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LARGEST EIGENVALUES AND EIGENVECTORS OF BAND MATRICES

FLORENT BENAYCH-GEORGES AND SANDRINE PÉCHÉ

ABSTRACT. In this text, we consider an $N \times N$ random matrix \mathbf{X} such that all but $o(N)$ rows of \mathbf{X} have W non identically zero entries (such as, for example, standard or cyclic band matrices). We always suppose that $1 \ll W \ll N$. We first prove that if the entries are independent, centered, have variance one, satisfy a certain tail upper-bound condition and $W \gg (\log N)^{6(1+\alpha)}$, where α is a positive parameter depending on the distribution of the entries, then the largest eigenvalue of \mathbf{X}/\sqrt{W} converges to the upper bound of its limit spectral distribution, that is 2, as for Wigner matrices. This extends some previous results by Khorunzhiy and Sodin [10, 12] where less hypotheses were made on W , but more hypotheses were made about the law of the entries and the structure of the matrix. Then, under the same hypotheses, we prove a delocalization result for the eigenvectors of \mathbf{X} , precisely that most of them cannot be essentially localized on less than $W/\log(N)$ entries. This lower bound on the localization length has to be compared to the upper bound $W^{7/5}$ recently given by Steinerberger in [14].

1. INTRODUCTION

Random band matrices (*i.e.* random Hermitian matrices with independent entries vanishing out of a band around the diagonal) have raised lots of attention recently, mainly due to the fact that they have been suggested as a model for the the so-called Anderson transition, between a strongly disordered regime, with localized eigenfunctions and weak eigenvalue correlation, and a weakly disordered regime, with extended eigenfunctions and strong eigenvalue repulsion. Indeed, such a model is believed to exhibit a phase transition, depending on the width W of the band. It is conjectured (and explained on a Physics level of rigor by Fyodorov and Mirlin in [9]) that for Gaussian band matrices, the *localization length* (*i.e.* the typical number of coordinates bearing most of the ℓ^2 mass) of a typical eigenvector in the bulk of the spectrum shall be of order $L \sim \min\{N, W^2\}$, so that eigenvectors of the bulk should be localized (resp. extended) if $W \ll \sqrt{N}$ (resp. $\gg \sqrt{N}$). The only rigorous result in the direction of localization is by Schenker [11]. Therein it is proved that $L \ll W^8$ for Gaussian band matrices. On the other hand, delocalization in the bulk is proved by Erdős, Knowles, Yau and Yin [7] when $W \gg N^{4/5}$. In both regimes, it is known from Erdős and Knowles [5, 6] that typically (*i.e.* disregarding a negligible proportion of eigenvectors) $L \gg \min\{W^{7/6}, N\}$ for a certain class of random band matrices (with

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sub-exponential tails and symmetric distribution). We refer the reader to Spencer [13] and Erdős, Schlein and Yau [8] for a more detailed discussion on the localized/delocalized regime.

Regarding the edges of the spectrum, little is known about the behavior of the extreme eigenvalues and the typical localization length of the associated eigenvectors. As far as the limit of the largest eigenvalue is concerned, Khorunzhiy proved in [10] that for Gaussian band matrices, if $(\log N)^{3/2} \ll W \ll N$, then the extreme eigenvalues converge to the bounds of the support of the limiting spectral measure (which is the semicircle law). For matrices with cyclic band structure and Bernoulli entries, Sodin extended this result to the case where $\log N \ll W \ll N$ in [12], where he proved important results about the fluctuations of the extreme eigenvalues around their limits. Concerning the localization length L of the eigenvectors associated to the extreme eigenvalues, Sodin's statement [12] combined with Erdős-Knowles-Yau-Yin's results [7] suggest that we should have $L \sim N$ as soon as $W \gg N^{5/6}$. Moreover, Steinerberger proved recently in [14] that for matrices with Bernoulli entries and cyclic band structure, $L \ll W^{7/5}$. Let us also mention that in the quite different framework of band matrices with heavy tailed entries, a transition between the localized and the delocalized regime at the edge was proved by the authors of the present text in [3].

In this text, we consider an random $N \times N$ Hermitian matrix \mathbf{X} such that all but $o(N)$ rows of \mathbf{X} have W non identically zero entries (such as, for example, standard or cyclic band matrices). We always suppose that $1 \ll W \ll N$. We first prove that if the entries are independent, centered, have variance one, satisfy a certain tail upper-bound condition and $W \gg (\log N)^{6(1+\alpha)}$, where α is a positive parameter depending on the distribution of the entries, then the largest eigenvalue of \mathbf{X}/\sqrt{W} converges to the upper bound of its limit spectral distribution, that is 2, as for Wigner matrices. This extends the above mentioned results by Khorunzhiy and Sodin [10, 12] where less hypotheses were made on W , but more hypotheses were made about the law of the entries (they use in a crucial way the fact that the entries are symmetrically distributed) and about the structure of the matrix (in our result, we only need that most rows have W non zero entries, no matter what position the entries have on the matrix). Then, under some close hypotheses, we prove a delocalization result for the eigenvectors of \mathbf{X} , precisely that most of them cannot be essentially localized on less than $W/\log(N)$ entries. This lower bound on the localization length has to be compared to the upper bound $W^{7/5}$ recently given by Steinerberger in [14].

The paper is organized as follows: our main results are stated in the next section, Theorem 2.4 is proved in Section 3, Theorem 2.9 is proved in Section 4, and some technical results needed here are proved in Section 5 and in the appendix.

Notation. Here, $A \ll B$ means that $A/B \rightarrow 0$ as $N \rightarrow \infty$. $\|\mathbf{X}\|$ denotes the spectral radius of the Hermitian matrix \mathbf{X} and $\lambda_{\max}(\mathbf{X})$ denotes its largest eigenvalue. At last, $\mathbf{a} \cdot \mathbf{b}$ denotes the standard scalar product of two complex vectors and $|\cdot|$ denotes the associated norm.

2. MAIN RESULTS

Let us first introduce some hypotheses needed below.

Hypothesis 2.1. *The matrix $\mathbf{X} = (X_{ij})$ is an $N \times N$ Hermitian random matrix (depending implicitly on N) with independent entries (up to the fact that the matrix is Hermitian).*

Hypothesis 2.2. *There is $W = W(N)$ such that*

$$(1) \quad 1 \ll W \ll N$$

and such that on each row of \mathbf{X} , the number of non identically zero entries is $\leq W$, with equality on all but $o(N)$ rows. All non identically zero entries of \mathbf{X} are centered with variance one. Moreover, there exist constants $C \in [0, +\infty)$ and $\alpha \in [0, +\infty)$ such that for all $k \geq 2$,

$$(2) \quad \mathbb{E}[|X_{ij}|^k] \leq (Ck)^{\alpha k},$$

uniformly on N, i, j .

Then the following theorem has been proved under some slightly different hypotheses in [4], but can easily be reproved using a standard moment method as in [1, 2].

Theorem 2.3. *Under Hypotheses 2.1 and 2.2, as $N \rightarrow \infty$, the empirical spectral law of $\frac{\mathbf{X}}{\sqrt{W}}$ converges weakly in probability towards the law $\frac{1}{2\pi} \sqrt{4 - x^2} dx$, with support $[-2, 2]$.*

Our first result is the following one.

Theorem 2.4. *Under Hypotheses 2.1 and 2.2, suppose that W satisfies*

$$(3) \quad W \gg (\log N)^{6(1+\alpha)},$$

with α the constant of (2). Then as $N \rightarrow \infty$, we have the convergence in probability

$$(4) \quad \frac{\lambda_{\max}(\mathbf{X})}{\sqrt{W}} \rightarrow 2.$$

Remark 2.5. This theorem extends some results of [10] and [12]. In these papers, the convergence of (4) is proved under the respective hypotheses $W \gg (\log N)^{3/2}$ and $W \gg \log N$, but for some particular models of matrices: in [10] the matrices considered are Gaussian and in [12], they have Bernoulli distributed entries. Both make a crucial use of the fact that the entries are symmetrically distributed. Moreover, in both papers, the authors also suppose and rely heavily on a particular position of the non zero entries of \mathbf{X} . We do not make such a hypothesis here.

To state our main result, a lower bound on the localization length of eigenvectors of \mathbf{X} , we slightly modify the hypotheses.

Let \mathbf{X} be an $N \times N$ Hermitian random matrix satisfying Hypothese 2.1. We make the following two assumptions.

Hypothesis 2.6. For a certain sequence $W = W(N) \gg 1$, we have the convergence in probability $\frac{\|\mathbf{X}\|}{\sqrt{W}} \rightarrow 2$ as $N \rightarrow \infty$.

For example, Hypothesis 2.6 is satisfied if \mathbf{X} satisfies the hypotheses of Theorem 2.4 or those the papers [10] and [12] (see Remark 2.5 above). However we emphasize that Hypothesis 2.6, focused on the extreme eigenvalues, does not make (at least directly) any assumption on the maximal number of non zero entries per row of \mathbf{X} (it may be N), neither on the relative growth of W with respect to N .

We also reinforce the assumption on the tail of the distribution of the entries. Let $\delta > 0$ and $K < \infty$ be fixed.

Hypothesis 2.7. The entries X_{ij} of \mathbf{X} belong to the set $\mathcal{E}_{\delta,K}$ of complex random variables Y such that

$$\mathbb{E}Y = 0, \quad \mathbb{E}e^{\delta|Y|^2} \leq K.$$

The following theorem is the main result of this text.

We use the following Definition 7.1 from Erdős, Schlein and Yau [8]: for L a positive integer and $\eta > 0$, a unit vector $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{C}^N$ is said to be (L, η) -localized if there exists a set $S \subset \{1, \dots, N\}$ such that $|S| = L$ and $\sum_{j \in S^c} |v_j|^2 \leq \eta$.

Remark 2.8. The largest L and η are, the strongest the statement “there is no (L, η) -localized eigenvector” is.

Theorem 2.9. We suppose Hypotheses 2.1, 2.6 and 2.7. Fix $\eta \in (0, 1/2)$ and choose $L = L(N)$ such that

$$(5) \quad L \ll \frac{W}{\log N}.$$

Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of \mathbf{X} and let $\mathbf{v}_1, \dots, \mathbf{v}_N$ be some associated normalized eigenvectors. Then for any κ such that $\sqrt{\eta/(1-\eta)} < \kappa < 1$,

$$\mathbb{P}(\exists i, |\lambda_i| \geq 2\kappa\sqrt{W} \text{ and } \mathbf{v}_i \text{ is } (L, \eta)\text{-localized}) \xrightarrow{N \rightarrow \infty} 0.$$

Remark 2.10. The same proof can also lead to a version of this theorem where $\eta = \eta(N) \rightarrow 0$. In this case, $\kappa = \kappa(N)$ is allowed to tend to zero, thus the theorem allows to lower bound the localization length of most eigenvectors of \mathbf{X} .

3. PROOF OF THEOREM 2.4

The proof goes along the same lines as the proof of Theorem 2.1.22 of [1]. First note that by Theorem 2.3, we already know that for any $\eta > 0$, $\mathbb{P}(\lambda_{\max}(\mathbf{X}) < (2 - \eta)\sqrt{W}) \rightarrow 0$.

For any $\eta > 0$, for any $k \geq 1$,

$$\mathbb{P}(\lambda_{\max}(\mathbf{X}) > (2 + \eta)\sqrt{W}) \leq \mathbb{P}(\text{Tr } \mathbf{X}^{2k} \geq (2 + \eta)^{2k} W^k) \leq W^{-k} (2 + \eta)^{-2k} \mathbb{E} \text{Tr } \mathbf{X}^{2k},$$

hence it suffices to find a sequence $k = k(N)$ such that for any $\eta > 0$,

$$(6) \quad \mathbb{E} \operatorname{Tr} \mathbf{X}^{2k} \ll W^k (2 + \eta)^{2k}.$$

We have

$$\mathbb{E} \operatorname{Tr} \mathbf{X}^{2k} = \sum \mathbb{E} X_{i_1 i_2} \cdots X_{i_{2k} i_1},$$

where the sum is over collections $\mathbf{i} = (i_1, \dots, i_{2k})$ such that for all ℓ , $i_\ell \in \{1, \dots, N\}$. For each \mathbf{i} , let $G_{\mathbf{i}}$ be the simple, non oriented graph with vertex set $\{i_1, \dots, i_{2k}\}$ and edges $\{i_\ell, i_{\ell+1}\}$ ($1 \leq \ell \leq 2k$, with the convention $i_{2k+1} = i_1$). For the expectation in the RHT above to be non zero, we need all edges to be visited at least twice by the path \mathbf{i} (because the X_{ij} 's are centered) and the edges $\{i_\ell, i_{\ell+1}\}$ to be such that $X_{i_\ell, i_{\ell+1}}$ is non identically zero. The symmetric group S_N acts on the set of \mathbf{i} 's by $\sigma \cdot (i_1, \dots, i_{2k}) := (\sigma(i_1), \dots, \sigma(i_{2k}))$. Following Section 2.1.3 of [1], we denote by $\mathcal{W}_{2k,t}$ the set of equivalence classes, under the action of S_N , of \mathbf{i} 's such that all edges of $G_{\mathbf{i}}$ are visited at least twice by the path \mathbf{i} and $G_{\mathbf{i}}$ has exactly t vertices (this set is actually stable under this action).

Note that for $\mathcal{W}_{2k,t}$ to be non empty, we need to have $t \leq k + 1$. Indeed, $G_{\mathbf{i}}$ is always connected hence its number of vertices is at most its number of edges plus one.

Note that for any $w \in \mathcal{W}_{2k,t}$, the number of \mathbf{i} 's in the class w is at most NW^{t-1} .

It follows from the previous remarks that

$$\mathbb{E} \operatorname{Tr} \mathbf{X}^{2k} \leq N \sum_{t=1}^{k+1} W^{t-1} \sum_{w \in \mathcal{W}_{2k,t}} \max_{\mathbf{i} \in w} \mathbb{E} X_{i_1 i_2} \cdots X_{i_{2k-1} i_1}.$$

Now, let us fix $t \in \{1, \dots, k + 1\}$, $w \in \mathcal{W}_{2k,t}$ and $\mathbf{i} \in w$. Let us denote by l (resp. m) the number of edges of $G_{\mathbf{i}}$ visited exactly twice (resp. at least three times). Obviously, $2l + 3m \leq 2k$. Moreover, as $l + m$ is the number of edges of the $G_{\mathbf{i}}$, hence by connectedness of $G_{\mathbf{i}}$ again, we have $t \leq l + m + 1$. So

$$6t \leq 6m + 6l + 6 = 2(3m + 2l) + 2l + 6 \leq 4k + 2l + 6,$$

so

$$2k - 2l \leq 6(k - t + 1).$$

Now, notice that as the X_{ij} 's have variance one, $\mathbb{E} X_{i_1 i_2} \cdots X_{i_{2k} i_1}$ can be reduced to the expectation of a product of $2k - 2l$ X_{ij} 's, hence by (2) and Hölder's inequality,

$$\mathbb{E} X_{i_1 i_2} \cdots X_{i_{2k} i_1} \leq \left(C(2k - 2l) \right)^{\alpha(2k - 2l)} \leq \{6C(k - t + 1)\}^{6\alpha(k - t + 1)}.$$

As a consequence,

$$\mathbb{E} \operatorname{Tr} \mathbf{X}^{2k} \leq N \sum_{t=1}^{k+1} W^{t-1} \#\mathcal{W}_{2k,t} \times \{6C(k - t + 1)\}^{6\alpha(k - t + 1)}.$$

Now, we shall use Lemma 2.1.23 of [1], which states that $\#\mathcal{W}_{2k,t} \leq 4^k (2k)^{6(k-t+1)}$ as soon as $t \leq k+1$ (the case $t = k+1$ is technically not contained in Lemma 2.1.23 of [1], but follows from Equation (2.1.20) and Lemma 2.1.3 of the same book). It follows that

$$\begin{aligned} \mathbb{E} \operatorname{Tr} \mathbf{X}^{2k} &\leq N 4^k \sum_{t=1}^{k+1} W^{t-1} (2k)^{6(k-t+1)} \{6C(k-t+1)\}^{6\alpha(k-t+1)} \\ &= N W^k 4^k \sum_{i=0}^k W^{-i} (2k)^{6i} (6Ci)^{6\alpha i} \\ &\leq N W^k 4^k \left(1 - \frac{(2k(6Ck)^\alpha)^6}{W} \right)^{-1}, \end{aligned}$$

where the last inequality is true as soon as $W > (2k(6Ck)^\alpha)^6$. Then it is easy to see that (6) holds for $k = k(N)$ such that $\log N \ll k \ll W^{\frac{1}{6(1+\alpha)}}$.

4. PROOF OF THEOREM 2.9

Before proving Theorem 2.9, we shall need the following theorem and its corollary. The proof of Theorem 4.1 is postponed to Section 5.

Theorem 4.1. *Under Hypotheses 2.1 and 2.7, there are constants $c_2 = c_2(\delta, K) > 0$ and $C_2 = C_2(\delta, K) < \infty$ independent of all the other parameters such that for all $t > 0$,*

$$(7) \quad \mathbb{P}(\|\mathbf{X}\| > t\sqrt{N}) \leq e^{-c_2(t^2 - C_2)N}.$$

Let us denote by $\rho(\mathbf{X})$ the spectral radius of \mathbf{X} and, for $L \geq 1$, by $\rho_L(\mathbf{X})$ the maximum spectral radius of its $L \times L$ principal submatrices (a *principal* submatrix is a submatrix chosen by extracting a certain set of columns and *the same* set of rows, but this set does not need to be an interval).

Corollary 4.2. *Under Hypotheses 2.1 and 2.7, there exists $t < \infty$ and $c_3 > 0$ such that*

$$(8) \quad \mathbb{P}(\rho_L(\mathbf{X}) \geq t\sqrt{L \log N}) \leq e^{-c_3 L \log N}.$$

Proof. The number of ways to choose an $L \times L$ principal submatrix is $\leq N^L = e^{L \log N}$. For each submatrix \mathbf{S} , $\mathbb{P}(\rho(\mathbf{S}) \geq t\sqrt{N \log N}) \leq \exp\{-c_2(t^2 \log N - C_2)L\}$, hence by the union bound,

$$\mathbb{P}(\rho_L(\mathbf{X}) \geq t\sqrt{L \log N}) \leq \exp\{[-c_2(t^2 \log N - C_2) + \log N]L\},$$

thus if $c_2 t^2 > 1$, then (8) holds for a certain $c_3 > 0$. \square

To prove Theorem 2.9, we shall also need the following lemma (see Lemma 4.2 in [3]).

Lemma 4.3. *For all i , if \mathbf{v}_i is (L, η) -localized, then $|\lambda_i| \leq \frac{\rho_L(\mathbf{X}) + \sqrt{\eta} \rho(\mathbf{X})}{\sqrt{1-\eta}}$.*

Let us now prove Theorem 2.9.

Proof. Let us choose $\varepsilon > 0$ such that $\sqrt{\eta/(1-\eta)}(1+\varepsilon) < \kappa$ and set

$$\delta := \kappa - \sqrt{\eta/(1-\eta)}(1+\varepsilon).$$

We know, by (2.6), that with probability tending to 1, $\rho(\mathbf{X}) \leq (1+\varepsilon)2\sqrt{W}$, *i.e.*

$$\frac{\sqrt{\eta}\rho(\mathbf{X})}{\sqrt{1-\eta}} \leq (\kappa - \delta)2\sqrt{W}.$$

Moreover, by Corollary 4.2, there is $t < \infty$ such that with probability tending to one, $\rho_L(\mathbf{X}) \leq t\sqrt{L \log N}$. But by (5), for N large enough,

$$\frac{t\sqrt{L \log N}}{\sqrt{1-\eta}} \leq 2\delta\sqrt{W},$$

so the theorem is proved. \square

5. PROOF OF THEOREM 4.1

5.1. A preliminary lemma.

Lemma 5.1. *Under Hypotheses 2.1 and 2.7, there are constants c_1, C_1 depending only on δ, K such that for any $\mathbf{z} \in \mathbb{C}^N$ with $|\mathbf{z}| \leq 1$,*

$$(9) \quad \mathbb{P}(\mathbf{z}^* \mathbf{X}^* \mathbf{X} \mathbf{z} \geq Nt) \leq e^{-c_1(t-C_1)N}.$$

Remark 5.2. If \mathbf{X} is a random $N \times N$ matrix whose maximum number of non identically zero entries per row is W (like for a *band matrix* with band width W), then (9) remains true with N replaced by W everywhere (for some constants still depending only on δ and K).

Proof. We denote by $\mathbf{X}_1, \dots, \mathbf{X}_N$ the columns of \mathbf{X} . We have, for any τ, C as in Lemma 6.2 of the appendix,

$$\mathbb{E}e^{\tau^2 \mathbf{z}^* \mathbf{X}^* \mathbf{X} \mathbf{z}} = \mathbb{E}e^{\tau^2 |\mathbf{X} \mathbf{z}|^2} = \mathbb{E}e^{\tau^2 \sum_j |\mathbf{X}_j \cdot \mathbf{z}|^2} \leq e^{NC\tau^2}$$

Hence

$$\mathbb{P}(\mathbf{z}^* \mathbf{X}^* \mathbf{X} \mathbf{z} \geq Nt) \leq \mathbb{E}e^{\tau^2 \mathbf{z}^* \mathbf{X}^* \mathbf{X} \mathbf{z}} e^{-\tau^2 Nt} \leq e^{-\tau^2(t-C)N}.$$

So the lemma is proved. \square

5.2. Proof of Theorem 4.1. The following lemma is well known.

Lemma 5.3. *Let $N \geq 1$. For any fixed $0 < \varepsilon < 1/4$, there exists a family $(\mathbf{z}_i)_{i \in I}$ of elements of the unit ball of \mathbb{C}^N such that $|I| \leq (2/\varepsilon)^{2N}$ and any element of the unit sphere of \mathbb{C}^N is within a distance at most ε of one of the \mathbf{z}_i 's. Moreover, for any positive $N \times N$ Hermitian matrix P ,*

$$\lambda_{\max}(P) \leq \frac{\max_i \mathbf{z}_i^* P \mathbf{z}_i}{1 - 2\varepsilon}.$$

Let us now prove Theorem 4.1.

Proof. By Lemma 5.1, we know that there are constants there are constants c_1, C_1 depending only on δ, K such that for any $\mathbf{z} \in \mathbb{C}^N$ with $\|\mathbf{z}\| \leq 1$,

$$\mathbb{P}(\mathbf{z}^* \mathbf{X}^* \mathbf{X} \mathbf{z} \geq Nt) \leq e^{-c_1(t-C_1)N}.$$

Now, by Lemma 5.3, we have

$$\begin{aligned} \mathbb{P}(\lambda_{\max}(\mathbf{X}^* \mathbf{X}) \geq Nt) &\leq \mathbb{P}(\max_i \mathbf{z}_i^* \mathbf{X}^* \mathbf{X} \mathbf{z}_i \geq Nt(1-2\varepsilon)) \\ &\leq (2/\varepsilon)^{2N} e^{-c_1(t(1-2\varepsilon)-C_1)N} \\ &= e^{(-c_1(1-2\varepsilon)(t-\frac{C_1}{1-2\varepsilon})+2\log(2/\varepsilon))N} \\ &= e^{-c_2(t-C_2)N} \end{aligned}$$

As a consequence,

$$\mathbb{P}(\|\mathbf{X}\| > t\sqrt{N}) = \mathbb{P}(\lambda_{\max}(\mathbf{X}^* \mathbf{X}) \geq t^2 N) \leq e^{-c_2(t^2-C_2)N}.$$

□

6. APPENDIX: TECHNICAL RESULTS

Lemma 6.1. *For any real centered random variable Y and any $r \in \mathbb{R}$, we have*

$$\mathbb{E}e^{rY} \leq 1 + 3\mathbb{E}[e^{\delta Y^2}](e^{r^2/\delta} - 1) \leq e^{3r^2\mathbb{E}[e^{\delta Y^2}]/\delta}$$

for any $\delta > 0$.

Proof. The second inequality follows from the fact that for any $u \geq 1$, we have the inequality $1 + u(e^{r^2/\delta} - 1) \leq e^{r^2 u/\delta}$ (this is obvious with the series expansion of \exp).

So let us prove the first inequality. Note that up to a replacement of Y by rY and of δ by δ/r^2 , we shall suppose that $r = 1$.

The case where $\mathbb{E}[e^{\delta Y^2}] = \infty$ is obvious, hence we focus on the other case, which allows to expand all sums with the moments of Y .

Claim : for all $x \geq 0$, $e^x \leq 1 + x + 3\frac{e^x + e^{-x} - 2}{2}$. Indeed, both terms are equal for $x = 0$ and the derivative of $2\text{RHT} - 2\text{LHT}$ is $e^x - 3e^{-x} + 2$, which is increasing, hence has the same sign as x .

It follows that

$$\mathbb{E}e^Y \leq 1 + 3 \sum_{n \geq 1} \frac{\mathbb{E}Y^{2n}}{(2n)!} \leq 1 + 3 \sum_{n \geq 1} \frac{n! \delta^{-n} \mathbb{E}e^{\delta Y^2}}{(2n)!} \leq 1 + 3\mathbb{E}[e^{\delta Y^2}](e^{1/\delta} - 1),$$

where we used first $\mathbb{E}Y = 0$, then $\frac{\delta^n \mathbb{E}Y^{2n}}{n!} \leq \mathbb{E}e^{\delta Y^2}$ and at last $\frac{n!}{(2n)!} \leq \frac{1}{n!}$. □

Lemma 6.2. *Let us fix $\delta, K > 0$. Then there is $\tau = \tau(\delta, K) > 0$ and $C = C(\delta, K) > 0$ such that for all $N \geq 1$, all $\mathbf{z} \in \mathbb{C}^N$ such that $|\mathbf{z}| \leq \tau$, for any \mathbf{Y} random vector taking values in \mathbb{C}^N with independent components in the set $\mathcal{E}_{\delta, K}$ defined at Hypothesis 2.7,*

$$\mathbb{E}e^{|\mathbf{Y} \cdot \mathbf{z}|^2} \leq e^{C|\mathbf{z}|^2}.$$

Proof. First step: Let us first prove the result for \mathbf{Y} having independent components in

$$\mathcal{E}_{\delta, K}^{\mathbb{R}} := \{Y \in \mathcal{E}_{\delta, K}; Y \text{ is real-valued}\}$$

and $\mathbf{z} \in \mathbb{R}^N$. Let $\tau_{\mathbb{R}} > 0$ be such that for any $t \in [0, \tau_{\mathbb{R}})$, we have

$$12t^2K/\delta < 1 \quad \text{and} \quad (1 - 12t^2K/\delta)^{-1/2} \leq e^{12t^2K/\delta}.$$

Let g be a standard real Gaussian variable, independent of the other variables, let \mathbb{E}_g denote the expectation with respect to g and let \mathbb{E} denote the expectation with respect to all other variables than g .

For any $\tau > 0$ and $\mathbf{z} \in \mathbb{R}^N$ such that $|\mathbf{z}| \leq \tau$, using the formula $e^{x^2} = \mathbb{E}_g e^{\sqrt{2}xg}$, we have

$$\mathbb{E}e^{|\mathbf{Y} \cdot \mathbf{z}|^2} = \mathbb{E}\mathbb{E}_g e^{\sqrt{2}g\mathbf{Y} \cdot \mathbf{z}} = \mathbb{E}_g \mathbb{E} e^{\sqrt{2}g\mathbf{Y} \cdot \mathbf{z}} = \mathbb{E}_g \prod_i \mathbb{E} e^{\sqrt{2}gY_i z_i}.$$

Hence by Lemma 6.1,

$$\mathbb{E}e^{|\mathbf{Y} \cdot \mathbf{z}|^2} \leq \mathbb{E}_g \prod_i e^{6g^2 z_i^2 K/\delta} = \mathbb{E}_g e^{6g^2 |\mathbf{z}|^2 K/\delta} = (1 - 12|\mathbf{z}|^2 K/\delta)^{-1/2} \leq e^{12|\mathbf{z}|^2 K/\delta}.$$

Second step: Let us now extend the result to the complex case. For $Y \in \mathcal{E}_{\delta, K}$ complex valued, we have $Y = Y_1 + iY_2$ with $Y_1, Y_2 \in \mathcal{E}_{\delta, K}^{\mathbb{R}}$ and $\mathbf{z} \in \mathbb{C}^N$, we have $\mathbf{z} = \mathbf{z}_1 + i\mathbf{z}_2$ with $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^N$ such that $|\mathbf{z}_1|, |\mathbf{z}_2| \leq |\mathbf{z}|$. Hence

$$\mathbf{Y} \cdot \mathbf{z} = (\mathbf{Y}_1 \cdot \mathbf{z}_1 + \mathbf{Y}_2 \cdot \mathbf{z}_2) + i(\mathbf{Y}_1 \cdot \mathbf{z}_2 - \mathbf{Y}_2 \cdot \mathbf{z}_1),$$

so, using the fact that for any $a, b \in \mathbb{R}$, $(a + b)^2 \leq 2a^2 + 2b^2$, we have

$$|\mathbf{Y} \cdot \mathbf{z}|^2 \leq 2(\mathbf{Y}_1 \cdot \mathbf{z}_1)^2 + 2(\mathbf{Y}_2 \cdot \mathbf{z}_2)^2 + 2(\mathbf{Y}_1 \cdot \mathbf{z}_2)^2 + 2(\mathbf{Y}_2 \cdot \mathbf{z}_1)^2.$$

It follows that if $\tau_{\mathbb{R}}$ and $C_{\mathbb{R}}$ convenient constants for the real case, we have, for any $\mathbf{z} \in \mathbb{C}^N$ such that $|\mathbf{z}| \leq \frac{\tau_{\mathbb{R}}}{\sqrt{8}}$,

$$\begin{aligned} \mathbb{E}e^{|\mathbf{Y} \cdot \mathbf{z}|^2} &\leq \mathbb{E}e^{2\{(\mathbf{Y}_1 \cdot \mathbf{z}_1)^2 + (\mathbf{Y}_2 \cdot \mathbf{z}_2)^2 + (\mathbf{Y}_1 \cdot \mathbf{z}_2)^2 + (\mathbf{Y}_2 \cdot \mathbf{z}_1)^2\}} \\ &\leq \left(\mathbb{E}e^{8(\mathbf{Y}_1 \cdot \mathbf{z}_1)^2} \mathbb{E}e^{8(\mathbf{Y}_2 \cdot \mathbf{z}_2)^2} \mathbb{E}e^{8(\mathbf{Y}_1 \cdot \mathbf{z}_2)^2} \mathbb{E}e^{8(\mathbf{Y}_2 \cdot \mathbf{z}_1)^2} \right)^{\frac{1}{4}} \\ &\leq e^{\frac{8C_{\mathbb{R}}}{4}(|\mathbf{z}_1|^2 + |\mathbf{z}_2|^2 + |\mathbf{z}_1|^2 + |\mathbf{z}_2|^2)} = e^{4C_{\mathbb{R}}|\mathbf{z}|^2}, \end{aligned}$$

so that the constants $\tau = \frac{\tau_{\mathbb{R}}}{\sqrt{8}}$ and $C = 4C_{\mathbb{R}}$ are convenient in the general case. \square

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MAP 5, UMR CNRS 8145 - UNIVERSITÉ PARIS DESCARTES, 45 RUE DES SAINTS-PÈRES 75270 PARIS CEDEX 6, FRANCE.

E-mail address: florent.benaych-georges@parisdescartes.fr

LPMA, UNIVERSITÉ PARIS DIDEROT, 5 RUE THOMAS MANN 75013 PARIS.

E-mail address: sandrine.peche@math.univ-paris-diderot.fr