Regulators of rank one quadratic twists

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REGULATORS OF RANK ONE QUADRATIC TWISTS

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Abstract. We investigate the regulators of elliptic curves with rank 1 in some families of quadratic twists of a fixed elliptic curve. In particular, we formulate some conjectures on the average size of these regulators. We also describe an efficient algorithm to compute explicitly some of the invariants of a rank one quadratic twist of an elliptic curve (regulator, order of the Tate-Shafarevich group, etc.) and we discuss the numerical data that we obtain and compare it with our predictions.

1. Introduction and notations

We study the regulators of elliptic curves of rank 1 in a family of quadratic twists of a fixed elliptic curve E defined over $\mathbb{Q}$. Methods coming from Random Matrix Theory, as developed in [K-S], [CKRS], [CFKRS], etc., allow us to derive precise conjectures for the moments of those regulators. Our hope is that these moments will help to make predictions for the number of curves with extra-rank (i.e. the number of even quadratic twists[1] with a Mordell-Weil rank ≥ 2, or the number of odd quadratic twists with Mordell-Weil rank ≥ 3). Then, we describe an efficient method, using the Heegner-point construction, for computing the regulator (and the order of the Tate-Shafarevich group) of an elliptic curve of rank 1 in a family of quadratic twists. Finally, we discuss and compare our extensive numerical data (for some families of odd quadratic twists of the curves $11a1$, $14a1$, $15a1$ and $17a1$) with our predictions.

From a numerical and experimental point of view, the situation of odd quadratic twists really differs from the one of even quadratic twists. Indeed, in the latter case, for each curve E_d in a family $(E_d)_d$ of even quadratic twists of a fixed elliptic curve E, one has to compute the special value $L(E_d, 1)$ of its L-function at $s = 1$ and determine if it is zero or not. If $L(E_d, 1) = 0$ then the curve E_d has extra-rank. Otherwise the curve has rank 0, the regulator is simply 1, and the Birch and Swinnerton-Dyer conjecture allows us to deduce the value of $|Ш(E_d)|$ from that of $L(E_d, 1)$. The computation of $L(E_d, 1)$ is done via a Waldspurger's formula which, roughly speaking, states that $L(E_d, 1)$ is, up to a fudge factor, the square of the $|d|$-th coefficient of a weight $3/2$ modular form given by an explicit linear combination of theta series. It follows that, in this case, computations are possible for very large families of quadratic twists (see for example [Rub], [Qua], etc.). Note that the numerical data coming from these computations are in close agreement with the well-known conjectures of [CKRS] about extra-vanishing (coming from

[1]An odd (resp. even) quadratic twist of E is a quadratic twist such that the sign of the functional equation of its L-function is -1 (resp. $+1$). By the Birch and Swinnerton-Dyer conjecture this is equivalent to say that its Mordell-Weil rank is odd (resp. even)

the models of Random Matrix Theory), or on the behavior of the Tate-Shafarevich groups $Ш(E_d)$ of E_d (see [Qua], [De1]).

In the rank 1 case, numerical investigation appears to be much more complicated and, as far as we know, has never been done before. In that case, we first have to compute the value of the derivative $L'(E_d, 1)$ for each curve E_d in the family of odd quadratic twists. However, there is no Waldspurger's formula to compute this value directly, and furthermore from this value one can only deduce (assuming it is non-zero and under the Birch and Swinnerton-Dyer conjecture) the value of the product $R(E_d)\,|Ш(E_d)|$ where $R(E_d)$ is the regulator of E_d. Thus we also need to be able to evaluate at least one of the two terms of this product.[2] The only (known) efficient way to do this is to write down a generator G_d of $E_d(\mathbb{Q})$ and to compute $R(E_d) = \hat{h}(G_d)$ where $\hat{h}$ is the canonical height[3] of E_d.

The method we used in this paper is to first adapt the Heegner-point construction to our situation in order to construct a generator G_d and then replace the Waldspurger's formula by the formula of Gross and Zagier. This allows us to compute directly the regulator $R(E_d)$ *and at the same time* the order of the Tate-Shafarevich group $|Ш(E_d)|$ (assuming the Birch and Swinnerton-Dyer conjecture).

Hypothesis. From now on, we assume the truth of the Birch and Swinnerton-Dyer conjecture.

We now give some notations. Fix an elliptic curve E defined over $\mathbb{Q}$ and let N be its conductor. The L-function of E is

$$L(E, s) = \sum_{n \geq 1} a(n) n^{-s}\ , \quad \Re(s) > 3/2$$

It is now a classical and deep result that $L(E, s)$ can be analytically continued to the whole complex plane and satisfies a functional equation:

$$\Lambda(E, s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(E, s) = w\Lambda(E, 2 - s)$$

where $w = \pm 1$ gives the parity of the order of vanishing of $L(E, s)$ at $s = 1$. Let d be a fundamental discriminant. We denote by E_d the quadratic twist of E by d. The curves E and E_d are isomorphic over the quadratic field $\mathbb{Q}(\sqrt{d})$ but not over $\mathbb{Q}$. We denote by ψ_d (ψ if d is clear in the context) the isomorphism between E and E_d defined in the following way. Assume that the curves E and E_d are given by:

$$\begin{aligned} E &: \quad y^2 = x^3 + Ax^2 + Bx + c \\ E_d &: \quad y^2 = x^3 + Adx + Bd^2x + Cd^3 \end{aligned}$$

[2] For some families of elliptic curves $(F_j)_j$, there exists a generic point in the Mordell-Weil group $F_j(\mathbb{Q})$, thus one can separate the terms in this product and a direct investigation is possible (see [De-Du]). However, such families for which we know in advance the regulator are very special and in particular are not quadratic families, although we must say that it is possible to get sometimes a generic point for some very specific and tiny sub-family of quadratic twists.

[3] This equality fixes once and for all our choice of the canonical height. Note that this height is *twice* the height in Silverman's book [Sil] or in Krir's paper [Kri] so this explains the difference of a factor 2 between the formulae in this paper and theirs.

then ψ_d is:

$$\begin{array}{rccc} \psi_d & : & E & \xrightarrow{\sim} & E_d \\ & & (x,y) & \longmapsto & (dx, d^{3/2}y) \end{array}$$

The non-trivial automorphism $x \mapsto \bar{x}$ of $\mathbb{Q}(\sqrt{d})$, which is the restriction of the complex conjugation if $d < 0$, acts by:

$$\psi_d(\overline{P}) = -\overline{\psi_d(P)} \tag{1.1}$$

Whenever d and N are coprime (and this will always be the case in our families), the conductor of E_d is Nd^2 and we have:

$$L(E_d, s) = \sum_{n \geq 1} a(n)\chi_d(n)n^{-s}$$

where $\chi_d(.) = \left(\frac{d}{.}\right)$ is the quadratic character associated to d. The sign of the functional equation satisfied by $L(E_d, s)$ is

$$w(E_d) = w \cdot \chi_d(-N).$$

In the odd rank case (i.e. $w(E_d) = -1$), we are interested in the values at $s = 1$ of the derivatives of the L-functions. We have:

$$L'(E_d, 1) = \frac{\Omega(E_d)\, c(E_d)}{|E_d(\mathbb{Q})_{\text{tors}}|^2}\, R(E_d)\, S(E_d)$$

where as usual $\Omega(E_d)$ is the real period, $R(E_d)$ is the regulator and $c(E_d)$ is the product of the local Tamagawa numbers $c_p(E)$ for $p \mid Nd$. The Birch and Swinnerton-Dyer conjecture predicts that $S(E_d) = |Ш(E_d)|$ if $L'(E_d, 1) \neq 0$ and $S(E_d) = 0$ otherwise.

2. Families of quadratic twists

For each prime p dividing the conductor N of E, we fix a sign $w_p = \pm 1$ so that $\prod_{p|N} w_p = w$. We then define the set:

$$\mathcal{F} = \left\{ d < 0,\ \text{fundamental discriminant with } \left(\frac{d}{p}\right) = w_p \text{ for all } p \mid N \right\}$$

and we let:

$$\mathcal{F}(T) = \left\{ d \in \mathcal{F},\ |d| < T \right\}$$

Then, our family of quadratic twists is the set $(E_d)_{d \in \mathcal{F}}$ and, for all these curves E_d, we have $w(E_d) = -1$ by the above assumption on the product of the w_p's. It will be convenient for us to partition the family $\mathcal{F}$ into two subfamilies corresponding to the odd and even discriminant cases. Therefore we define:

$$\mathcal{F}_{\text{odd}} = \left\{ d \in \mathcal{F},\ d \text{ odd} \right\} \quad \text{and} \quad \mathcal{F}_{\text{odd}}(T) = \left\{ d \in \mathcal{F}(T),\ d \text{ odd} \right\}$$

Note that we will not need to consider the subfamilies corresponding to the even discriminants.

For $d \in \mathcal{F}$ with $|d|$ large enough, it follows from Proposition 2 of [De2] that, if we denote by c_4 the usual invariant of E (cf. [Coh1, §7.1]), we have:

$$S(E_d)\, R(E_d) = \frac{\sqrt{|d|}\, L'(E_d, 1)}{\delta_8(d, c_4)\, \Omega_{\mathcal{F}} \prod_{p|d} c_p(E_d)} \tag{2.1}$$

where $\delta_8(d, c_4) = 2$ if $8 \mid d$ and $2 \mid c_4$, and $\delta_8(d, c_4) = 1$ otherwise, and $\Omega_{\mathcal{F}}$ is some positive number which does not depend on d. When $L'(E_d, 1)$ is not zero then $E_d(\mathbb{Q})$ has rank 1 and the regulator $R(E_d)$ is equal to the canonical height $\hat{h}(G_d)$ of a generator G_d of E_d. So, the problem of studying the behavior of $R(E_d)$ is roughly speaking the same as the one of studying the complexity of rational solutions of the associated Diophantine equations.

2.1. **On upper bounds for $h(G_d)$.** Lang's conjecture [Sil, Conjecture 10.2] predicts that for a general elliptic curve E:

$$R(E) \ll |\Delta_{\min}(E)|^{1/2+\epsilon}$$

where $\Delta_{\min}(E)$ is the minimal discriminant of E. In our family, we have $\Delta_{\min}(E_d) = d^6\Delta_{\min}(E)$ hence, this yields:

$$R(E_d) \ll |d|^{3+\epsilon}$$

Of course, this upper bound is very far from what we really expect for our family. Indeed, using equation (2.1) and the fact that $S(E_d)$ and $c_p(E_d)$ are positive integers (so greater or equal to 1), the Lindelöf hypothesis applied to $L'(E_d, 1)$ gives the following conditional upper bound:

$$R(E_d) \ll_\epsilon |d|^{1/2+\epsilon}$$

In some cases, this upper bound can be proved on average. Anticipating on the results and notations of Section 3.1, we prove:

Proposition 2.1. *Assume that N is square-free, $L(E, 1) \neq 0$ and $w_p = +1$ for all $p \mid N$. Then we have:*

$$\frac{1}{|\mathcal{F}_{odd}(T)|} \sum_{\substack{d \in \mathcal{F}_{odd}(T) \\ L'(E_d,1) \neq 0}} R(E_d) \ll T^{1/2} \log T \tag{2.2}$$

Proof. This is a direct corollary of a theorem of Ricotta and Vidick. Indeed, with the notations of section 3.1 we have

$$R(E_d) = \hat{h}(G_d) \leq \hat{h}(R_d) = 4\hat{h}_E(P_d),$$

where $\hat{h}_E$ is the canonical height on E and $P_d \in E(\mathbb{Q}\sqrt{d})$ is the Heegner point constructed in 3.1. Now, we apply the corollaire 3.2 of [Ri-Vi]. □

Remark. Classical conjectures predict that the number of discriminants d in our family for which $L'(E_d, 1) = 0$ should have density 0 (we will come back to this fact later), so $|\mathcal{F}_{odd}(T)|$ is roughly the number of terms in the sum of the formula above and hence the proposition really asserts that on average $R(E_d) \ll |d|^{1/2+\epsilon}$ for all $d \in \mathcal{F}_{odd}$.

2.2. **On lower bound for $R(E_d)$.** Another conjecture of Lang asserts that $\hat{h}(G_d) \gg \log|\Delta_{\min}(E_d)|$, thus we get:

$$\hat{h}(G_d) \gg \log|d| \tag{2.3}$$

In fact, we have the more precise result:

Proposition 2.2. *If $j(E) \neq 0, 1728$, then there is an explicit constant M, depending on E and on the w_p, such that we have for all $d \in \mathcal{F}$:*

$$\hat{h}(G_d) > \frac{1}{M} \log |d|$$

If $w_p = +1$ for all $p \mid N$, then one can take $M = 1296\, c(E)^2$.

Proof. We estimate $\mathrm{lcm}(c_p(E_d))_{p|Nd}$ where $c_p(E_d)$ is the local Tamagawa number at the prime p dividing Nd. If $p \mid N$, then $c_p(E_d)$ is either $c_p(E)$ if $w_p = +1$, or $c_p(E^*)$ if $w_p = -1$ where E^* is any fixed twist of E by a discriminant that is not a square in $\mathbb{Q}_p$. If $p \mid d$, then $c_p(E_d)$ is either 1, 2 or 4. Hence, we have

$$\mathrm{lcm}(c_p(E_d))_{p|N} \leq 4 \prod_{p|N,\, w_p=+1} c_p(E) \prod_{p|N,\, w_p=-1} c_p(E^*)$$

Now, the result follows using Corollaire 2.2 of [Kri] and the fact that $|\Delta_{\min}(E_d)| = |d|^6 |\Delta_{\min}(E)|$. □

Remark.

(1) With the same techniques, we can obtain similar results for $j(E) = 0$ or 1728.
(2) One can prove (see for example [Sil, exercise 8.17]) the following lower bound:

$$\hat{h}(G_d) \geq \frac{1}{3} \log |d| + C \tag{2.4}$$

where C is some constant depending on E. The factor $1/3$ in this formula is much better than the factor $1/M$ in Proposition 2.2. However, the constant C (which comes from the difference between the naive and the canonical heights) is negative and thus the estimate (2.4) is useless for small d (and in practice for all the d's we are dealing with). On the other hand, the estimate of Proposition 2.2 is good enough for our applications and has no consequence on the main complexity of our method.
(3) The lower bound in Proposition 2.2 is optimal in the following sense: suppose that E is given by the equation $y^2 = P(x)$ where $P(x)$ is a degree 3 polynomial. Then, one can easily check that the point $(rP(r), P(r)^2)$ belongs to $E_{P(r)}(\mathbb{Q})$ and that the height of this point is $\approx 4/3 \log |P(r)|$.

One expect much better lower bounds on average: indeed, it is proved in [De2] that predictions coming from Random Matrix Theory for derivatives of L-functions (see [Sna]) and Cohen-Lenstra type heuristics for Tate-Shafarevich groups (see [De1]) imply that for $k > 0$:

$$\frac{1}{|\mathcal{F}(T)|} \sum_{\substack{d \in \mathcal{F}(T) \\ L'(E_d,1) \neq 0}} \hat{h}(G_d)^k \gg T^{k/2-\epsilon} \tag{2.5}$$

where the implied constant depends on E, k, ϵ and w.

2.3. **Heuristics for the moments of $R(E_d)$.** For $k > 0$ we let:

$$M_k(T) = \frac{1}{|\mathcal{F}(T)|} \sum_{\substack{d \in \mathcal{F}(T) \\ L'(E_d,1) \neq 0}} R(E_d)^k$$

Equations (2.2) and (2.5) imply that on average $\hat{h}(G_d)$ should be of the size of $|d|^{1/2}$. In fact, one can make similar computations as in [De2] to estimate:

$$\sum_{\substack{d \in \mathcal{F}(T) \\ L'(E_d,1) \neq 0}} R(E_d)^k S(E_d)^k$$

Then, Cohen-Lenstra type heuristics for Tate-Shafarevich groups (see [De1]) predict that $\dfrac{1}{|\mathcal{F}(T)|} S(E_d)^k$ tends to a finite limit as $T \to \infty$ whenever $0 < k < 1$. Therefore, using an empirical argument, we replace the term $S(E_d)^k$ by a constant and deduce the following heuristics:

Heuristic for $M_k(T)$. For $0 < k < 1$ we have as $T \to \infty$:

$$M_k(T) \sim A_k \, T^{k/2} \log(T)^{k(k+1)/2 + a_k - 1} \tag{2.6}$$

for some constants A_k and a_k.

The number a_k comes from the contribution of the Tamagawa numbers in the Birch and Swinnerton-Dyer conjecture. More precisely we should have:

- If E (or an isogenous curve) has full rational 2-torsion then $a_k = 4^{-k}$.
- If E has exactly one rational 2-torsion point (and no isogenous curve has full 2-torsion) then $a_k = \frac{1}{2}(4^{-k} + 2^{-k})$.

For the other cases, we need to make the rather technical assumption that our restrictions on the discriminants are not incompatible with the use of the Chebotarev density theorem (see [De2]). Then we should have:

- If E has no rational 2-torsion point and its discriminant is not a square then $a_k = \frac{1}{6}\, 4^{-k} + \frac{1}{2}\, 2^{-k} + \frac{1}{3}$.
- If E has no rational 2-torsion point and its discriminant is a square then $a_k = \frac{1}{3}\, 4^{-k} + \frac{2}{3}$.

Indeed, the equivalence (2.6) depends only on the isogenous class of the curve, and this explains why we have to consider the curve in the class with the maximal rational 2-torsion point.

If we restrict our family to negative prime discriminants, the effect of the Tamagawa numbers disappears and we have $a_k = 1$. More precisely if we let:

$$\mathcal{F}' = \left\{ d < 0, \text{fund. disc. with } \left(\frac{d}{p}\right) = w_p \text{ for all } p \mid N \text{ and } |d| \text{ is prime} \right\}$$

$$\mathcal{F}'(T) = \left\{ d \in \mathcal{F}',\ |d| < T \right\}$$

and

$$M'_k(T) = \frac{1}{|\mathcal{F}'(T)|} \sum_{\substack{d \in \mathcal{F}'(T) \\ L'(E_d,1) \neq 0}} R(E_d)^k \ ,$$

we expect the following heuristic:

Heuristic for $M_k'(T)$. For $0 < k < 1$, we have as $T \to \infty$:

$$M_k'(T) \sim A_k' \, T^{k/2} \log(T)^{k(k+1)/2} \tag{2.7}$$

Remark. These two heuristics are supported by our numerical data for the elliptic curves of conductor $N \leq 17$ as we will see in the last section.

The asymptotics (2.6) and (2.7) imply that on average the regulators of $(E_d)_{d\in\mathcal{F}}$ behave as $\approx |d|^{1/2+\varepsilon}$ suggesting that $\theta = \varepsilon$ in the Saturday Night Conjecture (see [CRSW]). From this we get a density of $T^{1-\varepsilon}$ for the subset of $d \in \mathcal{F}(T)$ such that $L'(E_d, 1) = 0$, which is really surprising compared to the even-rank case. The numerical data seems to support this fact. On the other hand, extensive numerical computations by Watkins [Wat] seem to indicate otherwise. Indeed we want to emphasize that one has always to be careful with deducing too strong of statements from numerical investigations.

3. Computation of generators

We need to make a certain number of restrictions in order to be able to apply the method described in this section. First, we assume that E is the strong Weil curve in its isogeny class (in fact, we just need that the Manin's constant of E is equal to 1) and that $j(E) \neq 0$, 1728. These are just technical and not essential assumptions. Furthermore, we assume $L(E, 1) \neq 0$ which implies that $E(\mathbb{Q})$ has rank 0 and that $w = +1$. This is a fundamental assumption and the method would not work without it. Finally, the family of discriminants $\mathcal{F}$ is obtained by taking $w_p = +1$ for all $p \mid N$. Hence, $w(E_d) = -1$ and d is a square modulo $4N$ for all $d \in \mathcal{F}$.

The latter condition implies that one can apply the Heegner point construction to get a point $P_d \in E(\mathbb{Q}(\sqrt{d}))$ of infinite order if $L'(E_d, 1) \neq 0$.[4] For that one has to evaluate the modular parametrization at well chosen points $\tau \in X_0(N)$:

$$\begin{array}{rccccc} \varphi \ : & X_0(N) & \xrightarrow{\phi} & \mathbb{C}/\Lambda & \xrightarrow{\wp} & E(\mathbb{C}) \\ & \tau & \longmapsto & \sum\limits_{n\geq 1} \frac{a(n)}{n} e^{2i\pi n\tau} & & \end{array}$$

with $X_0(N) = \Gamma_0(N)\backslash\overline{\mathbb{H}}$ where $\Gamma_0(N)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ of matrices with lower left entry divisible by N, $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q}$ is the completed upper half plane, Λ is the period lattice associated to E and $\wp$ is the analytic isomorphism given by the Weierstrass function (and its derivative).

3.1. Description of the method.

We now briefly describe the algorithm step by step.

[4] Classically the Heegner point method is used to construct directly a rational point on $E_d(\mathbb{Q})$, see [Coh2, Chapter 8.5]. However the direct construction of a point in a quadratic extension has been already done in connection with the problem of congruent numbers by N. Elkies, see [Elk]. The main difference with the construction used in this article is that Elkies just wanted a strategy to compute efficiently a rational point of some quadratic twists of the elliptic curve $32a2$, whereas we want to compute a *generator* of *all* the $E_d(\mathbb{Q})$ for $d \in \mathcal{F}(T)$ of some large T. Hence, we really need to be careful in all the steps of the method in order to be as efficient as possible. We also have to use the full force of the Gross-Zagier formula and of the Birch and Swinnerton-Dyer conjecture in order to get as much information as possible all throughout our computations.

STEP 1. *For each ideal class $\mathcal{C}$ in the class group $\mathrm{Cl}(d)$ of $\mathbb{Q}(\sqrt{d})$, we choose an integral ideal $\mathfrak{a} \in \mathcal{C}$ such that:*

$$\mathfrak{a} = A\,\mathbb{Z} + \frac{-B+\sqrt{d}}{2}\,\mathbb{Z} \quad \text{with } N \mid A \text{ and } B \equiv \beta \pmod{2N} \tag{3.1}$$

where $\beta = \beta_d$ is a fixed integer such that $\beta^2 \equiv d \pmod{4N}$.
Then, to $\mathcal{C} = [\mathfrak{a}]$, we associate the Heegner point:

$$\tau_{[\mathfrak{a}]} = \frac{-B+\sqrt{d}}{2A}$$

Comments. The point $\tau_{[\mathfrak{a}]}$ lies in the upper half plane and is a well defined point in $X_0(N)$. Nevertheless, in order to make the computations as easy as possible, we need to choose $\mathfrak{a}$ such that A is as small as possible. Using classical algorithms (see [Coh1]), we can compute a set of ideals $\{\mathfrak{a}_i\}_i$ representing all the classes of $\mathrm{Cl}(d)$:

$$\mathfrak{a}_i = a_i\mathbb{Z} + \frac{-b_i+\sqrt{d}}{2}\mathbb{Z}$$

with $0 < a_i \ll |d|^{1/2}$ where the implied constant is explicit. We can assume without loss of generality that the a_i's are relatively prime with N. Then, the ideals $\mathfrak{a}_i\overline{\mathfrak{n}}$ satisfy (3.1) where

$$\overline{\mathfrak{n}} = N\,\mathbb{Z} + \frac{\beta-\sqrt{d}}{2}\mathbb{Z}$$

From this it follows that one can choose the ideals $\mathfrak{a}_i$'s in such a way that we have the following lower bound:

$$\Im(\tau_{[\mathfrak{a}_i]}) \gg 1/N \tag{3.2}$$

The complexity of this step is thus dominated by the class number of $\mathbb{Q}(\sqrt{d})$, hence is at most $O(|d|^{1/2}\log|d|)$.

STEP 2. *We compute*

$$z_d = \sum_{[\mathfrak{a}]} \phi(\tau_{[\mathfrak{a}]})$$

where the sum is over the classes of $\mathrm{Cl}(d)$, and then a complex approximation of $P_d = \wp(z_d) \in E(\mathbb{C})$. The theory of complex multiplication and of Heegner points imply that $P_d \in E(\mathbb{Q}(\sqrt{d}))$. Using this approximation, we try to recognize the four rational numbers r_1, s_1, r_2 and s_2 such that $P_d = (r_1 + s_1\sqrt{d}, r_2 + s_2\sqrt{d})$ and test if P_d is a point of infinite order.

Comments. This is the main step of the method. Note that one can reduce the number of evaluations of ϕ by 2 using the following trick. Once we have already computed $\varphi(\tau_{[\mathfrak{a}]})$, since $w = +1$ we can deduce from it $\varphi(\tau_{[\mathfrak{a}^{-1}n]})$ using the formula:

$$\overline{\varphi(\tau_{[\mathfrak{a}]})} = -\varphi(\tau_{[\mathfrak{a}^{-1}n]}) + Q \tag{3.3}$$

where Q is an explicit rational torsion point in $E(\mathbb{Q})$ depending only on E.

Given a complex number $\tilde{x}_{P_d}$ that is an approximation of the x-coordinate x_{P_d} of the point P_d computed as explained above, we need to recover from it the two rational numbers r_1 and s_1 such that $x_{P_d} = r_1 + s_1\sqrt{d}$. Note that for candidate

values r_1 and s_1, one can check if they are indeed correct by trying to compute two rationals r_2 and s_2 such that

$$(r_1 + s_1\sqrt{d}, r_2 + s_2\sqrt{d}) \in E(\mathbb{Q}(\sqrt{d})).$$

Let $\tilde{r} = \Re(\tilde{x}_{P_d})$ and $\tilde{s} = \Im(\tilde{x}_{P_d})/\sqrt{d}$. For $e \geq 1$ we look for a small integral relation (using the LLL-algorithm) between the columns C_1, C_2, C_3 of the matrix

$$\begin{pmatrix} -10^e & 0 & \lfloor 10^e\,\tilde{r}\,\rceil \\ 0 & -10^e & \lfloor 10^e\,\tilde{s}\,\rceil \\ 0 & 0 & 1 \end{pmatrix}$$

where $\lfloor . \rceil$ denotes the closest integer. Indeed, for such a relation, say

$$\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3$$

of norm M, we have that λ_1/λ_3, resp. λ_2/λ_3, is an approximation of $\tilde{r}$, resp. $\tilde{s}$, with an error less than $\sqrt{M}/10^e$, and the denominator λ_3 is smaller (in absolute value) than $\sqrt{M}$. In order for this method to work, we need to compute $\tilde{r}$ and $\tilde{s}$ at a suitably large enough precision and to choose e accordingly. More precisely, to recognize x_{P_d} as an element of $\mathbb{Q}(\sqrt{d})$ we need about $\hat{h}(P_d)$ digits. Bounding the coefficients $a(n)/n$ by 1 in the sum defining ϕ and using (3.2), we see that we need to sum approximatively $\hat{h}(P_d)$ coefficients for ϕ. The Gross-Zagier theorem [Gro-Zag] asserts that:[5]

$$\hat{h}(P_d) = \frac{L(E,1)\,L'(E_d,1)\,\sqrt{|d|}}{4\ \mathrm{vol}(E)} \tag{3.4}$$

Applying the Lindelöf hypothesis we deduce that $\hat{h}(P_d) \ll |d|^{1/2+\varepsilon}$. Hence, the complexity of this step is $\ll |d|^{1/2+\varepsilon}|\mathrm{Cl}(d)| \ll |d|^{1+\varepsilon}$. This step can fail in two ways. First case: the computation has not been done to a large enough precision. In that case we have to increase the precision and start over. Second case: the point P_d is a torsion point and in that case $L'(E_d,1) = 0$. If we suspect P_d to be in fact a torsion point, we can compute directly an approximation of $L'(E_d,1)$ and prove that it is indeed zero using the following proposition (whose proof we postpone to after the proof of the next proposition).

Proposition 3.1. *If*

$$L'(E_d,1) \leq \frac{\mathrm{vol}(E)}{1296\,c(E)^2\,L(E,1)}\ |d|^{-1/2}\log|d|$$

then $L'(E_d,1) = 0$.

STEP 3. *If P_d is a point of infinite order, i.e.* **STEP 2** *has succeeded, then the point $R_d = \psi(P_d - \overline{P_d})$ is a point of infinite order in $E_d(\mathbb{Q})$. We divide it in the Mordell-Weil group $E(\mathbb{Q})$ until we get a generator G_d of $E_d(\mathbb{Q})$ modulo torsion. We define the integer ℓ_d by $R_d = \ell_d\,G_d \pmod{E_d(\mathbb{Q})_{tor}}$.*

Comments. The point R_d is rational since $R_d = \psi(P_d) + \overline{\psi(P_d)}$ by (1.1). If $L'(E_d,1) \neq 0$ then we know that G_d is a generator of $E_d(\mathbb{Q})$ modulo torsion.

Proposition 3.2. *$\hat{h}(R_d) = 4\,\hat{h}_E(P_d)$, hence R_d is non-torsion if and only if P_d is non-torsion (that is if and only if $L'(E_d,1) \neq 0$).*

[5]Actually, the Gross-Zagier theorem only applies for odd d's. For even d's the formula is a conjecture of Hayashi [Hay].

Proof. The height does not depend on the model of the elliptic curve, hence $\hat{h}(R_d) = \hat{h}_E(P_d - \overline{P_d})$. Furthermore, equation (3.3) implies that $P_d = -\overline{P_d}$ plus a rational torsion point. □

Proof of proposition 3.1. We use the lower bound from proposition 2.2 for $\hat{h}(R_d)$ and equation (3.4). □

From Proposition 2.2 we know that:

$$|\ell_d| < 36\, c(E) \sqrt{\frac{\hat{h}(R_d)}{\log|d|}} \ll |d|^{1/4+\varepsilon} \tag{3.5}$$

Hence there are finitely many primes p for which we need to check p-divisibility. Also, it is well-known that $E_d(\mathbb{Q})_{\mathrm{tors}}$ does not depend upon d (for all d's except at most one) and can only be $\simeq \{0\}$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore we need to be careful about torsion only when we consider 2-divisibility which can be tested easily using 2-division polynomial. For an odd prime p, we use the following method to rule out p-divisibility. We find a prime r, of good reduction, such that the order α of the group $E_d(\mathbb{F}_r)$ is divisible by p. Then if $(\alpha/p)R_d$ is not zero in $E_d(\mathbb{F}_r)$, we know that R_d is not divisible by p in $E(\mathbb{F}_r)$, and thus in $E_d(\mathbb{Q})$ too. If after having performed a large number of such tests, we have not been able to prove that R_d is not divisible by p, then we "know" that the point must be divisible by p and we perform the division.[6]

STEP 4. *We compute the regulator of E_d (in the rank 1 case) which is equal to $\hat{h}(G_d)$ and the order of the Tate-Shafarevich group $|Ш(E_d)|$.*

Comments. We can compute the order of $|Ш(E_d)|$ using:

Proposition 3.3. *Under the Birch and Swinnerton-Dyer conjecture[7], the following equality holds*

$$|Ш(E_d)| = \frac{|E(\mathbb{Q})_{tor}|^2\, |E_d(\mathbb{Q})_{tor}|^2}{|Ш(E)|\, c(E)^2} \, \frac{\ell_d^2}{2sg(\Delta_{\min}(E))\, \delta_8(d, c_4) \prod_{p|d} c_p(E_d)}$$

where $sg(x) = 1$ if $x < 0$ and $sg(x) = 2$ otherwise.

Proof. Indeed, we have:

$$\ell_d^2\, \hat{h}(G_d) = 4\hat{h}(P_d) = \frac{L(E,1)\, L'(E_d,1)\, \sqrt{|d|}}{\mathrm{vol}(E)}$$

Now we replace $L(E,1)$ and $L'(E_d,1)$ by the values predicted by the Birch and Swinnerton-Dyer conjecture. After simplifying the regulator $\hat{h}(G_d)$ on both sides, we get:

$$\ell_d^2 = \frac{|Ш(E_d)|\, |Ш(E)|\, c(E)}{|E(\mathbb{Q})_{\mathrm{tor}}|^2\, |E_d(\mathbb{Q})_{\mathrm{tor}}|^2} \cdot c(E_d) \cdot \frac{\Omega(E)\, \Omega(E_d)\, \sqrt{|d|}}{\mathrm{vol}(E)} (\times 4 \text{ if } \Delta_{min}(E) > 0)$$

[6]Indeed, in all cases, either we could prove that the point is not divisible by p by such a test, or we could actually divide it by p.

[7]For even d's, we need again to assume the conjecture of Hayashi [Hay].

Since for all $p \mid N$ we have $w_p = +1$, the curves E and E_d are isomorphic over $\mathbb{Q}_p$ and thus $c_p(E_d) = c_p(E)$. So $c(E_d) = c(E) \prod_{p|d} c_p(E_d)$. Finally a computation of the periods of E_d shows that:

$$\frac{\Omega(E)\,\Omega(E_d)\,\sqrt{|d|}}{\mathrm{vol}(E)} = \begin{cases} 2\,\delta_8(d, c_4) & \text{if } \Delta_{\min}(E) < 0 \\ \delta_8(d, c_4) & \text{if } \Delta_{\min}(E) > 0 \end{cases}$$

and the proposition follows. □

Remark. The order of the Tate-Shafarevich group is a square, therefore the proposition implies that the following quantity must be a square:

$$2\,sg(\Delta_{min}(E))\,\delta_8(d, c_4) \prod_{p|d} c_p(E_d)$$

From the above we see that for each individual d the complexity for computing $\hat{h}(G_d)$ and $|Ш(E_d)|$ is at worst $O(|d|^{1+\varepsilon})$. From these values we can deduce the value of $L'(E_d, 1)$ at arbitrary precision. Note that the direct computation of $L'(E_d, 1)$ by the rapidly converging series needs also $O(|d|)$ terms.[8] Nevertheless, for large precisions, in practice, it is often much more efficient to compute $L'(E_d, 1)$ as a by product of our computations than to evaluate it directly. This is probably due to the fact (see the discussion on the computations) that the implied constant is small in the prediction $M_1(T) = O(T^{1/2}(\log T)^a)$.

3.2. **An example.** We take $E = 11a1 \ : \ y^2 + y = x^3 - x^2 - 10x - 20$ and $d = -79$ so that the curve E_d has minimal equation:

$$E_d \ : \ y^2 + y = x^3 + x^2 - 64490x + 11396008$$

We take $\beta = 3$ so that $\beta^2 \equiv -79 \pmod{44}$. The class group $\mathrm{Cl}(-79)$ of $\mathbb{Q}(\sqrt{-79})$ is cyclic of order 5, and the ideals:

$$\mathfrak{a} = 11\mathbb{Z} + \frac{-3+\sqrt{-79}}{2}\mathbb{Z}\,, \quad \mathfrak{b} = 22\mathbb{Z} + \frac{-3+\sqrt{-79}}{2}\mathbb{Z}\,,$$
$$\mathfrak{c} = 44\mathbb{Z} + \frac{-3+\sqrt{-79}}{2}\mathbb{Z}\,, \quad \mathfrak{a}^{-1}\mathfrak{n} \quad \text{and} \quad \mathfrak{b}^{-1}\mathfrak{n}$$

where

$$\mathfrak{n} = N\mathbb{Z} + \frac{\beta+\sqrt{d}}{2}\mathbb{Z} = 11\mathbb{Z} + \frac{3+\sqrt{-79}}{2}\mathbb{Z}$$

form a complete set of representatives of the ideal class group. We compute

$$z = 2\Re\big(\phi(\tau_{[\mathfrak{a}]}) + \phi(\tau_{[\mathfrak{b}]})\big) + \phi(\tau_{[\mathfrak{c}]}) \in \mathbb{C}/\Lambda$$

and we find

$$P_d = \wp(z) \approx (-3.5900\cdots + 0.2200\cdots\sqrt{-79}, 5.17600\cdots + 0.61600\cdots\sqrt{-79})$$

so we easily recognize

$$P_d = \left(\frac{-179 + 11\sqrt{-79}}{50}, \frac{647 + 77\sqrt{-79}}{125}\right) \in E(\mathbb{Q}(\sqrt{-79}))$$

From this, we get the point $R_d = \psi(P_d) + \overline{\psi(P_d)} = (47, 2910) \in E_d(\mathbb{Q})$. And Formula (3.5) says that $|\ell_d| \leq 293$ where $G_d = \ell_d R_d$. We find that the point R_d

[8] More generally, in order to compute $L'(E, 1)$ for an elliptic curve E, one needs to sum the first $O(\sqrt{N})$ terms of the series, where N is the conductor of E, and the constant in the "O" depends on the required accuracy.

is divisible by 2, more precisely $R_d = -2(26, 3120)$, so that $(26, 3120) = \ell'_d G_d$ with $|\ell'_d| \leq 73$. We then easily check that the point $(26, 3120)$ is not divisible by any prime ≤ 73 in the group $E_d(\mathbb{Q})$, hence one can take $G_d = (26, 3120)$ and $|\ell_d| = 2$. Proposition (3.3) gives:

$$|Ш(E_d)| = 1$$

4. Discussion and numerical data

We have computed, using the method described in the previous section, the regulators and the order of the Tate-Shafarevich groups of the twists E_d of E of the four elliptic curves $11a1$, $14a1$, $15a1$ and $17a1$, and for all available discriminants $d \in \mathcal{F}(1.5 \times 10^6)$ with $w_p = +1$ for all $p \mid N$. We discuss in this section the data we obtained and compare it with the heuristics. All the computations have been performed using the PARI/GP system [PARI] and the data is available at

`http://math.univ-lyon1.fr/~roblot/tables.html`

For each curve, we give several graphs.

- For the curves $11a1$ and $17a1$, two graphs of the regulators of the curve, one with all the regulators and one with the regulators less than 10 to illustrate Equation (2.4).
- Four different graphs comparing the moments of order 1/4, 1/2, 3/4 and 1 of the regulators with the functions given by the heuristics.
- One graph with the number of twists that have analytic rank at least 3 and one graph displaying the moments of order 1/4, 1/2, 3/4 and 1 for the order of the Tate-Shafarevich group of the twists. The heuristics suggest that the moments of order $k < 1$ tend to a constant (depending on k) whereas the moment of order 1 should tend to infinity.
- For the curves $11a1$ and $17a1$, two graphs comparing the moments of order 1/2 and 1 of the regulators of the twists by prime discriminants with the functions given by the heuristics.

We begin with the curves of prime conductor ($11a1$ and $17a1$) since for the last two curves ($14a1$ and $15a1$), the congruence conditions are more restrictive and therefore the number of discriminants in $\mathcal{F}(1.5 \times 10^6)$ is quite small compared to 1.5×10^6.

4.1. The curve 11a1.

The curve E is defined by

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

It has conductor $N = 11$ and rank 0 over $\mathbb{Q}$. We have $w_{11} = +1$.

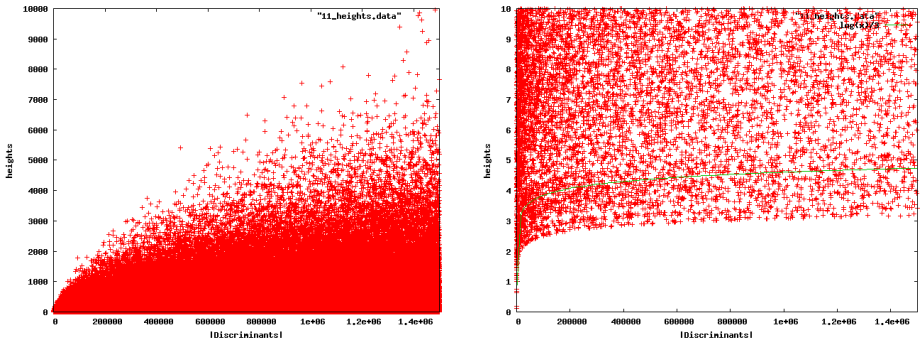
4.1.1. *Numerical results for all discriminants.*

- Number of discriminants: $|\mathcal{F}(1.5 \times 10^6)| = 208977$.
- Largest regulator: ≈ 9945 (for $d = -1482139$).
- Number of extra-vanishing: 638.

We have $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/5\mathbb{Z}$ and there is no curve in its isogeny class having rational 2-torsion. Hence the heuristics predict that:

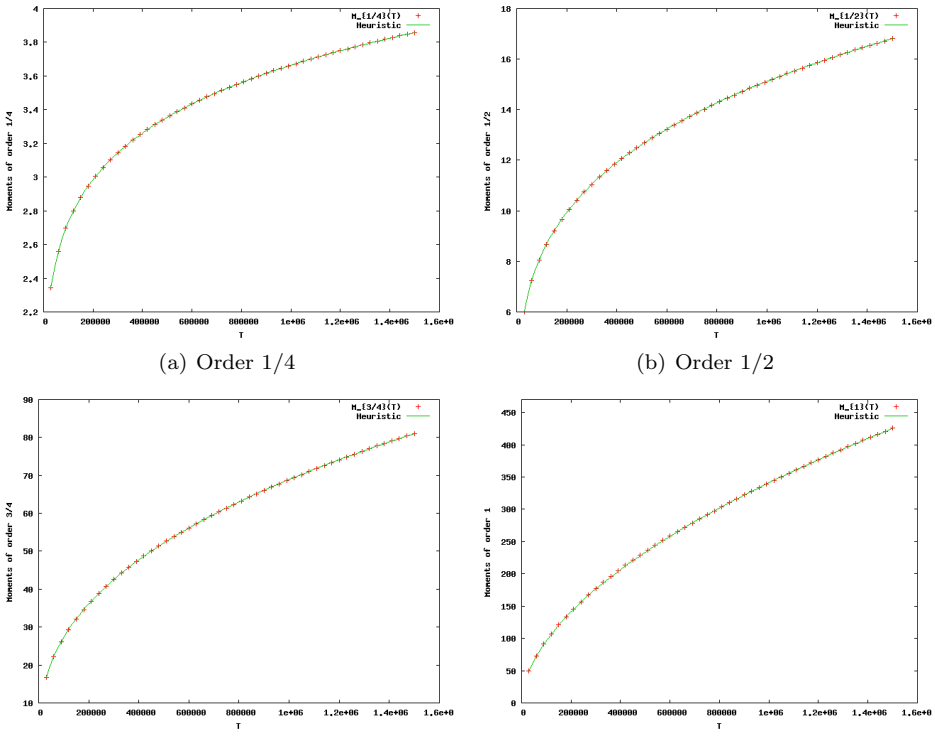
$$M_k(T) \sim A_k T^{k/2} \log(T)^{\frac{k(k+1)}{2} + \frac{1}{6 \cdot 4^k} + \frac{1}{2 \cdot 2^k} - \frac{2}{3}}$$

for some constant A_k. We computed A_k numerically to fit the data (values found: $A_{1/4} \approx 0.60$, $A_{1/2} \approx 0.33$, $A_{3/4} \approx 0.16$, $A_1 \approx 0.07$) and we plot the graph of the



(a) All the regulators. (b) Regulators less than 10.

FIGURE 1. Regulators of the twists of $11a1$.



(a) Order 1/4 (b) Order 1/2

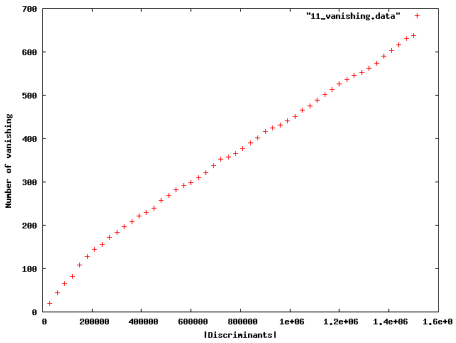
(c) Order 3/4 (d) Order 1

FIGURE 2. Moments of the regulators of the twists of $11a1$ and the function given by the heuristics.

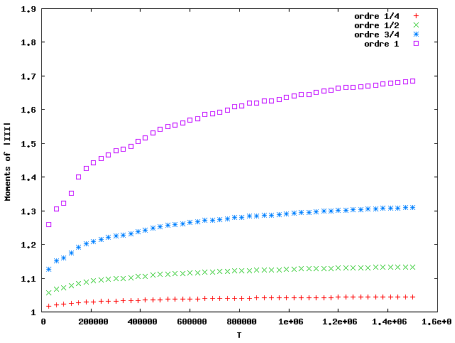
function given by the heuristics and the points $(T, M_k(T))$ for $T = 1, 2, \ldots, 150 \times 3 \cdot 10^4$ and for $k = 1/4$, $1/2$, $3/4$ and 1. As it can been seen the graphs (see Figure 2) are in close agreement.

4.1.2. *Numerical results for prime discriminants.*

- Number of prime discriminants: 28535.
- Largest regulator: ≈ 9250 (for $d = -1433539$).

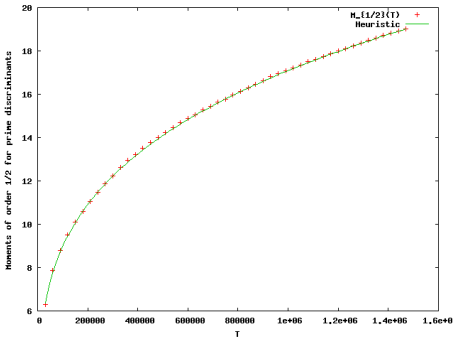


(a) Extra-vanishing of $L'(E_d, 1)$ for $E = 11a1$.

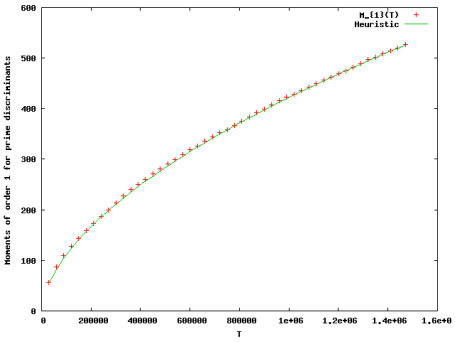


(b) Moments of different orders for the order of the Tate-Shafarevich groups of the twists of $11a1$.

FIGURE 3



(a) Order 1/2



(b) Order 1

FIGURE 4. Moments of the regulators of the twists of $11a1$ by prime discriminants and the functions given by the heuristics.

- Number of extra-vanishing: 0.[9]

The heuristics for prime discriminants predict that:

$$M'_k(T) \sim A'_k T^{k/2} \log(T)^{k(k+1)/2}$$

for some constant A'_k. We computed A'_k numerically to fit the data (values found: $A'_{1/2} \approx 0.20$, $A'_1 \approx 0.03$) and we plot the graph of the function given by the heuristics and the points $(T, M'_k(T))$ for $T = 1, 2, \dots 150 \times 3 \cdot 10^4$, and $k = 1/2$, 1 (see Figure 4).

4.2. **The curve 17a1.** The curve E is defined by

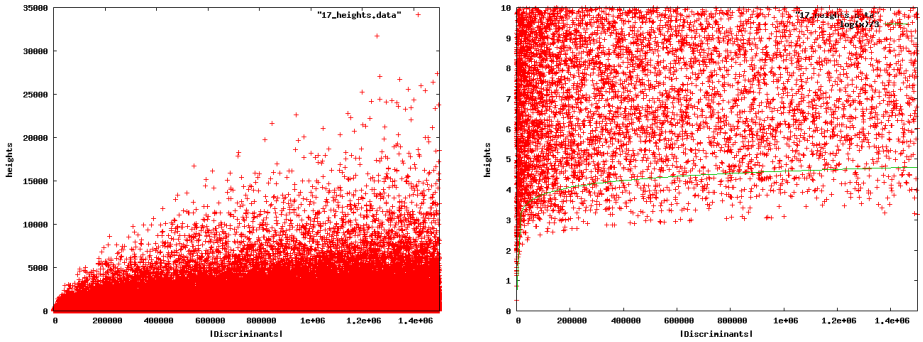
$$y^2 + xy + y = x^3 - x^2 - x - 14.$$

It has conductor 17 and rank 0 over $\mathbb{Q}$. We have $w_{17} = +1$.

[9]There is no extra-vanishing in this case using the results of [An-Bu-Fr].

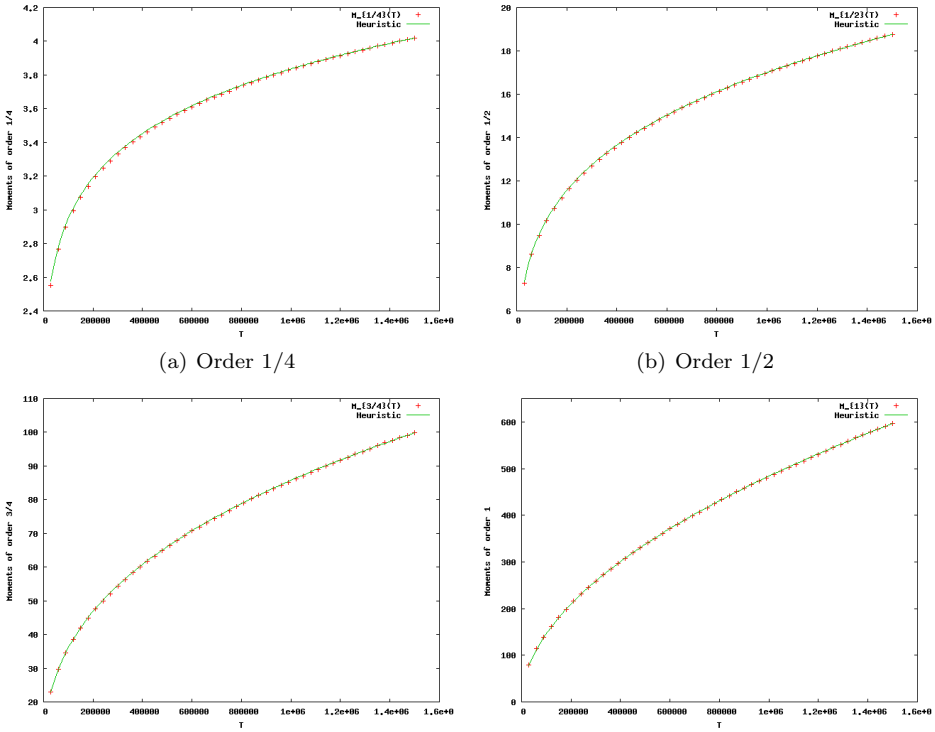
4.2.1. *Numerical results for all discriminants.*

- Number of discriminants: 215305.
- Largest regulator: ≈ 31746 (for $d = -1257787$).
- Number of extra-vanishing: 1140.



(a) All the regulators. (b) Regulators less than 10.

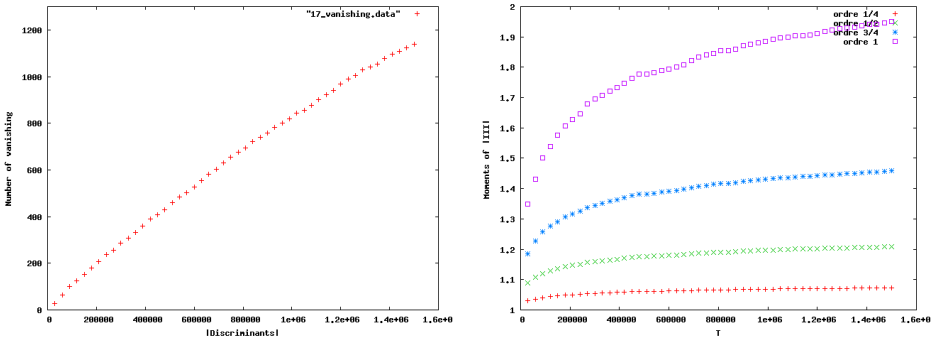
FIGURE 5. Regulators of the twists of $17a1$.



(a) Order 1/4 (b) Order 1/2

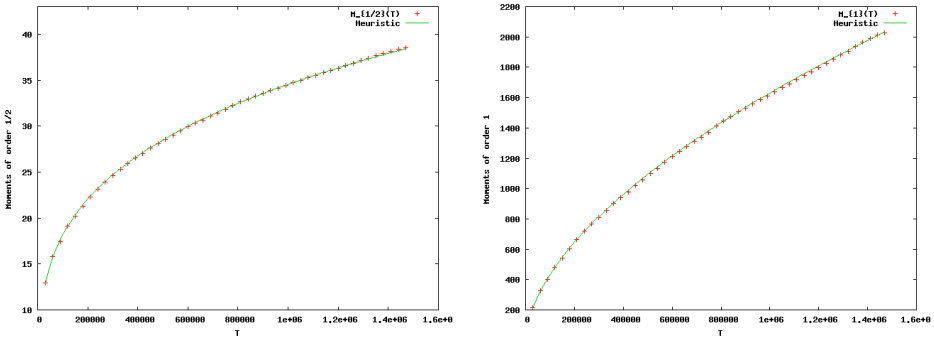
(c) Order 3/4 (d) Order 1

FIGURE 6. Moments of the regulators of the twists of $17a1$ and the function given by the heuristics.



(a) Extra-vanishing of $L'(E_d, 1)$ for $E = 17a1$. (b) Moments of different orders for the order of the Tate-Shafarevich groups of the twists of $17a1$.

FIGURE 7



(a) Order 1/2 (b) Order 1

FIGURE 8. Moments of the regulators of the twists of $17a1$ by prime discriminants and the functions given by the heuristics.

Remark. Note that the graphs of extra-vanishing for the curves 11a1 (Figure 3(a)) and 17a1 (Figure 7(a)) suggest that the density of extra-vanishing is larger for the twists of 17a1 than for those of 11a1. However the asymptotic for the moments of the regulators is smaller (as $T \to \infty$) for 17a1 than for 11a1 which suggest that there are more constraints on the regulators of the twists of 11a1 and thus imply in turn that we should have more extra-vanishing for this family. In fact, the constants A_k in the asymptotics of $M_k(T)$ are larger for the curve 17a1, but asymptotics of the functions $M_k(T)$ for the curve 11a1 are larger than for the curve 17a1 for very large values of T that are completely out of reach for computations. Therefore our guess is that the density of extra-vanishing for the twists of 11a1 will become greater than that for the twists of 17a1 for those very large values.

The curve $17a2$ has full rational 2-torsion, hence the heuristics predict that

$$M_k(T) \sim A_k T^{k/2} \log(T)^{\frac{k(k+1)}{2} + \frac{1}{4^k} - 1}$$

for some constant A_k. We computed A_k numerically to fit the data (values found: $A_{1/4} \approx 0.97$, $A_{1/2} \approx 0.75$, $A_{3/4} \approx 0.47$, $A_1 \approx 0.25$ and we plot the graph of the

function given by the heuristics and the points $(T, M_k(T))$ for $T = 1, 2, \dots 150 \times 3 \cdot 10^4$, and $k = 1/4$, $1/2$, $3/4$ and 1 (see Figure 6).

4.2.2. *Numerical results for prime discriminants.*

- Number of prime discriminants: 28601.
- Largest regulator: ≈ 31745 (for $d = -1257787$).
- Number of extra-vanishing: 0.[10]

The heuristics for prime discriminants predicts that:

$$M'_k(T) \sim A'_k T^{k/2} \log(T)^{k(k+1)/2}$$

for some constant A'_k. We computed A'_k numerically to fit the data (values found: $A'_{1/2} \approx 0.41$, $A'_1 \approx 0.12$) and we plot the graph of the function given by the heuristic and the points $(T, M'_k(T))$ for $T = 1, 2, \dots 150 \times 3 \cdot 10^4$, and $k = 1/2, 1$ (see Figure 8).

4.3. **The curve 14a1.** The curve E is defined by

$$y^2 + xy + y = x^3 + 4x - 6.$$

It has conductor $N = 14$ and rank 0 over $\mathbb{Q}$. We have $w_2 = w_7 = +1$.

- Number of discriminants: 66516.
- Largest regulator: ≈ 16937 (for $d = -1416631$).
- Number of extra-vanishing: 262.

We have $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$, and there is no curve in the isogeny class having full rational 2-torsion. Hence the heuristics predict that

$$M_k(T) \sim A_k T^{k/2} \log(T)^{\frac{k(k+1)}{2} + \frac{1}{2}(4^{-k} + 2^{-k}) - 1}$$

for some constant A_k. We computed A_k numerically to fit the data (values found: $A_{1/4} \approx 0.82$, $A_{1/2} \approx 0.56$, $A_{3/4} \approx 0.33$, $A_1 \approx 0.17$) and we plot the graph of the function given by the heuristics and the points $(T, M_k(T))$ for $T = 1, 2, \dots 150 \times 3 \cdot 10^4$, and $k = 1/4$, $1/2$, $3/4$ and 1 (see Figure 9).

4.4. **The curve 15a1.** The curve E is defined by

$$y^2 + xy + y = x^3 + x^2 - 10x - 10.$$

It has conductor $N = 15$ and rank 0 over $\mathbb{Q}$. We have $w_3 = w_5 = +1$.

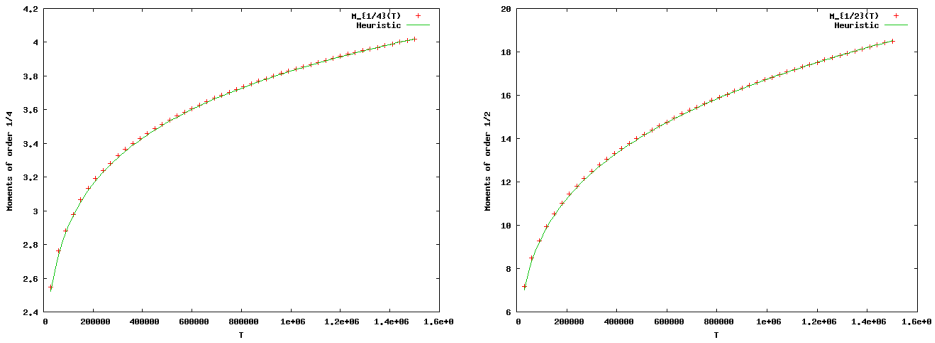
- Number of discriminants: 71254.
- Largest generator: ≈ 19352 (for $d = -1297619$).
- Number of extra-vanishing: 406.

We have $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, hence it has full 2-torsion. The heuristics predict that

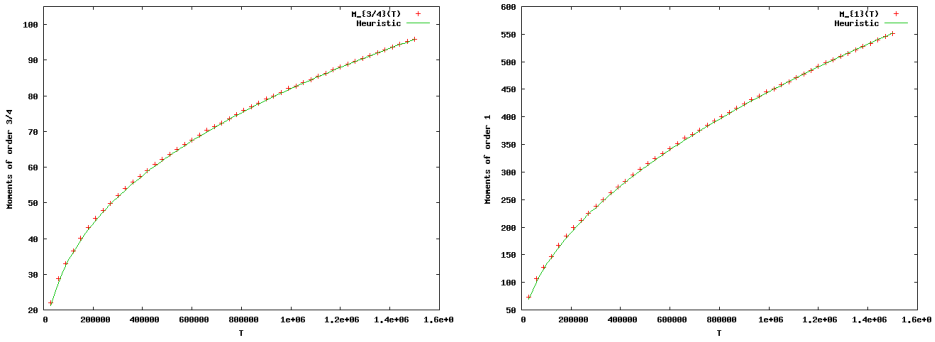
$$M_k(T) \sim A_k T^{k/2} \log(T)^{k(k+1)/2 + 4^{-k} - 1}$$

for some constant A_k. We computed A_k numerically to fit the data (values found: $A_{1/4} \approx 0.97$, $A_{1/2} \approx 0.75$, $A_{3/4} \approx 0.47$, $A_1 \approx 0.25$) and we plot the graph of the function given by the heuristic and the points $(T, M_k(T))$ for $T = 1, 2, \dots 150 \times 3 \cdot 10^4$, and $k = 1/4$, $1/2$, $3/4$ and 1 (see Figure 11).

[10] There is no extra-vanishing in this case using the results of [An-Bu-Fr].

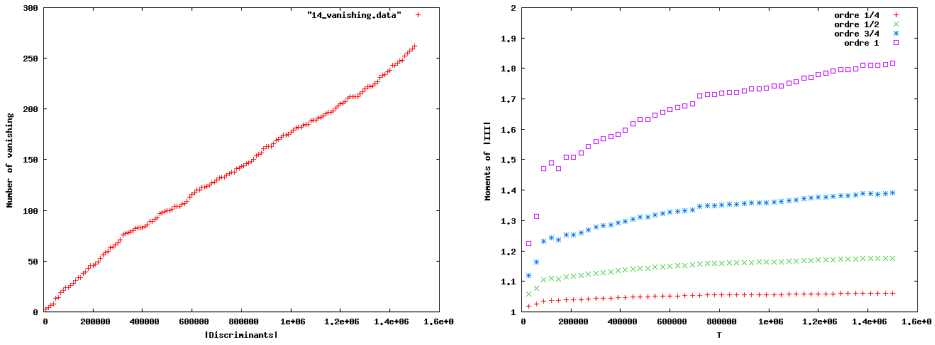


(a) Order 1/4 (b) Order 1/2



(c) Order 3/4 (d) Order 1

FIGURE 9. Moments of the regulators of the twists of $14a1$ and the function given by the heuristics.



(a) Extra-vanishing of $L'(E_d, 1)$ for $E = 14a1$. (b) Moments of different orders for the order of the Tate-Shafarevich groups of the twists of $14a1$.

FIGURE 10

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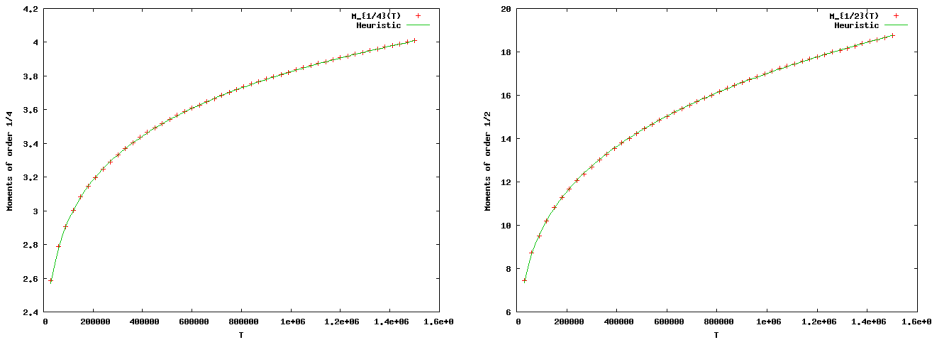
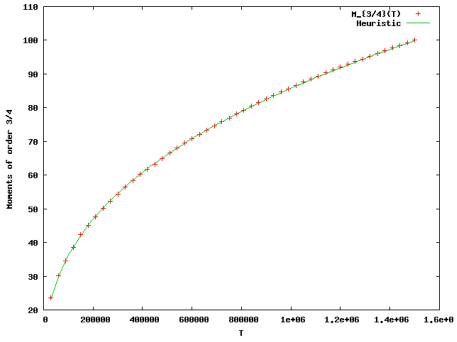
[Coh1] H. Cohen, *A course in Computational Algebraic Number Theory*, Graduate texts in Math. **138**, Springer-Verlag, New-York (1993).

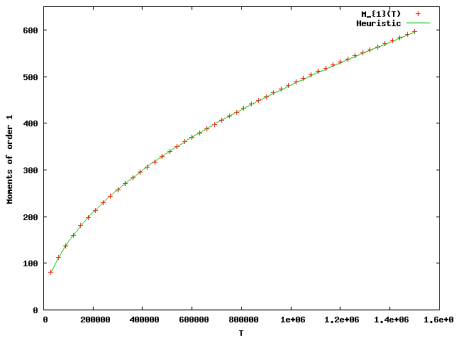


(a) Order 1/4

(b) Order 1/2

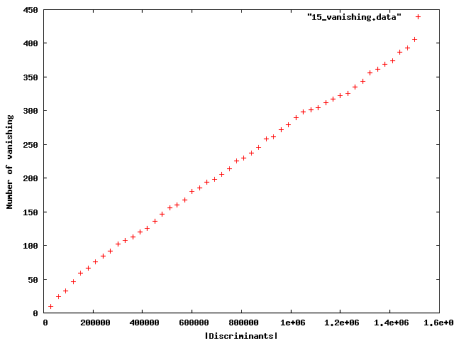


(c) Order 3/4

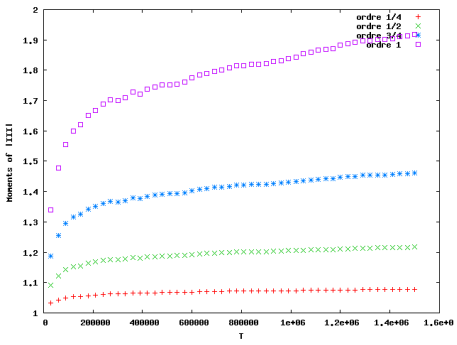


(d) Order 1

FIGURE 11. Moments of the regulators of the twists of $15a1$ and the function given by the heuristics.



(a) Extra-vanishing of $L'(E_d, 1)$ for $E = 15a1$.



(b) Moments of different orders for the order of the Tate-Shafarevich groups of the twists of $15a1$.

FIGURE 12

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