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Hamiltonian formulation of the modified Hasegawa Mima equation

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Abstract

We derive the Hamiltonian structure of the modified Hasegawa-Mima equation from the ion fluid equations applying Dirac's theory of constraints. We discuss the Casimirs obtained from the corresponding Poisson structure.

Zonal flows are believed to have a dramatic effect on the confinement of magnetized plasmas by suppressing the associated turbulent transport, notably the radial transport. Through the years various reduced models have been developed for describing transport in toroidal plasmas, and zonal flows have been observed in simulations of some of these models. An early example of such a reduced model is the Hasegawa-Mima (HM) equation [1], but more general electrostatic fluid models using both the FLR (e.g., [2]) and gyrofluid (e.g., [3, 4]) approaches have been available for many years. Similarly, electromagnetic gyrofluid models (e.g., [5, 6]) have been developed. Some of these models possess noncanonical Hamiltonian structure (see [7, 8, 9]), which has been used to guide construction and led to the identification of new and physically important terms (e.g., [10]), and has also been shown to be important for the consistent calculation of zonal flow dynamics (e.g., [11, 12]).

Zonal flows are also believed to be a nonlinear manifestation of drift wave modulational instability, the physics of which is contained in the HM equation. However, it was recognized in [13] that this physics is more accurately described by a modified form of the HM equation for which the adiabatic electron response is modified so as to take into account the geometry of magnetic surfaces. For example, it is shown in [14] that the modification enhances the generation of zonal flows.

The purpose of the present contribution is to demonstrate that the modified Hasegawa-Mima (mHM) equation possesses Hamiltonian structure by obtaining if from Dirac's theory of constrained Hamiltonian systems [15, 16, 17, 18], a technique used in previous derivations [19, 20, 21]. It is known that the modification of the HM equation applies for more general multifield theories (e.g., [22]); consequently, the methods we use and the results we obtain are of general utility and can be adapted to apply to a very large class of reduced fluid models.

For simplicity of our argumentation, we consider a slab geometry in which x corresponds to the radial direction and y to the poloidal angle. The zonal part of any field $\chi(x, y)$ is given by $\tilde{\chi} = \chi - \bar{\chi}$, where

$$\bar{\chi} = \frac{1}{2\pi} \int_0^{2\pi} dy\, \chi\,.$$

First we consider a parent model with two dynamical equations from which the mHM equation can be derived: one describing the transverse dynamics of the ion velocity field $\mathbf{v}(x, y, t)$ and the other describing the dynamics of the ion density field $n(x, y, t)$:

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\varphi + \mathbf{v} \times \mathbf{B}, \tag{1}$$

$$\dot{n} = -\nabla \cdot (n\mathbf{v}), \tag{2}$$

where the dot indicates the partial derivative with respect to time t. Here we use units such that the ion mass is $M = 1$, its charge $e = 1$, and the amplitude of the magnetic field $B = 1$. The usual HM derivation is

by a rather straightforward combination of the two equations for the density and the velocity field, assuming that the ion polarization velocity is much smaller than the $\mathbf{E}\times\mathbf{B}$ drift. In previous work [20] we have shown a different way of deriving this equation from the Hamiltonian structure of the parent model. This method of derivation allows one to derive the reduced equation with the Hamiltonian structure naturally provided.

The total energy of the ions, given by the sum of their kinetic energy plus the potential energy provided by the electrostatic potential φ, is a conserved quantity that is also the Hamiltonian of the system of Eqs. (1 and 2), viz.

$$H[n,\mathbf{v}] = \int d^2x \left[n\frac{v^2}{2} + n\varphi \right]. \tag{3}$$

The dynamics is determined by the Poisson bracket [23, 24]

$$\{F,G\} = -\int d^2x \left[F_{\mathbf{v}}\cdot\nabla G_n - \nabla F_n\cdot G_{\mathbf{v}} - \left(\frac{\nabla\times\mathbf{v}+\hat{\mathbf{z}}}{n}\right)\cdot F_{\mathbf{v}}\times G_{\mathbf{v}} \right], \tag{4}$$

where we denote the functional derivatives of a given observable $F[n,\mathbf{v}]$ by subscripts, i.e. $F_{\mathbf{v}} = \delta F/\delta\mathbf{v}$ and $F_n = \delta F/\delta n$. In the present context we assume that the electrostatic potential φ is determined by the dynamics of the electrons which leads to a function $\varphi(n_e)$, where n_e is the electron density. From the quasi-neutrality condition, $n = n_e$, the Hamiltonian becomes

$$H[n,\mathbf{v}] = \int d^2x \left[n\frac{v^2}{2} + \psi(n) \right], \tag{5}$$

where $\psi'(n) = \varphi(n)$. The usual HM equation is obtained by neglecting the inertia of the electrons so that their density obeys a Boltzmann law $n_e = n_0 \exp\varphi$, where $n_0 = n_0(x,y) = 1-\lambda(x,y)$ is the electron density at equilibrium. For the mHM equation, this adiabatic response has to take into account the prescription of [13], which here reads $n_e = n_0\exp\tilde{\varphi}$, where $\tilde{\varphi}$ is the zonal part of the potential.

Next we perform a change of variables $(n,\mathbf{v}) \mapsto (n,\phi,D)$ where

$$\Delta\phi = \hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}, \qquad \Delta D = \nabla\cdot\mathbf{v}\,, \tag{6}$$

where Δ denotes the Laplacian. In terms of the new variables (n,ϕ,D), the Hamiltonian of (5) becomes

$$H[n,\phi,D] = \int d^2x \left[n\left(\frac{|\nabla\phi|^2+|\nabla D|^2}{2} + [\phi,D]\right) + \psi(n) \right], \tag{7}$$

where $[f,g] = \hat{\mathbf{z}}\cdot\nabla f\times\nabla g$, and the bracket (4) becomes

$$\begin{aligned}\{F,G\} \;=\; & \int d^2x \Big[F_nG_D - F_DG_n - F_\phi\Delta^{-1}\mathcal{L}\Delta^{-1}G_\phi - F_D\Delta^{-1}\mathcal{L}\Delta^{-1}G_D \\ & + F_D\Delta^{-1}\Lambda\Delta^{-1}G_\phi - F_\phi\Delta^{-1}\Lambda\Delta^{-1}G_D \Big],\end{aligned} \tag{8}$$

where the two linear operators $\mathcal{L}$ and Λ acting on a function $f(\mathbf{x})$ are defined by

$$\mathcal{L}f = \left[\frac{\Delta\phi+1}{n}, f\right], \qquad \Lambda f = -\nabla\cdot\left(\frac{\Delta\phi+1}{n}\nabla f\right). \tag{9}$$

In [20] we showed that starting with the dynamical equations for the two-dimensional ionic fluid (with density $n(x,y)$ and velocity field $\mathbf{v}(x,y)$) and imposing a set of local constraints $\nabla\cdot\mathbf{v}=0$ and $n = N(\phi)$ around equilibrium, where ϕ is the streamfunction, the associated Dirac bracket is written as

$$\{F,G\}_* = \int d^2x\,(\Delta\phi - N(\phi))[(\Delta-\hat{N})^{-1}F_\phi, (\Delta-\hat{N})^{-1}G_\phi], \tag{10}$$

where $\hat{N}$ is the Fréchet derivative of the pseudo-differential function N. Here, the canonical Poisson bracket $[f,g] := \hat{\mathbf{z}} \cdot \nabla f \times \nabla g$. Then, the Hamiltonian is written as

$$H = \frac{1}{2} \int d^2x \, \left(|\nabla \phi|^2 + (N(\phi) + \lambda)^2 \right), \tag{11}$$

where $\lambda = \lambda(x)$ characterizes the equilibrium and the integration is over the two-dimensional cylinder $\mathbb{R} \times [0, 2\pi]$. Compared to the HM equation, where the second constraint was $N(\phi) = \phi - \lambda$, we impose here the following modified constraint:

$$N(\phi) = \phi - \frac{1}{2\pi} \int_0^{2\pi} \phi dy - \lambda. \tag{12}$$

Therefore, $\hat{N} = 1 - P$, with the operator P defined by

$$P(\chi) = \frac{1}{2\pi} \int_0^{2\pi} \chi dy \ ,$$

which evidently is symmetric in the sense that $\int f P g = \int g P f$.

The Poisson bracket for the mHM equation is given by

$$\{F, G\}_* = \int d^2x \, (\Delta\phi - (1-P)\phi + \lambda) \big[(\Delta - 1 + P)^{-1} F_\phi, (\Delta - 1 + P)^{-1} G_\phi \big], \tag{13}$$

and its Hamiltonian is

$$H = \frac{1}{2} \int d^2x \, \left(|\nabla \phi|^2 + (\phi - P\phi)^2 \right).$$

To see that this defines the correct Hamiltonian structure, note that $H_\phi = -(\Delta - 1 + P)\phi$ and the mHM equation for ϕ is given by $\dot{\phi} = \{\phi, H\}$ which reads as

$$(\Delta - 1 + P)\dot{\phi} = \big[\Delta\phi - \phi + P\phi + \lambda, \phi \big] .$$

In order to obtain the two dynamical equations for the mHM system, we perform the change of variables $\phi \mapsto (\bar{\phi}, \tilde{\phi})$ defined by $\phi = \bar{\phi} + \tilde{\phi}$ and $\bar{\phi} = P\phi$. Any observable $F(\phi) = \hat{F}(\bar{\phi}, \tilde{\phi})$ has the functional derivative chain rule relation

$$F_\phi = (1-P)\hat{F}_{\bar{\phi}} + P\hat{F}_{\tilde{\phi}},$$

where we notice that the part $P\hat{F}_{\bar{\phi}} - P\hat{F}_{\tilde{\phi}}$ is only a function of x. In what follows we drop the hats on F for simplicity. The Poisson bracket in these new field variables is

$$\begin{aligned}
\{F, G\}_* \quad = \quad & - \int dx \, \Delta^{-1} (PF_{\bar{\phi}} - PF_{\tilde{\phi}}) P[\Delta\tilde{\phi} - \tilde{\phi}, (\Delta - 1 + P)^{-1} G_{\tilde{\phi}}] \\
& + \int dx \, \Delta^{-1} (PG_{\bar{\phi}} - PG_{\tilde{\phi}}) P[\Delta\tilde{\phi} - \tilde{\phi}, (\Delta - 1 + P)^{-1} F_{\tilde{\phi}}] \\
& + \int dx \, (\Delta\bar{\phi} + \lambda) P[(\Delta - 1 + P)^{-1} F_{\tilde{\phi}}, (\Delta - 1 + P)^{-1} G_{\tilde{\phi}}] \\
& + \int d^2x \, (\Delta\tilde{\phi} - \tilde{\phi})[(\Delta - 1 + P)^{-1} F_{\tilde{\phi}}, (\Delta - 1 + P)^{-1} G_{\tilde{\phi}}],
\end{aligned} \tag{14}$$

and the Hamiltonian is

$$H = \frac{1}{2} \int d^2x \, \left(|\nabla \tilde{\phi}|^2 + \tilde{\phi}^2 + |\nabla \bar{\phi}|^2 \right), \tag{15}$$

where we have used the fact that $\int d^2x \nabla\tilde{\phi} \cdot \nabla\bar{\phi} = 0$. The two dynamical equations for the mHM system are

$$\begin{aligned}
\dot{\bar{\phi}} &= \{\bar{\phi}, H\} = \Delta^{-1} P[\Delta\tilde{\phi}, \tilde{\phi}], \\
\dot{\tilde{\phi}} &= \{\tilde{\phi}, H\} = (\Delta - 1)^{-1} \left((1-P)[\Delta\tilde{\phi}, \tilde{\phi}] + [\Delta\tilde{\phi} - \tilde{\phi}, \bar{\phi}] + [\Delta\bar{\phi} + \lambda, \tilde{\phi}] \right),
\end{aligned}$$

where we have used the fact that $H_{\bar\phi} - PH_{\tilde\phi} = -\Delta\bar\phi$ and $H_{\tilde\phi} = -\Delta\tilde\phi + \tilde\phi$.

The Casimir invariants of the Poisson bracket given by Eq. (10) are given by

$$C(\phi) = \int d^2x\, \alpha(\Delta\phi - N(\phi)),$$

where α is scalar function of one variable. For the Poisson bracket given by Eq. (14), the condition which determines the Casimir invariants is given by

$$C_{\bar\phi} - PC_{\tilde\phi} + C_{\tilde\phi} = (\Delta - 1 + P)\, \alpha'(\Delta\tilde\phi - \tilde\phi + \Delta\bar\phi + \lambda)\,,$$

whence we obtain

$$C(\tilde\phi, \bar\phi) = \int d^2x\, \alpha(\Delta\tilde\phi - \tilde\phi + \Delta\bar\phi + \lambda) + \int dx\, \beta(x) P\tilde\phi,$$

where α and β are two arbitrary scalar functions of one variable. Notice in particular that $P\tilde\phi(x,y)$ is a local Casimir invariant, obtained for $\alpha = 0$ and $\beta(x) = \delta(x - x')$. We have chosen here to restrict to $P\tilde\phi = 0$.

It should be pointed out that Dirac's theory ensures the Jacobi identity for the bracket (13) before the change of variables. Therefore if the change of variables is invertible, i.e., under the condition $P\tilde\phi = 0$, then the Jacobi identity is ensured too. However we show explicitly that the Poisson bracket of (14) satisfies the Jacobi identity unconditionally, i.e., for all field variables $(\bar\phi, \tilde\phi)$, not necessarily restricted to $P\tilde\phi = 0$. In order to demonstrate this, we write the bracket (14) as

$$\{F, G\}_* = \int d^2x \left((A - P)\tilde\phi + \Delta\bar\phi + \lambda\right)\left[A^{-1}((1-P)F_{\tilde\phi} + PF_{\bar\phi}), A^{-1}((1-P)G_{\tilde\phi} + PG_{\bar\phi})\right],$$

where $A = \Delta - 1 + P$. In the following, we let $f := (1-P)F_{\tilde\phi} + PF_{\bar\phi}$. As was shown in [7], only the functional derivatives of $\{F, G\}$ that take into account the explicit dependence of the bracket on the variables are needed. These are

$$\{F, G\}_{\tilde\phi} = (A - P)\left[A^{-1}f, A^{-1}g\right], \tag{16}$$

$$\{F, G\}_{\bar\phi} = P\Delta\left[A^{-1}f, A^{-1}g\right]. \tag{17}$$

The computation of $\{\{F, G\}, H\}$ leads to

$$\begin{aligned}\{\{F, G\}, H\} \quad = \quad & \int d^2x \left((A - P)\tilde\phi + \Delta\bar\phi + \lambda\right) \\ & \times\Big[A^{-1}\left((1-P)(A-P)[A^{-1}f, A^{-1}g] + P\Delta[A^{-1}f, A^{-1}g]\right), A^{-1}h\Big].\end{aligned} \tag{18}$$

Since $(1-P)(A-P) + P\Delta = A$, Eq. (18) becomes

$$\{\{F, G\}, H\} = \int d^2x((A - P)\tilde\phi + \Delta\bar\phi + \lambda)\left[[A^{-1}f, A^{-1}g], A^{-1}h\right]$$

Therefore, the Jacobi identity for the bracket $\{\cdot,\cdot\}$ follows from the Jacobi identity for the bracket $[\cdot,\cdot]$.

Given the Hamiltonian structure we are now set to use Hamiltonian techniques for instance for investigating equilibrium and stability (e.g., [8, 25, 26]). In addition, the proposed construction is sufficiently general that it could be used beyond mHM, in more general Hamiltonian zonal flow models.

[1] A. Hasegawa and K. Mima, Phys. Rev. Lett. **37**, 690 (1976).
[2] S. Hamaguchi and W. Horton, Phys. Fluids B **2**, 1833 (1990).
[3] R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison, Phys. Fluids **28**, 2466 (1985).
[4] G. W. Hammett, W. Dorland, and F. Perkins, Phys. Fluids B **4**, 2052 (1992).
[5] P. B. Snyder and G. W. Hammett, Phys. Plasmas **8**, 3199 (2001).
[6] F. L. Waelbroeck, R. D. Hazeltine, and P. J. Morrison, Phys.Plasmas **16**, 032109 (2009).
[7] P.J. Morrison, AIP Conference Proceedings **88**, 13 (1982).
[8] P.J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).

[9] P.J. Morrison, Phys. Plasmas **12**, 058102 (2005).
[10] R. D. Hazeltine, C. T. Hsu, and P. J. Morrison, Phys. Fluids **30**, 3204 (1987).
[11] J. A. Krommes and R. A. Kolesnikov, Phys. Plasmas **11**, L29 (2004).
[12] R. A. Kolesnikov and J. A. Krommes, Phys. Plasmas **12** 122302 (2005).
[13] W. Dorland and G. Hammett, Phys. Fluids B **5**, 812 (1993).
[14] R.L. Dewar and R.F. Abdullatif, Proceedings of the CSIRO/COSNet Workshop on Turbulence and Coherent Structures, Canberra, Australia, 10-13 January 2006; Frontiers in Turbulence and Coherent Structures - World Scientific Lecture Notes in Complex Systems, Vol. 6, eds. J.P. Denier and J.S. Frederiksen (World Scientific, 2007), pp 415-430.
[15] P.A.M. Dirac, Can. J. Math. **2**, 129 (1950).
[16] E.C.G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (John Wiley & Sons, New York, 1974).
[17] A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1976).
[18] K. Sundermeyer, *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).
[19] P.J. Morrison, N.R. Lebovitz and J.A. Biello, Ann. Phys. **324**, 1747 (2009).
[20] C. Chandre, E. Tassi and P.J. Morrison, Phys. Plasmas **17**, 042307 (2010).
[21] C. Chandre, L. de Guillebon, A. Back, E. Tassi, and P. J. Morrison, J. Phys. A **46**, 125203 (2013).
[22] F.L. Waelbroeck, P.J. Morrison, and W. Horton, Plasma Phys. Control. Fusion **46**, 1331 (2004).
[23] P.J. Morrison and J.M. Greene, Phys. Rev. Lett. **45**, 790 (1980)
[24] E. Tassi, C. Chandre, and P. J. Morrison, Phys. Plasmas **16**, 082301 (2009).
[25] D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, Phys. Rep. **123**, 1 (1985).
[26] T. Andreussi, P. J. Morrison, and F. Pegoraro, Phys. Plasmas **20**, 092104 (2013).