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► To cite this version:

Francois Ledrappier, Lin Shu. Differentiating the stochastic entropy for compact negatively curved spaces under conformal changes. 2013. hal-00863933v1

HAL Id: hal-00863933

<https://hal.science/hal-00863933v1>

Preprint submitted on 20 Sep 2013 (v1), last revised 11 Aug 2015 (v2)

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DIFFERENTIATING THE STOCHASTIC ENTROPY FOR COMPACT NEGATIVELY CURVED SPACES UNDER CONFORMAL CHANGES

FRANÇOIS LEDRAPPIER AND LIN SHU

ABSTRACT. We consider the universal cover of a closed connected Riemannian manifold of negative sectional curvature. We show that the linear drift and the stochastic entropy are differentiable under any C^3 one-parameter family of C^3 conformal changes of the original metric.

1. INTRODUCTION

Let (M, g) be an m -dimensional closed connected Riemannian manifold, and $\pi : (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ its universal cover endowed with the lifted Riemannian metric. The fundamental group $G = \pi_1(M)$ acts on \widetilde{M} as isometries such that $M = \widetilde{M}/G$.

We consider the Laplacian $\Delta := \text{Div} \nabla$ on smooth functions on $(\widetilde{M}, \widetilde{g})$ and the corresponding heat kernel function $p(t, x, y), t \in \mathbb{R}_+, x, y \in \widetilde{M}$, which is the fundamental solution to the heat equation $\frac{\partial u}{\partial t} = \Delta u$. Denote by Vol the Riemannian volume on \widetilde{M} . The following quantities were introduced by Guivarc'h [Gu] and Kaimanovich [K1], respectively, and are independent of $x \in \widetilde{M}$:

- the linear drift $\ell := \lim_{t \rightarrow +\infty} \frac{1}{t} \int d_{\widetilde{g}}(x, y) p(t, x, y) d\text{Vol}(y)$.
- the stochastic entropy $h := \lim_{t \rightarrow +\infty} -\frac{1}{t} \int p(t, x, y) \ln p(t, x, y) d\text{Vol}(y)$.

Let $\{g^\lambda = e^{2\varphi^\lambda} g : |\lambda| < 1\}$ be a one-parameter family of conformal changes of $g^0 = g$, where φ^λ 's are real valued functions on M such that $(\lambda, x) \mapsto \varphi^\lambda(x)$ is C^3 . Denote by ℓ_λ, h_λ the linear drift and the stochastic entropy for (M, g^λ) . We show

Theorem 1.1. *Let (M, g) be a negatively curved closed connected Riemannian manifold. With the above notation, the functions $\lambda \mapsto \ell_\lambda$ and $\lambda \mapsto h_\lambda$ are differentiable at 0.*

For each $\lambda \in (-1, 1)$, let Δ^λ be the Laplacian of $(\widetilde{M}, \widetilde{g}^\lambda)$ with heat kernel $p^\lambda(t, x, y), t \in \mathbb{R}_+, x, y \in \widetilde{M}$, and the associated Brownian motion $\omega_t^\lambda, t \geq 0$. The relation between Δ^λ

2000 *Mathematics Subject Classification.* 37D40, 58J65.

Key words and phrases. linear drift, locally symmetric space, stochastic entropy.

and Δ is easy to be formulated using $g^\lambda = e^{2\varphi^\lambda} g$:

$$\Delta^\lambda = e^{-2\varphi^\lambda} \left(\Delta + (m-2) \nabla^k \varphi^\lambda \nabla_k \right) =: e^{-2\varphi^\lambda} L^\lambda,$$

where we still denote φ^λ its lift to \widetilde{M} . Let $\widehat{p}^\lambda(t, x, y), t \in \mathbb{R}_+, x, y \in \widetilde{M}$, be the heat kernel of the diffusion process $\widehat{\omega}_t^\lambda, t \geq 0$, corresponding to the operator L^λ in $(\widetilde{M}, \widetilde{g})$. We define

- $\widehat{\ell}_\lambda := \lim_{t \rightarrow +\infty} \frac{1}{t} \int d\widetilde{g}(x, y) \widehat{p}^\lambda(t, x, y) d\text{Vol}(y)$.
- $\widehat{h}_\lambda := \lim_{t \rightarrow +\infty} -\frac{1}{t} \int \widehat{p}^\lambda(t, x, y) \ln \widehat{p}^\lambda(t, x, y) d\text{Vol}(y)$.

It is clear that the following hold true providing all the limits exist:

$$\begin{aligned} (d\ell_\lambda/d\lambda)|_{\lambda=0} &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\ell_\lambda - \widehat{\ell}_\lambda) + \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\widehat{\ell}_\lambda - \ell_0) =: (\mathbf{I})_\ell + (\mathbf{II})_\ell, \\ (dh_\lambda/d\lambda)|_{\lambda=0} &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (h_\lambda - \widehat{h}_\lambda) + \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\widehat{h}_\lambda - h_0) =: (\mathbf{I})_h + (\mathbf{II})_h. \end{aligned}$$

Here, loosely speaking, $(\mathbf{I})_\ell$ and $(\mathbf{I})_h$ are the infinitesimal drift and entropy affects of simultaneous metric change and time change of the diffusion, while $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ are the infinitesimal responses to the adding of drifts to ω_t^0 .

To analyze $(\mathbf{I})_\ell$ and $(\mathbf{I})_h$, we express the above linear drifts and entropies using the geodesic spray, the Martin kernel and the exit probability of the Brownian motion at infinity. It is known ([**K1**]) that

$$(1.1) \quad \ell_\lambda = \int_{M_0 \times \partial \widetilde{M}} \langle \overline{X}^\lambda, \nabla^\lambda \ln k_\xi^\lambda \rangle_\lambda d\widetilde{\mathbf{m}}^\lambda, \quad h_\lambda = \int_{M_0 \times \partial \widetilde{M}} \|\nabla^\lambda \ln k_\xi^\lambda\|_\lambda^2 d\widetilde{\mathbf{m}}^\lambda,$$

where M_0 is a fundamental domain of \widetilde{M} , $\partial \widetilde{M}$ is the geometric boundary of \widetilde{M} , \overline{X}^λ is the \widetilde{g}^λ -geodesic spray (i.e., $\overline{X}^\lambda(x, \xi)$ is the unit tangent vector of the \widetilde{g}^λ -geodesic starting from x pointing at ξ), $k_\xi^\lambda(x)$ is the Martin kernel function of ω_t^λ and $\widetilde{\mathbf{m}}^\lambda$ is the harmonic measure associated with Δ^λ . (Exact definitions will appear in Sec. 2.) Similar formulas also exist for $\widehat{\ell}_\lambda$ and \widehat{h}_λ (see Proposition 3.5 and the second part of the proof of Theorem 5.7):

$$(1.2) \quad \widehat{\ell}_\lambda = \int \langle \overline{X}^0, \nabla^0 \ln k_\xi^\lambda \rangle_0 d\widehat{\mathbf{m}}^\lambda, \quad \widehat{h}_\lambda = \int \|\nabla^0 \ln k_\xi^\lambda(x)\|_0^2 d\widehat{\mathbf{m}}^\lambda,$$

where $\widehat{\mathbf{m}}^\lambda$ is the harmonic measure related to the operator L^λ . The quantity $(\mathbf{I})_h$ turns out to be zero since the norm and the gradient changes cancel with the measure change, while the Martin kernel function remains the same under time rescaling of the diffusion process. But the metric variation is more involved in $(\mathbf{I})_\ell$ as we can see from the formulas in (1.1) and (1.2) for ℓ_λ and $\widehat{\ell}_\lambda$. Using the existing results of Jacobi fields and the (g, g^λ) -Morse correspondence maps (see [**Ano, Gro, Mor**] and [**FF**]), which are homeomorphisms between the unit tangent bundle spaces in g and g^λ metrics preserving the geodesics on M , we are able to identify the differential

$$(1.3) \quad \left(\overline{X}^\lambda \right)'_0(x, \xi) := \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left(\overline{X}^\lambda(x, \xi) - \overline{X}^0(x, \xi) \right)$$

using the stable and unstable Jacobi tensors and a family of Jacobi fields arising naturally from the infinitesimal Morse correspondence (Proposition 5.5 and Corollary 5.6). As a consequence, we can express $(\mathbf{I})_\ell$ using k_ξ , $\tilde{\mathbf{m}}$ and these terms (Theorem 5.7).

If we continue to analyze $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ using (1.1) and (1.2), we have the problem of showing the regularity in λ of the gradient of the Martin kernels. We avoid this by using an idea from Mathieu ([Ma]) to study $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ along the diffusion processes. For every point $x \in \widetilde{M}$ and almost every (a.e.) \tilde{g} -Brownian motion path ω^0 starting from x , it is known ([K1]) that

$$(1.4) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} d_{\tilde{g}}(x, \omega_t^0) = \ell_0, \quad \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln G(x, \omega_t^0) = h_0,$$

where $G(\cdot, \cdot)$ on $\widetilde{M} \times \widetilde{M}$ denotes the Green function for \tilde{g} -Brownian motion. A further study on the convergence of the limits of (1.4) in [L2] showed that there are positive numbers σ_0, σ_1 so that the distributions of the variables

$$Z_{\ell,t}(x) = \frac{1}{\sigma_0 \sqrt{t}} [d_{\tilde{g}}(x, \omega_t^0) - t\ell_0], \quad Z_{h,t}(x) = \frac{1}{\sigma_1 \sqrt{t}} [-\ln G(x, \omega_t^0) - th_0]$$

are asymptotically close to the normal distribution as t goes to infinity. Note that all $\hat{\omega}_t^\lambda$ starting from x can be simultaneously represented as random processes on the probability space (Θ, \mathbb{Q}) of a standard m -dimensional Euclidean Brownian motion. By using the Girsanov-Cameron-Martin formula on manifolds (cf. [El]), we show

$$(1.5) \quad (\mathbf{II})_\ell = \lim_{t \rightarrow +\infty} \mathbb{E}(Z_{\ell,t} M_t) \quad \text{and} \quad (\mathbf{II})_h = \lim_{t \rightarrow +\infty} \mathbb{E}(Z_{h,t} M_t),$$

where \mathbb{E} means the expectation and each M_t is a random process on (Θ, \mathbb{Q}) recording the change of metrics along the trajectories of Brownian motion to be specified in Sec. 5. We will further specify $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ in Theorem 5.7 using properties of martingales and the Central Limit Theorems for the linear drift and the stochastic entropy.

An immediate consequence of Theorem 1.1 is that $D_\lambda := h_\lambda / \ell_\lambda$, which is proportional¹ to the Hausdorff dimension of the distribution of the Brownian motion ω^λ at the infinity boundary of \widetilde{M} ([L1]), is also differentiable in λ . Let $\mathfrak{R}(M)$ be the manifold of negatively curved C^3 metrics on M . Another consequence of Theorem 1.1 is that

Theorem 1.2. *Let (M, g) be a negatively curved compact connected Riemannian manifold. If it is locally symmetric, then for any C^3 curve $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}(M)$ of conformal changes of the metric $g^0 = g$ with constant volume,*

$$(dh_\lambda/d\lambda)|_{\lambda=0} = 0, \quad (d\ell_\lambda/d\lambda)|_{\lambda=0} = 0.$$

In case M is a Riemannian surface, Theorem 1.2 for stochastic entropy does not provide any criterion information for locally symmetric spaces. This is because any $g \in \mathfrak{R}(M)$ is a conformal change of the metric with constant curvature by the Uniformization Theorem.

¹ D_λ is $\frac{1}{\ell}$ the Hausdorff dimension of the exit measure for the ι -Busemann distance (cf. Sec. 4.2).

The above calculation yields $dh_\lambda/d\lambda \equiv 0$, which implies the stochastic entropy remains the same for $g \in \mathfrak{R}(M)$ with constant volume.

When M has dimension at least 3, it is interesting to know whether the converse direction of Theorem 1.2 for the stochastic entropy holds. We have the following question.

Let (M, g) be a negatively curved compact connected Riemannian manifold with dimension greater than 3. Do we have that (M, g) is locally symmetric if and only if for any C^3 curve $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}(M)$ of constant volume with $g^0 = g$, the mapping $\lambda \mapsto h_\lambda$ is differentiable and has critical point at 0?

We will present the proof of Theorem 1.1 in a more general setting. In Sec. 2, we give some basic properties of harmonic measures for an operator \mathcal{L} subordinate to the stable foliation from [H2]. The corresponding formulas and the Central Limit Theorems for the linear drift and the stochastic entropy for \mathcal{L} will appear in Sec. 3 (Proposition 3.5 and Proposition 3.6). In Sec. 4, we proceed to study the regularity of the linear drift and the stochastic entropy under the change of the drift part of the operator \mathcal{L} (Theorem 4.3 and Theorem 4.9). As an application, we will obtain (1.5). In Sec. 5, we will first introduce a special Morse correspondence map and characterize it using stable and unstable Jacobi fields (Proposition 5.3 and Proposition 5.4). This step is important to valid the differential in (1.3) (Proposition 5.5 and Corollary 5.6). Following the argument mentioned above, we will obtain explicit formulas for $(d\ell_\lambda/d\lambda)|_{\lambda=0}$ and $(dh_\lambda/d\lambda)|_{\lambda=0}$ in Theorem 5.7, which, in particular, will imply Theorem 1.1. Finally, Theorem 1.2 can be deduced either using the formulas in Theorem 5.7 or merely using Theorem 1.1 and the existing results concerning the regularity of volume entropy for compact negatively curved spaces under conformal changes from [Ka, KKPW].

2. HARMONIC MEASURES FOR THE STABLE FOLIATION

Let $(\widetilde{M}, \widetilde{g})$ be the universal cover space of (M, g) , a negatively curved m -dimensional closed connected Riemannian manifold with fundamental group G .

Two geodesics in \widetilde{M} are said to be equivalent if they remain a bounded distance apart and the space of equivalent classes of unit speed geodesics is the geometric boundary $\partial\widetilde{M}$. For each $(x, \xi) \in \widetilde{M} \times \partial\widetilde{M}$, there is a unique unit speed geodesic $\gamma_{x, \xi}$ starting from x belonging to $[\xi]$, the equivalent class of ξ . The mapping $\xi \mapsto \dot{\gamma}_{x, \xi}(0)$ is a homeomorphism π_x^{-1} between $\partial\widetilde{M}$ and the unit sphere $S_x\widetilde{M}$ in the tangent space at x to \widetilde{M} . So we will identify $S\widetilde{M}$, the unit tangent bundle of \widetilde{M} , with $\widetilde{M} \times \partial\widetilde{M}$.

Consider the geodesic flow Φ_t on $S\widetilde{M}$. For each $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, its *stable manifold* with respect to Φ_t , denoted $W^s(\mathbf{v})$, is the collection of initial vectors \mathbf{w} of geodesics $\gamma_{\mathbf{w}} \in [\xi]$ and can be identified with $\widetilde{M} \times \{\xi\}$. Extend the action of G continuously to $\partial\widetilde{M}$. Then SM , the unit tangent bundle of M , can be identified with the quotient of $\widetilde{M} \times \partial\widetilde{M}$ under the diagonal action of G . Clearly, for $\psi \in G$, $\psi(W^s(\mathbf{v})) = W^s(D\psi(\mathbf{v}))$ so that the collection of

$W^s(\mathbf{v})$ defines a foliation \mathcal{W} on SM , the so-called *stable foliation* of SM . The leaves of the stable foliation \mathcal{W} are quotients of \widetilde{M} , they are naturally endowed with the Riemannian metric induced from \widetilde{g} . For $v \in SM$, let $W^s(v)$ be the leaf of \mathcal{W} containing v . Then $W^s(v)$ is a smoothly immersed submanifold of SM depending continuously on v in the C^3 -topology ([SFL]).

A differential operator \mathcal{L} on (the smooth functions on) SM with continuous coefficients and $\mathcal{L}1 = 0$ is said to be *subordinate to the stable foliation* \mathcal{W} , if for every smooth function f on SM the value of $\mathcal{L}(f)$ at $v \in SM$ only depends on the restriction of f to $W^s(v)$. A Borel measure \mathbf{m} on SM is called \mathcal{L} -harmonic if it satisfies

$$\int \mathcal{L}(f) d\mathbf{m} = 0$$

for every smooth function f on SM . If the restriction of \mathcal{L} to each leaf is elliptic, it is true by [Ga] that there always exist harmonic measures and the set of harmonic probability measures is a non-empty weak* compact convex set of measures on SM . A harmonic probability measure \mathbf{m} is *ergodic* if it is extremal among harmonic probability measures.

In this paper, we are interested in the case $\mathcal{L} = \Delta + Y$, where Δ is the laminated Laplacian and Y is a section of the tangent bundle of \mathcal{W} over SM of class $C_s^{k,\alpha}$ for some $k \geq 1$ and $\alpha \in [0, 1)$ in the sense that Y and its leafwise jets up to order k along the leaves of \mathcal{W} are Hölder continuous with exponent α ([H2]). Let \mathbf{m} be an \mathcal{L} -harmonic measure. We can characterize it by describing its lift on \widetilde{SM} .

Lift \mathcal{L} to an operator on $S\widetilde{M} = \widetilde{M} \times \partial\widetilde{M}$ which we shall denote with the same symbol. It defines a Markovian family of probabilities on Ω_+ , the space of paths of $\omega : [0, +\infty) \rightarrow S\widetilde{M}$, equipped with the smallest σ -algebra \mathcal{A} for which the projections $R_t : \omega \mapsto \omega(t)$ are measurable. Indeed, for $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, let $\mathcal{L}_{\mathbf{v}}$ denote the laminated operator of \mathcal{L} on $W^s(\mathbf{v})$. It can be regarded as an operator on \widetilde{M} with corresponding heat kernel $p_{\mathbf{v}}(t, y, z)$, $t \in \mathbb{R}_+$, $y, z \in \widetilde{M}$. Define

$$\mathbf{p}(t, (x, \xi), d(y, \eta)) = p_{\mathbf{v}}(t, x, y) d\text{Vol}(y) \delta_{\xi}(\eta),$$

where $\delta_{\xi}(\cdot)$ is the Dirac function at ξ . Then the diffusion process on $W^s(\mathbf{v})$ with infinitesimal operator $\mathcal{L}_{\mathbf{v}}$ is given by a Markovian family $\{\mathbb{P}_{\mathbf{w}}\}_{\mathbf{w} \in \widetilde{M} \times \{\xi\}}$, where for every $t > 0$ and every Borel set $A \in \widetilde{M} \times \partial\widetilde{M}$ we have

$$\mathbb{P}_{\mathbf{w}}(\{\omega : \omega(t) \in A\}) = \int_A \mathbf{p}(t, \mathbf{w}, d(y, \eta)).$$

The following concerning \mathcal{L} -harmonic measures holds true ([Ga, H2]).

Proposition 2.1. *Let $\widetilde{\mathbf{m}}$ be the G -invariant measure which extends an \mathcal{L} -harmonic measure \mathbf{m} on $\widetilde{M} \times \partial\widetilde{M}$. Then*

i) the measure $\tilde{\mathbf{m}}$ satisfies, for all $f \in C_c^2(\widetilde{M} \times \partial\widetilde{M})$,

$$\int_{\widetilde{M} \times \partial\widetilde{M}} \left(\int_{\widetilde{M} \times \partial\widetilde{M}} f(y, \eta) \mathbf{p}(t, (x, \xi), d(y, \eta)) \right) d\tilde{\mathbf{m}}(x, \xi) = \int_{\widetilde{M} \times \partial\widetilde{M}} f(x, \xi) d\tilde{\mathbf{m}}(x, \xi);$$

ii) the measure $\mathbb{P} = \int \mathbb{P}_{\mathbf{v}} d\tilde{\mathbf{m}}(\mathbf{v})$ on Ω_+ is invariant under the shift map $\{\sigma_t\}_{t \in \mathbb{R}_+}$ on Ω_+ , where $\sigma_t(\omega(s)) = \omega(s+t)$ for $s > 0$ and $\omega \in \Omega_+$;

iii) the measure $\tilde{\mathbf{m}}$ can be expressed locally at $\mathbf{v} = (x, \xi) \in S\widetilde{M}$ as $d\tilde{\mathbf{m}} = k(y, \eta)(dy \times d\nu(\eta))$, where ν is a finite measure on $\partial\widetilde{M}$ and, for ν -almost every η , $k(y, \eta)$ is a positive function on \widetilde{M} which satisfies $\Delta(k(y, \eta)) - \text{Div}(k(y, \eta)Y(y, \eta)) = 0$.

Let \mathbf{m} be an \mathcal{L} -harmonic measure and $\tilde{\mathbf{m}}$ be its G -invariant extension in $S\widetilde{M}$. Choose a fundamental domain M_0 of \widetilde{M} and identify SM with $M_0 \times \partial\widetilde{M}$. We normalize $\tilde{\mathbf{m}}$ so that $\tilde{\mathbf{m}}(M_0 \times \partial\widetilde{M}) = 1$. Let $d_{\mathcal{W}}$ denote the leafwise metric on the stable foliation of $S\widetilde{M}$. Then it can be identified with $d_{\widetilde{g}}$ on \widetilde{M} on each leaf. We define

$$\begin{aligned} \ell_{\mathcal{L}}(\mathbf{m}) &:= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_{M_0 \times \partial\widetilde{M}} d_{\mathcal{W}}((x, \xi), (y, \eta)) \mathbf{p}(t, (x, \xi), d(y, \eta)) d\tilde{\mathbf{m}}(x, \xi), \\ h_{\mathcal{L}}(\mathbf{m}) &:= \lim_{t \rightarrow +\infty} -\frac{1}{t} \int_{M_0 \times \partial\widetilde{M}} (\ln \mathbf{p}(t, (x, \xi), (y, \eta))) \mathbf{p}(t, (x, \xi), d(y, \eta)) d\tilde{\mathbf{m}}(x, \xi). \end{aligned}$$

Equivalently, by using \mathbb{P} in Proposition 2.1, we see that

$$\begin{aligned} \ell_{\mathcal{L}}(\mathbf{m}) &= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_{\omega(0) \in M_0 \times \partial\widetilde{M}} d_{\mathcal{W}}(\omega(0), \omega(t)) d\mathbb{P}(\omega), \\ h_{\mathcal{L}}(\mathbf{m}) &= \lim_{t \rightarrow +\infty} -\frac{1}{t} \int_{\omega(0) \in M_0 \times \partial\widetilde{M}} \ln \mathbf{p}(t, \omega(0), \omega(t)) d\mathbb{P}(\omega). \end{aligned}$$

Call $\ell_{\mathcal{L}}(\mathbf{m})$ the *linear drift* of \mathcal{L} for \mathbf{m} , and $h_{\mathcal{L}}(\mathbf{m})$ the *entropy* of \mathcal{L} for \mathbf{m} . In case there is a unique \mathcal{L} -harmonic measure \mathbf{m} , we will write $\ell_{\mathcal{L}} := \ell_{\mathcal{L}}(\mathbf{m})$ and $h_{\mathcal{L}} := h_{\mathcal{L}}(\mathbf{m})$ and call them the *linear drift* and the *entropy* for \mathcal{L} , respectively.

Clearly, $h_{\mathcal{L}}(\mathbf{m})$ is nonnegative by definition. We are interested in the case that $h_{\mathcal{L}}(\mathbf{m})$ is positive. Call \mathcal{L} *weakly coercive*, if $\mathcal{L}_{\mathbf{v}}$, $\mathbf{v} \in S\widetilde{M}$, are weakly coercive in the sense that there are a number $\varepsilon > 0$ (independent of \mathbf{v}) and a positive $(\mathcal{L}_{\mathbf{v}} + \varepsilon)$ -superharmonic function on \widetilde{M} . For instance, if $Y \equiv 0$, then $\mathcal{L} = \Delta$ is weakly coercive and it has a unique \mathcal{L} -harmonic measure \mathbf{m} , whose lift in $S\widetilde{M}$ satisfies $d\tilde{\mathbf{m}} = dx \times d\tilde{\mathbf{m}}_x$, where dx is proportional to the volume element and $\tilde{\mathbf{m}}_x$ is the hitting probability at $\partial\widetilde{M}$ of the Brownian motion starting at x . Note that $G = \pi_1(M)$ is non-amenable for any compact connected negatively curved M and hence λ_0 , the bottom of the spectrum of Laplacian, is positive by Brooks' result ([Br]). So, the entropy $h_{\mathcal{L}}(\mathbf{m})$, which is not smaller than $2\lambda_0$, is positive as well. In general, there exist weakly coercive \mathcal{L} 's which admit uncountably many harmonic measures with zero entropy ([H2]).

Let \mathcal{L} be such that Y^* , the dual of Y in the cotangent bundle of SM , satisfies $dY^* = 0$ leafwisely. Let X be the geodesic spray on M and let

$$\text{pr}(-\langle X, Y \rangle) := \sup \left\{ h_\mu - \int \langle X, Y \rangle d\mu : \mu \in \mathcal{M} \right\}$$

be the pressure of the function $-\langle X, Y \rangle$ on SM with respect to the geodesic flow Φ_t , where \mathcal{M} is the set of Φ_t -invariant probability measures on SM and h_μ is the entropy of μ with respect to Φ_t . It was shown in [H2] that $h_{\mathcal{L}}(\mathbf{m})$ is positive if and only if $\text{pr}(-\langle X, Y \rangle)$ is positive, and each one of the two positivity properties will imply that \mathcal{L} is weakly coercive, \mathbf{m} is the unique \mathcal{L} -harmonic measure and $\ell_{\mathcal{L}}(\mathbf{m})$ is positive.

3. A CENTRAL LIMIT THEOREM FOR THE LINEAR DRIFT AND THE STOCHASTIC ENTROPY

Let $\mathcal{L} = \Delta + Y$ be such that Y has closed dual and $\text{pr}(-\langle X, Y \rangle) > 0$. Let \mathbf{m} be the unique \mathcal{L} -harmonic measure. Since it is ergodic, we have for \mathbb{P} -almost all paths $\omega \in \Omega_+$,

$$(3.1) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} d_{\mathcal{W}}(\omega(0), \omega(t)) = \ell_{\mathcal{L}}.$$

Similarly, we can characterize $h_{\mathcal{L}}$ using the Green function along the trajectories. For each $\mathbf{v} = (x, \xi) \in \widetilde{M} \times \partial \widetilde{M}$, we can regard $\mathcal{L}_{\mathbf{v}}$ as an operator on \widetilde{M} . Since it is weakly coercive, there exists the corresponding Green function $G_{\mathbf{v}}(\cdot, \cdot)$ on $\widetilde{M} \times \widetilde{M}$. Define the *Green function* $\mathbf{G}(\cdot, \cdot)$ on $\widetilde{SM} \times \widetilde{SM}$ as being

$$\mathbf{G}((y, \eta), (z, \zeta)) := G_{(y, \eta)}(y, z) \delta_{\eta}(\zeta), \quad \text{for } (y, \eta), (z, \zeta) \in \widetilde{SM},$$

where $\delta_{\eta}(\cdot)$ is the Dirac function at η . We have the following proposition concerning $h_{\mathcal{L}}$.

Proposition 3.1. *Let $\mathcal{L} = \Delta + Y$ be such that Y has closed dual and $\text{pr}(-\langle X, Y \rangle) > 0$. Then for \mathbb{P} -a.e. paths $\omega \in \Omega_+$, we have*

$$(3.2) \quad h_{\mathcal{L}} = \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{p}(t, \omega(0), \omega(t))$$

$$(3.3) \quad = \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)).$$

The main difficulty to show the proposition comes from the lack of superadditivity of $-\ln \mathbf{p}$ along the trajectories. We will use the trick of [L3] to show that there exists a convex function $h_{\mathcal{L}}(s)$, $s > 0$, such that for \mathbb{P} -a.e. paths $\omega \in \Omega_+$, for any $s > 0$,

$$(3.4) \quad h_{\mathcal{L}}(s) = \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{p}(st, \omega(0), \omega(t)).$$

Setting $s = 1$ in (3.4) immediately gives (3.2). We observe

$$\mathbf{G}(\omega(0), \omega(t)) = t \int_0^{+\infty} \mathbf{p}(st, \omega(0), \omega(t)) ds,$$

so (3.3) will follow from (3.2) once we show that for \mathbb{P} -almost all paths $\omega \in \Omega_+$,

$$(3.5) \quad \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)) = \inf_{s > 0} \{h_{\mathcal{L}}(s)\} = h_{\mathcal{L}}.$$

To show (3.4) and (3.5), we need some detailed descriptions of $p_{\mathbf{v}}(t, x, y)$. First, we have a variant of Moser's parabolic Harnack inequality ([Mos]) (see [St, T] and also [Sa]).

Lemma 3.2. *There exist $A, \varsigma > 0$ such that for any $\mathbf{v} \in \widetilde{SM}$, $t \geq 1$, $\frac{1}{2} \leq t' \leq 1$, $x, x', y, y' \in \widetilde{M}$ with $d(x, x') \leq \varsigma$, $d(y, y') \leq \varsigma$,*

$$(3.6) \quad p_{\mathbf{v}}(t, x, y) \geq A p_{\mathbf{v}}(t - t', x', y').$$

Next, we have the exponential decay property of $p_{\mathbf{v}}(t, x, y)$ in time t .

Lemma 3.3. ([H2, p.76]) *There exist $B, \varepsilon > 0$ independent of \mathbf{v} such that*

$$(3.7) \quad p_{\mathbf{v}}(t, x, y) \leq B \cdot e^{-\varepsilon t}, \text{ for all } y \in \widetilde{M} \text{ and } t \geq 1.$$

Let $b > 0$ be an upper bound of $|Y|$. We have the following lower bound for $p_{\mathbf{v}}(t, x, y)$.

Lemma 3.4. ([W, Theorem 3.1]) *Let $\beta = \sqrt{K}(m - 1) + b$, where $K \geq 0$ is such that $\text{Ricci} \geq -K(m - 1)$. Then for any $\mathbf{v} \in \widetilde{SM}$, $t, \sigma > 0$ and $x, y \in \widetilde{M}$, we have*

$$(3.8) \quad p_{\mathbf{v}}(t, x, y) \geq (4\pi t)^{-\frac{m}{2}} \exp \left[-\left(\frac{1}{4t} + \frac{\sigma}{3\sqrt{2t}} \right) d^2(x, y) - \frac{\beta^2 t}{4} - \left(\frac{\beta^2}{4\sigma} + \frac{2m\sigma}{3} \right) \sqrt{2t} \right].$$

Proof of Proposition 3.1. We first show (3.4). Given $s > 0$, for $\omega \in \Omega_+$, define

$$F(s, \omega, t) := -\ln(\mathbf{p}(st - 1, \omega(0), \omega(t)) \cdot \widetilde{A}),$$

where $\widetilde{A} = A^2 \inf_{z \in \widetilde{M}} \text{Vol}(B(z, \varsigma))$ and A, ς are as in Lemma 3.2. Then for $t, t' \geq 1/s$, $\omega \in \Omega_+$,

$$F(s, \omega, t + t') \leq F(s, \omega, t) + F(s, \sigma_t(\omega), t').$$

This follows by the semi-group property of \mathbf{p} and (3.6) since

$$\begin{aligned} \mathbf{p}(s(t + t') - 1, \omega(0), \omega(t + t')) &= \int \mathbf{p}(st - \frac{1}{2}, \omega(0), z) \mathbf{p}(st' - \frac{1}{2}, z, \omega(t + t')) dz \\ &\geq \int_{B(\omega(t), \varsigma)} \mathbf{p}(st - \frac{1}{2}, \omega(0), z) \mathbf{p}(st' - \frac{1}{2}, z, \omega(t + t')) dz \\ &\geq \widetilde{A} \mathbf{p}(st - 1, \omega(0), \omega(t)) \mathbf{p}(st' - 1, \omega(t), \omega(t + t')). \end{aligned}$$

For $0 < t_1 < t_2 < +\infty$, by (3.8), there exists a constant $C > 0$, depending on t_1, t_2 and the curvature bounds, such that for any $\mathbf{v} \in \widetilde{SM}$, $x, y \in \widetilde{M}$, any $t, t_1 \leq t \leq t_2$,

$$C \exp \left[-\left(\frac{1}{4t_1} + \frac{\sigma}{3\sqrt{2t_1}} \right) d^2(x, y) \right] \leq p_{\mathbf{v}}(t, x, y).$$

As a consequence, we have

$$\mathbb{E} \left(\sup_{1+\frac{1}{s} \leq t \leq 2+\frac{1}{s}} F(s, \omega, t) \right) < \left(\frac{1}{4s} + \frac{\sigma}{3\sqrt{2s}} \right) \mathbb{E} \left(\sup_{1+\frac{1}{s} \leq t \leq 2+\frac{1}{s}} d^2(\omega(0), \omega(t)) \right) - \ln(C\tilde{A}),$$

where the second expectation term is bounded by a multiple of its value in a hyperbolic space with curvature the lower bound curvature of M and is finite (cf. [DGM]). So by the Subadditive Ergodic Theorem, there exists $h_{\mathcal{L}}(s)$ such that for \mathbb{P} -a.e. $\omega \in \Omega_+$,

$$(3.9) \quad h_{\mathcal{L}}(s) = \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{p}(st - 1, \omega(0), \omega(t)).$$

Using the semi-group property of \mathbf{p} and (3.6) again, we obtain that for $0 < a < 1$, $s_1, s_2 > 0$,

$$\begin{aligned} & \mathbf{p}((as_1 + (1-a)s_2)t - 1, \omega(0), \omega(t)) \\ & \geq \tilde{A} \mathbf{p}(as_1 t - 1, \omega(0), \omega(at)) \mathbf{p}((1-a)s_2 t - 1, \omega(at), \omega(t)). \end{aligned}$$

It follows that $h_{\mathcal{L}}(\cdot)$ is a convex function on \mathbb{R}_+ and hence is continuous.

Finally, let D be a countable dense subset of \mathbb{R}_+ . There is a measurable set $E \subset \Omega_+$ with $\mathbb{P}(E) = 1$ such that for $\omega \in E$, (3.9) holds true for any $s \in D$. Let $\omega \in \Omega_+$ be such an orbit. Given any $s_1 < s_2$ ($s_1, s_2 \in D$), let $t > 0$ be large, then we have by (3.6) that

$$\mathbf{p}(s_1 t, \omega(0), \omega(t)) \leq A^{(s_1 - s_2)t - 1} \mathbf{p}(s_2 t - 1, \omega(0), \omega(t)).$$

So for $s' < s < s''$ ($s', s'' \in D$), and $\omega \in E$,

$$\begin{aligned} h_{\mathcal{L}}(s'') + (s'' - s) \ln A & \leq \liminf_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{p}(st, \omega(0), \omega(t)) \\ & \leq \limsup_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{p}(st, \omega(0), \omega(t)) \\ & \leq h_{\mathcal{L}}(s') - (s - s') \ln A. \end{aligned}$$

Letting s', s'' go to s on both sides, it gives (3.4) by continuity of the function $h_{\mathcal{L}}$.

Next, we show (3.5). Since

$$\mathbf{G}(\omega(0), \omega(t)) = t \int_0^{+\infty} \mathbf{p}(st, \omega(0), \omega(t)) \, ds,$$

it is easy to see from (3.4) that for $\omega \in E$,

$$\limsup_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)) \leq \inf_{s > 0} \{h_{\mathcal{L}}(s)\}.$$

For the reverse inequality, we observe that for $s_1, t > 1$, we have by (3.7) that

$$\int_{s_1}^{+\infty} \mathbf{p}(st, \omega(0), \omega(t)) \, ds \leq B \int_{s_1}^{+\infty} e^{-\epsilon st} \, ds = \frac{1}{\epsilon t} B e^{-\epsilon s_1 t}.$$

So for any small $\epsilon_1 > 0$, if s_1 and t are large enough, then

$$\int_{s_1}^{+\infty} \mathbf{p}(st, \omega(0), \omega(t)) \, ds \leq e^{-(\inf_{s>0}\{h_{\mathcal{L}}(s)\}-\epsilon_1)t}.$$

Consequently, we see that for s_1 and t large,

$$\begin{aligned} \mathbf{G}(\omega(0), \omega(t)) &= t \left(\int_0^{s_1} \mathbf{p}(st, \omega(0), \omega(t)) \, ds + \int_{s_1}^{+\infty} \mathbf{p}(st, \omega(0), \omega(t)) \, ds \right) \\ &\leq t(s_1 + 1)e^{-(\inf_{s>0}\{h_{\mathcal{L}}(s)\}-\epsilon_1)t}, \end{aligned}$$

from which we derive that for $\omega \in E$,

$$\liminf_{t \rightarrow +\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)) \geq \inf_{s>0} \{h_{\mathcal{L}}(s)\}.$$

Finally, we have $\inf_{s>0}\{h_{\mathcal{L}}(s)\} = h_{\mathcal{L}}(1)$ since for any typical $\mathbf{v} \in SM$,

$$\begin{aligned} h_{\mathcal{L}}(s) - h_{\mathcal{L}}(1) &= \lim_{t \rightarrow +\infty} -\frac{1}{t} \int p_{\mathbf{v}}(t, x, y) \ln \frac{p_{\mathbf{v}}(st, x, y)}{p_{\mathbf{v}}(t, x, y)} \, dy \\ &\geq \lim_{t \rightarrow +\infty} \frac{1}{t} \int p_{\mathbf{v}}(t, x, y) \left(1 - \frac{p_{\mathbf{v}}(st, x, y)}{p_{\mathbf{v}}(t, x, y)} \right) \, dy \\ &\geq 0. \end{aligned}$$

□

Now we have that for \mathbb{P} -a.e. paths $\omega \in \Omega_+$, both (3.1) and (3.3) hold. For the corresponding Central Limit Theorems, we first express $\ell_{\mathcal{L}}$ and $h_{\mathcal{L}}$ using the Busemann function and the Martin kernel function at the geometric boundary and the \mathcal{L} -harmonic measure.

Let $x \in \widetilde{M}$ and define for $y \in \widetilde{M}$ the *Busemann function* $b_{x,y}(z)$ on \widetilde{M} by letting

$$b_{x,y}(z) := d(y, z) - d(y, x), \text{ for } z \in \widetilde{M}.$$

The assignment of $y \mapsto b_{x,y}$ is continuous, one-to-one and takes value in a relatively compact set of functions for the topology of uniform convergence on compact subsets of \widetilde{M} . The Busemann compactification of \widetilde{M} is the closure of \widetilde{M} for that topology. In the negative curvature case, the Busemann compactification coincides with the geometric compactification. So for each $\mathbf{v} = (x, \xi) \in \widetilde{M} \times \partial \widetilde{M}$, the *Busemann function at \mathbf{v}* , given by

$$b_{\mathbf{v}}(z) := \lim_{y \rightarrow \xi} b_{x,y}(z), \text{ for } z \in \widetilde{M},$$

is well-defined. Note that the *strong stable manifold at \mathbf{v}* , denoted $W^{ss}(\mathbf{v})$, is the collection of \mathbf{w} such that $d(\Phi^t(\mathbf{w}), \Phi^t(\mathbf{v}))$ tends to 0 as t goes to infinity (while the *strong unstable manifold at \mathbf{v}* , denoted $W^{su}(\mathbf{v})$, is defined by reversing the time). The quantity $-\Delta b_{\mathbf{v}}$ has its geometry meaning as being the mean curvature of the *strong stable horosphere*, which is the projection on \widetilde{M} of the strong stable manifold at \mathbf{v} .

Let $\mathbf{v} = (x, \xi) \in \widetilde{M} \times \partial\widetilde{M}$. A *Poisson kernel function* $k_{\mathbf{v}}(\cdot, \eta)$ of $\mathcal{L}_{\mathbf{v}}$ at $\eta \in \partial\widetilde{M}$ is a positive $\mathcal{L}_{\mathbf{v}}$ -harmonic function on \widetilde{M} such that

$$k_{\mathbf{v}}(x, \eta) = 1, k_{\mathbf{v}}(y, \eta) = O(G_{\mathbf{v}}(x, y)), \text{ as } y \rightarrow \eta' \neq \eta.$$

A point $\eta \in \partial\widetilde{M}$ is a *Martin point* of $\mathcal{L}_{\mathbf{v}}$ if it satisfies the following properties:

- i) there exists a Poisson kernel function $k_{\mathbf{v}}(\cdot, \eta)$ of $\mathcal{L}_{\mathbf{v}}$ at η ,
- ii) the Poisson kernel function is unique, and
- iii) if $y_n \rightarrow \eta$, then $\ln G_{\mathbf{v}}(\cdot, y_n) - \ln G_{\mathbf{v}}(x, y_n) \rightarrow \ln k_{\mathbf{v}}(\cdot, \eta)$ uniformly on compact sets.

Since (M, g) is negatively curved and $\mathcal{L}_{\mathbf{v}}$ is weakly coercive, every point η of the geometric boundary $\partial\widetilde{M}$ is a Martin point by Ancona [Anc].

Proposition 3.5. *Let $\mathcal{L} = \Delta + Y$ be such that Y has closed dual and $\text{pr}(-\langle X, Y \rangle) > 0$. Let \overline{X} denote the lift of X in \widetilde{SM} . Then we have*

$$(3.10) \quad \ell_{\mathcal{L}} = - \int_{M_0 \times \partial\widetilde{M}} (\text{Div} \overline{X} + \langle Y, \overline{X} \rangle) d\widetilde{\mathbf{m}},$$

$$(3.11) \quad h_{\mathcal{L}} = \int_{M_0 \times \partial\widetilde{M}} \|\nabla \ln k_{\mathbf{v}}(x, \xi)\|^2 d\widetilde{\mathbf{m}}.$$

Proof. We first show (3.10). For $\mathbb{P}_{\mathbf{v}}$ -a.e. path $\omega \in \Omega_+$, let $\widetilde{\omega}$ be its projection to \widetilde{M} and let $\eta = \lim_{t \rightarrow +\infty} \widetilde{\omega}(t) \in \partial\widetilde{M}$. We see that when t goes to infinity, the process $b_{\mathbf{v}}(\widetilde{\omega}(t)) - d(x, \widetilde{\omega}(t))$ converges $\mathbb{P}_{\mathbf{v}}$ a.e. to the a.e. finite number $-2(\xi|\eta)_x$, where

$$(3.12) \quad (\xi|\eta)_x := \lim_{y \rightarrow \xi, z \rightarrow \eta} (y|z)_x \text{ and } (y|z)_x := \frac{1}{2} (d(x, y) + d(x, z) - d(y, z)).$$

So for $\mathbb{P}_{\mathbf{v}}$ -a.e. $\omega \in \Omega_+$, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} b_{\mathbf{v}}(\widetilde{\omega}(t)) = \ell_{\mathcal{L}}.$$

Using the fact that the \mathcal{L} -diffusion has leafwise infinitesimal generator $\Delta + Y$, we obtain

$$\begin{aligned} \ell_{\mathcal{L}} &= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \frac{\partial}{\partial s} b_{\mathbf{v}}(\widetilde{\omega}(s)) ds \\ &= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t (\Delta + Y) b_{\mathbf{v}}(\widetilde{\omega}(s)) ds \\ &= - \int_{M_0 \times \partial\widetilde{M}} (\text{Div} \overline{X} + \langle Y, \overline{X} \rangle) d\widetilde{\mathbf{m}}. \end{aligned}$$

For (3.11), we first show for $\mathbb{P}_{\mathbf{v}}$ -a.e. $\omega \in \Omega_+$,

$$(3.13) \quad \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln k_{\mathbf{v}}(\widetilde{\omega}(t), \xi) = h_{\mathcal{L}}.$$

Let z_t be the point on the geodesic ray $\gamma_{\tilde{\omega}(t), \xi}$ closest to x . Then, as $t \rightarrow +\infty$,

$$(3.14) \quad G_{\mathbf{v}}(x, \tilde{\omega}(t)) \asymp G_{\mathbf{v}}(z_t, \tilde{\omega}(t)) \asymp \frac{G_{\mathbf{v}}(y, \tilde{\omega}(t))}{G_{\mathbf{v}}(y, z_t)}$$

for all y on the geodesic going from $\tilde{\omega}(t)$ to ξ , where \asymp means up to some constant independent of t . The first \asymp comes from Harnack inequality using the fact that $\sup_t d(x, z_t)$ is finite $\mathbb{P}_{\mathbf{v}}$ -almost everywhere. The second \asymp comes from Ancona's inequality ([**Anc**]). Replace $G_{\mathbf{v}}(y, \tilde{\omega}(t))/G_{\mathbf{v}}(y, z_t)$ by its limit as $y \rightarrow \xi$, which is $k_{(z_t, \xi)}(\tilde{\omega}(t), \xi)$, which is itself $\asymp k_{\mathbf{v}}(\tilde{\omega}(t), \xi)$ by Harnack inequality again. Altogether we may write that $\mathbb{P}_{\mathbf{v}}$ -a.e. we have

$$\limsup_{t \rightarrow +\infty} |\ln G_{\mathbf{v}}(x, \tilde{\omega}(t)) - \ln k_{\mathbf{v}}(\tilde{\omega}(t), \xi)| < +\infty,$$

which implies (3.13) by (3.3). Note that the Martin kernel function $k_{\mathbf{v}}(\cdot, \xi)$ satisfies $\mathcal{L}(k_{\mathbf{v}}(\cdot, \xi)) = 0$. Again, using the fact that the \mathcal{L} -diffusion has leafwise infinitesimal generator $\Delta + Y$, we obtain

$$\begin{aligned} h_{\mathcal{L}} &= \lim_{t \rightarrow +\infty} -\frac{1}{t} \int_0^t \frac{\partial}{\partial s} (\ln k_{\mathbf{v}}(\tilde{\omega}(s), \xi)) \, ds \\ &= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t -(\Delta + Y) (\ln k_{\mathbf{v}}(\tilde{\omega}(s), \xi)) \, ds \\ &= \int_{M_0 \times \partial \tilde{M}} \|\nabla \ln k_{\mathbf{v}}(\cdot, \xi)\|^2 \, d\tilde{\mathbf{m}}. \end{aligned}$$

□

Finally, we have the following Central Limit Theorem for $\ell_{\mathcal{L}}$ and $h_{\mathcal{L}}$ ([**H2**], see [**L2**]).

Proposition 3.6. *Let $\mathcal{L} = \Delta + Y$ be such that Y has closed dual and $\text{pr}(-\langle X, Y \rangle) > 0$. Then there are positive numbers σ_0 and σ_1 such that the distributions of the variables*

$$\frac{1}{\sigma_0 \sqrt{t}} [d_{\mathcal{W}}(\omega(0), \omega(t)) - t\ell_{\mathcal{L}}] \quad \text{and} \quad \frac{1}{\sigma_1 \sqrt{t}} [\ln \mathbf{G}(\omega(0), \omega(t)) + th_{\mathcal{L}}]$$

are asymptotically close to the normal distribution when t goes to infinity.

The proof of the proposition relies on the contraction property of the action of the diffusion process on a certain space of Hölder continuous functions. Let Q_t ($t \geq 0$) be the action of $[0, +\infty)$ on continuous functions f on SM which describes the \mathcal{L} -diffusion, i.e.,

$$Q_t(f)(x, \xi) = \int_{M_0 \times \partial \tilde{M}} \tilde{f}(y, \eta) \mathbf{p}(t, (x, \xi), d(y, \eta)),$$

where \tilde{f} denotes the lift of f to $M_0 \times \partial \tilde{M}$. For $\iota > 0$, define a norm $\|\cdot\|_{\iota}$ on the space of continuous functions f on SM by letting

$$\|f\|_{\iota} = \sup_{x, \xi} |\tilde{f}(x, \xi)| + \sup_{x, \xi_1, \xi_2} |\tilde{f}(x, \xi_1) - \tilde{f}(x, \xi_2)| \exp(\iota(\xi_1 | \xi_2)_x),$$

where $(\xi_1|\xi_2)_x$ is defined as in (3.12), and let \mathcal{H}_ι be the Banach space of continuous functions f on SM with $\|f\|_\iota < +\infty$. It was shown ([H2, Theorem 5.13]) that for sufficiently small $\iota > 0$, Q_t converges to the mapping $f \mapsto \int f \, d\mathbf{m}$ exponentially for $f \in \mathcal{H}_\iota$. As a consequence, one concludes that for any $f \in \mathcal{H}_\iota$ with $\int f \, d\mathbf{m} = 0$, $u = -\int_0^{+\infty} Q_t f \, dt$, is, up to an additive constant function, the unique element in \mathcal{H}_ι which solves $\mathcal{L}u = f$ ([H2, Corollary 5.14]). Applying this property to $b_{\mathbf{v}}$ and $k_{\mathbf{v}}(\cdot, \xi)$, where we observe that both $\mathbf{v} \mapsto \Delta b_{\mathbf{v}}$ and $\xi \mapsto \nabla \ln k_{\mathbf{v}}(\cdot, \xi)$ are G -equivalent and descend to Hölder continuous functions on SM (see [Ano, HPS] and [H1], respectively), we obtain two Hölder continuous functions u_0, u_1 on SM such that

$$\begin{aligned} \mathcal{L}(u_0 \circ \pi_{SM}) &= -(\text{Div}(\overline{X}) + \langle Y, \overline{X} \rangle) + \int_{M_0 \times \partial \widetilde{M}} (\text{Div}(\overline{X}) + \langle Y, \overline{X} \rangle) \, d\widetilde{\mathbf{m}} \\ &= -(\text{Div}(\overline{X}) + \langle Y, \overline{X} \rangle) - \ell_{\mathcal{L}}, \text{ by (3.10), and} \\ \mathcal{L}(u_1 \circ \pi_{SM}) &= \|\nabla \ln k_{\mathbf{v}}(\cdot, \xi)\|^2 - \int_{M_0 \times \partial \widetilde{M}} \|\nabla \ln k_{\mathbf{v}}(\cdot, \xi)\|^2 \, d\widetilde{\mathbf{m}} \\ &= \|\nabla \ln k_{\mathbf{v}}(\cdot, \xi)\|^2 - h_{\mathcal{L}}, \text{ by (3.11).} \end{aligned}$$

For each $\omega \in \Omega_+$ belonging to a stable leaf, let $\widetilde{\omega}$ be its projection to \widetilde{M} . Then for $f = -b_{\mathbf{v}} + u_0 \circ \pi_{SM}$ (or $\ln k_{\mathbf{v}}(\cdot, \xi) + u_1 \circ \pi_{SM}$), $f(\widetilde{\omega}(t)) - f(\widetilde{\omega}(0)) - \int_0^t (\mathcal{L}f)(\widetilde{\omega}(s)) \, ds$ is a martingale with increasing process $2\|\nabla f\|^2(\widetilde{\omega}(t)) \, dt$. In other words, we have the following.

Proposition 3.7. (cf. [L2, Corollary 3]) *For any $\mathbf{v} = (x, \xi)$, the process $(\mathbf{Z}_t^0)_{t \in \mathbb{R}_+}$ with $\omega(0) = \mathbf{v}$ [respectively, $(\mathbf{Z}_t^1)_{t \in \mathbb{R}_+}$ with $\omega(0) = \mathbf{v}$],*

$$\mathbf{Z}_t^0 := -b_{\omega(0)}(\widetilde{\omega}(t)) + t\ell_{\mathcal{L}} + u_0(\pi_{SM}(\omega(t))) - u_0(\pi_{SM}(\omega(0)))$$

[respectively,

$$\mathbf{Z}_t^1 := \ln k_{\mathbf{v}}(\widetilde{\omega}(t), \xi) + th_{\mathcal{L}} + u_1(\pi_{SM}(\omega(t))) - u_1(\pi_{SM}(\omega(0)))]$$

is a martingale with increasing process

$$2\|\overline{X} + \nabla u_0\|^2(\widetilde{\omega}(t)) \, dt \quad [\text{respectively, } 2\|\nabla \ln k_{\mathbf{v}}(\cdot, \xi) + \nabla u_1\|^2(\widetilde{\omega}(t)) \, dt].$$

To finish the proof of Proposition 3.6, let us recall a central limit theorem for martingales.

Lemma 3.8. *Let $\mathbf{M} = (\mathbf{M}_t)_{t \geq 0}$ be a right-continuous, square-integrable centered martingale with respect to an increasing filtration $(\mathfrak{F}_t)_{t \geq 0}$ of a probability space, with stationary increments. Assume that $\mathbf{M}_0 = 0$ and*

$$(3.15) \quad \lim_{t \rightarrow \infty} \mathbb{E} \left(\left| \frac{1}{t} \langle \mathbf{M}, \mathbf{M} \rangle_t - \sigma^2 \right| \right) = 0$$

for some real number σ^2 , where $\langle \mathbf{M}, \mathbf{M} \rangle_t$ denotes the quadratic variation of \mathbf{M}_t . Then the laws of \mathbf{M}_t/\sqrt{t} converge in distribution to a centered Gaussian law with variance σ^2 .

Now we see that both \mathbf{Z}_t^0 and \mathbf{Z}_t^1 are continuous and square integrable. The respective average variances converge to, respectively, σ_0^2 and σ_1^2 , where

$$\begin{aligned}\sigma_0^2 &= 2 \int_{M_0 \times \partial \widetilde{M}} \|\overline{X} + \nabla u_0\|^2 d\widetilde{\mathbf{m}}, \\ \sigma_1^2 &= 2 \int_{M_0 \times \partial \widetilde{M}} \|\nabla \ln k_{\mathbf{v}}(\cdot, \xi) + \nabla u_1\|^2 d\widetilde{\mathbf{m}}.\end{aligned}$$

Since the \mathcal{L} -diffusion system is weak mixing, (3.15) holds for \mathbf{Z}_t^0 and \mathbf{Z}_t^1 with $\sigma = \sigma_0$ or σ_1 , respectively. Hence both $(1/(\sigma_0\sqrt{t}))\mathbf{Z}_t^0$ and $(1/(\sigma_1\sqrt{t}))\mathbf{Z}_t^1$ will converge to the normal distribution as t tends to infinity. Note that in the proof of Proposition 3.5 we have shown that for $\mathbb{P}_{\mathbf{v}}$ -a.e. $\omega \in \Omega_+$, $b_{\mathbf{v}}(\widetilde{\omega}(t)) - d(x, \widetilde{\omega}(t))$ converges to a finite number and that

$$\limsup_{t \rightarrow +\infty} |\ln G_{\mathbf{v}}(x, \widetilde{\omega}(t)) - \ln k_{\mathbf{v}}(\widetilde{\omega}(t), \xi)| < +\infty.$$

As a consequence, we see from Proposition 3.7 that $(1/(\sigma_0\sqrt{t}))[d_{\mathcal{W}}(\omega(0), \omega(t)) - t\ell_{\mathcal{L}}]$ and $(1/(\sigma_0\sqrt{t}))\mathbf{Z}_t^0$ (respectively, $(1/(\sigma_1\sqrt{t}))[\ln \mathbf{G}(\omega(0), \omega(t)) + th_{\mathcal{L}}]$ and $(1/(\sigma_1\sqrt{t}))\mathbf{Z}_t^1$) have the same asymptotical distribution, which is normal, when t goes to infinity.

4. REGULARITY OF THE LINEAR DRIFT AND THE STOCHASTIC ENTROPY FOR $\Delta + Y$

Consider a one-parameter family of variations $\{\mathcal{L}^\lambda = \Delta + Y + Z^\lambda : |\lambda| < 1\}$ of \mathcal{L} with $Z^0 = 0$ and Z^λ twice differentiable in λ so that $\sup_{\lambda \in (-1, 1)} \max\{\|\frac{dZ^\lambda}{d\lambda}\|, \|\frac{d^2Z^\lambda}{d\lambda^2}\|\}$ is finite. Assume each \mathcal{L}^λ is subordinate to the stable foliation, $Y + Z^\lambda$ has closed dual and $\text{pr}(-\langle X, Y + Z^\lambda \rangle) > 0$. Then each \mathcal{L}^λ has a unique harmonic measure. Hence the linear drift for \mathcal{L}^λ , denoted $\bar{\ell}_\lambda := \ell_{\mathcal{L}^\lambda}$, and the stochastic entropy for \mathcal{L}^λ , denoted $\bar{h}_\lambda := h_{\mathcal{L}^\lambda}$, are well-defined. In this section, we show the differentiability of $\bar{\ell}_\lambda$ and \bar{h}_λ in λ at 0 (Theorem 4.3 and Theorem 4.9).

4.1. Distribution of the diffusion processes. In this subsection, we compare the distributions of the leafwise diffusion processes with infinitesimal generators \mathcal{L}^λ and \mathcal{L} , respectively, using techniques of stochastic differential equation (SDE).

We begin with the general theories of SDE. Let X_1, \dots, X_d, V be bounded C^1 vector fields on a C^3 Riemannian manifold $(\mathbf{N}, \langle \cdot, \cdot \rangle)$. Let $B_t = (B_t^1, \dots, B_t^d)$ be a d -dimensional Brownian motion on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with generator Δ . An \mathbf{N} -valued semimartingale $\mathbf{x} = (\mathbf{x}_t)_{t \in \mathbb{R}_+}$ defined up to a stopping time τ is said to be a solution of the following Stratonovich SDE

$$(4.1) \quad d\mathbf{x}_t = \sum_{i=1}^d X_i(\mathbf{x}_t) \circ dB_t^i + V(\mathbf{x}_t) dt$$

up to τ if for all $f \in C^\infty(\mathbf{N})$,

$$f(\mathbf{x}_t) = f(\mathbf{x}_0) + \int_0^t \sum_{i=1}^d X_i f(\mathbf{x}_s) \circ dB_s^i + \int_0^t V f(\mathbf{x}_s) ds, \quad 0 \leq t < \tau.$$

Call a second order differential operator \mathbf{A} the generator of \mathbf{x} if

$$f(\mathbf{x}_t) - f(\mathbf{x}_0) - \int_0^t \mathbf{A} f(\mathbf{x}_s) ds, \quad 0 \leq t < \tau,$$

is a local martingale for all $f \in C^\infty(\mathbf{N})$. It is known (cf. [Hs]) that (4.1) has a unique solution with a Hörmander type second order elliptic operator generator

$$\mathbf{A} = \sum_{i=1}^d X_i^2 + V.$$

If X_1, \dots, X_d, V are such that the corresponding \mathbf{A} is the Laplace operator on \mathbf{N} , then the solution of the SDE (4.1) generates the Brownian motion on \mathbf{N} . However, there is no general way obtaining such a collection of vector fields on a general Riemannian manifold.

To obtain the the Brownian motion $(\mathbf{x}_t)_{t \in \mathbb{R}_+}$ on \mathbf{N} , we can adopt the Eells-Elworthy-Malliavin approach (cf. [El]). Suppose \mathbf{N} has dimension m . Let $B_t = (B_t^1, \dots, B_t^m)$ be an m -dimensional Brownian motion on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with generator Δ . Let $\{e_i\}$ be the standard orthonormal basis on \mathbb{R}^m . Then, we consider the canonical Brownian motion on the orthonormal bundle $O(\mathbf{N})$ given by the solution \mathbf{w}_t of the Stratonovich SDE

$$\begin{aligned} d\mathbf{w}_t &= \sum_{i=1}^m H_i(\mathbf{w}_t) \circ dB_t^i, \\ \mathbf{w}_0 &= \mathbf{w}, \end{aligned}$$

where $H_i(\mathbf{w}_t)$ is the horizontal lift of $\mathbf{w}_t e_i$ to \mathbf{w}_t . The Brownian motion $\mathbf{x} = (\mathbf{x}_t)_{t \in \mathbb{R}_+}$ can be obtained as the projection on \mathbf{N} of \mathbf{w}_t for any choice of \mathbf{w}_0 which projects to \mathbf{x}_0 . We can regard $\mathbf{x}(\cdot)$ as a measurable map from Θ to $C_{\mathbf{x}_0}(\mathbb{R}_+, \mathbf{N})$, the space of continuous functions ρ from \mathbb{R}_+ to \mathbf{N} with $\rho(0) = \mathbf{x}_0$. So

$$\mathbb{P} := \mathbb{Q}(\mathbf{x}^{-1})$$

gives the probability distribution of the Brownian motion paths in Ω_+ . For any $\tau \in \mathbb{R}_+$, let $C_{\mathbf{x}_0}([0, \tau], \mathbf{N})$ denote the space of continuous functions ρ from $[0, \tau]$ to \mathbf{N} with $\rho(0) = \mathbf{x}_0$. Then \mathbf{x} also induces a measurable map $\mathbf{x}_{[0, \tau]} : \Theta \rightarrow C_{\mathbf{x}_0}([0, \tau], \mathbf{N})$ which sends $\underline{\omega}$ to $(\mathbf{x}_t(\underline{\omega}))_{t \in [0, \tau]}$. We see that

$$\mathbb{P}_\tau := \mathbb{Q}(\mathbf{x}_{[0, \tau]}^{-1})$$

describes the distribution probability of the Brownian motion paths on \mathbf{N} up to time τ .

More generally, let V_1 be a bounded C^1 vector field on \mathbf{N} . We denote by \bar{V}_1 the horizontal lift of V_1 in $O(\mathbf{N})$. To obtain the diffusion process $\mathbf{y} = (\mathbf{y}_t)_{t \in \mathbb{R}_+}$ on \mathbf{N} with infinitesimal

generator $\Delta_{\mathbf{N}} + V_1$, we consider the Stratonovich SDE

$$\begin{aligned} d\mathbf{u}_t &= \sum_{i=1}^m H_i(\mathbf{u}_t) \circ dB_t^i + \overline{V}_1(\mathbf{u}_t) dt, \\ \mathbf{u}_0 &= \mathbf{u}. \end{aligned}$$

Then \mathbf{y}_t is the projection on \mathbf{N} of the solution \mathbf{u}_t for any choice of \mathbf{u}_0 which projects to \mathbf{y}_0 . We call \mathbf{u}_t the horizontal lift of \mathbf{y}_t . Let \mathbb{P}^1 be the distribution of \mathbf{y} in $C_{\mathbf{y}_0}(\mathbb{R}_+, \mathbf{N})$ and let \mathbb{P}_τ^1 ($\tau \in \mathbb{R}_+$) be the distribution of $(\mathbf{y}_t(\underline{\omega}))_{t \in [0, \tau]}$ in $C_{\mathbf{y}_0}([0, \tau], \mathbf{N})$, respectively. Then

$$\mathbb{P}^1 = \mathbb{Q}(\mathbf{y}^{-1}), \quad \mathbb{P}_\tau^1 = \mathbb{Q}(\mathbf{y}_{[0, \tau]}^{-1}).$$

Let M_t^1 be the random process on \mathbb{R} satisfying $M_0^1 = 1$ and the Stratonovich SDE

$$dM_t^1 = M_t^1 \left\langle \frac{1}{2} V_1(\mathbf{x}_t), \mathbf{w}_t \circ dB_t \right\rangle_{\mathbf{x}_t} - M_t^1 \left(\left\| \frac{1}{2} V_1(\mathbf{x}_t) \right\|^2 + \text{Div} \left(\frac{1}{2} V_1(\mathbf{x}_t) \right) \right).$$

Then

$$M_t^1 = \exp \left\{ \int_0^t \left\langle \frac{1}{2} V_1(\mathbf{x}_s(\underline{\omega})), \mathbf{w}_s(\underline{\omega}) \circ dB_s(\underline{\omega}) \right\rangle_{\mathbf{x}_s} - \int_0^t \left(\left\| \frac{1}{2} V_1(\mathbf{x}_s(\underline{\omega})) \right\|^2 + \text{Div} \left(\frac{1}{2} V_1(\mathbf{x}_s(\underline{\omega})) \right) \right) ds \right\}.$$

In the more familiar Ito's stochastic integral form, we have

$$dM_t^1 = \frac{1}{2} M_t^1 \langle V_1(\mathbf{x}_t), \mathbf{w}_t dB_t \rangle_{\mathbf{x}_t}$$

and

$$(4.2) \quad M_t^1 = \exp \left\{ \frac{1}{2} \int_0^t \langle V_1(\mathbf{x}_s(\underline{\omega})), \mathbf{w}_s(\underline{\omega}) dB_s(\underline{\omega}) \rangle_{\mathbf{x}_s} - \frac{1}{4} \int_0^t \|V_1(\mathbf{x}_s(\underline{\omega}))\|^2 ds \right\}.$$

Since each $\mathbb{E}_{\mathbb{Q}} \left(\exp \left\{ \frac{1}{4} \int_0^t \|V_1(\mathbf{x}_s(\underline{\omega}))\|^2 ds \right\} \right)$ is finite, we have by Novikov [N], that $M_t^1, t \geq 0$, is a continuous (\mathcal{F}_t) -martingale, i.e.,

$$\mathbb{E}_{\mathbb{Q}}(M_t^1) = 1 \quad \text{for every } t \geq 0,$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation of a random variable with respect to \mathbb{Q} . For $\tau \in \mathbb{R}_+$, let \mathbb{Q}_τ^1 be a probability on Θ , which is absolutely continuous with respect to \mathbb{Q} with

$$\frac{d\mathbb{Q}_\tau^1}{d\mathbb{Q}}(\underline{\omega}) = M_\tau^1(\underline{\omega}).$$

Note that M_τ^1 is a martingale, so that the projection of \mathbb{Q}_τ^1 on the coordinates up to $\tau' < \tau$ is given by the same formula. A version of the Girsanov theorem (cf. [El, Theorem 11B]) says that $((\mathbf{y}_t)_{t \in [0, \tau]}, \mathbb{Q})$ is isonomous to $((\mathbf{x}_t)_{t \in [0, \tau]}, \mathbb{Q}_\tau^1)$ in the sense that for any finite numbers $\tau_1, \dots, \tau_s \in [0, \tau]$,

$$(4.3) \quad (\mathbb{Q}(\mathbf{y}_{\tau_1}^{-1}), \dots, \mathbb{Q}(\mathbf{y}_{\tau_s}^{-1})) = (\mathbb{Q}_\tau^1(\mathbf{x}_{\tau_1}^{-1}), \dots, \mathbb{Q}_\tau^1(\mathbf{x}_{\tau_s}^{-1})).$$

(The coefficients in (4.2) differ from the that in [El] because B_t has generator Δ .) Let \mathbb{Q}^1 be the probability on Θ associated with $\{\mathbb{Q}_\tau^1\}_{\tau \in \mathbb{R}_+}$. Then (4.3) intuitively means that

by changing the measure \mathbb{Q} on Θ to \mathbb{Q}^1 , \mathbf{x} has the same distribution as (\mathbf{y}, \mathbb{Q}) . As a consequence, we have $\mathbb{P}_\tau^1 = \mathbb{Q}_\tau^1(\mathbf{x}^{-1})$ for all $\tau \in \mathbb{R}_+$ and hence

$$\frac{d\mathbb{P}_\tau^1}{d\mathbb{P}_\tau}(\mathbf{x}_{[0,\tau]}) = \mathbb{E}_{\mathbb{Q}}(M_\tau^1 | \mathcal{F}(\mathbf{x}_{[0,\tau]})), \text{ a.s.},$$

where $\mathbb{E}_{\mathbb{Q}}(\cdot | \cdot)$ is the conditional expectation with respect to \mathbb{Q} and $\mathcal{F}(\mathbf{x}_{[0,\tau]})$ is the smallest σ -algebra on Θ for which the map $\mathbf{x}_{[0,\tau]}$ is measurable.

Let V_2 be another bounded C^1 vector field on \mathbf{N} . Consider the diffusion process $\mathbf{z} = (\mathbf{z}_t)_{t \in \mathbb{R}_+}$ on \mathbf{N} with the same initial point as \mathbf{y} , but with infinitesimal generator $\Delta_{\mathbf{N}} + V_1 + V_2$. Let \mathbb{P}^2 be the distribution of \mathbf{z} in Ω_+ and let $\mathbb{P}_\tau^2 (\tau \in \mathbb{R}_+)$ be the distribution of $(\mathbf{z}_t(\underline{\omega}))_{t \in [0,\tau]}$. The Girsanov-Cameron-Martin formula on manifolds (cf. [El, Theorem 11C]) says that \mathbb{P}_τ^2 is absolutely continuous with respect to \mathbb{P}_τ^1 with

$$(4.4) \quad \frac{d\mathbb{P}_\tau^2}{d\mathbb{P}_\tau^1}(\mathbf{y}_{[0,\tau]}) = \mathbb{E}_{\mathbb{Q}}(M_\tau^2 | \mathcal{F}(\mathbf{y}_{[0,\tau]})), \text{ a.s.},$$

where

$$M_\tau^2(\underline{\omega}) = \exp \left\{ \frac{1}{2} \int_0^\tau \langle V_2(\mathbf{y}_s(\underline{\omega})), \mathbf{u}_s(\underline{\omega}) dB_s(\underline{\omega}) \rangle_{\mathbf{y}_s} - \frac{1}{4} \int_0^\tau \|V_2(\mathbf{y}_s(\underline{\omega}))\|^2 ds \right\}$$

and $\mathcal{F}(\mathbf{y}_{[0,\tau]})$ is the smallest σ -algebra on Θ for which the map $\mathbf{y}_{[0,\tau]}$ is measurable.

Finally, we consider the diffusion process of the stable foliation of \widetilde{SM} corresponding to \mathcal{L}^λ ($\lambda \in (-1, 1)$). Let $B_t = (B_t^1, \dots, B_t^m)$ be an m -dimensional Brownian motion on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with generator Δ . For each $\mathbf{v} = (x, \xi) \in \widetilde{SM}$, $W^s(\mathbf{v})$ can be identified with $\widetilde{M} \times \{\xi\}$, or simply \widetilde{M} . So for each $\lambda \in (-1, 1)$, there exists the diffusion process $\mathbf{y}_\mathbf{v}^\lambda = (\mathbf{y}_{\mathbf{v},t}^\lambda)_{t \in \mathbb{R}_+}$ on $W^s(\mathbf{v})$ starting from \mathbf{v} with infinitesimal generator $\mathcal{L}_\mathbf{v}^\lambda$. Each $\mathbf{y}_\mathbf{v}^\lambda$ induces a measurable map from Θ to $C_\mathbf{v}(\mathbb{R}_+, W^s(\mathbf{v})) \subset \Omega_+$ and $\overline{\mathbb{P}}_\mathbf{v}^\lambda := \mathbb{Q}((\mathbf{y}_\mathbf{v}^\lambda)^{-1})$ gives the distribution probability of $\mathbf{y}_\mathbf{v}^\lambda$ in $C_\mathbf{v}(\mathbb{R}_+, W^s(\mathbf{v}))$. For any $\tau \in \mathbb{R}_+$, let $\overline{\mathbb{P}}_{\mathbf{v},\tau}^\lambda$ be the distribution of $(\mathbf{y}_{\mathbf{v},t}^\lambda)_{t \in [0,\tau]}$ in $C_\mathbf{v}([0,\tau], W^s(\mathbf{v}))$. We have by the Girsanov-Cameron-Martin formula on manifolds (4.4) that $\overline{\mathbb{P}}_{\mathbf{v},\tau}^\lambda$ is absolutely continuous with respect to $\overline{\mathbb{P}}_{\mathbf{v},\tau}^0$ with

$$(4.5) \quad \frac{d\overline{\mathbb{P}}_{\mathbf{v},\tau}^\lambda}{d\overline{\mathbb{P}}_{\mathbf{v},\tau}^0}(\mathbf{y}_{\mathbf{v},[0,\tau]}^0) = \mathbb{E}_{\mathbb{Q}}\left(\overline{M}_\tau^\lambda | \mathcal{F}(\mathbf{y}_{\mathbf{v},[0,\tau]}^0)\right), \text{ a.s.},$$

where

$$\overline{M}_\tau^\lambda(\underline{\omega}) = \exp \left\{ \frac{1}{2} \int_0^\tau \langle Z^\lambda(\mathbf{y}_{\mathbf{v},s}^0(\underline{\omega})), \mathbf{u}_{\mathbf{v},s}^0(\underline{\omega}) dB_s(\underline{\omega}) \rangle_{\mathbf{y}_{\mathbf{v},s}^0} - \frac{1}{4} \int_0^\tau \|Z^\lambda(\mathbf{y}_{\mathbf{v},s}^0(\underline{\omega}))\|^2 ds \right\},$$

$\mathbf{u}_{\mathbf{v},t}^0$ is the horizontal lift of $\mathbf{y}_{\mathbf{v},t}^0$ in the orthonormal bundle $O(W^s(\mathbf{v}))$ and $\mathcal{F}(\mathbf{y}_{\mathbf{v},[0,\tau]}^0)$ is the smallest σ -algebra on Θ for which the map $\mathbf{y}_{\mathbf{v},[0,\tau]}^0$ is measurable.

For each $\lambda \in (-1, 1)$, let \mathbf{m}^λ be the unique \mathcal{L}^λ -harmonic measure and $\widetilde{\mathbf{m}}^\lambda$ be its G -invariant extension in \widetilde{SM} . We see that $\overline{\mathbb{P}}^\lambda = \int \overline{\mathbb{P}}_\mathbf{v}^\lambda d\widetilde{\mathbf{m}}^\lambda(\mathbf{v})$ is the shift invariant measure

on Ω_+ corresponding to $\tilde{\mathbf{m}}^\lambda$. Consider the space $\bar{\Theta} = SM \times \Theta$ with product σ -algebra and probability $\bar{\mathbb{Q}}^\lambda$, $d\bar{\mathbb{Q}}^\lambda(\mathbf{v}, \underline{\omega}) = d\mathbb{Q}(\underline{\omega}) \times d\tilde{\mathbf{m}}^\lambda(\mathbf{v})$. Let $\mathbf{y}_t^\lambda : SM \times \Theta \rightarrow \widetilde{SM}$ be such that

$$\mathbf{y}_t^\lambda(\mathbf{v}, \underline{\omega}) = \mathbf{y}_{\mathbf{v},t}^\lambda(\underline{\omega}), \text{ for } (\mathbf{v}, \underline{\omega}) \in SM \times \Theta.$$

Then $\mathbf{y}^\lambda = (\mathbf{y}_t^\lambda)_{t \in \mathbb{R}_+}$ defines a random process on the probability space $(\bar{\Theta}, \bar{\mathbb{Q}}^\lambda)$ with images in the space of continuous paths on the stable leaves of \widetilde{SM} .

Simply write $\mathbf{y}_t = \mathbf{y}_t^0$ and let \mathbf{u}_t be such that $\mathbf{u}_t(\mathbf{v}, \underline{\omega}) = \mathbf{u}_{\mathbf{v},t}^0(\underline{\omega})$ for $(\mathbf{v}, \underline{\omega}) \in \bar{\Theta}$. Denote by $(Z^\lambda)'_0 := (dZ^\lambda/d\lambda)|_{\lambda=0}$. We consider three random variables on $(\bar{\Theta}, \bar{\mathbb{Q}}^0)$:

$$\begin{aligned} \mathbf{M}_t^0 &:= \frac{1}{2} \int_0^t \langle (Z^\lambda)'_0(\mathbf{y}_s), \mathbf{u}_s dB_s \rangle_{\mathbf{y}_s}, \\ \mathbf{Z}_{\ell,t}^0 &:= [d_{\mathcal{W}}(\mathbf{y}_0, \mathbf{y}_t) - t\ell_{\mathcal{L}^0}], \\ \mathbf{Z}_{h,t}^0 &:= -[\mathbf{1}_{\{d(\mathbf{y}_0, \mathbf{y}_t) \geq 1\}} \cdot \ln \mathbf{G}(\mathbf{y}_0, \mathbf{y}_t) + th_{\mathcal{L}^0}], \end{aligned}$$

where $\mathbf{1}_B$ is the characteristic function of the event B . We will prove the following two Propositions separately in Sec. 4.2 and Sec. 4.3.

Proposition 4.1. *The laws of the random vectors $(\mathbf{Z}_{\ell,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ under $\bar{\mathbb{Q}}^0$ converge in distribution as t tends to $+\infty$ to a bivariate centered Gaussian law with some covariance matrix Σ_ℓ . The covariance matrices of $(\mathbf{Z}_{\ell,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ under $\bar{\mathbb{Q}}^0$ converge to Σ_ℓ .*

Proposition 4.2. *The laws of the random vectors $(\mathbf{Z}_{h,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ under $\bar{\mathbb{Q}}^0$ converge in distribution as t tends to $+\infty$ to a bivariate centered Gaussian law with some covariance matrix Σ_h . The covariance matrices of $(\mathbf{Z}_{h,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ under $\bar{\mathbb{Q}}^0$ converge to Σ_h .*

4.2. The differential of the linear drift. For any $\lambda \in (-1, 1)$, let $\bar{\ell}_\lambda$ be the linear drift of \mathcal{L}^λ . The main result of this subsection is the following.

Theorem 4.3. *The function $\lambda \mapsto \bar{\ell}_\lambda$ is differentiable at 0 with*

$$(4.6) \quad \left. \frac{d\bar{\ell}_\lambda}{d\lambda} \right|_{\lambda=0} = \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\bar{\mathbb{Q}}^0}(\mathbf{Z}_{\ell,t}^0 \mathbf{M}_t^0).$$

For any $\tau \in \mathbb{R}_+$, recall that $\bar{\mathbb{P}}_{\mathbf{v},\tau}^\lambda$ is the distribution of $(\mathbf{y}_{\mathbf{v},t}^\lambda)_{t \in [0,\tau]}$ in $C_{\mathbf{v}}([0,\tau], W^s(\mathbf{v}))$. By an abuse of notation, we can also regard $\bar{\mathbb{P}}_{\mathbf{v},\tau}^\lambda$ as a measure on Ω_+ whose value only depends on $(\omega(t))_{t \in [0,\tau]}$ for $\omega = (\omega(t))_{t \in \mathbb{R}_+} \in \Omega_+$. Let $\bar{\mathbb{P}}_t^\lambda = \int \bar{\mathbb{P}}_{\mathbf{v},t}^\lambda d\tilde{\mathbf{m}}^\lambda(\mathbf{v})$. Then

$$\bar{\ell}_\lambda = \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\bar{\mathbb{P}}_t^\lambda} (d_{\mathcal{W}}(\omega(0), \omega(t))).$$

Let $\lambda = 1/\sqrt{t}$. The following holds true providing all the limits exist:

$$\begin{aligned}
 \left. \frac{d\bar{\ell}_\lambda}{d\lambda} \right|_{\lambda=0} &= \lim_{t \rightarrow +\infty} \left(\frac{1}{\lambda} (\bar{\ell}_\lambda - \bar{\ell}_0) - \frac{1}{\sqrt{t}} \left(\mathbb{E}_{\mathbb{P}_t^\lambda} (d_{\mathcal{W}}(\omega(0), \omega(t))) - \mathbb{E}_{\mathbb{P}_t^0} (d_{\mathcal{W}}(\omega(0), \omega(t))) \right) \right) \\
 &\quad + \lim_{t \rightarrow +\infty} \frac{1}{\sqrt{t}} \left(\mathbb{E}_{\mathbb{P}_t^\lambda} (d_{\mathcal{W}}(\omega(0), \omega(t))) - \mathbb{E}_{\mathbb{P}_t^0} (d_{\mathcal{W}}(\omega(0), \omega(t))) \right) \\
 (4.7) \quad &=: (\text{I})_\ell + (\text{II})_\ell.
 \end{aligned}$$

We establish (4.6) by showing $(\text{I})_\ell = 0$ and $(\text{II})_\ell = \lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\mathbb{Q}^0} (\mathbf{Z}_{\ell,t}^0 \mathbf{M}_t^0)$ successively.

For $(\text{I})_\ell = 0$, it suffices to find a finite number D_ℓ such that for all $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is from Lemma 4.4) and all $t > 0$,

$$(4.8) \quad \left| \mathbb{E}_{\mathbb{P}^\lambda} (d_{\mathcal{W}}(\omega(0), \omega(t))) - t\bar{\ell}_\lambda \right| \leq D_\ell.$$

Since the \mathcal{L}^λ -diffusion has leafwise infinitesimal generator \mathcal{L}_\vee^λ and \mathbb{P}^λ is stationary, we have

$$\begin{aligned}
 \mathbb{E}_{\mathbb{P}^\lambda} (b_{\omega(0)}(\tilde{\omega}(t))) &= \mathbb{E}_{\mathbb{P}^\lambda} \left(\int_0^t \frac{\partial}{\partial s} b_{\omega(0)}(\tilde{\omega}(s)) ds \right) \\
 &= \mathbb{E}_{\mathbb{P}^\lambda} \left(\int_0^t (\mathcal{L}_{\omega(0)}^\lambda b_{\omega(0)})(\tilde{\omega}(s)) ds \right) \\
 &= t \int_{M_0 \times \partial \widetilde{M}} \mathcal{L}_\vee^\lambda b_\vee d\widetilde{\mathbf{m}}^\lambda \\
 &= t\bar{\ell}_\lambda.
 \end{aligned}$$

So (4.8) is equivalent to that for all $\lambda \in [-\delta_1, \delta_1]$ and all $t > 0$,

$$(4.9) \quad \mathbb{E}_{\mathbb{P}^\lambda} (|d(\tilde{\omega}(0), \tilde{\omega}(t)) - b_{\omega(0)}(\tilde{\omega}(t))|) < D_\ell,$$

which intuitively means that for all λ , the \mathcal{L}^λ -diffusion orbits $\tilde{\omega}(t)$ travel almost along the geodesics connecting $\tilde{\omega}(0)$ and $\tilde{\omega}(\infty)$ on the average.

We first take a look at the distribution of $\tilde{\omega}(\infty)$ on the boundary. Let $x \in \widetilde{M}$ be a reference point and let $\iota > 0$ be a positive number. Define

$$d_x^\iota(\zeta, \eta) := \exp(-\iota(\zeta|\eta)_x), \quad \forall \zeta, \eta \in \partial \widetilde{M},$$

where $(\zeta|\eta)_x$ is defined as in (3.12). If ι_0 is small, each $d_x^\iota(\cdot, \cdot)$ ($x \in \widetilde{M}, \iota \in (0, \iota_0)$) defines a distance on $\partial \widetilde{M}$ ([**GH**]), the so-called ι -Busemann distance, which is related to the Busemann functions since

$$b_\vee(y) = \lim_{\zeta, \eta \rightarrow \xi} ((\zeta|\eta)_y - (\zeta|\eta)_x), \quad \text{for any } \vee = (x, \xi) \in S\widetilde{M}, y \in \widetilde{M}.$$

The following shadow lemma ([**Moh**, Lemma 2.14], see also [**PPS**]) says that the \mathcal{L}^λ -harmonic measure has a positive dimension on the boundary in a uniform way.

Lemma 4.4. *There are $D_1, \delta_1, \alpha_1, \iota_1 > 0$ such that for all $\lambda \in [-\delta_1, \delta_1]$, all $\mathbf{v} \in SM$ and all $\zeta \in \partial \widetilde{M}$, $t > 0$,*

$$\overline{\mathbb{P}}_{\mathbf{v}}^{\lambda} (d_x^{\iota_1}(\zeta, \widetilde{\omega}(\infty)) \leq e^{-t}) \leq D_1 e^{-\alpha_1 t},$$

where $\widetilde{\omega}(\infty) = \lim_{s \rightarrow +\infty} \widetilde{\omega}(s)$ and $\widetilde{\omega}(s)$ is the projection of $\omega(s)$ on \widetilde{M} .

As a consequence, we see that for $\overline{\mathbb{P}}^{\lambda}$ -almost all orbits ω , the distance between $\widetilde{\omega}(s)$ and $\gamma_{\widetilde{\omega}(0), \widetilde{\omega}(\infty)}$, the geodesic connecting $\widetilde{\omega}(0)$ and $\widetilde{\omega}(\infty)$, is bounded in the following sense.

Lemma 4.5. *There exists a finite number D_2 such that for all $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is as in Lemma 4.4) and $s \in \mathbb{R}_+$,*

$$\mathbb{E}_{\overline{\mathbb{P}}^{\lambda}} (d(\widetilde{\omega}(s), \gamma_{\widetilde{\omega}(0), \widetilde{\omega}(\infty)})) < D_2.$$

Proof. Extend $\overline{\mathbb{P}}^{\lambda}$ to a shift invariant probability measure $\check{\mathbb{P}}^{\lambda}$ on the set of trajectories from \mathbb{R} to SM , by

$$\check{\mathbb{P}}^{\lambda} = \int_{SM} \overline{\mathbb{P}}_{\mathbf{v}}^{\lambda} \otimes (\overline{\mathbb{P}}')_{\mathbf{v}}^{\lambda} d\mathbf{m}^{\lambda}(\mathbf{v}),$$

where $(\overline{\mathbb{P}}')_{\mathbf{v}}^{\lambda}$ is the probability describing the reversed $\mathcal{L}_{\mathbf{v}}^{\lambda}$ -diffusion starting from \mathbf{v} . Then we have by invariance of $\check{\mathbb{P}}^{\lambda}$ that

$$\begin{aligned} \mathbb{E}_{\overline{\mathbb{P}}^{\lambda}} (d(\widetilde{\omega}(s), \gamma_{\widetilde{\omega}(0), \widetilde{\omega}(\infty)})) &= \mathbb{E}_{\check{\mathbb{P}}^{\lambda}} (d(\widetilde{\omega}(0), \gamma_{\widetilde{\omega}(-s), \widetilde{\omega}(\infty)})) \\ (4.10) \qquad \qquad \qquad &= \int (d(x, \gamma_{\widetilde{\omega}(-s), \widetilde{\omega}(\infty)})) d\overline{\mathbb{P}}_{\mathbf{v}}^{\lambda}(\widetilde{\omega}) d(\overline{\mathbb{P}}')_{\mathbf{v}}^{\lambda}(\widetilde{\omega}(-s)) d\mathbf{m}^{\lambda}(\mathbf{v}). \end{aligned}$$

Fix $\widetilde{\omega}(-s) = z$ at distance D from x , and let $\zeta \in \partial \widetilde{M}$ be $\lim_{t \rightarrow +\infty} \gamma_{x, z}(t)$. We estimate

$$\int d(x, \gamma_{z, \widetilde{\omega}(\infty)}) d\overline{\mathbb{P}}_{\mathbf{v}}^{\lambda}(\widetilde{\omega}) = \int_0^{+\infty} \overline{\mathbb{P}}_{\mathbf{v}}^{\lambda}(d(x, \gamma_{z, \widetilde{\omega}(\infty)}) > t) dt.$$

For $t \geq D$, it is clear that $\overline{\mathbb{P}}_{\mathbf{v}}^{\lambda}(d(x, \gamma_{z, \widetilde{\omega}(\infty)}) > t) = 0$. For $t < D$, if $d(x, \gamma_{z, \widetilde{\omega}(\infty)}) > t$, then $d_x^{\iota_1}(\zeta, \widetilde{\omega}(\infty)) \leq C e^{-\iota_1 t}$ for some constant C and hence we have by Lemma 4.4 that

$$\overline{\mathbb{P}}_{\mathbf{v}}^{\lambda}(d(x, \gamma_{z, \widetilde{\omega}(\infty)}) > t) \leq C D_1 e^{-\alpha_1 \iota_1 t}.$$

Therefore,

$$\int d(x, \gamma_{z, \widetilde{\omega}(\infty)}) d\overline{\mathbb{P}}_{\mathbf{v}}^{\lambda}(\widetilde{\omega}) \leq \int_1^D C D_1 e^{-\alpha_1 \iota_1 t} dt + 1 \leq \frac{C D_1}{\alpha_1 \iota_1} e^{-\alpha_1 \iota_1} + 1 := D_2.$$

Using (4.10), we obtain that $\mathbb{E}_{\overline{\mathbb{P}}^{\lambda}} (d(\widetilde{\omega}(s), \gamma_{\widetilde{\omega}(0), \widetilde{\omega}(\infty)}))$ is bounded by D_2 as well. \square

Now, we can derive from Lemmas 4.4 and 4.5 that there is a bounded square integrable difference between $d(\widetilde{\omega}(0), \widetilde{\omega}(s))$ and $b_{\omega(0)}(\widetilde{\omega}(s))$ for all s (cf. [Ma, Lemma 3.4]). In particular, we will obtain (4.9) and finish the proof of (I) $_{\ell} = 0$.

Lemma 4.6. *There exists a finite number D_3 such that for all $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is as in Lemma 4.4) and $s \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}^\lambda} \left(|d(\tilde{\omega}(0), \tilde{\omega}(s)) - b_{\omega(0)}(\tilde{\omega}(s))|^2 \right) < D_3.$$

Proof. It is clear that

$$\mathbb{E}_{\mathbb{P}^\lambda} \left(|d(\tilde{\omega}(0), \tilde{\omega}(s)) - b_{\omega(0)}(\tilde{\omega}(s))|^2 \right) = 4 \int (\tilde{\omega}(s)|\xi)_x^2 d\mathbb{P}_\mathbf{v}^\lambda(\tilde{\omega}) d\mathbf{m}^\lambda(\mathbf{v}),$$

where $\mathbf{v} = \omega(0) = (x, \xi)$ and $(\tilde{\omega}(s)|\xi)_x := \lim_{y \rightarrow \xi} (\tilde{\omega}(s)|y)_x$ (see (3.12) for the definition of $(z|y)_x$ for $x, y, z \in \widetilde{M}$). So, it suffices to estimate

$$\int_0^{+\infty} \mathbb{P}_\mathbf{v}^\lambda((\tilde{\omega}(s)|\xi)_x^2 > t) dt = \int_0^{+\infty} \mathbb{P}_\mathbf{v}^\lambda((\tilde{\omega}(s)|\xi)_x > \sqrt{t}) dt.$$

For each $t > 0$, divide the event $\{\omega \in \Omega_+ : (\tilde{\omega}(s)|\xi)_x > \sqrt{t}\}$ into two sub-events

$$\begin{aligned} A_1(t) &:= \{\omega \in \Omega_+ : (\tilde{\omega}(s)|\xi)_x > \sqrt{t}, (\tilde{\omega}(s)|\tilde{\omega}(\infty))_x \geq \frac{1}{4}\sqrt{t}\}, \\ A_2(t) &:= \{\omega \in \Omega_+ : (\tilde{\omega}(s)|\xi)_x > \sqrt{t}, (\tilde{\omega}(s)|\tilde{\omega}(\infty))_x < \frac{1}{4}\sqrt{t}\}. \end{aligned}$$

We estimate $\mathbb{P}_\mathbf{v}^\lambda(A_i(t))$, $i = 1, 2$, successively. Since M is a closed connected negatively curved Riemannian manifold, its universal cover \widetilde{M} is Gromov hyperbolic in the sense that there exists $\delta > 0$ such that for any $x_1, x_2, x_3 \in \widetilde{M}$,

$$(x_1|x_2)_x \geq \min\{(x_1|x_3)_x, (x_2|x_3)_x\} - \delta.$$

So on each $A_1(t)$, where $t > 64\delta^2$, we have

$$(\xi|\tilde{\omega}(\infty))_x \geq \frac{1}{8}\sqrt{t}.$$

Hence, by Lemma 4.4,

$$\mathbb{P}_\mathbf{v}^\lambda(A_1(t)) \leq \mathbb{P}_\mathbf{v}^\lambda((\xi|\tilde{\omega}(\infty))_x \geq \frac{1}{8}\sqrt{t}) = \mathbb{P}_\mathbf{v}^\lambda(d_x^{\iota_1}(\tilde{\omega}(\infty), \xi) < e^{-\frac{1}{8}\iota_1\sqrt{t}}) \leq D_1 e^{-\frac{1}{8}\iota_1\alpha_1\sqrt{t}},$$

where the last quantity is integrable with respect to t , independent of s . For $\omega \in A_2(t)$,

$$d(\tilde{\omega}(0), \tilde{\omega}(s)) \geq (\tilde{\omega}(s)|\xi)_x > \sqrt{t}.$$

On the other hand, the point $y(s)$ on $\gamma_{\tilde{\omega}(0), \tilde{\omega}(\infty)}$ closest to $\tilde{\omega}(s)$ satisfies

$$(\tilde{\omega}(s)|y(s))_x \leq (\tilde{\omega}(s)|\tilde{\omega}(\infty))_x < \frac{1}{4}\sqrt{t}.$$

So we must have

$$d(\tilde{\omega}(s), \gamma_{\tilde{\omega}(0), \tilde{\omega}(\infty)}) > \frac{1}{2}\sqrt{t}.$$

Hence,

$$\int_0^\infty \mathbb{P}_\mathbf{v}^\lambda(A_2(t)) dt \leq \int \mathbb{P}_\mathbf{v}^\lambda \left(d(\tilde{\omega}(s), \gamma_{\tilde{\omega}(0), \tilde{\omega}(\infty)}) > \frac{1}{2}\sqrt{t} \right) dt d\mathbf{m}^\lambda(\mathbf{v}),$$

which, by the same argument as the one used in the proof of Lemma 4.5, is bounded from above by some constant independent of s . \square

To show $(\text{II})_\ell = \lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}_{\ell,t}^0 \mathbf{M}_t^0)$, we first prove Proposition 4.1.

Proof of Proposition 4.1. Let $(\mathbf{Z}_t^0)_{t \in \mathbb{R}_+}$, u_0 be as in Proposition 3.7. The process $(\mathbf{Z}_t^0)_{t \in \mathbb{R}_+}$ is a centered martingale with stationary increments and its law under $\overline{\mathbb{P}}^0$ is the same as the law of $(\overline{\mathbf{Z}}_t^0)_{t \in \mathbb{R}_+}$ under $\overline{\mathbb{Q}}^0$, where $(\overline{\mathbf{Z}}_t^0)_{t \in \mathbb{R}_+}$ on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ is given by

$$\overline{\mathbf{Z}}_t^0(\mathbf{v}, \underline{\omega}) = -b_{\mathbf{v}}(\pi_{\widetilde{M}}(\mathbf{y}_{\mathbf{v},t}(\underline{\omega}))) + t\bar{\ell}_0 + u_0(\pi_{SM}(\mathbf{y}_{\mathbf{v},t}(\underline{\omega}))) - u_0(\pi_{SM}(\mathbf{v})).$$

The pair $(-\overline{\mathbf{Z}}_t^0, \mathbf{M}_t^0)$ is a centered martingale on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ with stationary increments. To show $(-\overline{\mathbf{Z}}_t^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ converge in distribution to a bivariate centered Gaussian vector, it suffices to show for any $(a, b) \in \mathbb{R}^2$, the combination $-a\overline{\mathbf{Z}}_t^0/\sqrt{t} + b\mathbf{M}_t^0/\sqrt{t}$ converge to a centered Gaussian distribution. The martingales $\overline{\mathbf{Z}}_t^0$ and \mathbf{M}_t^0 on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ have integrable increasing process functions $2\|\overline{X} + \nabla u_0\|^2$ and $\|(Z^\lambda)'_0\|^2$, respectively. Using Schwarz inequality, we conclude that $-a\overline{\mathbf{Z}}_t^0 + b\mathbf{M}_t^0$ also has an integrable increasing process function. Now using the fact that the \mathcal{L}^0 -diffusion system on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ is weakly mixing, we see that (3.15) holds true for $-a\overline{\mathbf{Z}}_t^0 + b\mathbf{M}_t^0$. Hence, by Lemma 3.8, $-a\overline{\mathbf{Z}}_t^0/\sqrt{t} + b\mathbf{M}_t^0/\sqrt{t}$ converge in distribution in $\overline{\mathbb{Q}}^0$ to a centered Gaussian law with variance $\Sigma_\ell[a, b] = (a, b)\Sigma_\ell(a, b)^T$ for some matrix Σ_ℓ . Since both $\overline{\mathbf{Z}}_t^0$ and \mathbf{M}_t^0 have stationary increments, we also have

$$\Sigma_\ell[a, b] = \frac{1}{t} \mathbb{E}_{\overline{\mathbb{Q}}^0} \left[(-a\overline{\mathbf{Z}}_t^0 + b\mathbf{M}_t^0)^2 \right], \text{ for all } t \in \mathbb{R}_+.$$

Finally, for $\overline{\mathbb{P}}^0$ -a.e. $\omega \in \Omega_+$, $\widetilde{\omega}$, the projection of ω to \widetilde{M} , is such that $b_{\mathbf{v}}(\widetilde{\omega}(t)) - d(\pi_{\widetilde{M}}(\mathbf{v}), \widetilde{\omega}(t))$ converges to a finite number. Moreover, we have by Lemma 4.6 that

$$\sup_t \mathbb{E}_{\overline{\mathbb{P}}^\lambda}(|\mathbf{Z}_{\ell,t}^0 + \overline{\mathbf{Z}}_t^0|^2) < +\infty$$

and hence

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda} \left(\frac{1}{t} |\mathbf{Z}_{\ell,t}^0 + \overline{\mathbf{Z}}_t^0|^2 \right) \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

Consequently, $(\mathbf{Z}_{\ell,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ has the same limit Gaussian law as $(-\overline{\mathbf{Z}}_t^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ and its covariance matrix under $\overline{\mathbb{Q}}^0$ converges to Σ_ℓ as t goes to infinity. \square

We state one more lemma from [Bi] on the limit of the expectations of a class of random variables on a common probability space which converge in distribution.

Lemma 4.7. (cf. [Bi, Theorem 25.12]) *If the random variables X_t ($t \in \mathbb{R}$) on a common probability space converge to X in distribution, and there exists some $q > 1$ such that*

$\sup_t \mathbb{E}_\nu(|X_t|^q) < +\infty$, then X is integrable and

$$\lim_{t \rightarrow +\infty} \mathbb{E}_\nu(X_t) = \mathbb{E}_\nu(X).$$

Lemma 4.8. *The quantity $(\text{II})_\ell$ introduced in (4.7) equals to $\lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\overline{\mathbb{P}}^0}(\mathbf{Z}_{\ell,t}^0 \mathbf{M}_t^0)$.*

Proof. By Proposition 3.6, the distribution of $(d_{\mathcal{W}}(\omega(0), \omega(t)) - t\bar{\ell}_0) / \sqrt{t}$ under $\overline{\mathbb{P}}^0$ is asymptotic to a centered Gaussian distribution. Hence

$$\begin{aligned} (\text{II})_\ell &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\overline{\mathbb{P}}_t^\lambda} \left(\frac{1}{\sqrt{t}} (d_{\mathcal{W}}(\omega(0), \omega(t)) - t\bar{\ell}_0) \right) - \lim_{t \rightarrow +\infty} \mathbb{E}_{\overline{\mathbb{P}}^0} \left(\frac{1}{\sqrt{t}} (d_{\mathcal{W}}(\omega(0), \omega(t)) - t\bar{\ell}_0) \right) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\overline{\mathbb{P}}_t^\lambda} \left(\frac{1}{\sqrt{t}} (d_{\mathcal{W}}(\omega(0), \omega(t)) - t\bar{\ell}_0) \right) \\ &=: (\text{III})_\ell. \end{aligned}$$

Let $\mathbf{y} = (\mathbf{y}_t)_{t \in \mathbb{R}_+} = (\mathbf{y}_{\mathbf{v},t})_{\mathbf{v} \in SM, t \in \mathbb{R}_+}$ be the diffusion process on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ corresponding to \mathcal{L}^0 . We know from Sec. 4.1 that $\overline{\mathbb{P}}_{\mathbf{v},t}^\lambda$ is absolutely continuous with respect to $\overline{\mathbb{P}}_{\mathbf{v},t}^0$ with

$$\frac{d\overline{\mathbb{P}}_{\mathbf{v},t}^\lambda}{d\overline{\mathbb{P}}_{\mathbf{v},t}^0}(\mathbf{y}_{\mathbf{v},[0,t]}) = \mathbb{E}_{\mathbb{Q}} \left(\overline{\mathbf{M}}_t^\lambda | \mathcal{F}(\mathbf{y}_{\mathbf{v},[0,t]}) \right),$$

where

$$\overline{\mathbf{M}}_t^\lambda(\underline{\omega}) = \exp \left\{ \frac{1}{2} \int_0^t \langle Z^\lambda(\mathbf{y}_{\mathbf{v},s}(\underline{\omega})), \mathbf{u}_{\mathbf{v},s}(\underline{\omega}) dB_s(\underline{\omega}) \rangle_{\mathbf{y}_{\mathbf{v},s}(\underline{\omega})} - \frac{1}{4} \int_0^t \|Z^\lambda(\mathbf{y}_{\mathbf{v},s}(\underline{\omega}))\|^2 ds \right\}.$$

Consequently we have

$$\begin{aligned} (\text{III})_\ell &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\overline{\mathbb{P}}^0} \left(\frac{1}{\sqrt{t}} (d_{\mathcal{W}}(\omega(0), \omega(t)) - t\bar{\ell}_0) \frac{d\overline{\mathbb{P}}_{\omega(0),t}^\lambda}{d\overline{\mathbb{P}}_{\omega(0),t}^0} \right) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\overline{\mathbb{Q}}^0} \left(\frac{1}{\sqrt{t}} (d_{\mathcal{W}}(\mathbf{y}_0, \mathbf{y}_t) - t\bar{\ell}_0) \cdot e^{(\text{IV})_\ell^t} \right), \end{aligned}$$

where

$$(\text{IV})_\ell^t = \frac{1}{2} \int_0^t \langle Z^\lambda(\mathbf{y}_s(\mathbf{v}, \underline{\omega})), \mathbf{u}_s(\mathbf{v}, \underline{\omega}) dB_s \rangle_{\mathbf{y}_s(\mathbf{v}, \underline{\omega})} - \frac{1}{4} \int_0^t \|Z^\lambda(\mathbf{y}_s(\mathbf{v}, \underline{\omega}))\|^2 ds.$$

Let \overline{Z}^λ be such that $Z^\lambda = \lambda(Z^\lambda)'_0 + \lambda^2 \overline{Z}^\lambda$. Recall that $\lambda = 1/\sqrt{t}$. We calculate that

$$\begin{aligned}
(\text{IV})_\ell^t &= \frac{1}{2\sqrt{t}} \int_0^t \langle (Z^\lambda)'_0(\mathbf{y}_s), \mathbf{u}_s dB_s \rangle_{\mathbf{y}_s} - \frac{1}{4t} \int_0^t \|(Z^\lambda)'_0(\mathbf{y}_s)\|^2 ds \\
&\quad + \frac{1}{2t} \int_0^t \langle \overline{Z}^\lambda(\mathbf{y}_s), \mathbf{u}_s dB_s \rangle_{\mathbf{y}_s} \\
&\quad - \frac{1}{2t^{\frac{3}{2}}} \int_0^t \langle (Z^\lambda)'_0(\mathbf{y}_s), \overline{Z}^\lambda(\mathbf{y}_s) \rangle_{\mathbf{y}_s} ds - \frac{1}{4t^2} \int_0^t \|\overline{Z}^\lambda(\mathbf{y}_s)\|^2 ds \\
&=: \frac{1}{\sqrt{t}} \mathbf{M}_t^0 - \frac{1}{2t} \langle \mathbf{M}_t^0, \mathbf{M}_t^0 \rangle_t + (\text{V})_\ell^t + (\text{VI})_\ell^t,
\end{aligned}$$

where both $(\text{V})_\ell^t$ and $(\text{VI})_\ell^t$ converge almost surely to zero as t goes to infinity. Therefore, by Proposition 4.1, the variables $\frac{1}{\sqrt{t}} \mathbf{Z}_{\ell,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda$ converge in distribution to $\mathbf{Z}_\ell^0 e^{\mathbf{M}^0 - \frac{1}{2} \mathbb{E}_{\mathbb{Q}^0}((\mathbf{M}^0)^2)}$. To justify

$$(4.11) \quad (\text{III})_\ell = \lim_{t \rightarrow +\infty} \mathbb{E}_{\mathbb{Q}^0} \left(\frac{1}{\sqrt{t}} \mathbf{Z}_{\ell,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda \right) = \mathbb{E}_{\mathbb{Q}^0} \left(\mathbf{Z}_\ell^0 e^{\mathbf{M}^0 - \frac{1}{2} \mathbb{E}_{\mathbb{Q}^0}((\mathbf{M}^0)^2)} \right),$$

we have by Lemma 4.7 that it suffices to show for $q = \frac{3}{2}$,

$$\sup_t \mathbb{E}_{\mathbb{Q}^0} \left(\left| \frac{1}{\sqrt{t}} \mathbf{Z}_{\ell,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda \right|^q \right) < +\infty.$$

By Hölder's inequality, we calculate that

$$\begin{aligned}
\left(\mathbb{E}_{\mathbb{Q}^0} \left(\left| \frac{1}{\sqrt{t}} \mathbf{Z}_{\ell,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda \right|^{\frac{3}{2}} \right) \right)^4 &\leq \left(\mathbb{E}_{\mathbb{Q}^0} \left(\frac{1}{t} |\mathbf{Z}_{\ell,t}^0|^2 \right) \right)^3 \cdot \mathbb{E}_{\mathbb{Q}^0} \left(e^{6(\text{IV})_\ell^t} \right) \\
&=: (\text{VII})_\ell^t \cdot (\text{VIII})_\ell^t,
\end{aligned}$$

where $(\text{VII})_\ell^t$ is uniformly bounded in t by Proposition 4.1. For $(\text{VIII})_\ell^t$, we use the Girsanov-Cameron-Martin formula to conclude that

$$\begin{aligned}
(\text{VIII})_\ell^t &= \mathbb{E}_{\mathbb{Q}^0} \left(\exp \left(3 \int_0^t \langle Z^\lambda(\mathbf{y}_s(\mathbf{v}, \underline{\omega})), \mathbf{u}_s(\mathbf{v}, \underline{\omega}) dB_s \rangle_{\mathbf{y}_s(\mathbf{v}, \underline{\omega})} - \frac{3}{2} \int_0^t \|Z^\lambda(\mathbf{y}_s(\mathbf{v}, \underline{\omega}))\|^2 ds \right) \right) \\
&\leq \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\exp \left(\frac{15}{2} \int_0^t \|Z^\lambda(\mathbf{y}_s(\mathbf{v}, \underline{\omega}))\|^2 ds \right) \right)
\end{aligned}$$

for some probability measure $\tilde{\mathbb{Q}}$ on $\overline{\Theta}$. Using again $Z^\lambda = \lambda(Z^\lambda)'_0 + \lambda^2 \overline{Z}^\lambda$ and that $\lambda = 1/\sqrt{t}$, we see that

$$\int_0^t \|Z^\lambda(\mathbf{y}_s(\mathbf{v}, \underline{\omega}))\|^2 ds \leq \frac{2}{t} \int_0^t \|(Z^\lambda)'_0\|^2 ds + \frac{2}{t^2} \int_0^t \|\overline{Z}^\lambda\|^2 ds,$$

where the quantities on the right hand side of the inequality are uniformly bounded in t . So $(\text{VIII})_\ell^t$ is uniformly bounded in t as well. Now, (4.11) holds. Finally, since $(\mathbf{Z}_\ell^0, \mathbf{M}^0)$

has a bivariate normal distribution, we have

$$\mathbb{E}_{\mathbb{Q}^0} \left(\mathbf{Z}_\ell^0 e^{\mathbf{M}^0 - \frac{1}{2} \mathbb{E}_{\mathbb{Q}^0}((\mathbf{M}^0)^2)} \right) = \mathbb{E}_{\mathbb{Q}^0}(\mathbf{Z}_\ell^0 \mathbf{M}^0),$$

which is $\lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\mathbb{Q}^0}(\mathbf{Z}_{\ell,t}^0 \mathbf{M}_t^0)$ by Proposition 4.1. \square

4.3. The differential of the stochastic entropy. For any $\lambda \in (-1, 1)$, let \bar{h}_λ be the entropy of \mathcal{L}^λ . In this subsection, we establish the following formula for $(d\bar{h}_\lambda/d\lambda)|_{\lambda=0}$.

Theorem 4.9. *The function $\lambda \mapsto \bar{h}_\lambda$ is differentiable at 0 with*

$$\left. \frac{d\bar{h}_\lambda}{d\lambda} \right|_{\lambda=0} = \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\mathbb{Q}^0}(\mathbf{Z}_{h,t}^0 \mathbf{M}_t^0).$$

First of all, let us recall some classical results concerning Green functions.

Lemma 4.10. *Let $\mathcal{L} = \Delta + Y$ be such that Y has closed dual and $\text{pr}(-\langle X, Y \rangle) > 0$ and let $\mathbf{G}(\cdot, \cdot) = \{G_{\mathbf{v}}(\cdot, \cdot)\}_{\mathbf{v} \in S\widetilde{M}}$ be the Green function of \mathcal{L} . There exists a constant $c_0 \in (0, 1)$ such that for any $\mathbf{v} \in S\widetilde{M}$ and any $x, y, z \in \widetilde{M}$ with mutual distances greater than 1,*

$$(4.12) \quad G_{\mathbf{v}}(x, z) \geq c_0 G_{\mathbf{v}}(x, y) G_{\mathbf{v}}(y, z).$$

For $v, w \in S_x \widetilde{M}$, $x \in \widetilde{M}$, the angle $\angle_x(v, w)$ is the unique number $0 \leq \theta \leq \pi$ such that $\langle v, w \rangle = \cos \theta$. Given $v \in S_x \widetilde{M}$ and $0 < \theta < \pi$, the set

$$\Gamma_x(v, \theta) := \{y \in \widetilde{M} \cup \partial \widetilde{M} : \angle_x(v, \dot{\gamma}_{x,y}(0)) < \theta\}$$

is called the *cone of vertex x , axis v , and angle θ* , where $\gamma_{x,y}$ is the geodesic segment that starts at x and ends at y . For any $s > 0$, the cone Γ with vertex $\gamma_v(s)$ (where γ_v is the geodesic starting at x with initial speed v), axis $\dot{\gamma}_v(s)$ and angle θ is called the *s -shifted cone* of $\Gamma_x(v, \theta)$. The following is a special case of the Ancona's inequality at infinity ([**Anc**]).

Lemma 4.11. *Let \mathcal{L} and \mathbf{G} be as in Lemma 4.10. Let $\Gamma := \Gamma_{x_0}(v, \frac{\pi}{2})$ be a cone in \widetilde{M} with vertex x_0 , axis v and angle $\frac{\pi}{2}$. Let Γ_1 be the 1-shifted cone of Γ and x_1 be the vertex of Γ_1 . There exists a constant c_1 such that for any $\mathbf{v} \in S\widetilde{M}$, any Γ , all $x \in \widetilde{M} \setminus \Gamma$ and $z \in \Gamma_1$,*

$$(4.13) \quad G_{\mathbf{v}}(x, z) \leq c_1 G_{\mathbf{v}}(x, x_1) G_{\mathbf{v}}(x_0, z).$$

We may assume $c_1 = c_0^{-1}$, where c_0 is as in Lemma 4.10. As a consequence of Lemma 4.10 and Lemma 4.11, \mathbf{G} is related to the distance d in the following way.

Lemma 4.12. *Let \mathcal{L} and \mathbf{G} be as in Lemma 4.10. There exist positive numbers $c_2, c_3, \alpha_2, \alpha_3$ such that for any $\mathbf{v} \in S\widetilde{M}$ and any $x, z \in \widetilde{M}$ with $d(x, z) \geq 1$,*

$$(4.14) \quad c_2 e^{-\alpha_2 d(x, z)} \leq G_{\mathbf{v}}(x, z) \leq c_3 e^{-\alpha_3 d(x, z)}.$$

Proof. The upper bound of (4.14) was shown in [H2, Corollary 4.8] using Ancona's inequality at infinity (cf. Lemma 4.11). For the lower bound, we first observe that Lemma 4.10 also holds true if x, y, z satisfies $d(x, z) > 1$ and $d(x, y) = 1$. Indeed, by the classical Harnack inequality ([LY]), there exists $c_4 \in (0, 1)$ such that for any $\mathbf{v} \in S\widetilde{M}$ and $x, y, z \in \widetilde{M}$ with $d(x, z) > 1$ and $d(x, y) \leq 1$,

$$(4.15) \quad c_4 G_{\mathbf{v}}(y, z) \leq G_{\mathbf{v}}(x, z) \leq c_4^{-1} G_{\mathbf{v}}(y, z).$$

Since $d(x, y) = 1$, by [Anc, Proposition 7], there is $c_5 \in (0, 1)$ (independent of x, y) with

$$(4.16) \quad c_5 \leq G_{\mathbf{v}}(x, y) \leq c_5^{-1}.$$

So, if $c_0 \leq c_4 c_5$, then (4.12) holds true for $x, y, z \in \widetilde{M}$ with $d(x, z) > 1$ and $d(x, y) = 1$. Now, for $x, z \in \widetilde{M}$ with $d(x, z) > 1$, choose a sequence of points $x_i, 1 \leq i \leq n$, on the geodesic segment $\gamma_{x,z}$ with $x_0 = x, x_n = z, d(x_i, x_{i+1}) = 1, i = 0, \dots, n-2$, and $d(x_{n-1}, z) \in [1, 2)$. Applying (4.12) successively for x_i, x_{i+1}, z , we obtain

$$G_{\mathbf{v}}(x, z) \geq G_{\mathbf{v}}(x_{n-1}, z)(c_0 c_5)^{n-1} \geq c_4 c_5 (c_0 c_5)^{n-1} \geq c_4 c_5 (c_0 c_5)^{d(x,y)},$$

where, to derive the second inequality, we use (4.15) and the fact that the lower bound of (4.16) holds for any $x, y \in \widetilde{M}$ with $d(x, y) \leq 1$. The lower bound estimation of (4.14) follows for $c_2 = c_4 c_5$ and $\alpha_2 = -\ln c_0 c_5$. \square

We may assume the constants c_2, c_3 in Lemma 4.12 are such that c_2 is smaller than 1 and $c_3 = c_2^{-1}$. For each $\mathbf{v} \in S\widetilde{M}$, $x, z \in \widetilde{M}$, let

$$d_{G_{\mathbf{v}}}(x, z) := \begin{cases} -\ln(c_2 G_{\mathbf{v}}(x, z)), & \text{if } d(x, z) > 1; \\ -\ln c_2, & \text{otherwise.} \end{cases}$$

Although $d_{G_{\mathbf{v}}}$ is always greater than the positive number $\min\{\alpha_3, -\ln c_2\}$ by (4.14), we still call it a “Green metric” for $\mathcal{L}_{\mathbf{v}}$ (after [BHM] for the hyperbolic groups case) since it satisfies an almost triangle inequality in the following sense.

Lemma 4.13. *There exists a constant $c_6 \in (0, 1)$ such that for all $x, y, z \in \widetilde{M}$,*

$$(4.17) \quad d_{G_{\mathbf{v}}}(x, z) \leq d_{G_{\mathbf{v}}}(x, y) + d_{G_{\mathbf{v}}}(y, z) - \ln c_6.$$

Proof. If $d(x, z) \leq 1$, then (4.17) holds for $c_6 = c_2$. If x, y, z have mutual distances greater than 1, then (4.17) holds for $c_6 = c_0$ by Lemma 4.10. If $d(x, z) > 1$ and $d(y, z) \leq 1$, using the classical Harnack inequality (4.15), we have

$$G_{\mathbf{v}}(x, z) \geq c_4 G_{\mathbf{v}}(x, y)$$

and hence (4.17) holds with $c_6 = c_4$ if, furthermore, $d(x, y) > 1$ or with $c_6 = c_4 c_5$ otherwise. The case that $d(x, z) > 1, d(x, y) \leq 1$ can be treated similarly. \square

By Lemma 4.12, $d_{G_{\mathbf{v}}}$ is comparable to the metric d for any $x, z \in \widetilde{M}$ with $d(x, z) > 1$:

$$(4.18) \quad \alpha_3 d(x, z) \leq d_{G_{\mathbf{v}}}(x, z) \leq \alpha_2 d(x, z) - 2 \ln c_2.$$

Using Lemma 4.11, we can further obtain that $d_{G_{\mathbf{v}}}$ is almost additive along the geodesics.

Lemma 4.14. *Let \mathcal{L} and \mathbf{G} be as in Lemma 4.10. There exists a constant c_7 such that for any $\mathbf{v} \in \widetilde{SM}$, any $x, z \in \widetilde{M}$ and y in the geodesic segment $\gamma_{x,z}$ connecting x and z ,*

$$(4.19) \quad |d_{G_{\mathbf{v}}}(x, y) + d_{G_{\mathbf{v}}}(y, z) - d_{G_{\mathbf{v}}}(x, z)| \leq -\ln c_7.$$

Proof. Let $x, z \in \widetilde{M}$ and y belong to the geodesic segment $\gamma_{x,z}$. If $d(x, y), d(y, z) \leq 1$, then $d(x, z) \leq 2$ and, using (4.18), we obtain (4.19) with $c_7 = c_2^2 e^{-2\alpha_2}$. If $d(x, y) \leq 1$ and $d(y, z) > 1$ (or $d(y, z) \leq 1$ and $d(x, y) > 1$), using Harnack's inequality (4.15), we have (4.19) with $c_7 = c_2 c_4$. Finally, if x, y, z have mutual distances greater than 1, we have by Lemma 4.10 and Lemma 4.11 (where we can use Harnack's inequality to replace $G_{\mathbf{v}}(x, x_1)$ in (4.13) by $c_4^{-1} G_{\mathbf{v}}(x, x_0)$) that

$$|\ln G_{\mathbf{v}}(x, y) + \ln G_{\mathbf{v}}(y, z) - \ln G_{\mathbf{v}}(x, z)| \leq -\ln(c_1 c_4)$$

and consequently,

$$|d_{G_{\mathbf{v}}}(x, y) + d_{G_{\mathbf{v}}}(y, z) - d_{G_{\mathbf{v}}}(x, z)| \leq -\ln(c_1 c_2 c_4).$$

□

More is true, as we can see from Lemma 4.11 and Lemma 4.13 as well.

Lemma 4.15. *Let \mathcal{L} and \mathbf{G} be as in Lemma 4.10. There exists a constant c_8 such that for any $\mathbf{v} \in \widetilde{SM}$, if $x, y, z \in \widetilde{M}$ are such that x and z are separated by some cone Γ with vertex y and angle $\frac{\pi}{2}$, and Γ_1 , the 1-shifted cone of Γ , i.e., $x \in \widetilde{M} \setminus \Gamma$, $z \in \Gamma_1$, then*

$$|d_{G_{\mathbf{v}}}(x, y) + d_{G_{\mathbf{v}}}(y, z) - d_{G_{\mathbf{v}}}(x, z)| \leq -\ln c_8.$$

Since the constants $c_i, 0 \leq i \leq 8$, and α_2, α_3 only depend on the geometry of \widetilde{M} and the coefficients of \mathcal{L} , we may assume they are such that Lemmas 4.10–4.15 hold true for every couple $\mathcal{L}^\lambda, \mathbf{G}^\lambda$ with $\lambda \in (-1, 1)$.

For each $\lambda, \beta \in (-1, 1)$, following the proof of Proposition 3.1, we obtain a constant $\bar{h}_{\lambda,0}$ such that for $\tilde{\mathbf{m}}^\lambda$ -a.e. \mathbf{v} and $\bar{\mathbb{P}}_{\mathbf{v}}^\lambda$ -a.e. $\omega \in \Omega_+$,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) = \lim_{t \rightarrow +\infty} -\frac{1}{t} \ln G_{\mathbf{v}}^0(\tilde{\omega}(0), \tilde{\omega}(t)) = \bar{h}_{\lambda,\beta},$$

where the first equality holds since for $\bar{\mathbb{P}}_{\mathbf{v}}^\lambda$ -a.e. ω , $\tilde{\omega}(t)$ converges to a point in $\partial \widetilde{M}$ as t tends to infinity. Using (4.18), we see that

$$\mathbb{E}_{\bar{\mathbb{P}}^\lambda} \left(\sup_{t \in [0,1]} [d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t))]^2 \right) \leq 2\alpha_2^2 \mathbb{E}_{\bar{\mathbb{P}}^\lambda} \left(\sup_{t \in [0,1]} [d_{\mathcal{W}}(\omega(0), \omega(t))]^2 \right) + 8(\ln c_2)^2 < +\infty,$$

where $\bar{\mathbb{P}}^\lambda = \int \bar{\mathbb{P}}_{\mathbf{v}}^\lambda d\tilde{\mathbf{m}}^\lambda(\mathbf{v})$ is the shift invariant measure on Ω_+ corresponding to $\tilde{\mathbf{m}}^\lambda$. Using (4.17), we have by the Subadditive Ergodic Theorem that for $\lambda, \beta \in (-1, 1)$,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\bar{\mathbb{P}}^\lambda} \left(d_{G_{\mathbf{v}}^\beta}(\tilde{\omega}(0), \tilde{\omega}(t)) \right) = \bar{h}_{\lambda,\beta}.$$

By Proposition 3.1, $\bar{h}_\lambda = \bar{h}_{\lambda,\lambda}$ and $\bar{h}_0 = \bar{h}_{0,0}$. The main strategy to prove Theorem 4.9 is to split $(\bar{h}_\lambda - \bar{h}_0)/\lambda$ into two terms:

$$\frac{1}{\lambda}(\bar{h}_\lambda - \bar{h}_0) = \frac{1}{\lambda}(\bar{h}_{\lambda,\lambda} - \bar{h}_{\lambda,0}) + \frac{1}{\lambda}(\bar{h}_{\lambda,0} - \bar{h}_{0,0}) =: (\text{I})_h^\lambda + (\text{II})_h^\lambda,$$

then show $\lim_{\lambda \rightarrow 0} (\text{I})_h^\lambda = 0$ and $\lim_{\lambda \rightarrow 0} (\text{II})_h^\lambda = \lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\mathbb{Q}^0}(\mathbf{Z}_{h,t}^0 \mathbf{M}_t^0)$ successively.

Lemma 4.16. $\lim_{\lambda \rightarrow 0} (\text{I})_h^\lambda = 0$.

Proof. By the same the proof as for Proposition 3.1, we see that for each pair $\lambda, \beta \in (-1, 1)$,

$$\bar{h}_{\lambda,\beta} = \lim_{t \rightarrow +\infty} -\frac{1}{t} \int \left(\ln p_{\mathbf{v}}^\beta(t, x, y) \right) p_{\mathbf{v}}^\lambda(t, x, y) dy$$

holds true for $\widetilde{\mathbf{m}}^\lambda$ -almost every $\mathbf{v} = (x, \xi) \in S\widetilde{M}$. So we have $\limsup_{\lambda \rightarrow 0+} (\text{I})_h^\lambda \leq 0$ since

$$\begin{aligned} (4.20) \quad \bar{h}_{\lambda,\lambda} - \bar{h}_{\lambda,0} &= \lim_{t \rightarrow +\infty} -\frac{1}{t} \int \left(\ln \frac{p_{\mathbf{v}}^\lambda(t, x, y)}{p_{\mathbf{v}}^0(t, x, y)} \right) p_{\mathbf{v}}^\lambda(t, x, y) dy \\ &\leq \lim_{t \rightarrow +\infty} \frac{1}{t} \int \frac{p_{\mathbf{v}}^0(t, x, y)}{p_{\mathbf{v}}^\lambda(t, x, y)} p_{\mathbf{v}}^\lambda(t, x, y) dy \\ &= 0, \end{aligned}$$

where we use $-\ln a \leq a^{-1} - 1$ for $a > 0$ to derive the second inequality.

To show $\liminf_{\lambda \rightarrow 0+} (\text{I})_h^\lambda \geq 0$, we proceed to estimate $\ln(p_{\mathbf{v}}^\lambda(t, x, y)/p_{\mathbf{v}}^0(t, x, y))$ using the Girsanov-Cameron-Martin formula in Sec. 4.1. For $\mathbf{v}, \mathbf{w} \in S\widetilde{M}$, let $\Omega_{\mathbf{v}, \mathbf{w}, t}$ be the collection of $\omega \in \Omega_+$ such that $\omega(0) = \mathbf{v}, \omega(t) = \mathbf{w}$. Since the space Ω_+ is separable, the measure \mathbb{P}^λ disintegrates into a class of conditional probabilities $\{\mathbb{P}_{\mathbf{v}, \mathbf{w}, t}^\lambda\}_{\mathbf{v}, \mathbf{w} \in S\widetilde{M}}$ on $\Omega_{\mathbf{v}, \mathbf{w}, t}$'s such that

$$(4.21) \quad \mathbb{E}_{\mathbb{P}_{\mathbf{v}, \mathbf{w}, t}^\lambda} \left(\frac{d\mathbb{P}_t^0}{d\mathbb{P}_t^\lambda} \right) = \frac{\mathbf{p}^0(t, \mathbf{v}, \mathbf{w})}{\mathbf{p}^\lambda(t, \mathbf{v}, \mathbf{w})}.$$

Letting $\mathbf{v} = (x, \xi), \mathbf{w} = (y, \xi)$ in (4.21), we obtain

$$(4.22) \quad \ln \frac{p_{\mathbf{v}}^0(t, x, y)}{p_{\mathbf{v}}^\lambda(t, x, y)} = \ln \left(\mathbb{E}_{\mathbb{P}_{\mathbf{v}, \mathbf{w}, t}^\lambda} \left(\frac{d\mathbb{P}_{\mathbf{v}, t}^0}{d\mathbb{P}_{\mathbf{v}, t}^\lambda} \right) \right) \geq \mathbb{E}_{\mathbb{P}_{\mathbf{v}, \mathbf{w}, t}^\lambda} \left(\ln \left(\frac{d\mathbb{P}_{\mathbf{v}, t}^0}{d\mathbb{P}_{\mathbf{v}, t}^\lambda} \right) \right).$$

Recall that

$$\frac{d\overline{\mathbb{P}}_{\mathbf{v}, t}^0}{d\overline{\mathbb{P}}_{\mathbf{v}, t}^\lambda}(\mathbf{y}_{\mathbf{v}, [0, t]}^\lambda) = \mathbb{E}_{\mathbb{Q}^\lambda} \left(\overline{\mathbf{M}}_t^\lambda | \mathcal{F}(\mathbf{y}_{\mathbf{v}, [0, t]}^\lambda) \right),$$

where $\mathbf{y}^\lambda = (\mathbf{y}_{\mathbf{v}, t}^\lambda)_{\mathbf{v} \in S\widetilde{M}, t \in \mathbb{R}_+}$ is the diffusion process on $(\overline{\Theta}, \overline{\mathbb{Q}}^\lambda)$ corresponding to \mathcal{L}^λ and

$$\overline{\mathbf{M}}_t^\lambda(\underline{\omega}) = \exp \left\{ -\frac{1}{2} \int_0^t \langle Z^\lambda(\mathbf{y}_{\mathbf{v}, s}^\lambda(\underline{\omega})), \mathbf{u}_{\mathbf{v}, s}^\lambda(\underline{\omega}) dB_s(\underline{\omega}) \rangle_{\mathbf{y}_{\mathbf{v}, s}^\lambda(\underline{\omega})} - \frac{1}{4} \int_0^t \| -Z^\lambda(\mathbf{y}_{\mathbf{v}, s}^\lambda(\underline{\omega})) \|^2 ds \right\}.$$

So we can further deduce from (4.22) that

$$\begin{aligned}
\ln \frac{p_{\mathbf{v}}^0(t, x, y)}{p_{\mathbf{v}}^\lambda(t, x, y)} &\geq \mathbb{E}_{\mathbb{P}_{\mathbf{v}, \mathbf{w}, t}^\lambda} \left(\mathbb{E}_{\mathbb{Q}^\lambda} \left(\left(-\frac{1}{2} \int_0^t \langle Z^\lambda(\mathbf{y}_{\mathbf{v}, s}^\lambda), \mathbf{u}_{\mathbf{v}, s}^\lambda dB_s \rangle_{\mathbf{y}_{\mathbf{v}, s}^\lambda} - \frac{1}{4} \int_0^t \|Z^\lambda(\mathbf{y}_{\mathbf{v}, s}^\lambda)\|^2 ds \right) \middle| \mathcal{F}(\mathbf{y}_{\mathbf{v}, [0, t]}^\lambda) \right) \right) \\
&= -\mathbb{E}_{\mathbb{P}_{\mathbf{v}, \mathbf{w}, t}^\lambda} \left(\mathbb{E}_{\mathbb{Q}^\lambda} \left(\left(\frac{1}{4} \int_0^t \|Z^\lambda(\mathbf{y}_{\mathbf{v}, s}^\lambda)\|^2 ds \right) \middle| \mathcal{F}(\mathbf{y}_{\mathbf{v}, [0, t]}^\lambda) \right) \right) \\
&\geq -\frac{1}{4}(\lambda C)^2 t,
\end{aligned}$$

where the first equality holds true since $\int_0^t \langle Z^\lambda(\mathbf{y}_{\mathbf{v}, s}^\lambda), \mathbf{u}_{\mathbf{v}, s}^\lambda dB_s \rangle_{\mathbf{y}_{\mathbf{v}, s}^\lambda}$ is a centered martingale and C is some constant which bounds the norm of $dZ^\lambda/d\lambda$. Reporting this in (4.20) gives

$$\liminf_{\lambda \rightarrow 0+} (\text{I})_h^\lambda = \liminf_{\lambda \rightarrow 0+} \frac{1}{\lambda} (\bar{h}_{\lambda, \lambda} - \bar{h}_{\lambda, 0}) \geq -\frac{1}{4} \limsup_{\lambda \rightarrow 0+} (\lambda C^2) = 0.$$

□

The analysis of $(\text{II})_h^\lambda$ is analogous to that was used for $(\text{II})_\ell$. Let $\lambda = 1/\sqrt{t}$ and write

$$(4.23) \quad (\text{III})_h^t := \left(\mathbb{E}_{\mathbb{P}_t^\lambda} \left(\frac{1}{\sqrt{t}} d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) \right) - \mathbb{E}_{\mathbb{P}_t^0} \left(\frac{1}{\sqrt{t}} d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) \right) \right).$$

We claim that if $\lim_{t \rightarrow +\infty} (\text{III})_h^t$ exists, so does $\lim_{\lambda \rightarrow 0} (\text{II})_h^\lambda$ and the limits are equal. It suffices to find a finite number D_h such that for $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is from Lemma 4.4) and all $t \in \mathbb{R}_+$,

$$(4.24) \quad |\mathbb{E}_{\mathbb{P}^\lambda} (d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t))) - t\bar{h}_{\lambda, 0}| \leq D_h.$$

Using again the fact that the \mathcal{L}^λ -diffusion has leafwise infinitesimal generator $\mathcal{L}_{\mathbf{v}}^\lambda$ and \mathbb{P}^λ is stationary, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^\lambda} (-\ln k_{\mathbf{v}}^0(\tilde{\omega}(t), \xi)) &= \mathbb{E}_{\mathbb{P}^\lambda} \left(-\int_0^t \frac{\partial}{\partial s} (\ln k_{\mathbf{v}}^0(\tilde{\omega}(s), \xi)) ds \right) \\
&= \mathbb{E}_{\mathbb{P}^\lambda} \left(-\int_0^t \mathcal{L}_{\mathbf{v}}^\lambda (\ln k_{\mathbf{v}}^0)(\tilde{\omega}(s)) ds \right) \\
&= -t \int_{M_0 \times \partial \tilde{M}} \mathcal{L}_{\mathbf{v}}^\lambda (\ln k_{\mathbf{v}}^0) d\tilde{\mathbf{m}}^\lambda \\
&= t\bar{h}_{\lambda, 0}.
\end{aligned}$$

So (4.24) will be a simple consequence of the following lemma.

Lemma 4.17. *There exists a finite number \tilde{D}_3 such that for all $\lambda \in [-\delta_1, \delta_1]$ and $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}^\lambda} \left(|d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) + \ln k_{\mathbf{v}}^0(\tilde{\omega}(t), \xi)|^2 \right) < \tilde{D}_3.$$

Proof. For $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, $\omega \in \Omega_+$ starting from \mathbf{v} , $t \geq 0$, let $z_t(\widetilde{\omega})$ be the point on the geodesic ray $\gamma_{\widetilde{\omega}(t), \xi}$ closest to x . We will divide Ω_+ into four events $A'_i(t)$, $1 \leq i \leq 4$, and show there exists a finite \widetilde{D}'_3 such that

$$\mathbf{I}_i := \mathbb{E}_{\mathbb{P}^\lambda} \left(\left| d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(0), \widetilde{\omega}(t)) + \ln k_{\mathbf{v}}^0(\widetilde{\omega}(t), \xi) \right|^2 \cdot \mathbf{1}_{A'_i(t)} \right) \leq \widetilde{D}'_3.$$

Let $A'_1(t)$ be the event that $d(\widetilde{\omega}(0), \widetilde{\omega}(t)) > 1$ and $d(\widetilde{\omega}(0), z_t(\widetilde{\omega})) \leq 1$. For $\omega \in A'_1(t)$, using Harnack's inequality (4.15) and Lemma 4.14, we easily specify the constant ratios involved in (3.14) and obtain $\mathbf{I}_1 \leq (\ln(c_2 c_4^2 c_7))^2$.

Let $A'_2(t)$ be the collection of ω such that both $d(\widetilde{\omega}(0), \widetilde{\omega}(t))$ and $d(\widetilde{\omega}(0), z_t(\widetilde{\omega}))$ are greater than 1 and $z_t(\widetilde{\omega}) \neq \widetilde{\omega}(t)$. For such ω , we first have by Lemma 4.15 that

$$(4.25) \quad \left| d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(0), \widetilde{\omega}(t)) - d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(0), z_t(\widetilde{\omega})) - d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(t), z_t(\widetilde{\omega})) \right| \leq -\ln c_8.$$

For $d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(t), z_t(\widetilde{\omega}))$, it is true by Lemma 4.14 that

$$\left| d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(t), z_t(\widetilde{\omega})) + \ln G_{\mathbf{v}}^0(y, \widetilde{\omega}(t)) - \ln G_{\mathbf{v}}^0(y, z_t(\widetilde{\omega})) \right| \leq -\ln c_7,$$

where y is an arbitrary point on $\gamma_{z_t(\widetilde{\omega}), \xi}$ far away from $z_t(\widetilde{\omega})$. Then we can use Lemma 4.15 to replace $\ln G_{\mathbf{v}}^0(y, z_t(\widetilde{\omega}))$ by $\ln G_{\mathbf{v}}^0(y, \widetilde{\omega}(0)) - \ln G_{\mathbf{v}}^0(z_t(\widetilde{\omega}), \widetilde{\omega}(0))$, which, by letting y tend to ξ , gives

$$\left| d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(t), z_t(\widetilde{\omega})) + \ln k_{\mathbf{v}}^0(\widetilde{\omega}(t), \xi) \right| \leq -\ln(c_7 c_8) + \left| \ln G_{\mathbf{v}}^0(\widetilde{\omega}(0), z_t(\widetilde{\omega})) \right|.$$

This, together with (4.25), further implies

$$\begin{aligned} \left| d_{G_{\mathbf{v}}^0}(\widetilde{\omega}(0), \widetilde{\omega}(t)) + \ln k_{\mathbf{v}}^0(\widetilde{\omega}(t), \xi) \right| &\leq -\ln(c_2 c_7 c_8^2) + 2 \left| \ln G_{\mathbf{v}}^0(\widetilde{\omega}(0), z_t(\widetilde{\omega})) \right| \\ &\leq -\ln(c_2^5 c_7 c_8^2) + 2\alpha_2 d(\widetilde{\omega}(0), z_t(\widetilde{\omega})). \end{aligned}$$

Since \widetilde{M} is δ -Gromov hyperbolic for some $\delta > 0$, it is true (cf. [K2, Proposition 2.1]) that

$$d(x, \gamma_{y,z}) \leq (y|z)_x + 4\delta, \text{ for any } x, y, z \in \widetilde{M}.$$

Consequently, we have

$$(4.26) \quad d(\widetilde{\omega}(0), z_t(\widetilde{\omega})) \leq (\widetilde{\omega}(t)|\xi)_{\widetilde{\omega}(0)} + 4\delta = \frac{1}{2} |d(\widetilde{\omega}(0), \widetilde{\omega}(t)) - b_{\mathbf{v}}(\widetilde{\omega}(t))| + 4\delta.$$

Using Lemma 4.6, we finally obtain

$$\mathbf{I}_2 \leq 2 \left(8\alpha_2 \delta - \ln(c_2^5 c_7 c_8^2) \right)^2 + 2\alpha_2^2 D_3.$$

Let $A'_3(t)$ be the collection of ω such that $d(\widetilde{\omega}(0), \widetilde{\omega}(t)) > 1$ and $z_t(\widetilde{\omega}) = \widetilde{\omega}(t)$. Let $\gamma'_{\widetilde{\omega}(t), \xi}$ be the two sided extension of the geodesic $\gamma_{\widetilde{\omega}(t), \xi}$ and let $z'_t(\widetilde{\omega}) \in \gamma'_{\widetilde{\omega}(t), \xi}$ be the point closet to $\widetilde{\omega}(0)$. Then $z'_t(\widetilde{\omega}) \preceq z_t(\widetilde{\omega})$ on $\gamma'_{\widetilde{\omega}(t), \xi}$. For $\omega \in A'_3(t)$, using (4.15) if $d(z'_t(\widetilde{\omega}), \widetilde{\omega}(t)) < 1$ (or

using Lemma 4.15, otherwise), we see that

$$\begin{aligned}
d_{G_{\mathbf{V}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) &\leq d_{G_{\mathbf{V}}^0}(\tilde{\omega}(0), z'_t(\tilde{\omega})) + d_{G_{\mathbf{V}}^0}(z'_t(\tilde{\omega}), \tilde{\omega}(t)) - \ln(c_4 c_8) \\
&\leq \alpha_2 (d(\tilde{\omega}(0), z'_t(\tilde{\omega})) + d(z'_t(\tilde{\omega}), \tilde{\omega}(t))) - \ln(c_2^4 c_4 c_8) \\
&\leq 3\alpha_2 d(\tilde{\omega}(0), \gamma_{\tilde{\omega}(t), \xi}) - \ln(c_2^4 c_4 c_8) \\
&\leq \frac{3}{2}\alpha_2 |d(\tilde{\omega}(0), \tilde{\omega}(t)) - b_{\mathbf{V}}(\tilde{\omega}(t))| + 12\alpha_2 \delta - \ln(c_2^4 c_4 c_8),
\end{aligned}$$

where we use (4.26) to derive the last inequality. Choose $y \in \gamma_{\tilde{\omega}(t), \xi}$ with $d(\tilde{\omega}(0), y)$ and $d(\tilde{\omega}(t), y)$ are greater than 1. Similarly, using Lemma 4.15, and then Lemma 4.14, we have

$$\begin{aligned}
\left| \ln \frac{G_{\mathbf{V}}^0(\tilde{\omega}(t), y)}{G_{\mathbf{V}}^0(\tilde{\omega}(0), y)} \right| &= |d_{G_{\mathbf{V}}^0}(\tilde{\omega}(0), y) - d_{G_{\mathbf{V}}^0}(\tilde{\omega}(t), y)| \\
&\leq -\ln c_8 + |d_{G_{\mathbf{V}}^0}(\tilde{\omega}(0), z'_t(\tilde{\omega})) + d_{G_{\mathbf{V}}^0}(z'_t(\tilde{\omega}), y) - d_{G_{\mathbf{V}}^0}(\tilde{\omega}(t), y)| \\
&\leq -\ln(c_7 c_8) + d_{G_{\mathbf{V}}^0}(\tilde{\omega}(0), z'_t(\tilde{\omega})) + d_{G_{\mathbf{V}}^0}(z'_t(\tilde{\omega}), \tilde{\omega}(t)) \\
&\leq \frac{3}{2}\alpha_2 |d(\tilde{\omega}(0), \tilde{\omega}(t)) - b_{\mathbf{V}}(\tilde{\omega}(t))| + 12\alpha_2 \delta - \ln(c_2^4 c_7 c_8).
\end{aligned}$$

Letting y tend to ξ , we obtain

$$|\ln k_{\mathbf{V}}^0(\tilde{\omega}(t), \xi)| \leq \frac{3}{2}\alpha_2 |d(\tilde{\omega}(0), \tilde{\omega}(t)) - b_{\mathbf{V}}(\tilde{\omega}(t))| + 12\alpha_2 \delta - \ln(c_2^4 c_7 c_8).$$

Thus, using Lemma 4.6 again, we obtain

$$\begin{aligned}
\mathbf{I}_3 &\leq \mathbb{E}_{\mathbb{P}^\lambda} \left((3\alpha_2 |d(\tilde{\omega}(0), \tilde{\omega}(t)) - b_{\mathbf{V}}(\tilde{\omega}(t))| + 24\alpha_2 \delta - \ln(c_2^8 c_4 c_7 c_8^2))^2 \right) \\
&\leq 18\alpha_2^2 D_3 + 2 (24\alpha_2 \delta - \ln(c_2^8 c_4 c_7 c_8^2))^2.
\end{aligned}$$

Finally, let $A'_4(t)$ be the event that $d(\tilde{\omega}(0), \tilde{\omega}(t)) \leq 1$. Then $\mathbf{I}_4 \leq (-\ln(c_2 c_4))^2$ by the classical Harnack inequality (4.15). \square

We finish the proof of Theorem 4.9 by showing $\lim_{t \rightarrow +\infty} (\text{III})_h^t = \lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}_{h,t}^0 \mathbf{M}_t^0)$. The proof is completely parallel to the computation of $(\text{II})_\ell$. We prove Proposition 4.2 first.

Proof of Proposition 4.2. Let $(\mathbf{Z}_t^1)_{t \in \mathbb{R}_+}$, u_1 be as in Proposition 3.7. The process $(\mathbf{Z}_t^1)_{t \in \mathbb{R}_+}$ is a centered martingale with stationary increments and its law under $\overline{\mathbb{P}}^0$ is the same as the law of $(\overline{\mathbf{Z}}_t^1)_{t \in \mathbb{R}_+}$ under $\overline{\mathbb{Q}}^0$, where $(\overline{\mathbf{Z}}_t^1)_{t \in \mathbb{R}_+}$ on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ is given by

$$\overline{\mathbf{Z}}_t^1(\mathbf{v}, \underline{\omega}) = \ln k_{\mathbf{V}}(\pi_{\widetilde{M}}(\mathbf{y}_{\mathbf{V}, t}(\underline{\omega})), \xi) + t\overline{h}_0 + u_1(\pi_{SM}(\mathbf{y}_{\mathbf{V}, t}(\underline{\omega}))) - u_1(\pi_{SM}(\mathbf{v})).$$

The pair $(-\overline{\mathbf{Z}}_t^1, \mathbf{M}_t^0)$ is a centered martingale on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ with stationary increments and integrable increasing process function. As before, it follows that for $(a, b) \in \mathbb{R}^2$, $-a\overline{\mathbf{Z}}_t^1/\sqrt{t} + b\mathbf{M}_t^0/\sqrt{t}$ converge in distribution in $\overline{\mathbb{Q}}^0$ to a centered Gaussian law with variance $\Sigma_h[a, b] = (a, b)\Sigma_h(a, b)^T$ for some matrix Σ_h . Therefore, $(-\overline{\mathbf{Z}}_t^1/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ converge in

distribution to a centered Gaussian vector with covariance Σ_h . Since both $\bar{\mathbf{Z}}_t^1$ and \mathbf{M}_t^0 have stationary increments, we also have

$$\Sigma_h[a, b] = \frac{1}{t} \mathbb{E}_{\bar{\mathbb{Q}}^0} \left[(-a\bar{\mathbf{Z}}_t^1 + b\mathbf{M}_t^0)^2 \right], \text{ for all } t \in \mathbb{R}_+.$$

Finally, for $\bar{\mathbb{P}}_{\mathbf{v}}^0$ -a.e. orbits $\omega \in \Omega_+$, $\tilde{\omega}$, the projection of ω to \widetilde{M} , is such that

$$\limsup_{t \rightarrow +\infty} |\ln G_{\mathbf{v}}(x, \tilde{\omega}(t)) - \ln k_{\mathbf{v}}(\tilde{\omega}(t), \xi)| < +\infty.$$

Moreover, we have by Lemma 4.17 that

$$\sup_t \mathbb{E}_{\bar{\mathbb{P}}^\lambda} (|\mathbf{Z}_{h,t}^0 + \bar{\mathbf{Z}}_t^1|^2) < +\infty$$

and hence

$$\mathbb{E}_{\bar{\mathbb{P}}^\lambda} \left(\frac{1}{t} |\mathbf{Z}_{h,t}^0 + \bar{\mathbf{Z}}_t^1|^2 \right) \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

Consequently, $(\mathbf{Z}_{h,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ has the same limit Gaussian law as $(-\bar{\mathbf{Z}}_t^1/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ and its covariance matrix under $\bar{\mathbb{Q}}^0$ converges to Σ_h as t goes to infinity. \square

Lemma 4.18. *The $(\text{III})_h^t$ defined in (4.23) equals to $\lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\bar{\mathbb{Q}}^0} (\mathbf{Z}_{h,t}^0 \mathbf{M}_t^0)$.*

Proof. Since for $\bar{\mathbb{P}}^0$ -a.e. ω , $\tilde{\omega}(t)$ tends to a boundary point as t goes to infinity, so $(d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) - t\bar{h}_0)/\sqrt{t}$ and $-(\ln \mathbf{G}^0(\omega(0), \omega(t)) + t\bar{h}_0)/\sqrt{t}$ have the same asymptotic distribution, which is a centered Gaussian distribution by Proposition 3.6. Thus,

$$\begin{aligned} & \lim_{t \rightarrow +\infty} (\text{III})_h^t \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{\mathbb{P}}_t^\lambda} \left(\frac{1}{\sqrt{t}} (d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) - t\bar{h}_0) \right) - \lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{\mathbb{P}}^0} \left(\frac{1}{\sqrt{t}} (d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) - t\bar{h}_0) \right) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{\mathbb{P}}_t^\lambda} \left(\frac{1}{\sqrt{t}} (d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) - t\bar{h}_0) \right) \\ &=: (\text{IV})_h, \end{aligned}$$

providing the last limit exists. Let $\mathbf{y} = (\mathbf{y}_t)_{t \in \mathbb{R}_+} = (\mathbf{y}_{\mathbf{v},t})_{\mathbf{v} \in SM, t \in \mathbb{R}_+}$ be the diffusion process on $(\bar{\Theta}, \bar{\mathbb{Q}}^\lambda)$ corresponding to \mathcal{L}^λ defined in Sec. 4.1. Using the Girsanov-Cameron-Martin formula for $d\bar{\mathbb{P}}_{\mathbf{v},t}^\lambda/d\bar{\mathbb{P}}_{\mathbf{v},t}^0$ (see (4.5)), we have

$$\begin{aligned} (\text{IV})_h &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{\mathbb{P}}^0} \left(\frac{1}{\sqrt{t}} (d_{G_{\mathbf{v}}^0}(\tilde{\omega}(0), \tilde{\omega}(t)) - t\bar{h}_0) \frac{d\bar{\mathbb{P}}_{\omega(0),t}^\lambda}{d\bar{\mathbb{P}}_{\omega(0),t}^0} \right) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{\mathbb{Q}}^0} \left(\frac{1}{\sqrt{t}} (d_{G_{\mathbf{v}}^0}(\mathbf{y}_0(\mathbf{v}, \underline{\omega}), \mathbf{y}_t(\mathbf{v}, \underline{\omega})) - t\bar{h}_0) \cdot \bar{\mathbf{M}}_t^\lambda(\underline{\omega}) \right) \\ &= \lim_{t \rightarrow +\infty} \mathbb{E}_{\bar{\mathbb{Q}}^0} \left(\frac{1}{\sqrt{t}} \mathbf{Z}_{h,t}^0 \cdot \bar{\mathbf{M}}_t^\lambda \right), \end{aligned}$$

where we identify $\mathbf{y}_t(\mathbf{v}, \underline{\omega}) \in \widetilde{M} \times \{\xi\}$ with its projection point on \widetilde{M} . As before, by Proposition 4.2, the variables $\frac{1}{\sqrt{t}}\mathbf{Z}_{h,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda$ converge in distribution to $\mathbf{Z}_h^0 e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\mathbb{Q}^0}((\mathbf{M}^0)^2)}$. Again, we have by Proposition 4.2 and the same reasoning as in the proof of Lemma 4.8 that

$$\sup_t \mathbb{E}_{\mathbb{Q}^0} \left(\left| \frac{1}{\sqrt{t}} \mathbf{Z}_{h,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda \right|^{\frac{3}{2}} \right) < +\infty.$$

It follows from Lemma 4.7 that

$$\lim_{t \rightarrow +\infty} (\text{III})_h^t = \lim_{t \rightarrow +\infty} \mathbb{E}_{\mathbb{Q}^0} \left(\frac{1}{\sqrt{t}} \mathbf{Z}_{h,t}^0 \cdot \overline{\mathbf{M}}_t^\lambda \right) = \mathbb{E}_{\mathbb{Q}^0} \left(\mathbf{Z}_h^0 e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\mathbb{Q}^0}((\mathbf{M}^0)^2)} \right).$$

Finally, using the fact that $(\mathbf{Z}_h^0, \mathbf{M}^0)$ has a bivariate normal distribution, we have

$$\mathbb{E}_{\mathbb{Q}^0} \left(\mathbf{Z}_h^0 e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\mathbb{Q}^0}((\mathbf{M}^0)^2)} \right) = \mathbb{E}_{\mathbb{Q}^0}(\mathbf{Z}_h^0 \mathbf{M}^0),$$

which is $\lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\mathbb{Q}^0}(\mathbf{Z}_{h,t}^0 \mathbf{M}_t^0)$ by Proposition 4.2. \square

5. PROOF OF THE MAIN THEOREMS

In this section, we show Theorem 1.1 and Theorem 1.2 in the introduction.

Let (M, g) be a negatively curved closed connected m -dimensional Riemannian manifold as before. Let $\partial \widetilde{M}$ be the geometric boundary of the universal cover space $(\widetilde{M}, \widetilde{g})$. We can identify $\widetilde{M} \times \partial \widetilde{M}$ with $S\widetilde{M}_{\widetilde{g}}$, the unit tangent bundle of \widetilde{M} in metric \widetilde{g} , by sending (x, ξ) to $\overline{X}_{\widetilde{g}}(x, \xi)$, the unit tangent vector of the \widetilde{g} -geodesic starting at x pointing at ξ .

Let $\lambda \in (-1, 1) \mapsto g^\lambda$ be a one-parameter family of C^3 metrics on M of negative curvature with $g^0 = g$. Denote by \widetilde{g}^λ the lift of g^λ in \widetilde{M} . For each λ , the geometric boundary of $(\widetilde{M}, \widetilde{g}^\lambda)$, denoted $\partial \widetilde{M}_{\widetilde{g}^\lambda}$, can be identified with $\partial \widetilde{M}$ since the identity isomorphism from $G = \pi_1(M)$ to itself induces a homeomorphism between $\partial \widetilde{M}_{\widetilde{g}^\lambda}$ and $\partial \widetilde{M}$. So each $(x, \xi) \in \widetilde{M} \times \partial \widetilde{M}$ is also associated with a tangent vector $\overline{X}_{\widetilde{g}^\lambda}(x, \xi) \in T\widetilde{M}$, which is the unit tangent vector of the \widetilde{g}^λ -geodesic starting at x pointing at ξ . Our very first step to study the differentiability of the linear drift under a one-parameter family of conformal changes g^λ of g is to understand the differentiable dependence of $\overline{X}_{\widetilde{g}^\lambda}(x, \xi)$ on the parameter λ .

For each g^λ , there exist (g, g^λ) -Morse correspondence ([**Ano, Gro, Mor**]), the homeomorphisms from SM_g to SM_{g^λ} preserving the geodesics on M . The (g, g^λ) -Morse correspondence is not unique, but any two such maps only differ by shifts in the geodesic flow directions (i.e., if F_1, F_2 are two (g, g^λ) -Morse correspondence maps, then there exists a real valued function $t(\cdot)$ on SM_g such that $F_1^{-1} \circ F_2(v) = \Phi_{t(v)}(v)$ for $v \in SM_g$), where Φ is the geodesic flow map on SM_g ([**Ano, Gro, Mor**], see [**FF**, Theorem 1.1]).

Let us construct a (g, g^λ) -Morse correspondence map by lifting the systems to their universal cover spaces as in [**Gro**]. For an oriented geodesic γ in $(\widetilde{M}, \widetilde{g})$, denote by $\partial^+(\gamma) \in$

$\partial\widetilde{M}_{\widetilde{g}}$ and $\partial^-(\gamma) \in \partial\widetilde{M}_{\widetilde{g}}$ the asymptotic classes of its positive and negative directions. The map $\gamma \mapsto (\partial^+(\gamma), \partial^-(\gamma)) \in \partial\widetilde{M}_{\widetilde{g}} \times \partial\widetilde{M}_{\widetilde{g}}$ establishes a homeomorphism between the set of all oriented geodesics in $(\widetilde{M}, \widetilde{g})$ and $\partial^2(\widetilde{M}_{\widetilde{g}}) = (\partial\widetilde{M}_{\widetilde{g}} \times \partial\widetilde{M}_{\widetilde{g}}) \setminus \{(\xi, \xi) : \xi \in \partial\widetilde{M}_{\widetilde{g}}\}$. So the natural homeomorphism $D^\lambda : \partial^2(\widetilde{M}_{\widetilde{g}}) \rightarrow \partial^2(\widetilde{M}_{\widetilde{g}^\lambda})$ induced from the identity isomorphism from G to itself can be viewed as a homeomorphism between the sets of oriented geodesics in $(\widetilde{M}, \widetilde{g})$ and $(\widetilde{M}, \widetilde{g}^\lambda)$. Realize points from $SM_{\widetilde{g}}$ by pairs (γ, y) , where γ is an oriented geodesic and $y \in \gamma$, and define a map $\widetilde{F}^\lambda : SM_{\widetilde{g}} \rightarrow SM_{\widetilde{g}^\lambda}$ by sending $(\gamma, y) \in SM_{\widetilde{g}}$ to

$$\widetilde{F}^\lambda(\gamma, y) = (D^\lambda(\gamma), y'),$$

where y' is the intersection point of $D^\lambda(\gamma)$ and the hypersurface $\{\exp_{\widetilde{g}} Y : Y \perp v\}$, where v is the vector in $S_y \widetilde{M}_{\widetilde{g}}$ pointing at $\partial^+(\gamma)$. The map \widetilde{F}^λ is a homeomorphism since both g and g^λ are of negative curvature. Returning to SM_g and SM_{g^λ} , we obtain a map F^λ . Given any sufficiently small ϵ , if g^λ is in a sufficiently small C^3 -neighborhood of g , then F^λ is the only (g, g^λ) -Morse correspondence map such that the footpoint of $F^\lambda(v)$ belongs to the hypersurface of points $\{\exp_g Y : Y \perp v, \|Y\|_g < \epsilon\}$.

Regard SM_{g^λ} as a subset of TM and let $\pi^\lambda : SM_{g^\lambda} \rightarrow SM_g$ be the projection map sending v to $v/\|v\|_g$. The map π^λ records the direction information of the vectors of SM_{g^λ} in SM_g . Let $F^\lambda : SM_g \rightarrow SM_{g^\lambda}$ be the (g, g^λ) -Morse correspondence map obtained as above. We obtain a one-parameter family of homeomorphisms $\pi^\lambda \circ F^\lambda$ from SM_g to SM_g . By using the implicit function theorem, de la Llave-Marco-Moriyón [LMM, Theorem A.1] improved the differentiable dependence of $\pi^\lambda \circ F^\lambda$ on the parameter λ .

Theorem 5.1. (cf. [FF, Theorem 2.1]) *There exists a C^3 neighborhood of g so that for any C^3 one-parameter family of C^3 metrics $\lambda \in (-1, 1) \mapsto g^\lambda$ in it with $g^0 = g$, the map $\lambda \mapsto \pi^\lambda \circ F^\lambda$ is C^3 with values in the Banach manifold of continuous maps $SM_g \rightarrow SM_g$. The tangent to the curve $\pi^\lambda \circ F^\lambda$ is a continuous vector field Ξ_λ on SM_g .*

Following Fathi-Flaminio [FF], we will call $\Xi := \Xi_0$ in Theorem 5.1 the *infinitesimal Morse correspondence at g for the curve g^λ* . It was shown in [FF] that the vector field Ξ only depends on g and the differential of g^λ in λ at 0.

Theorem 5.2. ([FF, Proposition 2.7]) *Let Ξ be the infinitesimal Morse correspondence at g for the curve g^λ and let Ξ_γ be the restriction of the projection of Ξ in TM to a unit speed g -geodesic γ . Then Ξ_γ is the unique bounded solution of the equation*

$$(5.1) \quad \nabla_{\dot{\gamma}}^2 \Xi_\gamma + R(\Xi_\gamma, \dot{\gamma})\dot{\gamma} + \Gamma_{\dot{\gamma}}\dot{\gamma} - \langle \Gamma_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma} \rangle \dot{\gamma} = 0$$

satisfying $\langle \Xi_\gamma, \dot{\gamma} \rangle = 0$ along γ , where $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$, ∇ and R are the Levi-Civita connection and curvature tensor of metric g , ∇^λ is the Levi-Civita connection of the metric g^λ and $\Gamma = \partial_\lambda \nabla^\lambda|_{\lambda=0}$. The vertical component of Ξ in $T(SM_g)$ is given by $\nabla_{\dot{\gamma}} \Xi_\gamma$.

We will still denote Ξ the lift to $T(\widetilde{SM}_{\widetilde{g}})$ of the infinitesimal Morse-correspondence at g for the curve g^λ . For any geodesic γ in $(\widetilde{M}, \widetilde{g})$, let $N(\gamma)$ be the normal bundle of γ :

$$N(\gamma) = \cup_{t \in \mathbb{R}} N_t(\gamma), \text{ where } N_t(\gamma) = (\dot{\gamma}(t))^\perp = \{E \in T_{\gamma(t)}\widetilde{M} : \langle E, \dot{\gamma}(t) \rangle = 0\}.$$

The one-parameter family of vectors along γ arising from the infinitesimal Morse correspondence:

$$\Upsilon(t) := (\Gamma_{\dot{\gamma}}\dot{\gamma} - \langle \Gamma_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma} \rangle \dot{\gamma})(\gamma(t)), \quad t \in \mathbb{R},$$

is such that $\Upsilon(t)$ belongs to $N_t(\gamma)$ for all t . The restriction of the infinitesimal Morse correspondence to γ is $(\Xi_\gamma, \nabla_{\dot{\gamma}}\Xi_\gamma)$, with both Ξ_γ and $\nabla_{\dot{\gamma}}\Xi_\gamma$ belonging to $N(\gamma)$ as well. In the following, we will specify Ξ_γ and $\nabla_{\dot{\gamma}}\Xi_\gamma$ using Υ and a special coordinate system of $N_t(\gamma)$'s arising from the stable and unstable Jacobi fields along γ .

Let $\mathbf{v} = (x, v)$ be a point in \widetilde{TM} . The tangent vectors in $T_{\mathbf{v}}\widetilde{TM}$ correspond to variations of geodesics and can be represented by Jacobi fields along the geodesic $\gamma_{\mathbf{v}}$. A *Jacobi field* $J(t), t \in \mathbb{R}$, along $\gamma_{\mathbf{v}}$ satisfies the Jacobian equation

$$(5.2) \quad J'' + R(J, \dot{\gamma}_{\mathbf{v}})\dot{\gamma}_{\mathbf{v}} = 0,$$

with R being the curvature tensor, and is uniquely determined by the values of $J(0)$ and $J'(0)$. So we can describe tangent vectors in $T_{\mathbf{v}}\widetilde{TM}$ by the associated pair $(J(0), J'(0))$ of vectors in $T_x\widetilde{M}$. The metric on $T_{\mathbf{v}}\widetilde{TM}$ is given by $\|(J_0, J'_0)\|^2 = \|J_0\|^2 + \|J'_0\|^2$. Assume $\mathbf{v} \in \widetilde{SM}$. Horizontal vectors in $T_{\mathbf{v}}\widetilde{SM}$ correspond to pairs $(J(0), 0)$. In particular, the geodesic spray $\overline{X}_{\mathbf{v}}$ at \mathbf{v} is the horizontal vector associated with $(v, 0)$. A vertical vector in $T_{\mathbf{v}}\widetilde{SM}$ is a vector tangent to $S_x\widetilde{M}$. It corresponds to a pair $(0, J'(0))$, with $J'(0)$ orthogonal to v . The orthogonal space to $\overline{X}_{\mathbf{v}}$ is preserved by the differential $D\Phi_t$ of the geodesic flow. Indeed, the Jacobi fields representation of $TT\widetilde{M}$ satisfies $D_{\mathbf{v}}\Phi_t(J(0), J'(0)) = (J(t), J'(t))$. It is easy to deduce from (5.2) that the Wronskian of two Jacobi fields J and \tilde{J} along γ :

$$W(J, \tilde{J}) := \langle J', \tilde{J} \rangle - \langle J, \tilde{J}' \rangle$$

is constant. Hence $\langle J'(t), \dot{\gamma}(t) \rangle$ remains the same for a Jacobi field J along γ , which implies that if $J'(t_0)$ is in $N_{t_0}(\gamma)$ for some t_0 , then $J'(t)$ is in $N_t(\gamma)$ for all t . Similarly, if both $J(t_0), J'(t_0)$ are in $N_{t_0}(\gamma)$ for some t_0 , then $J(t), J'(t)$ are in $N_t(\gamma)$ for all t .

A (1,1) tensor along γ is a family $V = V(t), t \in \mathbb{R}$, where $V(t)$ is an endomorphism of N_t such that for any family E_t of parallel vectors along γ , the covariant derivative $V'(t)E_t := \nabla_{\dot{\gamma}(t)}V(t)E_t$ exists. Endow $N(\gamma)$ with Fermi orthonormal coordinates given by a parallel frame field along γ . A (1,1) tensor along γ is parallel if $V'(t) = 0$ for all t . It is then given by a constant matrix in Fermi coordinates. The curvature tensor R induces a symmetric (1,1) tensor along γ by $R(t)E = R(E, \dot{\gamma}(t))\dot{\gamma}(t)$. A (1,1) tensor $V(t)$ along γ is called a *Jacobi tensor* if it satisfies $V'' + RV = 0$. If $V(t)$ is a Jacobi tensor along γ , then $J(t) := V(t)E_t$ is a Jacobi field for any parallel field E_t . Consider the Jacobi tensors

determined by two initial conditions:

$$\begin{aligned} S_{\mathbf{v},s}(t) & \text{ is such that } S_{\mathbf{v},s}(0) = \text{Id and } S_{\mathbf{v},s}(s) = 0, \\ U_{\mathbf{v},s}(t) & \text{ is such that } U_{\mathbf{v},s}(0) = \text{Id and } U_{\mathbf{v},s}(-s) = 0. \end{aligned}$$

It is known ([Gre]) that the limits

$$S_{\mathbf{v}} = \lim_{s \rightarrow \infty} S_{\mathbf{v},s} \text{ and } U_{\mathbf{v}} = \lim_{s \rightarrow \infty} U_{\mathbf{v},s}$$

exist, which are called the *stable* and *unstable tensors*, respectively. For each $\mathbf{v} \in \widetilde{SM}$, the vectors $(Y, S'_{\mathbf{v}}(0)Y)$, $Y \in N_0(\gamma)$, (or $(Y, U'_{\mathbf{v}}(0)Y)$) generate $TW_{\mathbf{v}}^{ss}$ (or $TW_{\mathbf{v}}^{su}$).

Since the operator $(U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))$ is positive and symmetric, we can choose vectors $\vec{x}_1, \dots, \vec{x}_{m-1}$ to form a basis of $N_0(\gamma_{\mathbf{v}})$ so that

$$(5.3) \quad \langle (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))\vec{x}_i, \vec{x}_j \rangle = \delta_{ij}.$$

Let J_1, \dots, J_{2m-2} be the Jacobi fields with

$$(J_i(0), J'_i(0)) = \begin{cases} (\vec{x}_i, S'_{\mathbf{v}}(0)\vec{x}_i), & \text{if } i \in \{1, \dots, m-1\}; \\ (\vec{x}_{i+1-m}, U'_{\mathbf{v}}(0)\vec{x}_{i+1-m}), & \text{if } i \in \{m, \dots, 2m-2\}. \end{cases}$$

Since the Wronskian of two Jacobi fields remains constant along geodesics, we have

$$(5.4) \quad W(J_i, J_j) = \begin{cases} 0, & \text{if } i, j \in \{1, \dots, m-1\} \text{ or } i, j \in \{m, \dots, 2m-2\}; \\ -\delta_{i, j+1-m}, & \text{if } i \in \{1, \dots, m-1\} \text{ and } j \in \{m, \dots, 2m-2\}. \end{cases}$$

Equivalently, if we write $\mathbf{J}_s := (J_1, \dots, J_{m-1})$, $\mathbf{J}_u := (J_m, \dots, J_{2m-2})$, then (5.4) gives

$$(5.5) \quad \mathbf{J}_w^* \mathbf{J}'_w = (\mathbf{J}'_w)^* \mathbf{J}_w, \quad w = s \text{ or } u, \text{ and } \mathbf{J}_u^* \mathbf{J}'_s - (\mathbf{J}'_u)^* \mathbf{J}_s = -\text{Id}.$$

The collection $J_1(t), \dots, J_{2m-2}(t)$ provides a basis for each $N_t(\gamma) \times N_t(\gamma)$. Consequently, any $V(t) = (V_1(t), V_2(t)) \in TT\widetilde{M}$ along γ with $V_i(t) \in N_t(\gamma)$, $i = 1, 2$, can be expressed as $(\mathbf{J}_s(t)\vec{a}(t), \mathbf{J}'_s(t)\vec{a}(t)) + (\mathbf{J}_u(t)\vec{b}(t), \mathbf{J}'_u(t)\vec{b}(t))$ with $\vec{a}(t), \vec{b}(t)$ being two \mathbb{R}^{m-1} vector variables in t . To specify the infinitesimal Morse correspondence Ξ at g for the curve g^λ , it suffices to find the coefficients $\vec{a}(t), \vec{b}(t)$ for the restriction of Ξ along any \tilde{g} -geodesic γ .

Proposition 5.3. *Let Ξ be the infinitesimal Morse correspondence at g for a C^3 one-parameter family of C^3 metrics g^λ with $g^0 = g$. Then the restriction of Ξ to a \tilde{g} -geodesic γ is $(\mathbf{J}_s(t)\vec{a}(t), \mathbf{J}'_s(t)\vec{a}(t)) + (\mathbf{J}_u(t)\vec{b}(t), \mathbf{J}'_u(t)\vec{b}(t))$ with*

$$(5.6) \quad \vec{a}(t) = \int_{-\infty}^t \mathbf{J}_u^*(s) \Upsilon(s) \, ds, \quad \vec{b}(t) = \int_t^{+\infty} \mathbf{J}_s^*(s) \Upsilon(s) \, ds.$$

Proof. By the construction of Morse correspondence, for any \tilde{g} -geodesic γ , the value of Ξ along γ , denoted $\Xi(\gamma)$, belongs to $N(\gamma) \times N(\gamma)$. So, there are $\vec{a}(t), \vec{b}(t), t \in \mathbb{R}$, such that

$$(5.7) \quad \Xi(\gamma) = (\mathbf{J}_s(t)\vec{a}(t), \mathbf{J}'_s(t)\vec{a}(t)) + (\mathbf{J}_u(t)\vec{b}(t), \mathbf{J}'_u(t)\vec{b}(t)).$$

In particular, the vertical part of $\Xi(\gamma)$ is $\mathbf{J}'_s(t)\vec{a}(t) + \mathbf{J}'_u(t)\vec{b}(t)$, which, by Theorem 5.2, is also

$$\nabla_{\dot{\gamma}} \Xi_{\gamma} = \mathbf{J}'_s(t)\vec{a}(t) + \mathbf{J}'_u(t)\vec{b}(t) + \mathbf{J}_s(t)\vec{a}'(t) + \mathbf{J}_u(t)\vec{b}'(t).$$

So we must have

$$\mathbf{J}_s(t)\vec{a}'(t) + \mathbf{J}_u(t)\vec{b}'(t) = 0.$$

Differentiating $\nabla_{\dot{\gamma}}\Xi_{\gamma} = \mathbf{J}'_s(t)\vec{a}(t) + \mathbf{J}'_u(t)\vec{b}(t)$ along γ and reporting it in (5.1), we obtain

$$\mathbf{J}'_s(t)\vec{a}'(t) + \mathbf{J}'_u(t)\vec{b}'(t) + \mathbf{J}''_s(t)\vec{a}(t) + \mathbf{J}''_u(t)\vec{b}(t) + \mathbf{R}(t)\mathbf{J}_s(t)\vec{a}(t) + \mathbf{R}(t)\mathbf{J}_u(t)\vec{b}(t) = -\Upsilon(t),$$

which simplifies to

$$(5.8) \quad \mathbf{J}'_s(t)\vec{a}'(t) + \mathbf{J}'_u(t)\vec{b}'(t) = -\Upsilon(t)$$

by the defining property of Jacobi fields. Using (5.5), we solve \vec{a}', \vec{b}' from (5.7), (5.8) with

$$(5.9) \quad \vec{a}' = \mathbf{J}_u^* \Upsilon, \quad \vec{b}' = -\mathbf{J}_s^* \Upsilon.$$

Note that $\mathbf{J}_u(-\infty) = \mathbf{J}_s(\infty) = 0$. Finally, we recover $\vec{a}(t), \vec{b}(t)$ from (5.9) by integration. \square

For any $s \in \mathbb{R}$, let (K_s, K'_s) be the unique Jacobi field along a \tilde{g} -geodesic γ such that

$$K'_s(s) = \Upsilon(s) \quad \text{and} \quad K_s(s) = 0.$$

Then

$$(K_s(0), K'_s(0)) = (D\Phi_s)^{-1}(0, \Upsilon(s)).$$

We further express Ξ using K_s 's by specifying the value of $\Xi(\gamma(0))$ for any \tilde{g} -geodesic γ .

Proposition 5.4. *Let Ξ be the infinitesimal Morse correspondence at g for a C^3 one-parameter family of C^3 metrics g^λ with $g^0 = g$. Then for the \tilde{g} -geodesic γ with $\dot{\gamma}(0) = \mathbf{v}$:*

$$\begin{aligned} \Xi_{\gamma}(0) &= (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1} \left[\int_{-\infty}^0 (K'_s(0) - U'_{\mathbf{v}}(0)K_s(0)) \, ds \right. \\ &\quad \left. + \int_0^{+\infty} (K'_s(0) - S'_{\mathbf{v}}(0)K_s(0)) \, ds \right], \\ (\nabla_{\dot{\gamma}}\Xi_{\gamma})(0) &= S'_{\mathbf{v}}(0)(U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1} \int_{-\infty}^0 (K'_s(0) - U'_{\mathbf{v}}(0)K_s(0)) \, ds \\ &\quad + U'_{\mathbf{v}}(0)(U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1} \int_0^{+\infty} (K'_s(0) - S'_{\mathbf{v}}(0)K_s(0)) \, ds. \end{aligned}$$

Proof. By Proposition 5.3, for any \tilde{g} -geodesic γ ,

$$\Xi(\gamma(0)) = (\Xi_{\gamma}(0), (\nabla_{\dot{\gamma}}\Xi_{\gamma})(0)) = \left(\mathbf{J}_s(0)\vec{a}(0) + \mathbf{J}_u(0)\vec{b}(0), \mathbf{J}'_s(0)\vec{a}(0) + \mathbf{J}'_u(0)\vec{b}(0) \right),$$

where $\vec{a}(0), \vec{b}(0)$ are given by (5.6). We first express $\vec{a}(0)$ using K_s 's. Let $s \leq 0$. The Wronskian between K_s and any unstable Jacobi fields are preserved along the geodesics and must have the same value at $\gamma(s)$ and $\gamma(0)$. This gives

$$\mathbf{J}_u^*(s)\Upsilon(s) = \mathbf{J}_u^*(0)K'_s(0) - (\mathbf{J}'_u)^*(0)K_s(0).$$

Consequently,

$$(\mathbf{J}_u^*)^{-1}(0)\mathbf{J}_u^*(s)\Upsilon(s) = K'_s(0) - (\mathbf{J}_u^*)^{-1}(0)(\mathbf{J}'_u)^*(0)K_s(0) = K'_s(0) - U'_{\mathbf{v}}(0)K_s(0),$$

where we use the fact that $\mathbf{J}'_{\mathbf{u}}(0) = U'_{\mathbf{v}}(0)\mathbf{J}_{\mathbf{u}}(0)$ for the second equality. So we have

$$\vec{a}(0) = \mathbf{J}_{\mathbf{u}}^*(0) \int_{-\infty}^0 (\mathbf{K}'_s(0) - U'_{\mathbf{v}}(0)\mathbf{K}_s(0)) ds.$$

Similarly, for any $s \geq 0$, a comparison of the Wronskian between \mathbf{K}_s and any stable Jacobi fields at time s and 0 gives

$$\mathbf{J}_s^*(s)\Upsilon(s) = \mathbf{J}_s^*(0)\mathbf{K}'_s(0) - (\mathbf{J}'_s)^*(0)\mathbf{K}_s(0).$$

As a consequence, we have

$$(\mathbf{J}_s^*)^{-1}(0)\mathbf{J}_s^*(s)\Upsilon(s) = \mathbf{K}'_s(0) - (\mathbf{J}_s^*)^{-1}(0)(\mathbf{J}'_s)^*(0)\mathbf{K}_s(0) = \mathbf{K}'_s(0) - S'_{\mathbf{v}}(0)\mathbf{K}_s(0),$$

which gives

$$\vec{b}(0) = \mathbf{J}_s^*(0) \int_0^{+\infty} (\mathbf{K}'_s(0) - S'_{\mathbf{v}}(0)\mathbf{K}_s(0)) ds.$$

The formula for $\Xi(\gamma(0))$ follows by using $\mathbf{J}_s(0) = \mathbf{J}_{\mathbf{u}}(0)$ and $\mathbf{J}_{\mathbf{u}}(0)\mathbf{J}_{\mathbf{u}}^*(0) = (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1}$. \square

A dynamical point of view of the integrability of the integrals in Proposition 5.4 is that $(\mathbf{K}'_s(0) - U'_{\mathbf{v}}(0)\mathbf{K}_s(0))$ ($s \leq 0$) is the stable vertical part of $(D\Phi_s)^{-1}(0, \Upsilon(s))$ and hence decays exponentially when s goes to $-\infty$, while $(\mathbf{K}'_s(0) - S'_{\mathbf{v}}(0)\mathbf{K}_s(0))$ ($s \geq 0$) is the unstable vertical part of $(D\Phi_s)^{-1}(0, \Upsilon(s))$ and thus decays exponentially when s goes to ∞ .

For any curve $\lambda \in (-1, 1) \mapsto F_{\lambda} \in \mathbf{N}$ for some Riemannian manifold \mathbf{N} , we write $(F_{\lambda})'_0 := (dF_{\lambda}/d\lambda)|_{\lambda=0}$ whenever the differential exists and similarly for F^{λ} . We can put a formula concerning $(\overline{X}_{\tilde{g}^{\lambda}})'_0$ for any C^3 curve g^{λ} in $\mathfrak{R}(M)$ with $g^0 = g$.

Proposition 5.5. *Let (M, g) be a negatively curved closed connected m -dimensional Riemannian manifold. Then for any C^3 one-parameter family of C^3 metrics $\lambda \in (-1, 1) \mapsto g^{\lambda}$ in it with $g^0 = g$, the map $\lambda \mapsto \overline{X}_{\tilde{g}^{\lambda}}(x, \xi)$ is differentiable at $\lambda = 0$ for each $\mathbf{v} = (x, \xi)$ with*

$$(\overline{X}_{\tilde{g}^{\lambda}})'_0(x, \xi) = \left(0, (\|\overline{X}_{\tilde{g}^{\lambda}}\|_{\tilde{g}})'_0(\mathbf{v})\mathbf{v} + \int_0^{+\infty} (\mathbf{K}'_s(0) - S'_{\mathbf{v}}(0)\mathbf{K}_s(0)) ds \right).$$

Proof. Express the homeomorphism \tilde{F}^{λ} as a map from $\widetilde{M} \times \partial\widetilde{M}$ to $\widetilde{M} \times \partial\widetilde{M}_{\tilde{g}^{\lambda}}$ with

$$\tilde{F}^{\lambda}(x, \xi) = (f_{\xi}^{\lambda}(x), \xi), \quad \forall (x, \xi) \in S\widetilde{M},$$

where f_ξ^λ records the change of footpoint of the (g, g^λ) -Morse correspondence \tilde{F}^λ . We have

$$\begin{aligned} & \frac{1}{\lambda} (\overline{X}_{\tilde{g}^\lambda}(x, \xi) - \overline{X}_{\tilde{g}}(x, \xi)) \\ &= \frac{1}{\lambda} \left(\overline{X}_{\tilde{g}^\lambda}(x, \xi) - \frac{\overline{X}_{\tilde{g}^\lambda}(x, \xi)}{\|\overline{X}_{\tilde{g}^\lambda}(x, \xi)\|_{\tilde{g}}} \right) + \frac{1}{\lambda} \left(\frac{\overline{X}_{\tilde{g}^\lambda}(x, \xi)}{\|\overline{X}_{\tilde{g}^\lambda}(x, \xi)\|_{\tilde{g}}} - \overline{X}_{\tilde{g}}((f_\xi^\lambda)^{-1}(x), \xi) \right) \\ &+ \frac{1}{\lambda} (\overline{X}_{\tilde{g}}((f_\xi^\lambda)^{-1}(x), \xi) - \overline{X}_{\tilde{g}}(x, \xi)) \\ &=: (a)_\lambda + (b)_\lambda + (c)_\lambda. \end{aligned}$$

When λ tends to zero, $(a)_\lambda$ tends to $(0, (\|\overline{X}_{\tilde{g}^\lambda}\|'_0(\mathbf{v})\mathbf{v}))$. By Theorem 5.1 and Theorem 5.2, $(b)_\lambda$ tends to $(\Xi_{\gamma_{\mathbf{v}}}(0), \nabla_{\dot{\gamma}_{\mathbf{v}}} \Xi_{\gamma_{\mathbf{v}}}(0))$, which is also $(\mathbf{J}_s(0)\vec{a}(0), \mathbf{J}'_s(0)\vec{a}(0)) + (\mathbf{J}_u(0)\vec{b}(0), \mathbf{J}'_u(0)\vec{b}(0))$ with $\vec{a}(0), \vec{b}(0)$ from Proposition 5.3. Finally, the quantity $(c)_\lambda$ describes the variation of the unit tangent vectors of asymptotic \tilde{g} -geodesics starting at $f_\xi^\lambda(x)$ pointing at ξ and its limit (as λ goes to 0) is $-(\Xi_{\gamma_{\mathbf{v}}}(0), S'_{\mathbf{v}}(0)\Xi_{\gamma_{\mathbf{v}}}(0))$. So,

$$\lim_{\lambda \rightarrow 0} ((b)_\lambda + (c)_\lambda) = (\nabla_{\dot{\gamma}_{\mathbf{v}}} \Xi_{\gamma_{\mathbf{v}}})(0) - S'_{\mathbf{v}}(0)\Xi_{\gamma_{\mathbf{v}}}(0) = (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))\mathbf{J}_u(0)\vec{b}(0),$$

which, by our choice of $\mathbf{J}_u(0) = \mathbf{J}_s(0)$ and the defining property of $\mathbf{J}_u(0)$ in (5.3), is

$$(\mathbf{J}_s^*)^{-1}(0)\vec{b}(0) = \int_0^{+\infty} (K'_s(0) - S'_{\mathbf{v}}(0)K_s(0)) ds.$$

□

Corollary 5.6. *Let (M, g) be a negatively curved closed connected Riemannian manifold and let $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}(M)$ be a C^3 curve of C^3 conformal changes of the metric $g^0 = g$. The map $\lambda \mapsto \overline{X}_{\tilde{g}^\lambda}(x, \xi)$ is differentiable for each $\mathbf{v} = (x, \xi)$ with*

$$(\overline{X}_{\tilde{g}^\lambda})'_0(x, \xi) = \left(0, -\varphi\mathbf{v} + \int_0^{+\infty} (K'_s(0) - S'_{\mathbf{v}}(0)K_s(0)) ds \right),$$

where $\varphi : M \rightarrow \mathbb{R}$ is such that $g^\lambda = e^{2\lambda\varphi + O(\lambda^2)}g$ and $(K_s(0), K'_s(0)) = (D\Phi_s)^{-1}(0, \Upsilon(s))$ with $\Upsilon = -\nabla\varphi + \langle \nabla\varphi, \dot{\gamma}_{\mathbf{v}} \rangle \dot{\gamma}_{\mathbf{v}}$.

Proof. Let $\lambda \in (-1, 1) \mapsto \varphi^\lambda$ be such that $g^\lambda = e^{2\varphi^\lambda}g$. Clearly, $\|\overline{X}_{\tilde{g}^\lambda}\|_{\tilde{g}} = e^{-\varphi^\lambda}$ and hence $(\|\overline{X}_{\tilde{g}^\lambda}\|_{\tilde{g}})'_0(\mathbf{v})\mathbf{v} = -\varphi\mathbf{v}$. Let ∇^λ denote the Levi-Civita connection of $(\widetilde{M}, \tilde{g})$. It is true by Koszul's formula that

$$\nabla_X^\lambda Y - \nabla_X Y = (D_X \varphi^\lambda)Y + (D_Y \varphi^\lambda)X - \langle X, Y \rangle_g \nabla \varphi^\lambda$$

for any two smooth vector fields X, Y on \widetilde{M} . As a consequence, we have

$$\Gamma_X Y = (D_X \varphi)Y + (D_Y \varphi)X - \langle X, Y \rangle_g \nabla \varphi.$$

In particular, $\Gamma_{\dot{\gamma}} \dot{\gamma} = -\nabla \varphi + 2\langle \nabla \varphi, \dot{\gamma} \rangle \dot{\gamma}$ and the equation (5.1) reduces to

$$\nabla_{\dot{\gamma}}^2 \Xi_{\gamma} + \mathbf{R}(\Xi_{\gamma}, \dot{\gamma})\dot{\gamma} - \nabla \varphi + \langle \nabla \varphi, \dot{\gamma} \rangle \dot{\gamma} = 0.$$

The formula for $(\overline{X}_{\widetilde{g}^\lambda})'_0(x, \xi)$ follows immediately by Proposition 5.5. \square

Next, let us recall some basic notations for harmonic measures and Brownian motions to be used in this section. Let (M, g) be a negatively curved closed connected m -dimensional Riemannian manifold as before. The laminated Laplacian Δ is subordinate to the stable foliation and has a unique harmonic measure \mathbf{m} . The lift of \mathbf{m} to \widetilde{SM} , denoted $\widetilde{\mathbf{m}}$, has the expression $\widetilde{\mathbf{m}} = k_\xi(x)(dx \times \nu(d\xi))$ for some finite measure ν on $\partial\widetilde{M}$, where $k_\xi(x)$ is the Martin kernel function of the Laplacian and dx is proportional to the volume. We use \mathbb{P} to denote the invariant probability measure on Ω_+ associated with $\widetilde{\mathbf{m}}$ as in Proposition 2.1.

Let $B_t = (B_t^1, \dots, B_t^m)$ be an m -dimensional Brownian motion on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with generator Δ as before. For each $\mathbf{v} \in \widetilde{SM}$, the Brownian motion $(\mathbf{x}_{\mathbf{v}, t})_{t \in \mathbb{R}_+}$ on $W^s(\mathbf{v})$ is a diffusion process on (Θ, \mathbb{Q}) which can be obtained as the projection on $W^s(\mathbf{v})$ of the Brownian motion $\mathbf{w}_{\mathbf{v}, t}$ in $O(W^s(\mathbf{v}))$ for any choice of $\mathbf{w}_{\mathbf{v}, 0}$ which projects to \mathbf{v} . Consider the space $\overline{\Theta} = \widetilde{SM} \times \Theta$ with product σ -algebra and probability $\overline{\mathbb{Q}}$, $d\overline{\mathbb{Q}}(\mathbf{v}, \underline{\omega}) = d\mathbb{Q}(\underline{\omega}) \times d\widetilde{\mathbf{m}}(\mathbf{v})$. Let $\mathbf{x}_t : \widetilde{SM} \times \Theta \rightarrow \widetilde{SM}$ be such that $\mathbf{x}_t(\mathbf{v}, \underline{\omega}) = \mathbf{x}_{\mathbf{v}, t}(\underline{\omega})$, for $(\mathbf{v}, \underline{\omega}) \in \widetilde{SM} \times \Theta$ and let

$$\begin{aligned} \mathbf{Z}_{\ell, t} &:= [d_{\mathcal{W}}(\mathbf{x}_0, \mathbf{x}_t) - t\ell_0], \\ \mathbf{Z}_{h, t} &:= -[\mathbf{1}_{\{d(\mathbf{x}_0, \mathbf{x}_t) > 1\}} \cdot \ln \mathbf{G}(\mathbf{x}_0, \mathbf{x}_t) + th_0]. \end{aligned}$$

We have by Proposition 3.6 that both $\mathbf{Z}_{\ell, t}/\sqrt{t}$ and $\mathbf{Z}_{h, t}/\sqrt{t}$ are asymptotic to centered Gaussian distributions with respect to $\overline{\mathbb{Q}}$ as t goes to infinity.

Let $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}(M)$ be a C^3 curve of conformal changes of the metric $g^0 = g$. We simply use the superscript λ ($\lambda \neq 0$) for $\overline{X}, \mathbf{m}, \widetilde{\mathbf{m}}, k_\xi, \mathbb{P}$ to indicate that the metric used is g^λ , for instance, \mathbf{m}^λ is the harmonic measure for the laminated Laplacian in metric g^λ .

Let ℓ_λ and h_λ be the linear drift and entropy for (M, g^λ) as were defined in Sec. 1. Their differentiability in λ at 0 (Theorem 1.1) will be a consequence of the following Theorem.

Theorem 5.7. *Let (M, g) be a negatively curved compact connected m -dimensional Riemannian manifold and let $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}(M)$ be a C^3 curve of conformal changes of the metric $g^0 = g$ with constant volume. For each $\lambda \in (-1, 1)$, let φ be such that $g^\lambda = e^{2\lambda\varphi + O(\lambda^2)}g$. With the above notations, the following holds true.*

i) *The function $\lambda \mapsto \ell_\lambda$ is differentiable at 0 with*

$$\begin{aligned} (\ell_\lambda)'_0 &= \int_{M_0 \times \partial\widetilde{M}} \langle \varphi \overline{X} + \int_0^{+\infty} (K'_s(0) - S'_{(x, \xi)}(0)K_s(0)) \, ds, \nabla \ln k \rangle \, d\widetilde{\mathbf{m}} \\ (5.10) \quad &+ (m-2) \int_{M_0 \times \partial\widetilde{M}} \varphi \langle \nabla u_0 + \overline{X}, \nabla \ln k \rangle \, d\widetilde{\mathbf{m}}, \end{aligned}$$

where $(K_s(0), K'_s(0)) = (D\Phi_s)^{-1}(0, \Upsilon(s))$ with $\Upsilon = -\nabla\varphi + \langle \nabla\varphi, \dot{\gamma} \rangle \dot{\gamma}$ along the \widetilde{g} -geodesic γ with $\dot{\gamma}(0) = (x, \xi)$ and u_0 is the function defined before Proposition 3.7.

ii) The function $\lambda \mapsto h_\lambda$ is differentiable at 0 with

$$(5.11) \quad (h_\lambda)'_0 = (m-2) \int_{SM} \varphi \langle \nabla(u_1 + \ln k), \nabla \ln k \rangle d\mathbf{m},$$

where u_1 is the function defined before Proposition 3.7.

Proof. We derive the formula for $(h_\lambda)'_0$ first. We will omit the subscript of $\int_{M_0 \times \partial \widetilde{M}}$ whenever there is no ambiguity. Let $\lambda \in (-1, 1) \mapsto \varphi^\lambda$ be such that $g^\lambda = e^{2\varphi^\lambda} g$. For each λ , we have

$$\Delta^\lambda = e^{-2\varphi^\lambda} \left(\Delta + (m-2)\nabla\varphi^\lambda \right) =: e^{-2\varphi^\lambda} L_\lambda.$$

Let $\widehat{\mathcal{L}}^\lambda := \Delta + Z^\lambda$ with Z^λ being the horizontal lift of $(m-2)\nabla\varphi^\lambda$ to the tangent space of SM and let $\widehat{\mathbf{m}}^\lambda$ be the lift to $S\widetilde{M}$ of the harmonic measure corresponding to $\widehat{\mathcal{L}}^\lambda$ with respect to metric g . Then $d\widehat{\mathbf{m}}^\lambda = e^{-2\varphi^\lambda} d\widetilde{\mathbf{m}}^\lambda$, where φ^λ also denotes its lift to $\widetilde{M} \times \partial \widetilde{M}$. Moreover, since there is only a time change between the diffusion processes with infinitesimal operators Δ^λ and L^λ , the Martin kernel function corresponding to $\widehat{\mathcal{L}}^\lambda$ is the same as that of $\widehat{\mathcal{L}}^\lambda$ for the laminated Laplacian in the metric g^λ . Note that the \widehat{h}_λ defined in the introduction is just the stochastic entropy for the operator $\widehat{\mathcal{L}}^\lambda$ with respect to metric g . So, using Proposition 3.5, we obtain

$$(5.12) \quad \widehat{h}_\lambda = \int \|\nabla^0 \ln k_\xi^\lambda(x)\|_0^2 d\widehat{\mathbf{m}}^\lambda = \int e^{-2\varphi^\lambda} \|\nabla \ln k_\xi^\lambda(x)\|^2 d\widetilde{\mathbf{m}}^\lambda.$$

For $(h_\lambda)'_0$, we have

$$(h_\lambda)'_0 = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (h_\lambda - \widehat{h}_\lambda) + \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\widehat{h}_\lambda - h_0) =: (\mathbf{I})_h + (\mathbf{II})_h,$$

if both limits exist. It is easy to see $(\mathbf{I})_h = 0$ since by Proposition 3.5 and (5.12),

$$\begin{aligned} (\mathbf{I})_h &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left(\int \|\nabla^\lambda \ln k_\xi^\lambda(x)\|_\lambda^2 d\widetilde{\mathbf{m}}^\lambda - \int \|\nabla^0 \ln k_\xi^\lambda(x)\|_0^2 d\widetilde{\mathbf{m}}^\lambda \right) \\ &= \lim_{\lambda \rightarrow 0} \int \frac{1}{\lambda} (e^{-2\varphi^\lambda} - e^{-2\varphi^\lambda}) \|\nabla \ln k_\xi^\lambda(x)\|^2 d\widetilde{\mathbf{m}}^\lambda \\ &= 0, \end{aligned}$$

where we use $\nabla^\lambda \ln k_\xi^\lambda(x) = e^{-2\varphi^\lambda} \nabla \ln k_\xi^\lambda(x)$ and $\|\nabla^\lambda \ln k_\xi^\lambda(x)\|_\lambda^2 = e^{-2\varphi^\lambda} \|\nabla \ln k_\xi^\lambda(x)\|^2$. For $(\mathbf{II})_h$, we have by Theorem 4.9 that it equals to $\lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{h,t} \mathbf{M}_t)$. Let

$$(5.13) \quad \widetilde{\mathbf{Z}}_t^1 = f_1(\mathbf{x}_t) - f_1(\mathbf{x}_0) - \int_0^t (\Delta f_1)(\mathbf{x}_s) ds,$$

where $f_1 = -\ln k_\xi - u_1 \circ \pi_{SM}$ is the martingale with increasing process $2\|\nabla \ln k_\xi + \nabla u_1\|^2 (\pi_{\widetilde{M}}(\mathbf{x}_t)) dt$ and the function u_1 is such that

$$(5.14) \quad \Delta u_1 = \|\nabla \ln k_\xi\|^2 - h_0.$$

It is true by Proposition 4.2 that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\mathbb{Q}}(\mathbf{Z}_{h,t} \mathbf{M}_t) = \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\mathbb{Q}}(\tilde{\mathbf{Z}}_t^1 \mathbf{M}_t),$$

where

$$\mathbf{M}_t = \frac{1}{2} \int_0^t \langle (Z^\lambda)'_0(\mathbf{x}_s), \mathbf{w}_s dB_s \rangle_{\mathbf{x}_s}.$$

Note that $(Z^\lambda)'_0$, the lift of $(m-2)\nabla\varphi$ to the tangent space of SM , is a gradient field. So, if we write $\psi = \frac{1}{2}(m-2)\varphi$, we have by Ito's formula that

$$(5.15) \quad \mathbf{M}_t = \psi(\mathbf{x}_t) - \psi(\mathbf{x}_0) - \int_0^t (\Delta\psi)(\mathbf{x}_s) ds$$

is a martingale with increasing process $2\|\nabla\psi\|^2$. Using (5.13), (5.15) and a straightforward computation using integration by parts formula for $(a\tilde{\mathbf{Z}}_t^1 + b\mathbf{M}_t)^2$, $a, b = 0$ or 1 , we obtain

$$\tilde{\mathbf{Z}}_t^1 \mathbf{M}_t = 2 \int_0^t \langle \nabla f_1, \nabla\psi \rangle(\mathbf{x}_s) ds$$

and hence

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\mathbb{Q}}(\tilde{\mathbf{Z}}_t^1 \mathbf{M}_t) = 2 \int \langle \nabla f_1, \nabla\psi \rangle d\tilde{\mathbf{m}} = -2 \int \langle \nabla \ln k, \nabla\psi \rangle d\tilde{\mathbf{m}} - 2 \int \langle \nabla u_1, \nabla\psi \rangle d\tilde{\mathbf{m}},$$

where we still denote u_1, φ, ψ their lifts to $M_0 \times \partial\tilde{M}$. Here,

$$-2 \int \langle \nabla \ln k, \nabla\psi \rangle d\tilde{\mathbf{m}} = 2 \int_{SM} \text{Div}(\nabla\psi) d\mathbf{m} = (m-2) \int_{SM} \Delta\varphi d\mathbf{m} = 0,$$

where the first equality is the integration by parts formula and the last one holds because \mathbf{m} is Δ -harmonic. We finally obtain

$$(h_\lambda)'_0 = -(m-2) \int_{SM} \langle \nabla u_1, \nabla\varphi \rangle d\mathbf{m}.$$

Observe that:

$$(5.16) \quad \begin{aligned} 2\langle \nabla u_1, \nabla\varphi \rangle &= \Delta(u_1\varphi) - \Delta(u_1)\varphi - u_1\Delta\varphi \\ &= \Delta(u_1\varphi) - \varphi\|\nabla \ln k\|^2 + h_0\varphi - u_1\Delta\varphi, \end{aligned}$$

where we use the defining property (5.14) of u_1 . When we take the integral of (5.16) with respect to \mathbf{m} , the first term vanishes because \mathbf{m} is Δ harmonic, the second term gives $-\int \varphi\|\nabla \ln k\|^2 d\mathbf{m}$, the third term vanishes because the volume is constant. Finally for the last term, by using the integration by parts formula:

$$(5.17) \quad \int_{SM} u\Delta v d\mathbf{m} = \int_{SM} v\Delta u d\mathbf{m} + 2 \int_{SM} v\langle \nabla u, \nabla \ln k \rangle d\mathbf{m},$$

we have

$$\begin{aligned} \int_{SM} u_1 \Delta \varphi \, d\mathbf{m} &= \int_{SM} \varphi (\Delta u_1 + 2 \langle \nabla u_1, \nabla \ln k \rangle) \, d\mathbf{m} \\ &= \int_{SM} \varphi (\|\nabla \ln k\|^2 + 2 \langle \nabla u_1, \nabla \ln k \rangle) \, d\mathbf{m}. \end{aligned}$$

Next, we derive the formula for $(\ell_\lambda)'_0$. Clearly,

$$(\ell_\lambda)'_0 = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\ell_\lambda - \widehat{\ell}_\lambda) + \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\widehat{\ell}_\lambda - \ell_0) =: (\mathbf{I})_\ell + (\mathbf{II})_\ell,$$

if both limits exist. Here the $\widehat{\ell}_\lambda$ defined in the introduction is just the linear drift for the operator $\widehat{\mathcal{L}}^\lambda$ with respect to metric g . The $(\mathbf{II})_\ell$ term can be analyzed similarly as above for $(\mathbf{II})_h$. Indeed, by Theorem 4.3, $(\mathbf{II})_\ell = \lim_{t \rightarrow +\infty} (1/t) \mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{\ell,t} \mathbf{M}_t)$. Let

$$(5.18) \quad \widetilde{\mathbf{Z}}_t^0 = f_0(\mathbf{x}_t) - f_0(\mathbf{x}_0) - \int_0^t (\Delta f_0)(\mathbf{x}_s) \, ds,$$

where $f_0 = b_{\mathbf{v}} - u_0 \circ \pi_{SM}$, be the martingale with increasing process $2\|\overline{X} + \nabla u_0\|^2(\pi_{\widetilde{M}}(\mathbf{x}_t)) \, dt$ and the function u_0 is such that

$$(5.19) \quad \Delta u_0 = -\text{Div}(\overline{X}) - \ell_0.$$

It is true by Proposition 4.1 that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{h,t} \mathbf{M}_t) = \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\overline{\mathbb{Q}}}(\widetilde{\mathbf{Z}}_t^0 \mathbf{M}_t),$$

where \mathbf{M}_t , by (5.15), is a martingale with increasing process $2\|\nabla \psi\|^2$. So using (5.15), (5.18) and a straightforward computation using integration by parts formula for $(a\widetilde{\mathbf{Z}}_t^0 + b\mathbf{M}_t)^2$, $a, b = 0$ or 1 , we obtain

$$\widetilde{\mathbf{Z}}_t^0 \mathbf{M}_t = 2 \int_0^t \langle \nabla f_0, \nabla \psi \rangle(\mathbf{x}_s) \, ds$$

and hence

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}_{\overline{\mathbb{Q}}}(\widetilde{\mathbf{Z}}_t^0 \mathbf{M}_t) = 2 \int \langle \nabla f_0, \nabla \psi \rangle \, d\widetilde{\mathbf{m}} = -(m-2) \left(\int \langle \overline{X}, \nabla \varphi \rangle \, d\widetilde{\mathbf{m}} + \int \langle \nabla u_0, \nabla \varphi \rangle \, d\widetilde{\mathbf{m}} \right),$$

where we still denote u_0, φ, ψ their lifts to $M_0 \times \partial \widetilde{M}$. Using the formula $\text{Div}(\varphi \overline{X}) = \varphi \text{Div} \overline{X} + \langle \nabla \varphi, \overline{X} \rangle$, we obtain

$$\int \langle \overline{X}, \nabla \varphi \rangle \, d\widetilde{\mathbf{m}} = \int (\text{Div}(\varphi \overline{X}) - \varphi \text{Div} \overline{X}) \, d\widetilde{\mathbf{m}} = - \int \varphi (\langle \overline{X}, \nabla \ln k \rangle + \text{Div} \overline{X}) \, d\widetilde{\mathbf{m}}$$

Observe that:

$$\begin{aligned} 2 \langle \nabla u_0, \nabla \varphi \rangle &= \Delta(u_0 \varphi) - \Delta(u_0) \varphi - u_0 \Delta \varphi \\ &= \Delta(u_0 \varphi) + \varphi \text{Div}(\overline{X}) + \ell_0 \varphi - u_0 \Delta \varphi, \end{aligned}$$

where we use the defining property (5.19) of u_0 . When we report in the integration $2 \int \langle \nabla u_0, \nabla \varphi \rangle \, d\widetilde{\mathbf{m}}$, the first term vanishes because \mathbf{m} is Δ harmonic, the second term

is $-\int \varphi \Delta u_0 \, d\tilde{\mathbf{m}}$ by (5.19) and the third term vanishes because the volume is constant. Again, using the integration by parts formula (5.17) for $\int u_0 \Delta \varphi \, d\tilde{\mathbf{m}}$, we have

$$\int \langle \nabla u_0, \nabla \varphi \rangle \, d\tilde{\mathbf{m}} = - \int \varphi (\Delta u_0 + \langle \nabla u_0, \nabla \ln k \rangle) \, d\tilde{\mathbf{m}}$$

Finally, we obtain

$$\begin{aligned} (\mathbf{II})_\ell &= (m-2) \int \varphi (\Delta u_0 + \operatorname{Div} \bar{X} + \langle \nabla u_0 + \bar{X}, \nabla \ln k \rangle) \, d\tilde{\mathbf{m}} \\ &= (m-2) \int \varphi \langle \nabla u_0 + \bar{X}, \nabla \ln k \rangle \, d\tilde{\mathbf{m}}, \end{aligned}$$

where the last equality holds by using (5.19) and the fact that the volume is constant.

For $(\mathbf{I})_\ell$, we first observe the convergence of Martin kernels and harmonic measures. For any $\xi \in \partial \widetilde{M}$, the Martin kernel function $k_\xi^\lambda(x)$ converges to $k_\xi(x)$ pointwisely as λ goes to zero. For small λ , the function $\xi \mapsto \nabla \ln k_\xi^\lambda(x)$ is Hölder continuous on $\partial \widetilde{M}$ for some uniform exponent ([H1]). As a consequence, we have the convergence of $\nabla \ln k_\xi^\lambda(x)$ (and hence $\nabla^\lambda \ln k_\xi^\lambda(x)$) to $\nabla \ln k_\xi^0(x)$ when λ tends to zero. We also obtain the uniform convergence of $k_\xi^\lambda(x)$ to $k_\xi(x)$ and $\nabla \ln k_\xi^\lambda(x)$ to $\nabla \ln k_\xi^0(x)$ (as $\lambda \rightarrow 0$) in ξ . So, the harmonic measure $\tilde{\mathbf{m}}^\lambda$ converges weakly to $\tilde{\mathbf{m}}$ ($\lambda \rightarrow 0$) as well. By Proposition 3.5,

$$\ell_\lambda = \int \langle \bar{X}^\lambda, \nabla^\lambda \ln k_\xi^\lambda(x) \rangle_\lambda \, d\tilde{\mathbf{m}}^\lambda = \int \langle \bar{X}^\lambda, \nabla \ln k_\xi^\lambda(x) \rangle \, d\tilde{\mathbf{m}}^\lambda.$$

We have by Proposition 5.5 and the convergence of $\nabla \ln k^\lambda$ and $\tilde{\mathbf{m}}^\lambda$ stated above that

$$\begin{aligned} (\mathbf{I})_\ell &= \int \langle (\bar{X}^\lambda)'_0, \nabla \ln k \rangle \, d\tilde{\mathbf{m}} + \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left(\int \langle \bar{X}, \nabla \ln k_\xi^\lambda(x) \rangle \, d\tilde{\mathbf{m}}^\lambda - \widehat{\ell}_\lambda \right) \\ &=: (\mathbf{III})_\ell + (\mathbf{IV})_\ell \end{aligned}$$

if $(\mathbf{IV})_\ell$ exists. The quantity $(\mathbf{III})_\ell$, by Corollary 5.6, is

$$\int \langle -\varphi \bar{X} + \int_0^{+\infty} (K'_s(0) - S'_v(0) K_s(0)) \, ds, \nabla \ln k \rangle \, d\tilde{\mathbf{m}}.$$

By Proposition 3.5,

$$\widehat{\ell}_\lambda = - \int \left(\operatorname{Div} \bar{X} + \langle Z^\lambda, \bar{X} \rangle \right) \, d\widehat{\mathbf{m}}^\lambda.$$

For $(\mathbf{IV})_\ell$, let us first calculate $\int \operatorname{Div} \bar{X} \, d\widehat{\mathbf{m}}$. We have

$$\begin{aligned} \int \operatorname{Div} \bar{X} \, d\widehat{\mathbf{m}}^\lambda &= \int e^{-2\varphi^\lambda} \operatorname{Div} \bar{X} \, d\tilde{\mathbf{m}}^\lambda \\ &= \int e^{-2\varphi^\lambda} \operatorname{Div}^\lambda \bar{X} \, d\tilde{\mathbf{m}}^\lambda + \int \left(\operatorname{Div} \bar{X} - \operatorname{Div}^\lambda \bar{X} \right) \, d\widehat{\mathbf{m}}^\lambda \\ &= \int e^{-2\varphi^\lambda} \operatorname{Div}^\lambda \bar{X} \, d\tilde{\mathbf{m}}^\lambda - m \int \langle \nabla \varphi^\lambda, \bar{X} \rangle \, d\widehat{\mathbf{m}}^\lambda, \end{aligned}$$

where the last equality holds since $(\text{Div}^\lambda - \text{Div})(\cdot) = m\langle \nabla \varphi^\lambda, \cdot \rangle$ for $g^\lambda = e^{2\varphi^\lambda} g$. Note that

$$\text{Div}^\lambda(e^{-2\varphi^\lambda} \overline{X}) = e^{-2\varphi^\lambda} \text{Div}^\lambda \overline{X} - 2e^{-2\varphi^\lambda} \langle \nabla^\lambda \varphi^\lambda, \overline{X} \rangle_\lambda.$$

So we have

$$\begin{aligned} \int \text{Div} \overline{X} \, d\widehat{\mathbf{m}}^\lambda &= \int \text{Div}^\lambda(e^{-2\varphi^\lambda} \overline{X}) \, d\widehat{\mathbf{m}}^\lambda + \int 2e^{-2\varphi^\lambda} \langle \nabla^\lambda \varphi^\lambda, \overline{X} \rangle_\lambda \, d\widehat{\mathbf{m}}^\lambda - m \int \langle \nabla \varphi^\lambda, \overline{X} \rangle \, d\widehat{\mathbf{m}}^\lambda \\ &= - \int \langle \overline{X}, \nabla^\lambda \ln k_\xi^\lambda \rangle_\lambda \, d\widehat{\mathbf{m}}^\lambda - (m-2) \int \langle \nabla \varphi^\lambda, \overline{X} \rangle \, d\widehat{\mathbf{m}}^\lambda \\ &= - \int \langle \overline{X}, \nabla \ln k_\xi^\lambda \rangle \, d\widehat{\mathbf{m}}^\lambda - (m-2) \int \langle \nabla \varphi^\lambda, \overline{X} \rangle \, d\widehat{\mathbf{m}}^\lambda, \end{aligned}$$

which gives

$$\widehat{\ell}_\lambda = \int \langle \overline{X}, \nabla \ln k_\xi^\lambda \rangle \, d\widehat{\mathbf{m}}^\lambda.$$

Finally, we obtain

$$(\mathbf{IV})_\ell = \lim_{\lambda \rightarrow 0} \int \frac{1}{\lambda} (e^{2\varphi^\lambda} - 1) \langle \overline{X}, \nabla \ln k_\xi^\lambda \rangle \, d\widehat{\mathbf{m}}^\lambda = 2 \int \varphi \langle \overline{X}, \nabla \ln k \rangle \, d\widehat{\mathbf{m}}.$$

□

Proof of Theorem 1.2. Let (M, g) be a negatively curved compact connected Riemannian manifold. Define the *volume entropy* v_g by:

$$v_g = \lim_{r \rightarrow +\infty} \frac{\ln \text{Vol}(B(x, r))}{r},$$

where $B(x, r)$ is the ball of radius r in \widetilde{M} . we have $\ell_g \leq v_g$, $h_g \leq v_g^2$ (see [LS] and the references within). In particular, if $\lambda \in (-1, 1) \mapsto g^\lambda \in \mathfrak{R}(M)$ is a C^3 curve of conformal changes of the metric $g^0 = g$,

$$\ell_{g^\lambda} \leq v_{g^\lambda}, \quad h_{g^\lambda} \leq v_{g^\lambda}^2.$$

Assume (M, g^0) is locally symmetric. Then $\ell_{g^0} = v_{g^0}$ and $h_{g^0} = v_{g^0}^2$. Moreover it is known (Katok [Ka]) that v_0 is a global minimum of the volume entropy among metrics g which are conformal to g^0 and have the same volume and (Katok-Knieper-Pollicott-Weiss [KKPW]) that $\lambda \mapsto v_{g^\lambda}$ is differentiable. In particular v_{g^λ} is critical at $\lambda = 0$. Since, by Theorem 1.1, ℓ_{g^λ} and h_{g^λ} are differentiable at $\lambda = 0$, they have to be critical as well. □

Remark 5.8. We can also show Theorem 1.2 using the formulas in Theorem 5.7. Indeed, the conclusion for the stochastic entropy follows from (5.11) since for a locally symmetric space, the solutions u_1 to (5.14) are constant ([L2]). We also see that the stochastic entropy depends only on the volume for surfaces ($m = 2$). For the drift ℓ , it is true that for a locally

symmetric space, $\nabla \ln k_\xi = -\ell \nabla b_\xi$ everywhere. The solutions u_0 to (5.19) are constant for a locally symmetric space as well ([L2]). So (5.10) reduces to

$$(\ell_\lambda)'_0 = - \int_{M_0 \times \partial \widetilde{M}} \varphi \left(\int_0^{+\infty} (K'_s(0) - S'_v(0) K_s(0)) \, ds, \nabla \ln k \right) d\widetilde{\mathbf{m}},$$

which is zero because the vector $\int_0^{+\infty} (K'_s(0) - S'_v(0) K_s(0)) \, ds$ is orthogonal to \mathbf{v} and hence is orthogonal to $\nabla \ln k$.

Acknowledgments The second author was partially supported by Major Program of National Natural Science Foundation of China (No. 11331007). She would also like to thank LPMA and the Department of Mathematics of the University of Notre Dame for hospitality during her stays.

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FRANÇOIS LEDRAPPIER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, IN 46556-4618, USA AND LPMA, BOÎTE COURRIER 188, 4, PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE

E-mail address: fledrapp@nd.edu

LIN SHU, BICMR AND LMAM, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, PEOPLE’S REPUBLIC OF CHINA

E-mail address: lshu@math.pku.edu.cn