi_{C P L}\left(t_{1}\right) ш \ldots ш \Phi_{C P L}\left(t_{k}\right)=\Phi_{C P L}\left(t_{1} \ldots t_{k}\right)$.

Let us consider now two $\sigma$-admissible biwords $u$ and $u^{\prime}$. There exists admissible forests $t$ and $t^{\prime}$ such that $u$ appears in $\Phi_{C P L}(t)$ and $u^{\prime}$ appears in $\Phi_{C P L}\left(t^{\prime}\right)$. Then any word appearing in $u ш u^{\prime}$ or in $u \bullet u^{\prime}$ appears in $\Phi_{C P L}(t) \boldsymbol{\omega} \Phi_{C P L}\left(t^{\prime}\right)=\Phi_{C P L}\left(t ш t^{\prime}\right)$ or in $\Phi_{C P L}(t) \bullet \Phi_{C P L}\left(t^{\prime}\right)=\Phi_{C P L}\left(t \bullet t^{\prime}\right)$, so is $\sigma$-admissible by proposition 45 .

## References

[1] D. J. Broadhurst and D. Kreimer, Towards cohomology of renormalization: bigrading the combinatorial Hopf algebra of rooted trees, Comm. Math. Phys. 215 (2000), no. 1, 217-236, arXiv:hep-th/00 01202.
[2] Frédéric Chapoton and Muriel Livernet, Pre-Lie algebras and the rooted trees operad, Internat. Math. Res. Notices (2001), no. 8, 395-408, arXiv:math/0002069.
[3] A. Connes and H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem, Comm. Math. Phys. 198 (1998), no. 1, 199-246.
[4] Alain Connes and Dirk Kreimer, Hopf algebras, Renormalization and Noncommutative geometry, Comm. Math. Phys 199 (1998), no. 1, 203-242, arXiv:hep-th/98 08042.
[5] Ioannis Dokas, Zinbiel algebras and commutative algebras with divided powers, Glasg. Math. J. 52 (2010), no. 2, 303-313.
[6] Samuel Eilenberg and Saunders Mac Lane, On the groups of $H(\Pi, n)$. I, Ann. of Math. (2) 58 (1953), 55-106.
[7] Héctor Figueroa and José M. Gracia-Bondía, On the antipode of Kreimer's Hopf algebra, Modern Phys. Lett. A 16 (2001), no. 22, 1427-1434.
[8] Loïc Foissy, Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson-Schwinger equations, Adv. Math. 218 (2008), no. 1, 136-162, arXiv:0707.1204.
[9] , Hopf subalgebras of rooted trees from Dyson-Schwinger equations, Motives, quantum field theory, and pseudodifferential operators, Clay Math. Proc., vol. 12, Amer. Math. Soc., Providence, RI, 2010, pp. 189-210.
[10] _, The Hopf algebra of Fliess operators and its dual pre-Lie algebra, arXiv:1304.1726, 2013.
[11] W. Steven Gray and Luis A. Duffaut Espinosa, A Faà di Bruno Hopf algebra for a group of Fliess operators with applications to feedback, Systems Control Lett. 60 (2011), no. 7, 441-449.
[12] Michael E. Hoffman, Combinatorics of rooted trees and Hopf algebras, Trans. Amer. Math. Soc. 355 (2003), no. 9, 3795-3811.
[13] Jean-Louis Loday, Cup-product for Leibniz cohomology and dual Leibniz algebras, Math. Scand. 77 (1995), no. 2, 189-196.
[14] _, Scindement d'associativité et algèbres de Hopf, Actes des Journées Mathématiques à la Mémoire de Jean Leray, Sémin. Congr., vol. 9, Soc. Math. France, Paris, 2004, arXiv:math/0402084, pp. 155-172.
[15] Jean-Louis Loday and María Ronco, Combinatorial Hopf algebras, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, arXiv:0810.0435, pp. 347-383.
[16] Saunders MacLane, The homology products in $K(\Pi, n)$, Proc. Amer. Math. Soc. 5 (1954), 642-651.
[17] Anthony Mansuy, Preordered forests, packed words and contraction algebras, arXiv:1305.0343, 2013.
[18] J.-M. Oudom and D. Guin, On the Lie enveloping algebra of a pre-Lie algebra, J. K-Theory 2 (2008), no. 1, 147-167, arXiv:math/0404457.
[19] Jean-Michel Oudom and Daniel Guin, Sur l'algèbre enveloppante d'une algèbre pré-Lie, C. R. Math. Acad. Sci. Paris 340 (2005), no. 5, 331-336.
[20] Florin Panaite, Relating the Connes-Kreimer and Grossman-Larson Hopf algebras built on rooted trees, Lett. Math. Phys. 51 (2000), no. 3, 211-219.
[21] Christophe Reutenauer, Free Lie algebras, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications.
[22] N. J. A Sloane, On-line encyclopedia of integer sequences, avalaible at http://oeis.org/.

