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ABSTRACT. We prove that W^1_p is a real interpolation space between $W^1_{p_1}$ and $W^1_{p_2}$ for $p > q_0$ and $1 \leq p_1 < p < p_2 \leq \infty$ on some classes of manifolds and general metric spaces, where q_0 depends on our hypotheses.

CONTENTS

1. INTRODUCTION

Do the Sobolev spaces W^1_p form a real interpolation scale for $1 < p < \infty$? The aim of the present work is to provide a positive answer for Sobolev spaces on some metric spaces. Let us state here our main theorems for non-homogeneous Sobolev spaces (resp. homogeneous Sobolev spaces) on Riemannian manifolds.

Theorem 1.1. *Let M be a complete non-compact Riemannian manifold satisfying the local doubling property (D_{loc}) and a local Poincaré inequality (P_{qloc}), for some*

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$1 \leq q < \infty$. *Then for* $1 \leq r \leq q < p < \infty$, W^1_p *is a real interpolation space between* W^1_r *and* W^1_∞.

To prove Theorem 1.1, we characterize the K-functional of real interpolation for non-homogeneous Sobolev spaces:

Theorem 1.2. *Let* M *be as in Theorem 1.1. Then*

1. *there exists* $C_1 > 0$ *such that for all* $f \in W^1_r + W^1_\infty$ *and* $t > 0$
$$K(f, t^{\frac{1}{r}}, W^1_r, W^1_\infty) \geq C_1 t^{\frac{1}{r}} \Big(|f|^{r**\frac{1}{r}}(t) + |\nabla f|^{r**\frac{1}{r}}(t) \Big);$$
2. *for* $r \leq q \leq p < \infty$, *there is* $C_2 > 0$ *such that for all* $f \in W^1_p$ *and* $t > 0$
$$K(f, t^{\frac{1}{r}}, W^1_r, W^1_\infty) \leq C_2 t^{\frac{1}{r}} \Big(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t) \Big).$$

In the special case $r = q$, *we obtain the upper bound of* K *in point 2. for every* $f \in W^1_q + W^1_\infty$ *and hence get a true characterization of* K.

The proof of this theorem relies on a Calderón-Zygmund decomposition for Sobolev functions (Proposition 3.5).
Above and from now on, $|g|^{q**\frac{1}{q}}$ means $(|g|^{q**})^{\frac{1}{q}}$ –see section 2 for the definition of g^{**}–.

The reiteration theorem ([6], Chapter 5, Theorem 2.4 p.311) and an improvement result for the exponent of a Poincaré inequality due to Keith-Zhong yield a more general version of Theorem 1.1. Define $q_0 = \inf \{ q \in [1, \infty[: \ (P_{qloc}) \text{ holds } \}$.

Corollary 1.3. *For* $1 \leq p_1 < p < p_2 \leq \infty$ *with* $p > q_0$, W^1_p *is a real interpolation space between* $W^1_{p_1}$ *and* $W^1_{p_2}$. *More precisely*
$$W^1_p = (W^1_{p_1}, W^1_{p_2})_{\theta,p}$$
where $0 < \theta < 1$ *such that* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$.

However, if $p \leq q_0$, we only know that $(W^1_{p_1}, W^1_{p_2})_{\theta,p} \subset W^1_p$.
For the homogeneous Sobolev spaces, a weak form of Theorem 1.2 is available. This result is presented in section 5. The consequence for the interpolation problem is stated as follows.

Theorem 1.4. *Let* M *be a complete non-compact Riemannian manifold satisfying the global doubling property* (D) *and a global Poincaré inequality* (P_q) *for some* $1 \leq q < \infty$. *Then, for* $1 \leq r \leq q < p < \infty$, $\dot{W}^1_p$ *is a real interpolation space between* $\dot{W}^1_r$ *and* $\dot{W}^1_\infty$.

Again, the reiteration theorem implies another version of Theorem 1.4; see section 5 below.

For $\mathbb{R}^n$ and the non-homogeneous Sobolev spaces, our interpolation result follows from the leading work of Devore-Scherer [14]. The method of [14] is based on spline functions. Later, simpler proofs were given by Calderón-Milman [9] and Bennett-Sharpley [6], based on the Whitney extension and covering theorems. Since $\mathbb{R}^n$ admits (D) and (P_1), we recover this result by our method. Moreover, applying Theorem 1.4, we obtain the interpolation of the homogeneous Sobolev spaces on $\mathbb{R}^n$. Notice that this result is not covered by the existing references.

The interested reader may find a wealth of examples of spaces satisfying doubling and Poincaré inequalities –to which our results apply– in [1], [4], [15], [18], [23].

Some comments about the generality of Theorem 1.1- 1.4 are in order. First of all, completeness of the Riemannian manifold is not necessary (see Remark 4.3). Also, our technique can be adapted to more general metric-measure spaces, see sections 7-8. Finally it is possible to build examples where interpolation *without* a Poincaré inequality *is* possible. The question of the necessity of a Poincaré inequality for a general statement arises. This is discussed in the Appendix.

The initial motivation of this work was to provide an answer for the interpolation question for $\dot{W}^1_p$. This problem was explicitly posed in [3], where the authors interpolate inequalities of type $\|\Delta^{\frac{1}{2}} f\|_p \leq C_p \| \, |\nabla f| \, \|_p$ on Riemannian manifolds.

Let us briefly describe the structure of this paper. In section 2 we review the notions of a doubling property as well as the real K interpolation method. In sections 3 to 5, we study in detail the interpolation of Sobolev spaces in the case of a complete non-compact Riemannian manifold M satisfying (D) and (P_q) (resp. (D_{loc}) and (P_{qloc})). We briefly mention the case where M is a compact manifold in section 6. In section 7, we explain how our results extend to more general metric-measure spaces. We apply this interpolation result to Carnot-Carathéodory spaces, weighted Sobolev spaces and to Lie groups in section 8. Finally, the Appendix is devoted to an example where the Poincaré inequality is not necessary to interpolate Sobolev spaces.

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2. Preliminaries

Throughout this paper we will denote by $\mathbb{1}_E$ the characteristic function of a set E and E^c the complement of E. If X is a metric space, Lip will be the set of real Lipschitz functions on X and Lip_0 the set of real, compactly supported Lipschitz functions on X. For a ball B in a metric space, λB denotes the ball co-centered with B and with radius λ times that of B. Finally, C will be a constant that may change from an inequality to another and we will use $u \sim v$ to say that there exists two constants C_1, $C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$.

2.1. The doubling property.

By a metric-measure space, we mean a triple (X, d, μ) where (X, d) is a metric space and μ a non negative Borel measure. Denote by $B(x, r)$ the open ball of center $x \in X$ and radius $r > 0$.

Definition 2.1. *Let (X, d, μ) be a metric-measure space. One says that X satisfies the local doubling property (D_{loc}) if there exist constants $r_0 > 0$, $0 < C = C(r_0) < \infty$,*

such that for all $x \in X$, $0 < r < r_0$ *we have*

$$(D_{loc}) \qquad \mu(B(x,2r)) \leq C\mu(B(x,r)).$$

Furthermore X *satisfies a global doubling property or simply doubling property* (D) *if one can take* $r_0 = \infty$. *We also say that* μ *is a locally (resp. globally) doubling Borel measure.*

Observe that if X is a metric-measure space satisfying (D) then

$$diam(X) < \infty \Leftrightarrow \mu(X) < \infty \ ([1]).$$

Theorem 2.2 (Maximal theorem)**.** *([11]) Let* (X, d, μ) *be a metric-measure space satisfying* (D). *Denote by* $\mathcal{M}$ *the uncentered Hardy-Littlewood maximal function over open balls of* X *defined by*

$$\mathcal{M}f(x) = \sup_{B: x \in B} |f|_B$$

where $f_E := \fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu$. *Then*

1. $\mu(\{x : \mathcal{M}f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X |f| d\mu$ *for every* $\lambda > 0$;
2. $\|\mathcal{M}f\|_{L_p} \leq C_p \|f\|_{L_p}$, *for* $1 < p \leq \infty$.

2.2. **The K-method of real interpolation.** The reader can refer to [6], [7] for details on the development of this theory. Here we only recall the essentials to be used in the sequel.

Let A_0, A_1 be two normed vector spaces embedded in a topological Hausdorff vector space V. For each $a \in A_0 + A_1$ and $t > 0$, we define the K-functional of interpolation by

$$K(a, t, A_0, A_1) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For $0 < \theta < 1$, $1 \leq q \leq \infty$, we denote by $(A_0, A_1)_{\theta,q}$ the interpolation space between A_0 and A_1:

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(a, t, A_0, A_1))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

It is an exact interpolation space of exponent θ between A_0 and A_1, see [7], Chapter II.

Definition 2.3. *Let* f *be a measurable function on a measure space* (X, μ). *The decreasing rearrangement of* f *is the function* f^* *defined for every* $t \geq 0$ *by*

$$f^*(t) = \inf \{\lambda : \mu(\{x : |f(x)| > \lambda\}) \leq t\}.$$

The maximal decreasing rearrangement of f *is the function* f^{**} *defined for every* $t > 0$ *by*

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

It is known that $(\mathcal{M}f)^* \sim f^{**}$ and $\mu(\{x : |f(x)| > f^*(t)\}) \leq t$ for all $t > 0$. We refer to [6], [7], [8] for other properties of f^* and f^{**}.

We conclude the preliminaries by quoting the following classical result ([7] p.109):

Theorem 2.4. *Let (X,μ) be a measure space where μ is a totally σ-finite positive measure. Let $f \in L_p + L_\infty$, $0 < p < \infty$ where $L_p = L_p(X, d\mu)$. We then have*

1. $K(f,t,L_p,L_\infty) \sim \left(\int_0^{t^p} (f^*(s))^p ds\right)^{\frac{1}{p}}$ *and equality holds for $p = 1$;*
2. *for $0 < p_0 < p < p_1 \leq \infty$, $(L_{p_0}, L_{p_1})_{\theta,p} = L_p$ with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with $0 < \theta < 1$.*

3. Non-homogeneous Sobolev spaces on Riemannian manifolds

In this section M denotes a complete non-compact Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L_p(M,\mu)$, $1 \leq p \leq +\infty$. Our goal is to prove Theorem 1.2.

3.1. Non-homogeneous Sobolev spaces.

Definition 3.1 ([2])**.** *Let M be a C^∞ Riemannian manifold of dimension n. Write E_p^1 for the vector space of C^∞ functions φ such that φ and $|\nabla\varphi| \in L_p$, $1 \leq p < \infty$. We define the Sobolev space W_p^1 as the completion of E_p^1 for the norm*

$$\|\varphi\|_{W_p^1} = \|\varphi\|_p + \| \, |\nabla\varphi| \, \|_p.$$

We denote W_∞^1 for the set of all bounded Lipschitz functions on M.

Proposition 3.2. *([2], [20]) Let M be a complete Riemannian manifold. Then C_0^∞ and in particular Lip_0 is dense in W_p^1 for $1 \leq p < \infty$.*

Definition 3.3 (Poincaré inequality on M)**.** *We say that a complete Riemannian manifold M admits **a local Poincaré inequality** (P_{qloc}) for some $1 \leq q < \infty$ if there exist constants $r_1 > 0$, $C = C(q, r_1) > 0$ such that, for every function $f \in \mathrm{Lip}_0$ and every ball B of M of radius $0 < r < r_1$, we have*

$$(P_{qloc}) \qquad \fint_B |f - f_B|^q d\mu \leq C r^q \fint_B |\nabla f|^q d\mu.$$

M admits a global Poincaré inequality (P_q) if we can take $r_1 = \infty$ in this definition.

Remark 3.4. *By density of C_0^∞ in W_p^1, we can replace* Lip_0 *by C_0^∞.*

3.2. Estimation of the K-functional of interpolation.

In the first step, we prove Theorem 1.2 in the global case. This will help us to understand the proof of the more general local case.

3.2.1. *The global case.* Let M be a complete Riemannian manifold satisfying (D) and (P_q), for some $1 \leq q < \infty$. Before we prove Theorem 1.2, we make a Calderón-Zygmund decomposition for Sobolev functions inspired by the one done in [3]. To achieve our aims, we state it for more general spaces (in [3], the authors only needed the decomposition for the functions f in C_0^∞). This will be the principal tool in the estimation of the functional K.

Proposition 3.5 (Calderón-Zygmund lemma for Sobolev functions)**.** *Let M be a complete non-compact Riemannian manifold satisfying (D). Let $1 \leq q < \infty$ and assume that M satisfies (P_q). Let $q \leq p < \infty$, $f \in W_p^1$ and $\alpha > 0$. Then one can find*

a collection of balls $(B_i)_i$, functions $b_i \in W^1_q$ and a Lipschitz function g such that the following properties hold:

$$f = g + \sum_i b_i \tag{3.1}$$

$$|g(x)| \leq C\alpha \ \text{ and } \ |\nabla g(x)| \leq C\alpha \quad \mu - a.e\ x \in M \tag{3.2}$$

$$\operatorname{supp} b_i \subset B_i, \ \int_{B_i} (|b_i|^q + |\nabla b_i|^q) d\mu \leq C\alpha^q \mu(B_i) \tag{3.3}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int (|f| + |\nabla f|)^p d\mu \tag{3.4}$$

$$\sum_i \chi_{B_i} \leq N. \tag{3.5}$$

The constants C and N only depend on q, p and on the constants in (D) and (P_q).

Proof. Let $f \in W^1_p$, $\alpha > 0$. Consider $\Omega = \{x \in M : \mathcal{M}(|f| + |\nabla f|)^q(x) > \alpha^q\}$. If $\Omega = \emptyset$, then set

$$g = f \ , \quad b_i = 0 \ \text{ for all } i$$

so that (3.2) is satisfied according to the Lebesgue differentiation theorem. Otherwise the maximal theorem –Theorem 2.2– gives us

$$\begin{aligned} \mu(\Omega) &\leq C\alpha^{-p} \| (|f| + |\nabla f|)^q \|_{\frac{p}{q}}^{\frac{p}{q}} \\ &\leq C\alpha^{-p} \Big(\int |f|^p d\mu + \int |\nabla f|^p d\mu \Big) \\ &< +\infty. \end{aligned} \tag{3.6}$$

In particular $\Omega \neq M$ as $\mu(M) = +\infty$. Let F be the complement of Ω. Since Ω is an open set distinct of M, let $(\underline{B_i})$ be a Whitney decomposition of Ω ([12]). The balls $\underline{B_i}$ are pairwise disjoint and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \cup_i B_i$ with $B_i = C_1 \underline{B_i}$ and the balls B_i have the bounded overlap property;
2. $r_i = r(B_i) = \frac{1}{2} d(x_i, F)$ and x_i is the center of B_i;
3. each ball $\overline{B_i} = C_2 B_i$ intersects F ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i : x \in B_i\}$. By the bounded overlap property of the balls B_i, we have that $\sharp I_x \leq N$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3} r_i \leq r_j \leq 3 r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (3.5) is nothing but the bounded overlap property of the B_i's and (3.4) follows from (3.5) and (3.6). The doubling property and the fact that $\overline{B_i} \cap F \neq \emptyset$ yield

$$\int_{B_i} (|f|^q + |\nabla f|^q) d\mu \leq \int_{\overline{B_i}} (|f| + |\nabla f|)^q d\mu \leq \alpha^q \mu(\overline{B_i}) \leq C\alpha^q \mu(B_i). \tag{3.7}$$

Let us now define the functions b_i. Let $(\chi_i)_i$ be a partition of unity of Ω subordinated to the covering $(\underline{B_i})$, such that for all i, χ_i is a Lipschitz function supported in B_i with $\| \, |\nabla \chi_i| \, \|_\infty \leq \dfrac{C}{r_i}$. To this end it is enough to choose $\chi_i(x) =$

$\psi(\frac{C_1 d(x_i,x)}{r_i})\Big(\sum_k \psi(\frac{C_1 d(x_k,x)}{r_k})\Big)^{-1}$, where ψ is a smooth function, $\psi = 1$ on $[0,1]$, $\psi = 0$ on $[\frac{1+C_1}{2}, +\infty[$ and $0 \le \psi \le 1$. We set $b_i = (f - f_{B_i})\chi_i$. It is clear that $\operatorname{supp} b_i \subset B_i$. Let us estimate $\int_{B_i} |b_i|^q d\mu$ and $\int_{B_i} |\nabla b_i|^q d\mu$. We have

$$\begin{aligned}\int_{B_i} |b_i|^q d\mu &= \int_{B_i} |(f - f_{B_i})\chi_i|^q d\mu \\ &\le C\Big(\int_{B_i} |f|^q d\mu + \int_{B_i} |f_{B_i}|^q d\mu\Big) \\ &\le C \int_{B_i} |f|^q d\mu \\ &\le C\alpha^q \mu(B_i).\end{aligned}$$

We applied Jensen's inequality in the second estimate, and (3.7) in the last one. Since $\nabla\Big((f - f_{B_i})\chi_i\Big) = \chi_i \nabla f + (f - f_{B_i})\nabla\chi_i$, the Poincaré inequality (P_q) and (3.7) yield

$$\begin{aligned}\int_{B_i} |\nabla b_i|^q d\mu &\le C\left(\int_{B_i} |\chi_i \nabla f|^q d\mu + \int_{B_i} |f - f_{B_i}|^q |\nabla\chi_i|^q d\mu\right) \\ &\le C\alpha^q \mu(B_i) + C\frac{C^q}{r_i^q} r_i^q \int_{B_i} |\nabla f|^q d\mu \\ &\le C\alpha^q \mu(B_i).\end{aligned}$$

Therefore (3.3) is proved.

Set $g = f - \sum_i b_i$. Since the sum is locally finite on Ω, g is defined almost everywhere on M and $g = f$ on F. Observe that g is a locally integrable function on M. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x,F) \ge r_i$ for $x \in \operatorname{supp} b_i$, we obtain

$$\int \sum_i |b_i|\,|\varphi|\,d\mu \le \Big(\int \sum_i \frac{|b_i|}{r_i}\,d\mu\Big) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$$

and

$$\begin{aligned}\int \frac{|b_i|}{r_i} d\mu &= \int_{B_i} \frac{|f - f_{B_i}|}{r_i}\chi_i d\mu \\ &\le \Big(\mu(B_i)\Big)^{\frac{1}{q'}}\Big(\int_{B_i} |\nabla f|^q d\mu\Big)^{\frac{1}{q}} \\ &\le C\alpha\mu(B_i).\end{aligned}$$

We used the Hölder inequality, (P_q) and the estimate (3.7), q' being the conjugate of q. Hence $\int \sum_i |b_i||\varphi| d\mu \le C\alpha\mu(\Omega) \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$. Since $f \in L_{1,loc}$, we deduce that $g \in L_{1,loc}$. (Note that since $b \in L_1$ in our case, we can say directly that $g \in L_{1,loc}$. However, for the homogeneous case –section 5– we need this observation to conclude that $g \in L_{1,loc}$.) It remains to prove (3.2). Note that $\sum_i \chi_i(x) = 1$ and $\sum_i \nabla\chi_i(x) = 0$

for all $x \in \Omega$. We have

$$\begin{aligned}\nabla g &= \nabla f - \sum_i \nabla b_i \\ &= \nabla f - (\sum_i \chi_i)\nabla f - \sum_i (f - f_{B_i})\nabla \chi_i \\ &= \mathbb{1}_F(\nabla f) + \sum_i f_{B_i}\nabla \chi_i.\end{aligned}$$

From the definition of F and the Lebesgue differentiation theorem, we have that $\mathbb{1}_F(|f|+|\nabla f|) \leq \alpha$ $\mu-$a.e.. We claim that a similar estimate holds for $h = \sum_i f_{B_i}\nabla\chi_i$. We have $|h(x)| \leq C\alpha$ for all $x \in M$. For this, note first that h vanishes on F and is locally finite on Ω. Then fix $x \in \Omega$ and let B_j be some Whitney ball containing x. For all $i \in I_x$, we have $|f_{B_i} - f_{B_j}| \leq Cr_j\alpha$. Indeed, since $B_i \subset 7B_j$, we get

$$\begin{aligned}|f_{B_i} - f_{7B_j}| &\leq \frac{1}{\mu(B_i)}\int_{B_i} |f - f_{7B_j}|d\mu \\ &\leq \frac{C}{\mu(B_j)}\int_{7B_j} |f - f_{7B_j}|d\mu \\ &\leq Cr_j(\fint_{7B_j} |\nabla f|^q d\mu)^{\frac{1}{q}} \\ &\leq Cr_j\alpha \qquad (3.8)\end{aligned}$$

where we used Hölder inequality, (D), (P_q) and (3.7). Analogously $|f_{7B_j} - f_{B_j}| \leq Cr_j\alpha$. Hence

$$\begin{aligned}|h(x)| &= |\sum_{i\in I_x}(f_{B_i} - f_{B_j})\nabla\chi_i(x)| \\ &\leq C\sum_{i\in I_x}|f_{B_i} - f_{B_j}|r_i^{-1} \\ &\leq CN\alpha.\end{aligned}$$

From these estimates we deduce that $|\nabla g(x)| \leq C\alpha$ $\mu - a.e.$. Let us now estimate $\|g\|_\infty$. We have $g = f\mathbb{1}_F + \sum_i f_{B_i}\chi_i$. Since $|f|\mathbb{1}_F \leq \alpha$, still need to estimate $\|\sum_i f_{B_i}\chi_i\|_\infty$. Note that

$$\begin{aligned}|f_{B_i}|^q &\leq C\Big(\frac{1}{\mu(\overline{B_i})}\int_{\overline{B_i}} |f|d\mu\Big)^q \\ &\leq \Big(\mathcal{M}(|f| + |\nabla f|)\Big)^q(y) \\ &\leq \mathcal{M}(|f| + |\nabla f|)^q(y) \\ &\leq \alpha^q \qquad (3.9)\end{aligned}$$

where $y \in \overline{B_i} \cap F$ since $\overline{B_i} \cap F \neq \emptyset$. The second inequality follows from the fact that $(\mathcal{M}f)^q \leq \mathcal{M}f^q$ for $q \geq 1$.
Let $x \in \Omega$. Inequality (3.9) and the fact that $\sharp I_x \leq N$ yield

$$|g(x)| = |\sum_{i\in I_x} f_{B_i}\chi_i|$$

$$\leq \sum_{i\in I_x} |f_{B_i}|$$
$$\leq N\alpha.$$

We conclude that $\|g\|_\infty \leq C\,\alpha \quad \mu - a.e.$ and the proof of Proposition 3.5 is therefore complete. □

Remark 3.6. *1- It is a straightforward consequence of (3.3) that $b_i \in W^1_r$ for all $1 \leq r \leq q$ with $\|b_i\|_{W^1_r} \leq C\alpha\mu(B_i)^{\frac{1}{r}}$.*
2- From the construction of the functions b_i, we see that $\sum_i b_i \in W^1_p$, with $\|\sum_i b_i\|_{W^1_p} \leq C\|f\|_{W^1_p}$. It follows that $g \in W^1_p$. Hence $(g, |\nabla g|)$ satisfies the Poincaré inequality (P_p). Theorem 3.2 of [23] *asserts that for $\mu - a.e.$ x, $y \in M$*

$$|g(x) - g(y)| \leq Cd(x,y)\left((\mathcal{M}|\nabla g|^p)^{\frac{1}{p}}(x) + (\mathcal{M}|\nabla g|^p)^{\frac{1}{p}}(y)\right).$$

From Theorem 2.2 with $p = \infty$ and the inequality $\| \, |\nabla g| \, \|_\infty \leq C\alpha$, we deduce that g has a Lipschitz representative. Moreover, the Lipschitz constant is controlled by $C\alpha$.
3- We also deduce from this Calderón-Zygmund decomposition that $g \in W^1_s$ for $p \leq s \leq \infty$. We have $\left(\int_\Omega (|g|^s + |\nabla g|^s)d\mu\right)^{\frac{1}{s}} \leq C\alpha\mu(\Omega)^{\frac{1}{s}}$ and

$$\begin{aligned}\int_F (|g|^s + |\nabla g|^s)d\mu &= \int_F (|f|^s + |\nabla f|^s)d\mu \\ &\leq \int_F (|f|^p|f|^{s-p} + |\nabla f|^p|\nabla f|^{s-p})d\mu \\ &\leq \alpha^{s-p}\|f\|^p_{W^1_p} < \infty.\end{aligned}$$

Corollary 3.7. *Under the same hypotheses as in the Calderón-Zygmund lemma, we have*

$$W^1_p \subset W^1_r + W^1_s \quad \textit{for } 1 \leq r \leq q \leq p \leq s < \infty.$$

Proof of Theorem 1.2. To prove part 1., we begin applying Theorem 2.4, part 1. We have

$$K(f, t^{\frac{1}{r}}, L_r, L_\infty) \sim \left(\int_0^t (f^*(s))^r ds\right)^{\frac{1}{r}}.$$

On the other hand

$$\begin{aligned}\left(\int_0^t f^*(s)^r ds\right)^{\frac{1}{r}} &= \left(\int_0^t |f(s)|^{r*} ds\right)^{\frac{1}{r}} \\ &= \left(t|f|^{r**}(t)\right)^{\frac{1}{r}}\end{aligned}$$

where in the first equality we used the fact that $f^{*r} = (|f|^r)^*$ and the second follows from the definition of f^{**}. We thus get $K(f, t^{\frac{1}{r}}, L_r, L_\infty) \sim t^{\frac{1}{r}}(|f|^{r**})^{\frac{1}{r}}(t)$. Moreover,

$$K(f, t^{\frac{1}{r}}, W^1_r, W^1_\infty) \geq K(f, t^{\frac{1}{r}}, L_r, L_\infty) + K(|\nabla f|, t^{\frac{1}{r}}, L_r, L_\infty)$$

since the linear operator

$$(I,\, \nabla) : W^1_s(M) \to (L_s(M; \mathbb{C} \times TM))$$

is bounded for every $1 \leq s \leq \infty$. These two points yield the desired inequality.

We will now prove part 2.. We treat the case when $f \in W^1_p$, $q \leq p < \infty$. Let $t > 0$. We consider the Calderón-Zygmund decomposition of f of Proposition 3.5

with $\alpha = \alpha(t) = \left(\mathcal{M}(|f| + |\nabla f|)^q\right)^{*\frac{1}{q}}(t)$. We write $f = \sum_i b_i + g = b + g$ where $(b_i)_i$, g satisfy the properties of the proposition. From the bounded overlap property of the B_i's, it follows that for all $r \leq q$

$$\begin{aligned}\|b\|_r^r &\leq \int_M (\sum_i |b_i|)^r d\mu \\ &\leq N \sum_i \int_{B_i} |b_i|^r d\mu \\ &\leq C\alpha^r(t) \sum_i \mu(B_i) \\ &\leq C\alpha^r(t)\mu(\Omega).\end{aligned}$$

Similarly we have $\| \, |\nabla b| \, \|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$.

Moreover, since $(\mathcal{M}f)^* \sim f^{**}$ and $(f+g)^{**} \leq f^{**} + g^{**}$, we get

$$\alpha(t) = (\mathcal{M}(|f| + |\nabla f|)^q)^{*\frac{1}{q}}(t) \leq C\left(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\right).$$

Noting that $\mu(\Omega) \leq t$, we deduce that

$$K(f, t^{\frac{1}{r}}, W_r^1, W_\infty^1) \leq Ct^{\frac{1}{r}}\left(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\right) \tag{3.10}$$

for all $t > 0$ and obtain the desired inequality for $f \in W_p^1$, $q \leq p < \infty$.

Note that in the special case where $r = q$, we have the upper bound of K for $f \in W_q^1$. Applying a similar argument to that of [14] –Euclidean case– we get (3.10) for $f \in W_q^1 + W_\infty$. Here we will omit the details. □

We were not able to show this characterization when $r < q$ since we could not show its validity even for $f \in W_r^1$. Nevertheless this theorem is enough to achieve interpolation (see the next section).

3.2.2. *The local case.* Let M be a complete non-compact Riemannian manifold satisfying a local doubling property (D_{loc}) and a local Poincaré inequality (P_{qloc}) for some $1 \leq q < \infty$.

Denote by $\mathcal{M}_E$ the Hardy-Littlewood maximal operator relative to a measurable subset E of M, that is, for $x \in E$ and every locally integrable function f on M

$$\mathcal{M}_E f(x) = \sup_{B:\, x \in B} \frac{1}{\mu(B \cap E)} \int_{B \cap E} |f| d\mu$$

where B ranges over all open balls of M containing x and centered in E. We say that a measurable subset E of M has the relative doubling property if there exists a constant C_E such that for all $x \in E$ and $r > 0$ we have

$$\mu(B(x, 2r) \cap E) \leq C_E \mu(B(x, r) \cap E).$$

This is equivalent to saying that the metric-measure space $(E, d|_E, \mu|_E)$ has the doubling property. On such a set $\mathcal{M}_E$ is of weak type $(1,1)$ and bounded on $L_p(E, \mu)$, $1 < p \leq \infty$.

Proof of Theorem 1.2. To fix ideas, we assume without loss of generality $r_0 = 5$, $r_1 = 8$. The lower bound of K is trivial (same proof as for the global case). It remains to prove the upper bound.

For all $t > 0$, take $\alpha = \alpha(t) = \left(\mathcal{M}(|f| + |\nabla f|)^q\right)^{*\frac{1}{q}}(t)$. Consider

$$\Omega = \{x \in M : \mathcal{M}(|f| + |\nabla f|)^q(x) > \alpha^q(t)\}.$$

We have $\mu(\Omega) \leq t$. If $\Omega = M$ then

$$\begin{aligned}\int_M |f|^r d\mu + \int_M |\nabla f|^r d\mu &= \int_\Omega |f|^r d\mu + \int_\Omega |\nabla f|^r d\mu \\ &\leq \int_0^{\mu(\Omega)} |f|^{r*}(s)ds + \int_0^{\mu(\Omega)} |\nabla f|^{r*}(s)ds \\ &\leq \int_0^t |f|^{r*}(s)ds + \int_0^t |\nabla f|^{r*}(s)ds \\ &= t\left(|f|^{r**}(t) + |\nabla f|^{r**}(t)\right).\end{aligned}$$

Therefore

$$\begin{aligned}K(f, t^{\frac{1}{r}}, W_r^1, W_\infty^1) &\leq \|f\|_{W_r^1} \\ &\leq Ct^{\frac{1}{r}}\left(|f|^{r**\frac{1}{r}}(t) + |\nabla f|^{r**\frac{1}{r}}(t)\right) \\ &\leq Ct^{\frac{1}{r}}\left(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t)\right)\end{aligned}$$

since $r \leq q$. We thus obtain the upper bound in this case.

Now assume $\Omega \neq M$. Pick a countable set $\{x_j\}_{j\in J} \subset M$, such that $M = \bigcup_{j\in J} B(x_j, \frac{1}{2})$ and for all $x \in M$, x does not belong to more than N_1 balls $B^j := B(x_j, 1)$. Consider a C^∞ partition of unity $(\varphi_j)_{j\in J}$ subordinated to the balls $\frac{1}{2}B^j$ such that $0 \leq \varphi_j \leq 1$, $\operatorname{supp}\varphi_j \subset B^j$ and $\| \, |\nabla\varphi_j| \, \|_\infty \leq C$ uniformly with respect to j. Consider $f \in W_p^1$, $q \leq p < \infty$. Let $f_j = f\varphi_j$ so that $f = \sum_{j\in J} f_j$. We have for $j \in J$, $f_j \in L_p$ and $\nabla f_j = f\nabla\varphi_j + \nabla f\varphi_j \in L_p$. Hence $f_j \in W_p^1(B^j)$. The balls B^j satisfy the relative doubling property with constant independent of the balls B^j. This follows from the next lemma quoted from [4] p.947.

Lemma 3.8. *Let M be a complete Riemannian manifold satisfying (D_{loc}). Then the balls B^j above, equipped with the induced distance and measure, satisfy the relative doubling property (D), with the doubling constant that may be chosen independently of j. More precisely, there exists $C \geq 0$ such that for all $j \in J$*

$$\mu(B(x, 2r) \cap B^j) \leq C\,\mu(B(x, r) \cap B^j) \quad \forall x \in B^j,\ r > 0, \tag{3.11}$$

and

$$\mu(B(x, r)) \leq C\mu(B(x, r) \cap B^j) \quad \forall x \in B^j,\ 0 < r \leq 2. \tag{3.12}$$

Remark 3.9. *Noting that the proof in [4] only used the fact that M is a length space, we observe that Lemma 3.8 still holds for any length space. Recall that a length space X is a metric space such that the distance between any two points x, $y \in X$ is equal to the infimum of the lengths of all paths joining x to y (we implicitly assume that there is at least one such path). Here a path from x to y is a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Let us return to the proof of the theorem. For any $x \in B^j$ we have

$$\begin{aligned}\mathcal{M}_{B^j}(|f_j| + |\nabla f_j|)^q(x) &= \sup_{B:\, x\in B,\, r(B)\le 2} \frac{1}{\mu(B^j \cap B)} \int_{B^j\cap B} (|f_j| + |\nabla f_j|)^q d\mu \\ &\le \sup_{B:\, x\in B,\, r(B)\le 2} C\frac{\mu(B)}{\mu(B^j \cap B)} \frac{1}{\mu(B)} \int_B (|f| + |\nabla f|)^q d\mu \\ &\le C\mathcal{M}(|f| + |\nabla f|)^q(x) \qquad (3.13)\end{aligned}$$

where we used (3.12) of Lemma 3.8. Consider now

$$\Omega_j = \big\{x \in B^j : \mathcal{M}_{B^j}(|f_j| + |\nabla f_j|)^q(x) > C\alpha^q(t)\big\}$$

where C is the constant in (3.13). Ω_j is an open subset of B^j, hence of M, and $\Omega_j \subset \Omega \neq M$ for all $j \in J$. For the f_j's, and for all $t > 0$, we have a Calderón-Zygmund decomposition similar to the one done in Proposition 3.5: there exist b_{jk}, g_j supported in B^j, and balls $(B_{jk})_k$ of M, contained in Ω_j, such that

$$f_j = g_j + \sum_k b_{jk} \qquad (3.14)$$

$$|g_j(x)| \le C\alpha(t) \text{ and } |\nabla g_j(x)| \le C\alpha(t) \quad \text{for } \mu - a.e.\, x \in M \qquad (3.15)$$

$$\operatorname{supp} b_{jk} \subset B_{jk}, \text{ for } 1 \le r \le q \int_{B_{jk}} (|b_{jk}|^r + |\nabla b_{jk}|^r) d\mu \le C\alpha^r(t)\mu(B_{jk}) \qquad (3.16)$$

$$\sum_k \mu(B_{jk}) \le C\alpha^{-p}(t) \int_{B^j} (|f_j| + |\nabla f_j|)^p d\mu \qquad (3.17)$$

$$\sum_k \chi_{B_{jk}} \le N \qquad (3.18)$$

with C and N depending only on q, p and the constants in (D_{loc}) and (P_{qloc}). The proof of this decomposition will be the same as in Proposition 3.5, taking for all $j \in J$ a Whitney decomposition $(B_{jk})_k$ of $\Omega_j \neq M$ and using the doubling property for balls whose radii do not exceed $3 < r_0$ and the Poincaré inequality for balls whose radii do not exceed $7 < r_1$. For the bounded overlap property (3.18), just note that the radius of every ball B_{jk} is less than 1. Then apply the same argument as for the bounded overlap property of a Whitney decomposition for an homogeneous space, using the doubling property for balls with sufficiently small radii.

By the above decomposition we can write $f = \sum\limits_{j\in J}\sum\limits_k b_{jk} + \sum\limits_{j\in J} g_j = b + g$. Let us now estimate $\|b\|_{W^1_r}$ and $\|g\|_{W^1_\infty}$.

$$\begin{aligned}\|b\|_r^r &\le N_1 N \sum_j \sum_k \|b_{jk}\|_r^r \\ &\le C\alpha^r(t) \sum_j \sum_k (\mu(B_{jk})) \\ &\le NC\alpha^r(t)\Big(\sum_j \mu(\Omega_j)\Big) \\ &\le N_1 C\alpha^r(t)\mu(\Omega).\end{aligned}$$

We used the bounded overlap property of the $(\Omega_j)_{j\in J}$'s and that of the $(B_{jk})_k$'s for all $j \in J$. It follows that $\|b\|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$. Similarly we get $\| \, |\nabla b| \, \|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$.

For g we have

$$\begin{aligned}\|g\|_\infty &\leq \sup_x \sum_{j\in J} |g_j(x)| \\ &\leq \sup_x N_1 \sup_{j\in J} |g_j(x)| \\ &\leq N_1 \sup_{j\in J} \|g_j\|_\infty \\ &\leq C\alpha(t).\end{aligned}$$

Analogously $\| \, |\nabla g| \, \|_\infty \leq C\alpha(t)$. We conclude that

$$\begin{aligned}K(f, t^{\frac{1}{r}}, W_r^1, W_\infty^1) &\leq \|b\|_{W_r^1} + t^{\frac{1}{r}}\|g\|_{W_\infty^1} \\ &\leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}} + Ct^{\frac{1}{r}}\alpha(t) \\ &\leq Ct^{\frac{1}{r}}\alpha(t) \\ &\sim Ct^{\frac{1}{r}}(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t))\end{aligned}$$

which completes the proof of Theorem 1.2 in the case $r < q$. When $r = q$ we get the characterization of K for every $f \in W_q^1 + W_\infty^1$ by applying again a similar argument to that of [14]. □

4. Interpolation Theorems

In this section we establish our interpolation Theorem 1.1 and some consequences for non-homogeneous Sobolev spaces on a complete non-compact Riemannian manifold M satisfying (D_{loc}) and (P_{qloc}) for some $1 \leq q < \infty$.

For $1 \leq r \leq q < p < \infty$, we define the real interpolation space $W_{p,r}^1$ between W_r^1 and W_∞^1 by

$$W_{p,r}^1 = (W_r^1, W_\infty^1)_{1-\frac{r}{p},p}.$$

From the previous results we know that for $f \in W_r^1 + W_\infty^1$

$$\|f\|_{1-\frac{r}{p},p} \geq C_1 \left\{ \int_0^\infty \left(t^{\frac{1}{p}}(|f|^{r**\frac{1}{r}} + |\nabla f|^{r**\frac{1}{r}})(t) \right)^p \frac{dt}{t} \right\}^{\frac{1}{p}}$$

and for $f \in W_p^1$

$$\|f\|_{1-\frac{r}{p},p} \leq C_2 \left\{ \int_0^\infty \left(t^{\frac{1}{p}}(|f|^{q**\frac{1}{q}} + |\nabla f|^{q**\frac{1}{q}})(t) \right)^p \frac{dt}{t} \right\}^{\frac{1}{p}}.$$

We claim that $W_{p,r}^1 = W_p^1$, with equivalent norms. Indeed,

$$\begin{aligned}\|f\|_{1-\frac{r}{p},p} &\geq C_1 \left\{ \int_0^\infty \left(|f|^{r**\frac{1}{r}}(t) + |\nabla f|^{r**\frac{1}{r}}(t) \right)^p dt \right\}^{\frac{1}{p}} \\ &\geq C \left(\|f^{r**}\|_{\frac{p}{r}}^{\frac{1}{r}} + \||\nabla f|^{r**}\|_{\frac{p}{r}}^{\frac{1}{r}} \right) \\ &\geq C \left(\|f^r\|_{\frac{p}{r}}^{\frac{1}{r}} + \| \, |\nabla f|^r \, \|_{\frac{p}{r}}^{\frac{1}{r}} \right)\end{aligned}$$

$$= C\left(\|f\|_p + \| \, |\nabla f| \, \|_p\right)$$
$$= C\|f\|_{W^1_p},$$

and

$$\|f\|_{1-\frac{r}{p},p} \leq C_2 \left\{ \int_0^\infty \left(|f|^{q**\frac{1}{q}}(t) + |\nabla f|^{q**\frac{1}{q}}(t) \right)^p dt \right\}^{\frac{1}{p}}$$
$$\leq C \left(\|f^{q**}\|^{\frac{1}{q}}_{\frac{p}{q}} + \| \, |\nabla f|^{q**} \, \|^{\frac{1}{q}}_{\frac{p}{q}} \right)$$
$$\leq C \left(\|f^{q}\|^{\frac{1}{q}}_{\frac{p}{q}} + \| \, |\nabla f|^{q} \, \|^{\frac{1}{q}}_{\frac{p}{q}} \right)$$
$$= C\left(\|f\|_p + \| \, |\nabla f| \, \|_p\right)$$
$$= C\|f\|_{W^1_p},$$

where we used that for $l > 1$, $\|f^{**}\|_l \sim \|f\|_l$ (see [34], Chapter V: Lemma 3.21 p.191 and Theorem 3.21, p.201). Moreover, from Corollary 3.7, we have $W^1_p \subset W^1_r + W^1_\infty$ for $r < p < \infty$. Therefore W^1_p is a real interpolation space between W^1_r and W^1_∞ for $r < p < \infty$.

Let us recall some known facts about Poincaré inequalities with varying q.
It is known that (P_{qloc}) implies (P_{ploc}) when $p \geq q$ (see [23]). Thus if the set of q such that (P_{qloc}) holds is not empty, then it is an interval unbounded on the right. A recent result of Keith and Zhong [28] asserts that this interval is open in $[1, +\infty[$.

Theorem 4.1. *Let (X, d, μ) be a complete metric-measure space with μ locally doubling and admitting a local Poincaré inequality (P_{qloc}), for some $1 < q < \infty$. Then there exists $\epsilon > 0$ such that (X, d, μ) admits (P_{ploc}) for every $p > q - \epsilon$.*

Here, the definition of (P_{qloc}) is that of section 7. It reduces to the one of section 3 when the metric space is a Riemannian manifold.

Comment on the proof of this theorem. The proof goes as in [28] where this theorem is proved for X satisfying (D) and admitting a global Poincaré inequality (P_q). By using the same argument and choosing sufficiently small radii for the considered balls, (P_{qloc}) will give us $(P_{(q-\epsilon)loc})$ for every ball of radius less than r_2, for some $r_2 < \min(r_0, r_1)$, r_0, r_1 being the constants given in the definitions of local doubling property and local Poincaré inequality. □

Define $A_M = \{q \in [1, \infty[: (P_{qloc}) \text{ holds } \}$ and $q_{0_M} = \inf A_M$. When no confusion arises, we write q_0 instead of q_{0_M}. As we mentioned in the introduction, this improvement of the exponent of a Poincaré inequality together with the reiteration theorem yield another version of our interpolation result: Corollary 1.3.

Proof of Corollary 1.3. Let $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$.

1. *Case when $p_1 > q_0$.* Since $p_1 > q_0$, there exists $q \in A_M$ such that $q_0 < q < p_1$. Then $1 - \frac{q}{p} = (1-\theta)(1-\frac{q}{p_1}) + \theta(1-\frac{q}{p_2})$. The reiteration theorem –[6], Theorem 2.4 p. 311– yields

$$(W^1_{p_1}, W^1_{p_2})_{\theta,p} = (W^1_{p_1,q}, W^1_{p_2,q})_{\theta,p}$$
$$= (W^1_q, W^1_\infty)_{1-\frac{q}{p},p}$$

$$= W^1_{p,q}$$
$$= W^1_p.$$

2. Case when $1 \leq p_1 \leq q_0$. Let $\theta' = \theta(1 - \frac{p_1}{p_2}) = 1 - \frac{p_1}{p}$. The reiteration theorem applied this time only to the second exponent yields
$$(W^1_{p_1}, W^1_{p_2})_{\theta,p} = (W^1_{p_1}, W^1_{p_2,p_1})_{\theta,p}$$
$$= (W^1_{p_1}, W^1_{\infty})_{\theta',p}$$
$$= W^1_{p,p_1}$$
$$= W^1_p.$$

□

Theorem 4.2. *Let M and N be two complete non-compact Riemannian manifolds satisfying (D_{loc}). Assume that q_{0_M} and q_{0_N} are well defined. Take $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq r_1, r, r_2 \leq \infty$. Let T be a bounded linear operator from $W^1_{p_i}(M)$ to $W^1_{r_i}(N)$ of norm L_i, $i = 1, 2$. Then for every couple (p, r) such that $p \leq r$, $p > q_{0_M}$, $r > q_{0_N}$ and $(\frac{1}{p}, \frac{1}{r}) = (1-\theta)(\frac{1}{p_1}, \frac{1}{r_1}) + \theta(\frac{1}{p_2}, \frac{1}{r_2})$, $0 < \theta < 1$, T is bounded from $W^1_p(M)$ to $W^1_r(N)$ with norm $L \leq CL_0^{1-\theta}L_1^{\theta}$.*

Proof.
$$\|Tf\|_{W^1_r(N)} \leq C\|Tf\|_{(W^1_{r_1}(N), W^1_{r_2}(N))_{\theta,r}}$$
$$\leq CL_0^{1-\theta}L_1^{\theta}\|f\|_{(W^1_{p_1}(M), W^1_{p_2}(M))_{\theta,r}}$$
$$\leq CL_0^{1-\theta}L_1^{\theta}\|f\|_{(W^1_{p_1}(M), W^1_{p_2}(M))_{\theta,p}}$$
$$\leq CL_0^{1-\theta}L_1^{\theta}\|f\|_{W^1_p(M)}.$$

We used the fact that $K_{\theta,q}$ is an exact interpolation functor of exponent θ, that $W^1_p(M) = (W^1_{p_1}(M), W^1_{p_2}(M))_{\theta,p}$, $W^1_r(N) = (W^1_{r_1}(N), W^1_{r_2}(N))_{\theta,r}$ with equivalent norms and that $(W^1_{p_1}(M), W^1_{p_2}(M))_{\theta,p} \subset (W^1_{p_1}(M), W^1_{p_2}(M))_{\theta,r}$ if $p \leq r$. □

Remark 4.3. *Let M be a Riemannian manifold, not necessarily complete, satisfying (D_{loc}). Assume that for some $1 \leq q < \infty$, a weak local Poincaré inequality holds for all C^∞ functions, that is there exists $r_1 > 0$, $C = C(q, r_1)$, $\lambda \geq 1$ such that for all $f \in C^\infty$ and all ball B of radius $r < r_1$ we have*
$$\Big(\fint_B |f - f_B|^q d\mu\Big)^{\frac{1}{q}} \leq Cr\Big(\fint_{\lambda B} |\nabla f|^q d\mu\Big)^{\frac{1}{q}}.$$
Then, we obtain the characterization of K as in Theorem 1.2 and we get by interpolating a result analogous to Theorem 1.1.

5. Homogeneous Sobolev spaces on Riemannian manifolds

Definition 5.1. *Let M be a C^∞ Riemannian manifold of dimension n. For $1 \leq p \leq \infty$, we define $\dot{E}^1_p$ to be the vector space of distributions φ with $|\nabla\varphi| \in L_p$, where $\nabla\varphi$ is the distributional gradient of φ. It is well known that the elements of $\dot{E}^1_p$ are in L_{ploc}. We equip $\dot{E}^1_p$ with the semi norm*
$$\|\varphi\|_{\dot{E}^1_p} = \| \, |\nabla\varphi| \, \|_p.$$

Definition 5.2. *We define the homogeneous Sobolev space* $\dot{W}^1_p$ *as the quotient space* $\dot{E}^1_p/\mathbb{R}$.

Remark 5.3. *For all* $\varphi \in \dot{E}^1_p$, $\|\overline{\varphi}\|_{\dot{W}^1_p} = \| \, |\nabla\varphi| \, \|_p$, *where* $\overline{\varphi}$ *denotes the class of* φ.

Proposition 5.4. *1. ([20]) $\dot{W}^1_p$ is a Banach space.*
2. Assume that M satisfies (D) and (P_q) for some $1 \leq q < \infty$ and for all $f \in Lip$, that is there exists a constant $C > 0$ such that for all $f \in Lip$ and for every ball B of M of radius $r > 0$ we have

$$(P_q) \qquad \left(\fint_B |f - f_B|^q d\mu \right)^{\frac{1}{q}} \leq Cr \left(\fint_B |\nabla f|^q d\mu \right)^{\frac{1}{q}}.$$

Then $Lip(M) \cap \dot{W}^1_p$ is dense in $\dot{W}^1_p$ for $q \leq p < \infty$.

Proof. The proof of item 2. is implicit in the proof of Theorem 9 in [17]. □

We obtain for the K-functional of the homogeneous Sobolev spaces the following homogeneous form of Theorem 1.2, weaker in the particular case $r = q$, but again sufficient for us to interpolate.

Theorem 5.5. *Let M be a complete Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Let $1 \leq r \leq q$. Then*

1. *there exists C_1 such that for every $F \in \dot{W}^1_r + \dot{W}^1_\infty$ and $t > 0$*

$$K(F, t^{\frac{1}{r}}, \dot{W}^1_r, \dot{W}^1_\infty) \geq C_1 t^{\frac{1}{r}} |\nabla f|^{r**\frac{1}{r}}(t) \text{ where } f \in \dot{E}^1_r + \dot{E}^1_\infty \text{ and } \overline{f} = F;$$

2. *for $q \leq p < \infty$, there exists C_2 such that for every $F \in \dot{W}^1_p$ and $t > 0$*

$$K(F, t^{\frac{1}{r}}, \dot{W}^1_r, \dot{W}^1_\infty) \leq C_2 t^{\frac{1}{r}} |\nabla f|^{q**\frac{1}{q}}(t) \text{ where } f \in \dot{E}^1_p \text{ and } \overline{f} = F.$$

Before we prove Theorem 5.5, we give the following Calderón-Zygmund decomposition that will be also in this case our principal tool to estimate K.

Proposition 5.6 (Calderón-Zygmund lemma for Sobolev functions)**.** *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Let $q \leq p < \infty$, $f \in \dot{E}^1_p$ and $\alpha > 0$. Then there is a collection of balls $(B_i)_i$, functions $b_i \in \dot{E}^1_q$ and a Lipschitz function g such that the following properties hold :*

$$f = g + \sum_i b_i \tag{5.1}$$

$$|\nabla g(x)| \leq C\,\alpha \quad \mu - a.e. \tag{5.2}$$

$$\operatorname{supp} b_i \subset B_i \text{ and for } 1 \leq r \leq q \int_{B_i} |\nabla b_i|^r d\mu \leq C\alpha^r \mu(B_i) \tag{5.3}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int |\nabla f|^p d\mu \tag{5.4}$$

$$\sum_i \chi_{B_i} \leq N. \tag{5.5}$$

The constants C and N depend only on q, p and the constant in (D).

Proof. The proof goes as in the case of non-homogeneous Sobolev spaces, but taking $\Omega = \{x \in M : \mathcal{M}(|\nabla f|^q)(x) > \alpha^q\}$ as $\|f\|_p$ is not under control. We note that in the non-homogeneous case, we used that $f \in L_p$ only to control $g \in L_\infty$ and $b \in L_r$. □

Remark 5.7. *It is sufficient for us that the Poincaré inequality holds for all $f \in \dot{E}^1_p$.*

Corollary 5.8. *Under the same hypotheses as in the Calderón-Zygmund lemma, we have*

$$\dot{W}^1_p \subset \dot{W}^1_r + \dot{W}^1_\infty \quad \textit{for } 1 \leq r \leq q \leq p < \infty\,.$$

Proof of Theorem 5.5. The proof of item 1. is the same as in the non-homogeneous case. Let us turn to inequality 2.. For $F \in \dot{W}^1_p$ we take $f \in \dot{E}^1_p$ with $\overline{f} = F$. Let $t > 0$ and $\alpha(t) = \left(\mathcal{M}(|\nabla f|^q)\right)^{*\frac{1}{q}}(t)$. By the Calderón-Zygmund decomposition with $\alpha = \alpha(t)$, f can be written $f = b + g$, hence $F = \overline{b} + \overline{g}$, with $\|\overline{b}\|_{\dot{W}^1_r} = \| \, |\nabla b| \, \|_r \leq C\alpha(t)\mu(\Omega)^{\frac{1}{r}}$ and $\|\overline{g}\|_{\dot{W}^1_\infty} = \| \, |\nabla g| \, \|_\infty \leq C\alpha(t)$. Since for $\alpha = \alpha(t)$ we have $\mu(\Omega) \leq t$, then we get $K(F, t^{\frac{1}{r}}, \dot{W}^1_r, \dot{W}^1_\infty) \leq Ct^{\frac{1}{r}}|\nabla f|^{q**\frac{1}{q}}(t)$. □

We can now prove our interpolation result for the homogeneous Sobolev spaces.

Proof of Theorem 1.4. The proof follows directly from Theorem 5.5. Indeed, item 1. of Theorem 5.5 yields

$$(\dot{W}^1_r, \dot{W}^1_\infty)_{1-\frac{r}{p},p} \subset \dot{W}^1_p$$

with $\|F\|_{\dot{W}^1_p} \leq C\|F\|_{1-\frac{r}{p},p}$, while item 2. gives us that

$$\dot{W}^1_p \subset (\dot{W}^1_r, \dot{W}^1_\infty)_{1-\frac{r}{p},p}$$

with $\|F\|_{1-\frac{r}{p},p} \leq C\|F\|_{\dot{W}^1_p}$. We conclude that

$$\dot{W}^1_p = (\dot{W}^1_r, \dot{W}^1_\infty)_{1-\frac{r}{p},p}$$

with equivalent norms. □

Corollary 5.9 (The reiteration theorem)**.** *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \leq q < \infty$. Define $q_0 = \inf\{q \in [1, \infty[: (P_q) \textit{ holds }\}$. Then for $p > q_0$ and $1 \leq p_1 < p < p_2 \leq \infty$, $\dot{W}^1_p$ is a real interpolation space between $\dot{W}^1_{p_1}$ and $\dot{W}^1_{p_2}$.*

Application. Consider a complete non-compact Riemannian manifold M satisfying (D) and (P_q) for some $1 \leq q < 2$. Let Δ be the Laplace-Beltrami operator. Consider the linear operator $\Delta^{\frac{1}{2}}$ with the following resolution

$$\Delta^{\frac{1}{2}} f = c \int_0^\infty \Delta e^{-t\Delta} f \frac{dt}{\sqrt{t}}, \quad f \in C_0^\infty$$

where $c = \pi^{-\frac{1}{2}}$. Here $\Delta^{\frac{1}{2}} f$ can be defined for $f \in \mathrm{Lip}$ as a measurable function (see [3]).

In [3], Auscher and Coulhon proved that on such a manifold, we have

$$\mu\left\{x \in M : |\Delta^{\frac{1}{2}} f(x)| > \alpha\right\} \leq \frac{C}{\alpha^q} \| \, |\nabla f| \, \|_q$$

for $f \in C_0^\infty$, with $q \in [1, 2[$. In fact one can check that the argument applies to all $f \in \mathrm{Lip} \cap \dot{E}_q^1$ and since $\Delta^{\frac{1}{2}} 1 = 0$, $\Delta^{\frac{1}{2}}$ can be defined on $\mathrm{Lip} \cap \dot{W}_q^1$ by taking quotient which we keep calling $\Delta^{\frac{1}{2}}$. Moreover, Proposition 5.4 gives us that $\Delta^{\frac{1}{2}}$ has a bounded extension from $\dot{W}_q^1$ to $L_{q,\infty}$. Since we already have

$$\|\Delta^{\frac{1}{2}} f\|_2 \leq \| \, |\nabla f| \, \|_2$$

then by Corollary 5.9, we see at once

$$\|\Delta^{\frac{1}{2}} f\|_p \leq C_p \| \, |\nabla f| \, \|_p \tag{5.6}$$

for all $q < p \leq 2$ and $f \in \dot{W}_p^1$, without using the argument in [3].

6. Sobolev spaces on compact manifolds

Let M be a C^∞ compact manifold equipped with a Riemannian metric. Then M satisfies the doubling property (D) and the Poincaré inequality (P_1).

Theorem 6.1. *Let M be a C^∞ compact Riemannian manifold. There exist C_1, C_2 such that for all $f \in W_1^1 + W_\infty^1$ and all $t > 0$ we have*

$$(*_{\mathrm{comp}}) \quad C_1 t\Big(|f|^{**}(t) + |\nabla f|^{**}(t)\Big) \leq K(f, t, W_1^1, W_\infty^1) \leq C_2 t\Big(|f|^{**}(t) + |\nabla f|^{**}(t)\Big).$$

Proof. It remains to prove the upper bound for K as the lower bound is trivial. Indeed, let us consider for all $t > 0$ and for $\alpha(t) = \left(\mathcal{M}(|f| + |\nabla f|)\right)^*(t)$, $\Omega = \{x \in M; \mathcal{M}(|f| + |\nabla f|)(x) \geq \alpha(t)\}$. If $\Omega \neq M$, we have the Calderón-Zygmund decomposition as in Proposition 3.5 with $q = 1$ and the proof will be the same as the proof of Theorem 1.2 in the global case. Now if $\Omega = M$, we prove the upper bound by the same argument used in the proof of Theorem 1.2 in the local case. Thus, in the two cases we obtain the right hand inequality of $(*_{\mathrm{comp}})$ for all $f \in W_1^1 + W_\infty^1$. □

It follows that

Theorem 6.2. *For all $1 \leq p_1 < p < p_2 \leq \infty$, W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$.*

7. Metric-measure spaces

In this section we consider (X, d, μ) a metric-measure space with μ doubling.

7.1. Upper gradients and Poincaré inequality.

Definition 7.1 (Upper gradient [26])**.** *Let $u : X \to \mathbb{R}$ be a Borel function. We say that a Borel function $g : X \to [0, +\infty]$ is an upper gradient of u if $|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))dt$ for all 1-Lipschitz curve $\gamma : [a, b] \to X$* [1].

[1] Since every rectifiable curve admits an arc-length parametrization that makes the curve 1-Lipschitz, the class of 1-Lipschitz curves coincides with the class of rectifiable curves, modulo a parameter change.

Remark 7.2. *If X is a Riemannian manifold, $|\nabla u|$ is an upper gradient of $u \in \mathrm{Lip}$ and $|\nabla u| \leq g$ for all upper gradients g of u.*

Definition 7.3. *For every locally Lipschitz continuous function u defined on a open set of X, we define*

$$\mathrm{Lip}\, u(x) = \begin{cases} \limsup_{\substack{y \to x \\ y \neq x}} \frac{|u(y)-u(x)|}{d(y,x)} & \text{if } x \text{ is not isolated,} \\ 0 \text{ otherwise.} \end{cases}$$

Remark 7.4. $\mathrm{Lip}\, u$ *is an upper gradient of u.*

Definition 7.5 (Poincaré Inequality)**.** *A metric-measure space (X, d, μ) admits a weak local Poincaré inequality (P_{qloc}) for some $1 \leq q < \infty$, if there exist $r_1 > 0$, $\lambda \geq 1$, $C = C(q, r_1) > 0$, such that for every continuous function u and upper gradient g of u, and for every ball B of radius $0 < r < r_1$ the following inequality holds:*

$$(P_{qloc}) \qquad \Big(\fint_B |u - u_B|^q d\mu \Big)^{\frac{1}{q}} \leq Cr \Big(\fint_{\lambda B} g^q d\mu \Big)^{\frac{1}{q}}.$$

If $\lambda = 1$, we say that we have a strong local Poincaré inequality.
Moreover, X admits a global Poincaré inequality or simply a Poincaré inequality (P_q) if one can take $r_1 = \infty$.

7.2. **Interpolation of the Sobolev spaces H^1_p.** Before defining the Sobolev spaces H^1_p it is convenient to recall the following proposition.

Proposition 7.6. *(see* [22] *and* [10] *Theorem 4.38) Let (X, d, μ) be a complete metric-measure space, with μ doubling and satisfying a weak Poincaré inequality (P_q) for some $1 < q < \infty$. Then there exist an integer N, $C \geq 1$ and a linear operator D which associates to each locally Lipschitz function u a measurable function $Du : X \to \mathbb{R}^N$ such that :*

1. *if u is L-Lipschitz, then $|Du| \leq CL$ $\mu - a.e.$;*
2. *if u is locally Lipschitz and constant on a measurable set $E \subset X$, then $Du = 0$ $\mu - a.e.$ on E;*
3. *for locally Lipschitz functions u and v, $D(uv) = uDv + vDu$;*
4. *for each locally Lipschitz function u, $\mathrm{Lip}\, u \leq |Du| \leq C\ \mathrm{Lip}\, u$, and hence $(u, |Du|)$ satisfies the weak Poincaré inequality (P_q) .*

We define now $H^1_p = H^1_p(X, d, \mu)$ for $1 \leq p < \infty$ as the closure of locally Lipschitz functions for the norm

$$\|u\|_{H^1_p} = \|u\|_p + \| \, |Du| \, \|_p \equiv \|u\|_p + \| \, \mathrm{Lip}\, u\|_p.$$

We denote H^1_∞ for the set of all bounded Lipschitz functions on X.

Remark 7.7. *Under the hypotheses of Proposition 7.6, the uniqueness of the gradient holds for every $f \in H^1_p$ with $p \geq q$. By uniqueness of gradient we mean that if u_n is a locally Lipschitz sequence such that $u_n \to 0$ in L_p and $Du_n \to g \in L_p$ then $g = 0$ a.e.. Then D extends to a bounded linear operator from H^1_p to L_p.*

In the remaining part of this section, we consider a complete non-compact metric-measure space (X, d, μ) with μ doubling. We also assume that X admits a Poincaré inequality (P_q) for some $1 < q < \infty$ as defined in Definition 7.5. By [27] Theorem

1.3.4, this is equivalent to say that there exists $C > 0$ such that for all $f \in \mathrm{Lip}$ and for every ball B of X of radius $r > 0$ we have

$$(P_q) \qquad \int_B |f - f_B|^q d\mu \leq C r^q \int_B |\operatorname{Lip} f|^q d\mu.$$

Define $q_0 = \inf \{q \in]1, \infty[: (P_q) \text{ holds } \}$.

Lemma 7.8. *Under these hypotheses, and for $q_0 < p < \infty$, $Lip \cap H^1_p$ is dense in H^1_p.*

Proof. See the proof of Theorem 9 in [17]. □

Proposition 7.9. ***Calderón-Zygmund lemma for Sobolev functions***
Let (X, d, μ) be a complete non-compact metric-measure space with μ doubling, admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then, the Calderón-Zygmund decomposition of Proposition 3.5 still holds in the present situation for $f \in \mathrm{Lip} \cap H^1_p$, $q \leq p < \infty$, replacing ∇f by Df.

Proof. The proof is similar, replacing ∇f by Df, using that D of Proposition 7.6 is linear. Since the χ_i are $\frac{C}{r_i}$ Lipschitz then $\|D\chi_i\|_\infty \leq \frac{C}{r_i}$ by item 1. of Theorem 7.6 and the b_i's are Lipschitz. We can see that g is also Lipschitz. Moreover, using the finite additivity of D and the property 2. of Proposition 7.6, we get the equality $\mu - a.e.$

$$Dg = Df - D(\sum_i b_i) = Df - (\sum_i Db_i).$$

The rest of the proof goes as in Proposition 3.5. □

Theorem 7.10. *Let (X, d, μ) be a complete non-compact metric-measure space with μ doubling, admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then, there exist C_1, C_2 such that for all $f \in H^1_q + H^1_\infty$ and all $t > 0$ we have*

$(*_{\mathrm{met}})$

$$C_1 t^{\frac{1}{q}} \Big(|f|^{q**\frac{1}{q}}(t) + |Df|^{q**\frac{1}{q}}(t) \Big) \leq K(f, t^{\frac{1}{q}}, H^1_q, H^1_\infty) \leq C_2 t^{\frac{1}{q}} \Big(|f|^{q**\frac{1}{q}}(t) + |Df|^{q**\frac{1}{q}}(t) \Big).$$

Proof. We have $(*_{\mathrm{met}})$ for all $f \in \mathrm{Lip} \cap H^1_q$ from the Calderón-Zygmund decomposition that we have done. Now for $f \in H^1_q$, by Lemma 7.8, $f = \lim_n f_n$ in H^1_q, with f_n Lipschitz and $\|f - f_n\|_{H^1_q} < \frac{1}{n}$ for all n. Since for all n, $f_n \in \mathrm{Lip}$, there exist g_n, h_n such that $f_n = h_n + g_n$ and $\|h_n\|_{H^1_q} + t^{\frac{1}{q}} \|g_n\|_{H^1_\infty} \leq C t^{\frac{1}{q}} \Big(|f_n|^{q**\frac{1}{q}}(t) + |Df_n|^{q**\frac{1}{q}}(t) \Big)$. Therefore we find

$$\begin{aligned} \|f - g_n\|_{H^1_q} + t^{\frac{1}{q}} \|g_n\|_{H^1_\infty} &\leq \|f - f_n\|_{H^1_q} + (\|h_n\|_{H^1_q} + t^{\frac{1}{q}} \|g_n\|_{H^1_\infty}) \\ &\leq \frac{1}{n} + C t^{\frac{1}{q}} \Big(|f_n|^{q**\frac{1}{q}}(t) + |Df_n|^{q**\frac{1}{q}}(t) \Big). \end{aligned}$$

Letting $n \to \infty$, since $|f_n|^q \underset{n\to\infty}{\longrightarrow} |f|^q$ in L_1 and $|Df_n|^q \underset{n\to\infty}{\longrightarrow} |Df|^q$ in L_1, it comes $|f_n|^{q**}(t) \underset{n\to\infty}{\longrightarrow} |f|^{q**}(t)$ and $|Df_n|^{q**}(t) \underset{n\to\infty}{\longrightarrow} |Df|^{q**}(t)$ for all $t > 0$. Hence $(*_{\mathrm{met}})$ holds for $f \in H^1_q$. We prove $(*_{\mathrm{met}})$ for $f \in H^1_q + H^1_\infty$ by the same argument of [14]. □

Theorem 7.11 (Interpolation Theorem)**.** *Let (X, d, μ) be a complete non-compact metric-measure space with μ doubling, admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then, for $q_0 < p_1 < p < p_2 \leq \infty$[2], H^1_p is an interpolation space between $H^1_{p_1}$ and $H^1_{p_2}$.*

[2]We allow $p_1 = 1$ if $q_0 = 1$.

Proof. Theorem 7.10 provides us with all the tools needed for interpolating, as we did in the Riemannian case. In particular, we get Theorem 7.11. □

Remark 7.12. *We were not able to get our interpolation result as in the Riemmanian case for $p_1 \leq q_0$. Since we do not have Poincaré inequality (P_{p_1}), the uniqueness of the gradient D does not hold in general in $H^1_{p_1}$.*

Remark 7.13. *Other Sobolev spaces on metric-measure spaces were introduced in the last few years, for instance M^1_p, N^1_p, C^1_p, P^1_p. If X is a complete metric-measure space satisfying (D) and (P_q) for some $1 < q < \infty$, it can be shown that for $q_0 < p \leq \infty$, all the mentioned spaces are equal to H^1_p with equivalent norms (see* [23]*). In conclusion our interpolation result carries over to those Sobolev spaces.*

Remark 7.14. *The purpose of this remark is to extend our results to local assumptions. Assume that (X, d, μ) is a complete metric-measure space, with μ locally doubling, and admitting a local Poincaré inequality (P_{qloc}) for some $1 < q < \infty$. Since X is complete and (X, μ) satisfies a local doubling condition and a local Poincaré inequality (P_{qloc}), then according to an observation of David and Semmes (see the introduction in* [10]*), every ball $B(z, r)$, with $0 < r < \min(r_0, r_1)$, is $\lambda = \lambda(C(r_0), C(r_1))$ quasi-convex, $C(r_0)$ and $C(r_1)$ being the constants appearing in the local doubling property and in the local Poincaré inequality. Then, for $0 < r < \min(r_0, r_1)$, $B(z, r)$ is λ bi-Lipschitz to a length space (one can associate, canonically, to a λ-quasi-convex metric space a length metric space, which is λ-bi-Lipschitz to the original one). Hence, we get a result similar to the one in Theorem 7.10. Indeed, the proof goes as that of Theorem 1.2 in the local case noting that the B^j's considered there are then λ bi-Lipschitz to a length space with λ independent of j. Thus Lemma 3.8 still holds (see Remark 3.9). Therefore, we get the characterization $(*_{met})$ of K and by interpolating, we obtain the correspondance analogue of Theorem 7.11.*

8. Applications

8.1. **Carnot-Carathéodory spaces.** An important application of the theory of Sobolev spaces on metric-measure spaces is to a Carnot-Carathéodory space. We refer to [23] for a survey on the theory of Carnot-Carathéodory spaces.
Let $\Omega \subset \mathbb{R}^n$ be a connected open set, $X = (X_1, ..., X_k)$ a family of vector fields defined on Ω, with real locally Lipschitz continuous coefficients and $|Xu(x)| = \Big(\sum_{j=1}^{k} |X_j u(x)|^2\Big)^{\frac{1}{2}}$.
We equip Ω with the Lebesgue measure $\mathcal{L}^n$ and the Carnot-Carathéodory metric ρ associated to the X_i. We assume that ρ defines a distance. Then, the metric space (Ω, ρ) is a length space.

Definition 8.1. *Let $1 \leq p < \infty$. We define $H^1_{p,X}(\Omega)$ as the completion of locally metric* [3] *Lipschitz functions (equivalently of C^∞ functions) for the norm*

$$\|f\|_{H^1_{p,X}} = \|f\|_{L_p(\Omega)} + \| \, |Xf| \, \|_{L_p(\Omega)}$$

We denote $H^1_{\infty,X}$ for the set of bounded metric Lipschitz function.

Remark 8.2. *For all $1 \leq p \leq \infty$, $H^1_{p,X} = W^1_{p,X}(\Omega) := \{f \in L_p(\Omega) : |Xf| \in L_p(\Omega)\}$, where Xf is defined in the distributional sense (see for example* [19] *Lemma 7.6).*

[3] that is relative to the metric ρ of Carnot-Carathéodory.

Adapting the same method, we obtain the following interpolation theorem for the $H^1_{p,X}$.

Theorem 8.3. *Consider $(\Omega, \rho, \mathcal{L}^n)$ where Ω is a connected open subset of $\mathbb{R}^n$. We assume that $\mathcal{L}^n$ is locally doubling, that the identity map $id : (\Omega, \rho) \to (\Omega, |.|)$ is an homeomorphism. Moreover, we suppose that the space admits a local weak Poincaré inequality (P_{qloc}) for some $1 \leq q < \infty$. Then, for $1 \leq p_1 < p < p_2 \leq \infty$ with $p > q_0$, $H^1_{p,X}$ is a real interpolation space between $H^1_{p_1,X}$ and $H^1_{p_2,X}$.*

8.2. **Weighted Sobolev spaces.** We refer to [24], [29] for the definitions used in this subsection. Let Ω be an open subset of $\mathbb{R}^n$ equipped with the Euclidean distance, $w \in L_{1,loc}(\mathbb{R}^n)$ with $w > 0$, $d\mu = wdx$. We assume that μ is q-admissible for some $1 < q < \infty$ (see [25] for the definition). This is equivalent to say, (see [23]), that μ is doubling and there exists $C > 0$ such that for every ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and for every function $\varphi \in C^\infty(B)$,

$$(P_q) \qquad \int_B |\varphi - \varphi_B|^q d\mu \leq Cr^q \int_B |\nabla \varphi|^q d\mu$$

with $\varphi_B = \frac{1}{\mu(B)} \int_B \varphi d\mu$. The A_q weights, $q > 1$, satisfy these two conditions (see [25], Chapter 15).

Definition 8.4. *For $q \leq p < \infty$, we define the Sobolev space $H^1_p(\Omega, \mu)$ to be the closure of $C^\infty(\Omega)$ for the norm*

$$\|u\|_{H^1_p(\Omega,\mu)} = \|u\|_{L_p(\mu)} + \| \, |\nabla u| \, \|_{L_p(\mu)}.$$

We denote $H^1_\infty(\Omega, \mu)$ for the set of all bounded Lipschitz functions on Ω.

Using our method, we obtain the following interpolation theorem for the Sobolev spaces $H^1_p(\Omega, \mu)$:

Theorem 8.5. *Let Ω be as in above. Then for $q_0 < p_1 < p < p_2 \leq \infty$, $H^1_p(\Omega, \mu)$ is a real interpolation space between $H^1_{p_1}(\Omega, \mu)$ and $H^1_{p_2}(\Omega, \mu)$.*

As in section 7, we were not able to get our interpolation result for $p_1 \leq q_0$ since again in this case the uniqueness of the gradient does not hold for $p_1 \leq q_0$.

Remark 8.6. *Equip Ω with the Carnot-Carathéodory distance associated to a family of vector fields with real locally Lipschitz continuous coefficients instead of the Euclidean distance. Under the same hypotheses used in the beginning of this section, just replacing the balls B by the balls $\tilde{B}$ with respect to ρ, and ∇ by X and assuming that $id : (\Omega, \rho) \to (\Omega, |.|)$ is an homeomorphism, we obtain our interpolation result. As an example we take vectors fields satisfying a Hörmander condition or vectors fields of Grushin type* [16].

8.3. **Lie Groups.** In all this subsection, we consider G a connected unimodular Lie group equipped with a Haar measure $d\mu$ and a family of left invariant vector fields $X_1, ..., X_k$ such that the X_i's satisfy a Hörmander condition. In this case the Carnot-Carathéodory metric ρ is is a distance, and G equipped with the distance ρ is complete and defines the same topology as that of G as a manifold (see [13] page 1148). From the results in [21], [32], it is known that G satisfies (D_{loc}). Moreover, if G has polynomial growth it satisfies (D). From the results in [33], [35], G admits a local Poincaré

inequality (P_{1loc}). If G has polynomial growth, then it admits a global Poincaré inequality (P_1).

Interpolation of non-homogeneous Sobolev spaces. We define the non-homogeneous Sobolev spaces on a Lie group W_p^1 in the same manner as in section 3 on a Riemannian manifold replacing ∇ by X (see Definition 3.1 and Proposition 3.2).

To interpolate the $W_{p_i}^1$, we distinguish between the polynomial and the exponential growth cases. If G has polynomial growth, then we are in the global case. If G has exponential growth, we are in the local case. In the two cases we obtain the following theorem.

Theorem 8.7. *Let G be as above. Then, for all $1 \leq p_1 < p < p_2 \leq \infty$, W_p^1 is an interpolation space between $W_{p_1}^1$ and $W_{p_2}^1$, ($q_0 = 1$ here). Therefore, we get all the interpolation theorems of section 4.*

Interpolation of homogeneous Sobolev spaces. Let G be a connected Lie group as before. We define the homogeneous Sobolev space $\dot{W}_p^1$ in the same manner as in section 5 on Riemannian manifolds replacing ∇ by X.

For these spaces we have the following interpolation theorem.

Theorem 8.8. *Let G be as above and assume that G has polynomial growth. Then for $1 \leq p_1 < p < p_2 \leq \infty$, $\dot{W}_p^1$ is a real interpolation space between $\dot{W}_{p_1}^1$ and $\dot{W}_{p_2}^1$.*

9. Appendix

In view of the hypotheses in the previous interpolation results, a naturel question to address is whether the properties (D) and (P_q) are necessary. The aim of the appendix is to exhibit an example where at least Poincaré is not needed. Consider

$$X = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n;\ x_1^2 + ... + x_{n-1}^2 \leq x_n^2\right\}$$

equipped with the Euclidean metric of $\mathbb{R}^n$ and with the Lebesgue measure. X consists of two infinite closed cones with a common vertex. X satisfies the doubling property and admits (P_q) in the sense of metric-measure spaces if and only if $q > n$ ([23] p.17). Denote by Ω the interior of X. Let $H_p^1(X)$ be the closure of $\mathrm{Lip}_0(X)$ for the norm

$$\|f\|_{H_p^1(X)} = \|f\|_{L_p(\Omega)} + \| \, |\nabla f| \, \|_{L_p(\Omega)}.$$

We define $W_p^1(\Omega)$ as the set of all functions $f \in L_p(\Omega)$ such that $\nabla f \in L_p(\Omega)$ and equip this space with the norm

$$\|f\|_{W_p^1(\Omega)} = \|f\|_{H_p^1(X)}.$$

The gradient is always defined in the distributional sense on Ω.

Using our method, it is easy to check that the $W_p^1(\Omega)$ interpolate for all $1 \leq p \leq \infty$. Also our interpolation result asserts that $H_p^1(X)$ is an interpolation space between $H_{p_1}^1(X)$ and $H_{p_2}^1(X)$ for $1 \leq p_1 < p < p_2 \leq \infty$ with $p > n$. It can be shown that $H_p^1(X) \subsetneq W_p^1(\Omega)$ for $p > n$ and $H_p^1(X) = W_p^1(\Omega)$ for $1 \leq p < n$. Hence $H_p^1(X)$ is an interpolation space between $H_{p_1}^1(X)$ and $H_{p_2}^1(X)$ for $1 \leq p_1 < p < p_2 < n$ although the Poincaré inequality does not hold on X for those p. However, we do not know if

the H^1_p interpolate for all $1 \leq p \leq \infty$ (see [5], Chapter 4 for more details).

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