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# AMPLE HIERARCHY

ANDREAS BAUDISCH, AMADOR MARTIN-PIZARRO AND MARTIN ZIEGLER

ABSTRACT. The ample hierarchy of geometries of stables theories is strict. We generalise the construction of the free pseudospace to higher dimensions and show that the  $n$ -dimensional free pseudospace is  $\omega$ -stable  $n$ -ample yet not  $(n + 1)$ -ample. In particular, the free pseudospace is not 3-ample. A thorough study of forking is conducted and an explicit description of canonical bases is exhibited.

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## 1. INTRODUCTION

Morley's renowned categoricity theorem [9] described any model of an uncountably categorical theory in terms of basic foundational bricks, so-called strongly minimal sets. A long-standing conjecture aimed to understand the geometry of a strongly minimal set in terms of three archetypal examples: a trivial set, a vector space over a division ring and an irreducible curve over an algebraically closed field. The conjecture was proven wrong [7] by obtaining in a clever fashion a non-trivial strongly minimal set which does not interpret a group. In particular, Hrushovski's new strongly minimal set does not interpret any infinite field, which follows from the fact that the obtained structure is CM-trivial. Recall that CM-triviality is a generalisation of 1-basedness and it prohibits a certain point-line-plane configuration which is present in Euclidian geometry. The simplest example of a CM-trivial theory that is not 1-based is the *free pseudoplane*: an infinite forest with infinite

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branching at every node. CM-trivial theories are rather rigid and in particular definable groups of finite Morley rank are nilpotent-by-finite [10].

Taking the pseudoplane as a guideline, a non CM-trivial  $\omega$ -stable theory which does not interpret an infinite field was constructed in a pure combinatorial way [2]. The structure so obtained is of infinite rank, and it remains still open whether the construction could be modified to produce one of finite Morley rank. In [11, 4] a whole hierarchy of new geometries (called  $n$ -ample) was exhibited, infinite fields being at the top of the classification. Evans suggested that his construction could be used to show that the hierarchy is strict, though no proof was given.

The goal of this article is to generalise the aforementioned construction to higher dimensions in order to show that the  $N$ -dimensional pseudospace is  $N$ -ample yet not  $(N + 1)$ -ample, showing therefore that the ample hierarchy is proper. After a thorough study of the pseudospace, we were able to simplify the combinatorics behind the original construction. In particular, we characterise non-forking and give explicit descriptions of canonical basis of finitary types over certain substructures. Moreover, we show that the theory of the pseudospace has weak elimination of imaginaries.

Tent has obtained the same result [12] independently; however, we present a different construction and axiomatisation of the free pseudospace for higher dimensions. We are indebted to her as she pointed out that the prime model of the 2-dimensional free pseudospace could be seen as a building. We would like to express our gratitude to Yoneda for a careful reading of a first version of this work.

## 2. AMPLE CONCEPTS

Throughout this article, we assume a certain knowledge of stability theory, in particular nonforking and canonical bases. We refer the reader to [13] for a gentle and careful explanation of these notions. All throughout this article, we work inside a sufficiently saturated model of a first-order theory  $T$  and all sets are small subsets of it.

We first state a fact, which we believe is common knowledge, that will be used repeatedly.

**Fact 2.1.** Given a stable theory  $T$  and sets  $A, B, C$  and  $D$ , if  $\text{acl}^{\text{eq}}(B) \cap \text{acl}^{\text{eq}}(C) = \text{acl}^{\text{eq}}(A)$  and  $D \downarrow_A BC$ , then

$$\text{acl}^{\text{eq}}(DB) \cap \text{acl}^{\text{eq}}(DC) = \text{acl}^{\text{eq}}(DA).$$

*Proof.* In order to show that  $\text{acl}^{\text{eq}}(DB) \cap \text{acl}^{\text{eq}}(DC) \subset \text{acl}^{\text{eq}}(DA)$ , pick an element  $e$  in  $\text{acl}^{\text{eq}}(DB) \cap \text{acl}^{\text{eq}}(DC)$ . The independence  $D \downarrow_A BC$  yields that  $\text{Cb}(De/BC)$  lies in  $\text{acl}^{\text{eq}}(B) \cap \text{acl}^{\text{eq}}(C) = \text{acl}^{\text{eq}}(A)$ , so  $e$  lies in  $\text{acl}^{\text{eq}}(DA)$ .  $\square$

Recall now the definition of CM-triviality and  $n$ -ampleness [11, 4].

**Definition 2.2.** Let  $T$  be a stable theory.

The theory  $T$  is *1-based* if for every pair of algebraically closed (in  $T^{\text{eq}}$ ) subsets  $A \subset B$  and every real tuple  $c$ , we have that  $\text{Cb}(c/A)$  is algebraic over  $\text{Cb}(c/B)$ . Equivalently, for every algebraically closed set  $A$  (in  $T^{\text{eq}}$ ) and every real tuple  $c$ , the canonical base  $\text{Cb}(c/A)$  is algebraic over  $c$ .

The theory  $T$  is *CM-trivial* if for every pair of algebraically closed (in  $T^{\text{eq}}$ ) subsets  $A \subset B$  and every real tuple  $c$ , if  $\text{acl}^{\text{eq}}(Ac) \cap B = A$ , then  $\text{Cb}(c/A)$  is algebraic over  $\text{Cb}(c/B)$ .

The theory  $T$  is called  $n$ -ample if there are  $n + 1$  real tuples satisfying the following conditions (possibly working over parameters):

- (1)  $\text{acl}^{\text{eq}}(a_0, \dots, a_i) \cap \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1}, a_{i+1}) = \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1})$  for every  $0 \leq i < n$ ,
- (2)  $a_{i+1} \downarrow_{a_i} a_0, \dots, a_{i-1}$  for every  $1 \leq i < n$ ,
- (3)  $a_n \not\downarrow a_0$ .

By inductively choosing models  $M_i \supset a_i$  such that

$$M_i \downarrow_{a_i} M_0, \dots, M_{i-1}, a_{i+1}, \dots, a_n,$$

Fact 2.1 allows us to deduce the following, which was already remarked in [10, Corollary 2.5] in the case of CM-triviality.

**Remark 2.3.** In the definition of  $n$ -ampleness, we can replace all tuples by models.

**Corollary 2.4.** *A stable theory  $T$  is  $n$ -ample if and only if  $T^{\text{eq}}$  is.*

Clearly, every 1-based theory is CM-trivial. Furthermore, a theory is 1-based if and only if it is not 1-ample; it is CM-trivial if and only if it is not 2-ample [11]. Also, to be  $n$ -ample implies  $(n-1)$ -ampleness: by construction, if  $a_0, \dots, a_n$  witness that  $T$  is  $n$ -ample, the sequence  $a_0, \dots, a_{n-1}$  witnesses that  $T$  is  $(n-1)$ -ample. In order to see this, we need only show that

$$a_{n-1} \not\downarrow a_0,$$

which follows from

$$a_n \not\downarrow a_0$$

and

$$a_n \downarrow_{a_{n-1}} a_0,$$

by transitivity.

In order to prove that the  $N$ -dimensional free pseudospace is not  $(N+1)$ -ample, we need only consider some of the consequences from the conditions listed above. Therefore, we will isolate such conditions for Section 8.

**Remark 2.5.** If the (possibly infinite) tuples  $a_0, \dots, a_n$  witness that  $T$  is  $n$ -ample, they satisfy the following conditions:

- (a)  $a_n \downarrow_{a_i} a_{i-1}$  for every  $1 \leq i < n$ .
- (b)  $\text{acl}^{\text{eq}}(a_i, a_{i+1}) \cap \text{acl}^{\text{eq}}(a_i, a_n) = \text{acl}^{\text{eq}}(a_i)$  for every  $0 \leq i < n-1$ .
- (c)  $a_n \not\downarrow_{\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1})} a_i$  for every  $0 \leq i < n-1$ .

If the tuples  $a_0, \dots, a_n$  witness that  $T$  is  $n$ -ample over some set of parameters  $A$ , by adding all elements of  $A$  to each of the tuples, then we may assume that all the conditions hold with  $A = \emptyset$ .

*Proof.* Let  $a_0, \dots, a_n$  witness that  $T$  is  $n$ -ample.

First, note that  $\text{acl}^{\text{eq}}(a_1) \cap \text{acl}^{\text{eq}}(a_2) \subset \text{acl}^{\text{eq}}(a_0)$  by property (1). For  $i \leq 2$ , the set  $\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1})$  is contained in  $\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1})$  again by (1). Now, condition (2) implies that  $\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1})$  is a subset of  $\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i-1})$ . By induction, we have that

$$\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1}) \subset \text{acl}^{\text{eq}}(a_0).$$

The independence  $a_n \downarrow_{a_i} a_{i-1}$  follows directly from property (2) and yields (a). Since  $a_n \downarrow_{a_{i+2}} a_0, \dots, a_{i+1}$ , we have that

$$a_n \downarrow_{a_i, a_{i+2}} a_{i+1}.$$

Hence,

$$\text{acl}^{\text{eq}}(a_i, a_{i+1}) \cap \text{acl}^{\text{eq}}(a_i, a_n) \subset \text{acl}^{\text{eq}}(a_i, a_{i+1}) \cap \text{acl}^{\text{eq}}(a_i, a_{i+2}),$$

and thus in  $\text{acl}^{\text{eq}}(a_0, \dots, a_i)$  by (1). Since

$$a_{i+1} \downarrow_{a_i} a_0, \dots, a_{i-1},$$

we get (b).

If

$$a_n \downarrow_{\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1})} a_i$$

for some  $0 \leq i < n-1$ , then  $i > 0$  by (3). Since  $a_n \downarrow_{a_i} a_0, \dots, a_{i-1}$ , transitivity gives that

$$a_n \downarrow_{\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1})} a_0, \dots, a_i.$$

Thus, we obtain the independence  $a_n \downarrow_{a_0} a_0, \dots, a_i$  and in particular  $a_n \downarrow_{a_0} a_1$ . Since  $a_n \downarrow_{a_1} a_0$  by (2) and  $\text{acl}^{\text{eq}}(a_0) \cap \text{acl}^{\text{eq}}(a_1) = \emptyset$  by (1), this implies that

$$a_n \downarrow a_0,$$

which contradicts (3).  $\square$

In [3], a weakening of CM-triviality was introduced, following the spirit of [8], where some of the consequences for definable groups in 1-based theories were extended to type-definable groups in theories with the *Canonical Base Property*. For the purpose of this article, we extend the definition to all values of  $n$ . However, we do not know of any definability properties for groups that may follow from the general definition.

Let  $\Sigma$  be an  $\emptyset$ -invariant family of partial types. Recall that a type  $p$  over  $A$  is *internal to  $\Sigma$* , or  $\Sigma$ -*internal*, if for every realisation  $a$  of  $p$  there is some superset  $B \supset A$  with  $a \downarrow_A B$ , and realisations  $b_1, \dots, b_r$  of types in  $\Sigma$  based on  $B$  such that  $a$  is definable over  $B, b_1, \dots, b_r$ . If we replace definable by algebraic, then we say that  $p$  is *almost internal to  $\Sigma$*  or *almost  $\Sigma$ -internal*.

**Definition 2.6.** A stable theory  $T$  is called  *$n$ -tight* (possibly working over parameters) with respect to the family  $\Sigma$  if, whenever there are  $n+1$  real tuples  $a_0, \dots, a_n$  satisfying the following conditions:

- (1)  $\text{acl}^{\text{eq}}(a_0, \dots, a_i) \cap \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1}, a_{i+1}) = \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1})$  for every  $0 \leq i < n$ .
- (2)  $a_{i+1} \downarrow_{a_i} a_0, \dots, a_{i-1}$  for every  $1 \leq i < n$ ,

then  $\text{Cb}(a_n/a_0)$  is almost  $\Sigma$ -internal over  $a_1$ .

**Remark 2.7.** As before, we may assume that all tuples are models. In particular, the theory  $T$  is  $n$ -tight if and only if  $T^{\text{eq}}$  is.

A theory  $T$  is 2-tight with respect to  $\Sigma$  if for every pair of sets  $A \subset B$  and every tuple  $c$ , if  $\text{acl}^{\text{eq}}(Ac) \cap \text{acl}^{\text{eq}}(B) = \text{acl}^{\text{eq}}(A)$ , then  $\text{Cb}(c/A)$  is almost  $\Sigma$ -internal over  $\text{Cb}(c/B)$ . In particular, this notion agrees with [3, Definition 3.1]

If  $T$  is not  $n$ -ample, it is  $n$ -tight with respect to any family  $\Sigma$ . Furthermore, if  $T$  is  $(n - 1)$ -tight, it is  $n$ -tight.

*Proof.* The equivalence between both definitions is a standard reformulation by setting  $a_0 = A$ ,  $a_1 = \text{Cb}(c/B)$  and  $a_2 = c$  for one direction (working over  $\text{acl}^{\text{eq}}(a_0) \cap \text{acl}^{\text{eq}}(a_1)$ ), and  $A = a_0$ ,  $B = a_0 \cup \text{Cb}(a_2/a_1)$  and  $c = a_2$  for the other.

If  $T$  is not  $n$ -ample, it is clearly  $n$ -tight, since algebraic types are always almost  $\Sigma$ -internal for any  $\Sigma$ .

Suppose now that  $T$  is  $(n - 1)$ -tight, and consider  $n + 1$  tuples  $a_0, \dots, a_n$  witnessing (1) and (2). So do  $a_0, \dots, a_{n-1}$  as well. Hence, the canonical base  $\text{Cb}(a_{n-1}/a_0)$  is almost  $\Sigma$ -internal over  $a_1$ .

Since  $a_n \downarrow_{a_{n-1}} a_0$ , it follows by transitivity that  $\text{Cb}(a_n/a_0)$  is algebraic over  $\text{Cb}(a_{n-1}/a_0)$  and therefore the former is also almost  $\Sigma$ -internal over  $a_1$ . □

In this article, we will show that the free  $N$ -dimensional pseudospace is  $N$ -ample yet not  $(N + 1)$ -ample. Furthermore, if  $N \geq 2$ , it is  $N$ -tight with respect to the family of Lascar rank 1 types.

### 3. FRAÏSSÉ LIMITS

The results in this section were obtained by the third author in an unpublished note [15] (in a slightly more general context). We include them here for the sake of completeness.

Throughout this section, let  $\mathcal{K}$  denote a class of structures closed under isomorphisms in a fixed language  $L$ . We assume that the empty structure  $0$  is in  $\mathcal{K}$ . Furthermore, a class  $\mathcal{S}$  of embeddings between elements of  $\mathcal{K}$  is given, called *strong embeddings*, containing all isomorphisms and closed under composition. We also assume that the empty map  $0 \rightarrow A$  is in  $\mathcal{S}$  for every  $A \in \mathcal{K}$ .

We call a substructure  $A$  of  $B$  *strong* if the inclusion map is in  $\mathcal{S}$ . We denote this by  $A \leq B$ .

**Definition 3.1.** Given an infinite cardinal  $\kappa$ , an increasing chain of strong substructures  $\{A_i\}_{i < \kappa}$  is *rich* if, for all  $i < \kappa$  and all strong  $f : A_i \rightarrow B$ , there is some  $i \leq j < \kappa$  and a strong  $g : B \rightarrow A_j$  such that  $gf : A_i \rightarrow A_j$  is the inclusion map.

A *Fraïssé limit* of  $(\mathcal{K}, \mathcal{S})$  of length  $\kappa$  is the union of a rich sequence of length  $\kappa$ .

**Theorem 3.2.** *Suppose  $(\mathcal{K}, \mathcal{S})$  satisfies the following conditions:*

- (1) *There are at most countably many isomorphism types in  $\mathcal{K}$ .*
- (2) *For each  $A$  and  $B$  in  $\mathcal{K}$ , there are at most countably many strong embeddings  $A \rightarrow B$ .*
- (3)  *$\mathcal{K}$  has the amalgamation property with respect to strong embeddings.*

*Then rich sequences exist and all Fraïssé limits of the same length are isomorphic.*

The existence of rich sequences is easy to show. The uniqueness for countable Fraïssé limits will follow from the next lemma. For that, let us say that  $A$  is *r-strong* in a Fraïssé limit  $M$ , denoted by  $A \leq_r M$ , if  $M$  is the union of a rich sequence starting with  $A$ .

**Lemma 3.3.** *A Fraïssé limit  $M$  has the following properties:*

- (a)  $\emptyset \leq_r M$
- (b) *for every finite  $A \leq_r M$  and every  $B$  in  $\mathcal{K}$  such that  $A \leq B$ , there is an  $r$ -strong subset  $B'$  of  $M$  containing  $A$  and isomorphic to  $B$  over  $A$ .*

*Proof.* We observe first that if  $A_0 \leq A_1 \leq \dots$  is a rich sequence and  $B \leq A_0$ , then the sequence  $B \leq A_0 \leq A_1 \leq \dots$  is also rich. This implies (a). For (b), choose a rich sequence  $A = A_0 \leq A_1 \leq \dots$  with union  $M$ . If  $B \geq A$  is given, there exists, by richness, some index  $j$  and  $B' \leq A_j$  isomorphic to  $B$  over  $A$ . The set  $B'$  is  $r$ -strong in  $M$ , since the subsequence  $B' \leq A_j \leq A_{j+1} \leq \dots$  is again rich.  $\square$

The lemma implies that countable Fraïssé limits are isomorphic by a standard back-and-forth argument: given two Fraïssé limits  $M$  and  $M'$  with rich sequences  $A_0 \leq A_1 \leq \dots$  and  $A'_0 \leq A'_1 \leq \dots$ , consider an isomorphism  $B \rightarrow B'$ , where  $B$  is strong in  $A_i$  and  $B'$  is strong in  $A'_i$ . Then there is an extension to an isomorphism  $C \rightarrow C'$  such that  $A_i \leq C \leq A_j$  and  $A'_i \leq C' \leq A'_j$  for some  $j > i$ . This results in an ascending sequence of isomorphisms whose union yields an isomorphism  $M \rightarrow M'$ .

**Corollary 3.4.** *Assume that  $M$  and  $M'$  are Fraïssé limits of the same length. Given sets  $B \leq_r M$  and  $B' \leq_r M'$ , every isomorphism  $B \rightarrow B'$  extends to an isomorphism  $M \rightarrow M'$ .*

The convention that  $\mathcal{S}$  contains all isomorphisms and is closed under composition represents no obstacle, thanks to the following easy remark.

**Remark 3.5.** Let  $\mathcal{S}$  be a set of embeddings between elements of  $\mathcal{K}$  with the amalgamation property. The closure of  $\mathcal{S}$  together with all isomorphisms under composition has again the amalgamation property.

#### 4. THE FREE PSEUDOSPACE

In this section, we will construct and axiomatise the  $N$ -dimensional free pseudospace, which is a generalisation of [2], based on the free pseudoplane. An alternative axiomatisation, in terms of *flags*, may be found in [1].

**Remark 4.1.** Recall that the (free) pseudoplane is a bicolored graph with infinite branching and no loops. These elementary properties describe a complete  $\omega$ -stable theory of Morley rank  $\omega$ .

Quantifier elimination is obtained after adding the collection of binary predicates:

$$d_n(x, y) \iff \text{the distance between } x \text{ and } y \text{ is exactly } n.$$

In particular, since there are no loops, the set  $d_1(x, a)$  is strongly minimal. Morley rank for this theory is additive and agrees with Lascar rank. Given the type of an element  $c$  over an algebraically closed set  $A$ , its canonical base  $\text{Cb}(c/A)$  is the unique point  $a$  in  $A$  whose distance to  $c$  is smallest possible (or empty if there is no path between  $c$  and  $A$ ). It follows that the theory has weak elimination of imaginaries and is moreover  $CM$ -trivial but not 1-based.

The idea behind the construction of the free pseudospace [2] is to take a free pseudoplane, whose vertices of one color are called *planes* and vertices of the other are referred to as *lines*, and on each line put an infinite set of *points*, such that, for each plane, the lines which are incident with it, together with the points on them form again a free pseudoplane. Nevertheless, the actual construction was

rather combinatorial and therefore less intuitive. Instead, our approach consists in building a model out of some basic operations and study the complete theory of such a structure, in order to show that it agrees with the free pseudospace in [2] for dimension  $N = 2$ .

**Definition 4.2.** For  $N \geq 1$ , a *colored  $N$ -space*  $A$  is a colored graph with colors (or *levels*)  $\mathcal{A}_0, \dots, \mathcal{A}_N$  such that an element in  $\mathcal{A}_i$  can only be linked to vertices in  $\mathcal{A}_{i-1} \cup \mathcal{A}_{i+1}$ . We will furthermore consider two (invisible) levels  $\mathcal{A}_{-1}$  and  $\mathcal{A}_{N+1}$ , consisting of a single *imaginary* element  $a_{-1}$  and  $a_{N+1}$  respectively, which are connected to all vertices in  $\mathcal{A}_0$  and  $\mathcal{A}_N$  respectively. Given such a graph  $A$  and a subset  $s$  of  $\{0, \dots, N\}$ , we set

$$\mathcal{A}_s(A) = \bigcup_{i \in s} \mathcal{A}_i(A).$$

Given  $x$  and  $y$  in  $\mathcal{A}_s(A)$ , its distance in  $\mathcal{A}_s(A)$  is denoted by  $d_s^A(x, y)$ .

Given a colored  $N$ -space  $A$  and vertices  $a$  in  $\mathcal{A}_l(A)$  and  $b \in \mathcal{A}_r(A)$ , we say that  $b$  *lies over*  $a$  (or  $a$  *lies beneath*  $b$ ) if  $l < r$  and there is a path of the form  $a = a_l, a_{l+1}, \dots, a_r = b$ . Note that  $a_k$  must be in  $\mathcal{A}_k(A)$ . By convention, the point  $a_{N+1}$  lies over all other vertices (including  $a_{-1}$ ) and  $a_{-1}$  lies beneath all other vertices.

With  $A$ ,  $a$  and  $b$  as above, we denote by  $A_a$  the subgraph of  $A$  consisting of all the elements of  $A$  lying over  $a$ . Similarly  $A^b$  denotes the subgraph of all the elements lying beneath  $b$ . The subgraph  $A_a^b = (A_a)^b$  consists of all the elements of  $A$  lying between  $a$  and  $b$ , if  $a$  lies beneath  $b$ .

Observe that, after a suitable renumbering of levels, the subgraph  $A_a$  becomes a colored  $(N - l - 1)$ -space, whereas  $A^b$  becomes a colored  $(r - 1)$ -space and  $A_a^b$  a colored  $(r - l - 2)$ -space.

**Notation.** Intervals are assumed to be non-empty

**Definition 4.3.** Given an interval  $s = (l_s, r_s)$  (where  $-1$  and  $N + 1$  are possible values) in  $\{0, \dots, N\}$  and a colored  $N$ -space  $A$  with two distinguished vertices  $a_{l_s}$  in  $\mathcal{A}_{l_s}(A)$  beneath  $a_{r_s}$  in  $\mathcal{A}_{r_s}(A)$ , we say that  $B = A \cup \{b_i \mid i \in s\}$  with  $b_i \in \mathcal{A}_i(B)$  is obtained from  $A$  by applying *the operation*  $\alpha_s$  on  $(a_{l_s}, a_{r_s})$  if

- (a) The sequence  $a_{l_s}, b_{l_s+1}, \dots, b_{r_s-1}, a_{r_s}$  is a path in  $B$ .
- (b)  $B$  has no new edges besides the aforementioned (and those of  $A$ ).

If either  $l_s = -1$  or  $r_s = N + 1$ , then  $a_{l_s}$  lies automatically beneath  $a_{r_s}$ .

The  $N$ -dimensional pseudospace will now be obtained by iterating countably many times all operations  $\alpha_s$  for  $s$  varying over all intervals in  $[0, N]$ . Clearly, we have the following.

**Remark 4.4.** If both  $B_1$  and  $B_2$  are obtained from  $A$  by applying respectively  $\alpha_{s_1}$  and  $\alpha_{s_2}$ , then the graph-theoretic amalgam  $C = B_1 \otimes_A B_2$  is obtained by applying  $\alpha_{s_1}$  to  $B_2$  and  $\alpha_{s_2}$  to  $B_1$ .

**Definition 4.5.** Given two colored  $N$ -spaces  $A$  and  $B$ , we say that  $A$  a *strong subspace* of  $B$  if  $A$  is a subgraph of  $B$  and  $B$  can be obtained from  $A$  by a (possibly infinite) sequence of operations  $\alpha_s$  for varying  $s$ . We denote this by  $A \leq B$ .

A *strong embedding*  $A \rightarrow B$  is an isomorphism of  $A$  with a strong subspace of  $B$ . Let  $\mathcal{K}_\infty$  be the class of all finite colored  $N$ -spaces  $A$  with  $\emptyset \leq A$ . By the last



remark and Remark 3.5, the class  $\mathcal{K}_\infty$  has the amalgamation property with respect to strong embeddings. Clearly, there are only countably many isomorphism types in  $\mathcal{K}_\infty$  and only finitely many maps between two structures of  $\mathcal{K}_\infty$ . We can consider the subclass  $\mathcal{K}_0$ , where by a 0-strong embedding we only allow operations  $\alpha_s$ , for singleton  $s$ . Again, the class  $\mathcal{K}_0$  has the amalgamation property.

By Theorem 3.2, we define the following structures:

**Definition 4.6.** Let  $M_\infty^N$  be the Fraïssé limit of  $\mathcal{K}_\infty$  with strong embeddings and  $M_0^N$  be the Fraïssé limit of  $\mathcal{K}_0$  with 0-strong embeddings, starting from a given (fixed) path  $a_0 - \dots - a_N$ , where  $a_i \in \mathcal{A}_i$ .

We will drop the superindex  $N$  in  $M_\infty^N$  or  $M_0^N$  when they are clear from the context.

In particular, the structure  $M_0^2$  so obtained agrees with the prime model constructed in [2], as Theorem 4.14 will show.

**Remark 4.7.** Let  $p$  be either 0 or  $\infty$ . Consider  $a$  in  $\mathcal{A}_l(M_p^N)$  and  $b$  be in  $\mathcal{A}_r(M_p^N)$  lying over  $a$ . Then,

$$\begin{aligned} (M_p^N)_a &\cong M_p^{N-l-1}, \\ (M_p^N)_b &\cong M_p^{r-1}, \\ (M_p^N)_{ab} &\cong M_p^{r-l-2}. \end{aligned}$$

Furthermore, given  $-1 \leq l < r \leq N+1$ , we have that  $\mathcal{A}_{[l,r]}(M_p^N) \cong M_p^{r-l-1}$ .

*Proof.* Given a colored  $N$ -space  $M$  and corresponding vertices  $a$  and  $b$ , every operation in  $M_a$  can be extended to an operation on  $M$ . Moreover, if an operation on  $M$  has no meaning restricted to  $M_a$ , then  $M_a$  does not change. The other statements can be proved in a similar fashion.  $\square$

We will now introduce a notion, *simply connectedness*, which traditionally implies path-connectedness topologically. Despite this abuse of notation, we will use this term since it implies that loops are not punctured (*cf.* Remark 4.9(2) and Corollary 6.16).

**Definition 4.8.** A colored  $N$ -space  $M$  is simply connected if, whenever we are given  $l < r$  in  $[-1, N+1]$ , an interval  $t \subset [l, r]$ , vertices  $a$  in  $\mathcal{A}_l(M)$  beneath  $b$  in  $\mathcal{A}_r(M)$  and  $x$  and  $y$  in  $\mathcal{A}_t(M)$  lying between  $a$  and  $b$  which are  $t$ -connected by a path of length  $k$  not passing through  $a$  nor  $b$ , then there is a path in  $\mathcal{A}_t(M)$  of length at most  $k$  connecting  $x$  and  $y$  such that every vertex in the path lies between  $a$  and  $b$ .

Note that simply connectedness is an empty condition for  $l = -1$  and  $r = N+1$ .

**Remark 4.9.** Let  $M$  be a simply connected colored  $N$ -space. The following hold:

- (1) The subgraph  $\mathcal{A}_{[l,l+1]}(M)$  has no closed paths with no repetitions.
- (2) In a closed path  $P$  in  $\mathcal{A}_{[l,r]}(M)$ , all elements in  $P \cap \mathcal{A}_{[l,r]}$  are connected (in  $\mathcal{A}_{[l,r]}(M)$ ). Likewise for the dual statement.

*Proof.* For (1), set  $r = N+1$ ,  $l = l$  and take  $t = [l, l+1]$  in the definition of simply connectedness.

For (2), given  $x$  and  $y$  in  $P \cap \mathcal{A}_{[l,r]}$ , if they are connected using an arch of  $P$  in  $\mathcal{A}_{[l,r]}(M)$ , there is nothing to prove. Otherwise, replace successively every

occurrence of a vertex  $z$  in  $P \cap \mathcal{A}_r(M) \cap P$  by a subpath in  $\mathcal{A}_{[l,r]}(M)$  connecting the immediate neighbours of  $z$  in  $P$ .  $\square$

As the following Lemma shows, simply connectedness is preserved under application of the operations  $\alpha_s$ 's,

**Lemma 4.10.** *Let  $A$  be a simply connected colored  $N$ -space. If  $B$  is obtained from  $A$  by applying  $\alpha_s$  on  $(a_{l_s}, a_{r_s})$ , then  $B$  is simply connected as well.*

*Proof.* By hypothesis, the set  $B$  equals  $A \cup S_B$ , where  $S_B$  is the path

$$a_{l_s}, b_{l_s+1}, \dots, b_{r_s-1}, a_{r_s}.$$

Let now  $t \subset [l, r]$  be given, as well as  $a$  in  $\mathcal{A}_l$  beneath  $b$  in  $\mathcal{A}_r$  and vertices  $x$  and  $y$  in  $\mathcal{A}_t$  lying between  $a$  and  $b$  connected by a path  $P$  in  $\mathcal{A}_t(B)$  of length  $k$ . We consider the following cases:

- (a) Both  $a$  and  $b$  lie in  $B \setminus A$ . Take the direct path between  $x$  and  $y$ .
- (b) Both  $a$  and  $b$  lie in  $A$ . We consider the following mutually exclusive sub-cases:
  - (i) Both  $x$  and  $y$  lie in  $A$ : We can replace all repetitions in  $P$  to transform it into a path fully contained in  $A$  of length at most  $k$ . Since  $A$  is simply connected, the result follows.
  - (ii) Both  $x$  and  $y$  lie in  $S_B$ . Again, take the direct path between  $x$  and  $y$ .
  - (iii) Exactly one vertex, say  $y$ , lies in  $A$ . The path  $P$  must contain either  $a_{l_s}$  or  $a_{r_s}$ . Suppose that  $P$  contains  $a_{r_s}$ . Hence, we can decompose  $P$  into the direct connection (which lies between  $a$  and  $b$ ) from  $x$  to  $a_{r_s}$  and a path  $P'$  in  $\mathcal{A}_t(A)$  from  $a_{r_s}$  to  $y$ . As  $A$  is simply connected, we obtain a path in  $\mathcal{A}_t(A)$  between  $a$  and  $b$  connecting  $y$  and  $a_{r_s}$  whose length is bounded by the length of  $P'$ . This yields a path from  $y$  to  $x$  between  $a$  and  $b$  of the appropriate length.
- (c) Exactly one vertex in  $\{a, b\}$  lies in  $A$ . Suppose that  $a$  lies in  $A \setminus B$  and  $b$  lies in  $S_B \setminus A$ . In particular, the vertex  $a$  lies beneath  $a_{l_s}$ . Consider the following mutually exclusive cases:
  - (i) Both  $x$  and  $y$  lie in  $S_B$ . The direct path between them in  $S_B$  yields again the result.
  - (ii) Both  $x$  and  $y$  lie in  $A$ : If either  $x$  or  $y$  equals  $a_{l_s}$ , then one of them lies over the other and the direct connection between them yields the result. Otherwise, we may assume that both  $x$  and  $y$  lie beneath  $a_{l_s}$ . Let  $Q$  be the path consisting of the direct connection from  $x$  to  $a_{l_s}$  and from  $a_{l_s}$  to  $y$ . If the path  $P$  connecting  $x$  and  $y$  necessarily passes through  $a_{l_s}$ , then its length is at least the length of  $Q$  and the result follows. Otherwise, since  $A$  is simply connected, there is a path connecting  $x$  and  $y$  of length at most  $k$  between  $a$  and  $a_{l_s}$ , and thus, between  $a$  and  $b$ .
  - (iii) Exactly one, say  $y$ , is in  $A$ . Then  $y$  must lie beneath  $x$  and the direct path between them yields the result.

$\square$

Since the only moment a vertex from  $\mathcal{A}_l \cup \mathcal{A}_r$  was added was in case (c)(ii), namely  $a_{l_s}$  (though only if the original path passed through it), a careful analysis of the previous proof yields the following, which corresponds to Axiom  $(\Sigma 4)$  in [2]; though we will not require its full strength.

**Corollary 4.11.** *A colored  $N$ -space  $B$  with  $\emptyset \leq B$  has the following property. Given  $t = [l_t, r_t] \subset [l, r]$ , as well as  $a$  in  $\mathcal{A}_l(B)$  beneath  $b$  in  $\mathcal{A}_r(B)$ , vertices  $x$  and  $y$  in  $\mathcal{A}_t(B)$  lying between  $a$  and  $b$  and a path in  $\mathcal{A}_t(B)$  of length  $k$  connecting them, there is a path  $P$  in  $\mathcal{A}_t(B)$  between  $a$  and  $b$  connecting  $x$  and  $y$  of length at most  $k$  such that all vertices in  $P$  with levels  $\mathcal{A}_{l_t} \cup \mathcal{A}_{r_t}$  come from the original path.*

By iterating Lemma 4.10, we obtain the following:

**Corollary 4.12.** *If  $A$  is simply connected, then so is every strong extension of  $A$ .*

The following observation can be easily shown.

**Lemma 4.13.** *Let  $B$  be obtained from  $A$  by applying the operation  $\alpha_s$ . Then, for every  $t \subset \{0, \dots, N\}$  and every  $x$  and  $y$  in  $\mathcal{A}_t(A)$ ,*

$$d_t^A(x, y) = d_t^B(x, y).$$

**Theorem 4.14 (Axioms).** *Both Fraïssé limits  $M_\infty$  and  $M_0$  have the following elementary properties:*

- (1) *simply connectedness.*
- (2) *Given a finite subset  $A$  and a non-empty interval  $s = (l, r)$ , for any two elements  $a_l$  and  $a_r$  in  $A$  with  $a_r$  over  $a_l$ , there are paths*

$$a_l, b_{l+1}, \dots, b_{r-1}, a_r$$

*such that the  $s$ -distance of  $b_i$  to  $\mathcal{A}_s(A)$  is arbitrarily large. In particular, if  $s = \{i\}$ , there is a new vertex  $b_i$  not contained in  $A$ .*

*Proof.* (1): This follows from Corollary 4.12.

(2): After enlarging  $A$ , we may assume that  $A \leq M_\infty$ . One single application of  $\alpha_s$  on  $(a_l, a_r)$  yields that  $s$ -distance of  $b_i$  to  $A$  is infinite and remains so at the end of the construction by Lemma 4.13.

If we are considering  $M_0$ , we may assume as well that  $A \leq M_0$ . Furthermore, we may suppose that in order to build up  $M_0$  from  $A$ , each of the operations  $\alpha_i$ , for  $i$  in  $s$ , was applied  $k$  many times consecutively on each of the new vertices in  $\mathcal{A}_{i+1}$  and  $\mathcal{A}_{i-1}$  between  $a_l$  and  $a_r$ . Lemma 4.13 yields now the desired result.  $\square$

**Definition 4.15.** We will denote by  $\text{PS}_N$  the collection of sentences expressing properties (1) and (2) in Theorem 4.14.

**Definition 4.16.** A *flag* is a subgraph of a colored  $N$ -space  $M$  of the form

$$a_0 - \dots - a_N,$$

where  $a_i$  belongs to  $\mathcal{A}_i(M)$  and they form a path.

A set  $D$  of a colored  $N$ -space  $M$  is *complete* if every point in  $D$  is contained in a flag in  $D$ .

Observe that, if  $D$  satisfies Axiom (2), it is complete.

**Definition 4.17.** A subset  $D$  of a colored  $N$ -space  $M$  is *nice* if it satisfies the following conditions:

- (1) For any two (possibly imaginary) points  $a$  and  $b$  in  $D$ ,

$$D_a^b = D \cap M_a^b.$$

(2) for all intervals  $t \subset \{0, \dots, N\}$  and all  $x$  and  $y$  in  $\mathcal{A}_t(D)$ ,

$$d_t^M(x, y) < \infty \Rightarrow d_t^D(x, y) < \infty.$$

A set  $D$  is *wunderbar* in  $M$  if it satisfies the following:

(1) For any two (possibly imaginary) points  $a$  and  $b$  in  $D$ ,

$$D_a^b = D \cap M_a^b.$$

(2) for all intervals  $t \subset \{0, \dots, N\}$  and all  $x$  and  $y$  in  $\mathcal{A}_t(D)$ ,

$$d_t^M(x, y) = d_t^D(x, y).$$

Clearly, wunderbar sets are nice. As an application of the operation  $\alpha_s$  on  $A$  does not yield connections between the points of  $A$  unless there was already one, the following result follows immediately from Lemma 4.13.

**Lemma 4.18.** *If  $A \leq B$ , then  $A$  is wunderbar in  $B$ .*

**Lemma 4.19.** *Let  $M$  be a simply connected colored  $N$ -space and  $D$  nice in  $M$ . Given an interval  $s = [l, r]$  in  $\{-1, \dots, N+1\}$  and  $a_l \in \mathcal{A}_l(D)$  beneath  $a_r \in \mathcal{A}_r(D)$ , the set  $D_{a_l}^{a_r}$  is nice in  $\mathcal{A}_s(M)$ .*

*Proof.* Since  $D_a^b = D \cap M_a^b$  for any  $a$  and  $b$  in  $D$ , the first condition of niceness holds for  $D_{a_l}^{a_r}$ .

For the second condition, we may assume that  $a_l = -1$  by Remark 4.7. Let  $t \subset (-1, r]$  be an interval and vertices  $x$  and  $y$  in  $\mathcal{A}_t(D)$  beneath  $a_r$ . We need only show that, if  $x$  and  $y$  are connected in  $\mathcal{A}_t(D)$ , then they are connected in  $\mathcal{A}_t(D)$  beneath  $a_r$ . Let  $P$  be a path in  $\mathcal{A}_t(D)$  connecting  $x$  and  $y$ , but not necessarily running beneath  $a_r$ . We call a vertex in  $P$  avoidable if it does not lie beneath  $a_r$ . Let  $\mathcal{A}_n$  be the largest level containing an avoidable vertex in  $P$ . Let  $m$  be the number of avoidable vertices in  $P$  of level  $n$ . Choose  $P$  such that the pair  $(n, m)$  is minimal for the lexicographical order.

Given an avoidable vertex  $b$  in  $\mathcal{A}_n \cap P$ , denote by  $a'_1$  in  $\mathcal{A}_{l_1}$  the first non-avoidable vertex in  $P$  between  $b$  and  $x$ . Likewise, let  $a'_2$  in  $\mathcal{A}_{l_2}$  be the first non-avoidable vertex in  $P$  between  $b$  and  $y$ . Note that  $l_1$  and  $l_2$  are both smaller than  $n$ , by maximality of  $n$ . Furthermore, since every avoidable direct neighbour of a non-avoidable vertex lies necessarily in a larger level, by definition, it follows that both  $l_1$  and  $l_2$  are strictly smaller than  $n$ . Hence, the subpath  $P'$  of  $P$  between  $a'_1$  and  $a'_2$  yields a connection in  $\mathcal{A}_{t'}$ , where  $t' = t \cap (-1, n]$  not passing through  $a_r$ . As  $M$  is simply connected, there is a path  $Q$  (with no repetitions) connecting  $a'_1$  and  $a'_2$  running beneath  $a_r$ . Now, the paths  $Q$  and  $P'$  have only  $a'_1$  and  $a'_2$  as common vertices and they induce a loop. Remark 4.9(2) yields that  $a'_1$  and  $a'_2$  are  $t_1$ -connected, where  $t_1 = t \cap (-1, n)$ . Since  $D$  is nice, there is also a  $t_1$ -connection  $R$  in  $D$ . Replacing  $P'$  by  $R$ , we have a path whose avoidable vertices are still contained in  $(-1, n]$  and with fewer avoidable vertices of level  $n$ . Minimality of  $(n, m)$  shows that this path runs beneath  $a_r$ , as desired.  $\square$

**Corollary 4.20.** *Let  $D$  be nice in a colored  $N$ -space  $M$ . If  $M$  is simply connected, then so is  $D$ .*

**Lemma 4.21.** *Let  $A$  be a nice subset of a simply connected colored  $N$ -space  $M$ . Consider a non-empty interval  $s = (l, r)$  and two vertices  $a_{l_s}$  in  $\mathcal{A}_{l_s}(A)$  and  $a_{r_s}$  in*

$\mathcal{A}_{r_s}(A)$  such that  $a_{r_s}$  lies over  $a_{l_s}$ . Let  $B \subset M$  be an extension of  $A$  given by new vertices  $b_{l_s+1}, \dots, b_{r_s-1}$  such that the sequence

$$a_l, b_{l+1}, \dots, b_{r-1}, a_r$$

is a path. The following are equivalent:

- (a) The set  $B$  is nice and obtained from  $A$  by applying  $\alpha_s$  on  $(a_{l_s}, a_{r_s})$ .
- (b) For some (equivalently, all)  $i$  in  $s$ , we have that  $d_s^M(b_i, A) = \infty$ .
- (c) For some (equivalently, all)  $i$  in  $s$ , we have that  $d^{M_{a_l}^{a_r}}(b_i, A) = \infty$ .

Note that simply connectedness yields that

$$d^{M_{a_l}^{a_r}}(b_i, A) = d_{(l,r)}^M(b_i, A_{a_l}^{a_r}).$$

We say that  $B$  is obtained from  $A$  by a *global application* of  $\alpha_s$  if it satisfies (any of) the above conditions. In particular, the set  $B$  is nice.

*Proof.* (a)  $\rightarrow$  (b): By the definition of  $\alpha_s$  the distance  $d_s^B(b_i, A)$  is infinite for every  $i$  in  $s$ . Since  $B$  is nice in  $M$ , so is  $d_s^M(b_i, A) = \infty$ .

(b)  $\rightarrow$  (c): Obvious.

(c)  $\rightarrow$  (b) If both  $a_l$  and  $a_r$  are imaginary, then there is nothing to prove. Thus, may assume that  $a_r$  is real. Furthermore, suppose that there is a path  $P$  connecting some  $b_i$  with some  $a$  in  $\mathcal{A}_s(A)$  in  $\mathcal{A}_s(M)$ . Take  $P$  of shortest possible length.

We need to show that

$$d^{M_{a_l}^{a_r}}(b_i, A) < \infty.$$

Note that  $a$  and  $a_r$  are connected in  $\mathcal{A}_{(l,r]}(M)$  and, since  $A$  is nice, there is a shortest path  $Q$  in  $\mathcal{A}_{(l,r]}(A)$  witnessing this. In particular, let  $a_{r-1}$  be the direct neighbour of  $a_r$  in  $Q$ . Connecting  $Q$  and  $P$ , we have that  $a_{r-1}$  and  $b_i$  lie beneath  $a_r$  and are connected in  $\mathcal{A}_{(l,r]}$  by a path disjoint from  $a_r$ . Simply connectedness yields a path  $Q_1$  beneath  $a_r$  in  $\mathcal{A}_{(l,r]}$  connecting them. If  $a_l$  is imaginary, we are done. Otherwise, the vertices  $a_{r-1}$  and  $a_l$  are connected through  $b_i$ . Again by simply connectedness, there is a path  $Q'$  connecting them below  $a_r$  in  $[l, r)$ . Let now  $a_{l+1}$  be the direct neighbour in  $Q'$  above  $a_l$ . Note that  $a_{l+1}$  and  $b_i$  lie between  $a_l$  and  $a_r$ . Simply connectedness of  $M$  yields that there is a path in  $M_{a_l}^{a_r}$  between  $b_i$  and  $a_{l+1}$ . Hence

$$d^{M_{a_l}^{a_r}}(b_i, A) < \infty.$$

(b)  $\rightarrow$  (a): If both  $a_l$  and  $a_r$  are imaginary, then there are clearly no new connections between any  $b_i$  and  $A$ , and thus  $B$  is obtained by applying  $\alpha_{[0,N]}$  to  $A$ . Hence, we may assume that  $a_r$  is real.

We first need to show that no  $b_i$  is in relation to an element in  $A$  besides  $a_r$  and  $a_l$ . This implies that  $B$  is obtained from  $A$  by application of  $\alpha_s$ . Assume first that  $b_{r-1}$  is connected with some other element  $a'_r$  in  $\mathcal{A}_r(A)$ . Since  $A$  is nice, there is a path in  $\mathcal{A}_{\{r-1,r\}}(A)$  connecting  $a_r$  and  $a'_r$ . This, together with the extra connection to  $b_{r-1}$  yields a loop in  $\mathcal{A}_{\{r-1,r\}}$ , which contradicts Remark 4.9 (2). Likewise for  $b_{l+1}$ . Finally, by assumption, no  $b_i$  in  $\mathcal{A}_{(l+1,r-1)}$  is in relation with an element in  $\mathcal{A}_s(A)$ .

Now, in order to show that  $B$  is nice, consider  $x$  and  $y$  in  $B$  with finite  $t$ -distance in  $M$ . If both  $x$  and  $y$  lie in  $A$ , we are done, since  $A$  is nice. Likewise, if both  $x$  and  $y$  lie in the path  $a_l, b_{l+1}, \dots, b_{r-1}, a_r$ , the direct connection works as well. Therefore, assume that  $x$  lies in  $A$  and  $y$  does not. By the assumption it follows that  $t \not\subseteq s$ . Suppose that  $l$  lies in  $t$ . Since  $y$  and  $a_l$  are  $t$ -connected (in  $M$ ), so

are  $x$  and  $a_l$ . As  $A$  is nice, there is a connection between  $x$  and  $a_l$  in  $\mathcal{A}_t(A)$ . In particular, there is a connection between  $x$  and  $y$  in  $\mathcal{A}_t(B)$ .  $\square$

**Theorem 4.22.** *Let  $M$  be complete and simply connected. Given a nice subset  $A$  and  $b$  in  $M$ , there is a nice subset  $B$  of  $M$  containing  $b$  such that  $A \leq B$  in finitely many steps.*

*Proof.* We may clearly assume that  $b$  does not lie in  $A$ .

Let  $r$  be minimal such that there exists an element  $a_r$  in  $\mathcal{A}_r(A)$  lying over  $b$  (if  $r = N + 1$ , set  $a_r = a_{N+1}$ ). Likewise, choose  $l$  maximal such that there exists an element  $a_l$  in  $\mathcal{A}_l(A)$  beneath  $b$  (if  $l = -1$ , then set  $a_l = a_{-1}$ ). We call the interval  $s = (l, r)$  the width of  $b$  over  $A$ . Define as well the distance from  $b$  to  $A$  as

$$d_s(b, A_{a_l}^{a_r}).$$

We prove the theorem by induction on the width and the distance from  $b$  to  $A$ : If the distance is infinite, by completeness of  $M$ , choose a path

$$a_l, b_{l+1}, \dots, b_{r-1}, a_r,$$

passing through  $b$ . By Lemma 4.21, the set  $A \cup \{b_{l+1}, \dots, b_{r-1}\}$  obtained from  $A$  by applying  $\alpha_s$  is nice and contains  $b$ .

Otherwise, let  $P$  be a path of minimal length lying between  $a_l$  and  $a_r$  connecting  $b$  to  $A$ . Let  $b'$  be the last element in  $P$  before  $b$ . By assumption, the distance from  $b'$  to  $A$  is strictly smaller than the length of  $P$ . Thus, there is a nice set  $B' \geq A$  containing  $b'$ . Either the width or the distance of  $b$  to  $B'$  has become smaller and we can now finish by induction.  $\square$

In particular, we can now prove that the notions of nice and wunderbar agree.

**Corollary 4.23.** *A nice subset  $A$  of a complete simply connected set  $M$  is wunderbar.*

*Proof.* Suppose we are given two points  $a$  and  $b$  in  $A$  and an  $s$ -path  $P$  in  $M$  of length  $n$  connecting them. By Theorem 4.22, we can obtain a nice set  $B$  such that  $A \leq B$  and  $B$  contains the path  $P$ . By Lemma 4.18, the set  $A$  is wunderbar in  $B$ , so there is an  $s$ -path of length  $n$  in  $A$  connecting  $a$  and  $b$ . Thus, the set  $A$  is wunderbar.  $\square$

Combining the previous results, we obtain the following.

**Corollary 4.24.** *Let  $M$  be complete and simply connected and  $A$  be a nice subset. The following hold:*

- (a) *If  $M \setminus A$  is countable, then  $A \leq M$ .*
- (b)  *$A$  is simply connected.*
- (c)  *$A$  is wunderbar.*
- (d) *If  $A$  is countable, then  $\emptyset \leq A$ .*

*Proof.* Theorem 4.22 yields (a). Now, Corollary 4.20 yields (b). In order to prove (c), it is sufficient to consider countable nice subsets  $A$ . Replace  $M$  by a countable elementary substructure  $M'$  that contains  $A$ . Then  $A$  is nice in  $M'$  and  $A \leq M'$  by a. Lemma 4.18 yields that  $A$  is wunderbar in  $M'$  and hence in  $M$ . Since  $\emptyset$  is nice, clearly (d) follows from (a) and (b).  $\square$

It follows that, for countable  $A$ , we have  $\emptyset \leq A$  if and only if  $A$  is simply connected and complete. And for simply connected complete countable  $B$ , we have that  $A \leq B$  if and only if  $A$  is nice in  $B$ . Therefore

**Corollary 4.25.** *The model  $M_\infty$  is the Fraïssé limit of the class of finite complete simply connected colored  $N$ -spaces together with nice embeddings.*

The construction is actually simpler than the general construction given in Section 3, since if a finite set  $B$  satisfies that  $B_a^b = B \cap M_a^b$  for all  $a$  and  $b$  in  $B$ , then  $B$  is  $r$ -strong in  $M_\infty$  if and only if it is nice in  $M_\infty$ . Indeed, consider a rich sequence  $A_0 \leq A_1 \leq \dots$  with union  $M_\infty$ . Then  $B$  is contained some  $A_i$ . But  $B$  is also nice in  $A_i$ , which implies  $B \leq A_i$ , and therefore  $B$  is  $r$ -strong in  $M_\infty$ .

Having  $M_\infty$  as a model, the theory  $\text{PS}_N$  is consistent. It will follow from the next proposition that it is complete. In particular, the stronger version of Axiom (1) stated in Corollary 4.11 follows formally from our axioms.

**Proposition 4.26.** *Any two  $\omega$ -saturated models of  $\text{PS}_N$  have the back-and-forth property with respect to partial isomorphisms between finite nice substructures.*

*Proof.* Let  $M$  and  $M'$  be two  $\omega$ -saturated models and consider a partial isomorphism  $f : A \rightarrow A'$ , where  $A$  is nice in  $M$  and  $A'$  is nice in  $M'$ .

Given  $b$  in  $M$ , Theorem 4.22 yields a nice finite subset  $B \geq A$  containing it. Thus, we may assume that  $B$  is obtained from  $A$  by applying  $\alpha_s$  on  $(a_l, a_r)$ . Since  $M'$  is an  $\omega$ -saturated model of Axiom (2), there is a path  $a'_l, b'_{l+1}, \dots, b'_{r-1}, a'_r$  in  $M'$  such that the  $s$ -distance of  $b'_i$  to  $A'$  is infinite. By Lemma 4.21) the set  $B' = A' \cup \{b'_{l+1}, \dots, b'_{r-1}\}$  is nice and  $f$  extends to an isomorphism between  $B$  and  $B'$ .  $\square$

**Theorem 4.27.** *Any partial isomorphism  $f : A \rightarrow A'$  between two finite nice subsets of two models of  $\text{PS}_N$  is elementary.*

*Proof.* Replace the models  $M$  and  $M'$  by two  $\omega$ -saturated extensions  $M_1$  and  $M'_1$ . Note that  $A$  and  $A'$  remain nice in the corresponding extensions. Lemma 4.26 yields that  $f$  is elementary with respect to  $M_1$  and  $M'_1$  and thus its restriction to  $M$  and  $M'$  is elementary as well.  $\square$

**Corollary 4.28.** *The theory  $\text{PS}_N$  is complete.*

*Proof.* Note that set  $\emptyset$  is nice in any colored  $N$ -space and apply Theorem 4.27.  $\square$

**Corollary 4.29.** *The type of a nice set  $A$  is determined by its quantifier-free type.*

**Corollary 4.30.** *The model  $M_\infty$  is  $\omega$ -saturated.*

*Proof.* Let  $M$  be any  $\omega$ -saturated model of  $\text{PS}_N$ . It follows from Lemma 3.3 and the equality of nice and  $r$ -strong that the family of isomorphisms between finite nice subset of  $M$  and  $M_\infty$  has the back-and-forth property. This implies that  $M_\infty$  is also  $\omega$ -saturated.  $\square$

**Corollary 4.31.** *The Fraïssé limit  $M_0$  is the prime model of  $\text{PS}_N$ .*

*Proof.* Consider any finite  $A \subset M$  which can be obtained from some fixed flag by a sequence of applications of  $\alpha_{\{i\}}$  for varying  $i \in [0, N]$ . Since the  $d_{\{i\}}$ -distances are either 0 or  $\infty$ , it follows inductively from Lemma 4.21 that all intermediate sets are nice. So the quantifier-free type of  $A$  implies that  $A$  is nice and therefore implies the type of  $A$ . Whence  $A$  is atomic. This shows that  $M_0$  is atomic.  $\square$

**Corollary 4.32.** *Nice sets are algebraically closed.*

*Proof.* By Corollary 4.30, we may assume that the nice set  $A$  is a subset of  $M_\infty$ . By Corollary 4.24 (a), we have that  $M$  is an increasing union of nice sets containing  $A$ . Thus, we may reduce the statement to showing that if  $B = A \cup \{b_{l_s+1}, \dots, b_{r_s-1}\}$  is obtained by applying the operation  $\alpha_s$  on  $a_{l_s}, a_{r_s}$  in  $A$ , then the tuple  $(b_{l_s+1}, \dots, b_{r_s-1})$  has infinitely many  $A$ -conjugates. This is now clear, as any two sets resulting from applying the operation  $\alpha_s$  on  $a_{l_s}, a_{r_s}$  in  $A$  have the same type over  $A$ , by Lemma 4.21 and Corollary 4.29.  $\square$

## 5. WORDS AND LETTERS

In this section, we will study the semigroup  $\text{Cox}(N)$  generated by the operations  $\alpha_s$ , where  $s$  stands for a non-empty interval in  $[0, N]$ . Such intervals will be then called *letters*. We will exhibit a normal reduced form for words in  $\text{Cox}(N)$  and describe the possible interactions between words when multiplying them.

Two letters  $s$  and  $t$  in  $[0, N]$  *commute* if their distance is at least 2. That is, either  $r_s \leq l_t$  or  $r_t \leq l_s$ , where  $s = (l_s, r_s)$  and  $t = (l_t, r_t)$ . By definition, no letter commutes with itself nor with any proper subletter.

**Definition 5.1.** We define  $\text{Cox}(N)$  to be the monoid generated by all letters in  $[0, N]$  modulo the following relations:

- $ts = st = s$  if  $t \subset s$ ,
- $ts = st$  if  $s$  and  $t$  commute.

We denote by 1 the empty word.

The *inversion*  $u \mapsto u^{-1}$  of words defines an antiautomorphism of  $\text{Cox}(N)$ . All concepts introduced from now on will be invariant under inversion.

The *centraliser*  $C(u)$  of a word  $u$  in  $\text{Cox}(N)$  is the collection of all indexes in  $[0, N]$  commuting with every letter in  $u$ . Clearly, a letter  $s$  commutes with  $u$  in  $\text{Cox}(N)$  if and only if  $s \subset C(u)$ .

In order to obtain a normal form for elements in  $\text{Cox}(N)$ , we say that a word  $s_1 \cdots s_n$  is *reduced* if there is no pair  $i \neq j$  of indices such that  $s_i \subset s_j$  and  $s_i$  commutes with all  $s_k$  with  $k$  between  $i$  and  $j$ .

**Definition 5.2.** The word  $u$  can be *reduced to*  $v$ , denoted by  $u \rightarrow v$ , if  $v$  is obtained from  $u$  by finitely many iterations of the following rules:

- COMMUTATION: Replace an occurrence of  $s \cdot t$  by  $t \cdot s$ , if  $s$  and  $t$  commute.
- CANCELLATION: Replace an occurrence of  $s \cdot t$  or  $t \cdot s$  by  $s$ , if  $t \subset s$ .

Two words  $u$  and  $v$  are *equivalent* (or  $u$  is a *permutation* of  $v$ ), denoted by  $u \approx v$ , if  $u \rightarrow v$  by exclusively applying the commutation rule.

It is easy to see that permutations of reduced words remain reduced. In particular, a word is reduced if and only if the cancellation rule cannot be applied to any permutation.

Clearly, two word  $u$  and  $v$  represent the same element in  $\text{Cox}(N)$  if  $u \rightarrow v$ . The following proposition yields in particular that the converse is true: Two words have a common reduction if they represent the same element in  $\text{Cox}(N)$  (cf. Corollary 5.4).



**Proposition 5.3.** *Every word  $u$  can be reduced to a unique (up to equivalence) reduced word  $v$ . We refer to  $v$  as the reduct of  $u$ .*

*Proof.* Among all possible reductions of the word  $u$ , choose  $v$  of minimal length. Clearly, cancellation cannot be applied any further to a permutation of  $v$ , thus  $v$  is reduced. We need only show that  $v$  is unique such.

For that, we first introduce the following rule:

GENERALISED CANCELLATION: Given a word  $s_1 \cdots s_n$  and a pair of indices  $i \neq j$  such that  $s_i \subset s_j$  and  $s_i$  commutes with all  $s_k$ 's with  $k$  between  $i$  and  $j$ , then delete the letter  $s_i$ .

If the situation described above occurs, we say that  $s_i$  is *absorbed by  $s_j$* . Note that a generalised cancellation is obtained by successive commutations and one single cancellation. Furthermore, one single cancellation applied to some permutation of  $u$  can be obtained as some permutation of a generalised cancellation applied to  $u$ . This implies that every reduct can be obtained by a sequence of generalised cancellations followed by a permutation.

Assume now that  $u \rightarrow v_1$  and  $u \rightarrow v_2$ , where both  $v_1$  and  $v_2$  are reduced. We will show, by induction on the length of  $u$ , that  $v_2$  is a permutation of  $v_1$ . If  $u$  is itself reduced, then  $v_1$  and  $v_2$  are permutations of  $u$  and hence the result follows. Otherwise, there are two words  $u_1$  and  $u_2$  obtained from  $u$  by one single generalised cancellation such that  $u_i \rightarrow v_i$  for  $i = 1, 2$ .

We claim that there is a word  $u'$  such that  $u_i \rightarrow u'$  for  $i = 1, 2$ , either by permutation or by a single generalised cancellation. This is immediate except for the case where there are indices  $i, j$  and  $k$  (for  $i \neq k$ ) such that  $u_1$  is obtained from  $u$  because the letter  $s_i$  is absorbed by  $s_j$  and  $u_2$  is obtained from  $u$  in which the same letter  $s_j$  is absorbed by  $s_k$ . In this case, set  $u'$  to be the word obtained from  $u$  by having both  $s_i$  and  $s_j$  absorbed by  $s_k$ . Clearly, we have that  $u_1 \rightarrow u'$ . Also, since  $s_i \subset s_j$ , it follows that  $s_i$  commutes also with all letters between  $s_j$  and  $s_k$ . Hence, the word  $u'$  is obtained from  $u_2$  in which  $s_k$  absorbs  $s_i$ . Let  $v'$  be a reduct of  $u'$ . Induction applied to  $u_1$  and  $u_2$  implies that  $v'$  is a permutation of both  $v_1$  and  $v_2$ . Hence, the word  $v_1$  is a permutation of  $v_2$ .  $\square$

**Corollary 5.4.** *Every element of  $\text{Cox}(N)$  is represented by a reduced word, which is unique up to equivalence.*

*Proof.* Let  $C$  be the collection of equivalence classes of reduced words. From the previous result, it follows that there is a natural surjection  $C \rightarrow \text{Cox}(N)$ . Represent by  $[u]$  the equivalence class of the word  $u$ . Set

$$[u] \cdot [v] = [w] \quad \text{iff} \quad u \cdot v \rightarrow w.$$

Then  $C$  has a natural semigroup structure. Since  $C$  satisfies the defining relations of  $\text{Cox}(N)$ , the map  $C \rightarrow \text{Cox}(N)$  is an isomorphism.  $\square$

In order to exhibit a canonical representative of the equivalence class  $[u]$ , we introduce the following partial ordering on letters:

$$(l_s, r_s) < (l_t, r_t) \quad \text{iff} \quad r_s \leq l_t.$$

A reduced word  $s_1 \cdots s_n$  is in *normal form* if for all  $i < n$ , if  $s_i$  and  $s_{i+1}$  commute, then  $s_i < s_{i+1}$ .

**Remark 5.5.** Every reduced word is equivalent to a unique word in normal form.

*Proof.* We will actually prove a more general result: Let  $S$  be any set equipped with a partial order  $<$ . We say that  $s$  and  $t$  commute if either  $s < t$  or  $t < s$ . Let  $S^*$  be the semigroup generated by  $S$  modulo commutation. Two words in  $S^*$  are equivalent if they can be transformed into each other by successive commutations of adjacent elements. A word  $s_1 \cdots s_n$  is in normal form if  $s_i \not\prec s_{i+1}$  for all  $i < n$ . We have the following.

**Claim.** *Every word  $u$  in  $S^*$  is equivalent to a unique word  $v$  in normal form.*

For existence, start with  $u$  and swap successively every pair  $s_i > s_{i+1}$ . This process must stop since the number of inversions  $\{(i, j) \mid i < j \text{ and } s_i > s_j\}$  is decreased by 1 at every step. The resulting  $v$  is in normal form.

For uniqueness, consider two equivalent words in normal form  $u = s_1 \cdots s_n$  and  $v = t_1 \cdots t_n$ . Let  $\pi$  be some permutation transforming  $u$  into  $v$ . Suppose for a contradiction that  $\pi(1) = k \neq 1$ . Then  $t_k = s_1$  commutes with  $t_i$  for  $i < k$ . By hypothesis, we have  $t_{k-1} < t_k$ . Note that there is no  $i < k$  with  $t_i < t_k$  and  $t_k < t_{i-1}$ . Hence, for all  $i < k$ , we have that  $t_i < t_k$  and thus  $t_1 < t_k$ , that is,  $t_1 < s_1$ . By means of the permutation  $\pi^{-1}$ , we conclude that  $s_1 < t_1$ , which yields a contradiction. Thus  $\pi(1) = 1$  and hence  $s_2 \cdots s_n$  is equivalent to  $t_2 \cdots t_n$ . Induction on  $n$  yields the desired result.  $\square$

It is an easy exercise to show that, for  $S$  and  $S^*$  as before, we have

$$r \cdot t_2 \cdots t_n \approx r \cdot s_2 \cdots s_n \Rightarrow t_2 \cdots t_n \approx s_2 \cdots s_n.$$

Therefore, we obtain the following result.

**Remark 5.6.**  $u \cdot v \approx u \cdot v'$  implies  $v \approx v'$ .

Given two reduced words  $u = s_1 \cdots s_m$  and  $v = t_1 \cdots t_n$ , their product  $u \cdot v$  is not reduced if and only if one of the two following cases occurs:

- There are  $i \leq m$  and  $j \leq n$  such that  $s_i$  commutes with  $s_{i+1} \cdots s_m$  and with  $t_1 \cdots t_{j-1}$  and it is contained in  $t_j$ .
- There are  $j \leq n$  and  $i \leq m$  such that  $t_j$  commutes with  $t_1 \cdots t_{j-1}$  and with  $s_{i+1} \cdots s_m$  and it is contained in  $s_i$ .

Based on the previous observation, we introduce the following definition.

**Definition 5.7.** Given two words  $u = s_1 \cdots s_m$  and  $v = t_1 \cdots t_n$  words, we say that:

- (1)  $s_i$  belongs to the *final segment* of  $u$  if  $s_i$  commutes with  $s_{i+1} \cdots s_m$ .
- (2) The letter  $s$  is (properly) *left-absorbed* by  $v$  if it commutes with  $t_1 \cdots t_{j-1}$  and is a (proper) subset of  $t_j$  for some  $j \leq n$ . A word is (properly) left-absorbed by  $v$  if all its letters are (properly) left absorbed by  $v$ .
- (3)  $v$  *bites  $u$  from the right* if  $v$  left-absorbs some element in the final segment of  $u$ .

The concepts *initial segment*, *right-absorbed* and *left-biting* are defined likewise.

Clearly, these notions depend only on the equivalence class of  $u$  and  $v$ . Thus, the following lemma follows.

**Lemma 5.8.** *Given two reduced words  $u$  and  $v$ , the product  $u \cdot v$  is reduced if and only if none of them bites the other one (in the corresponding directions).*

If both  $u$  and  $v$  are reduced and  $u$  is absorbed by  $v$ , then  $u \cdot v$  reduces to  $v$ . Corollary 5.14 will show that the converse also holds.

The following observations will be often used throughout this article.

**Lemma 5.9** (Absorption Lemma). *Let  $v$  be a (possibly non-reduced) word.*

- (1) *If a letter  $s$  is left-absorbed by  $v$ , then there is a unique letter in  $v$  witnessing it.*
- (2) *If two non-commuting letters are absorbed by  $v$ , then they are absorbed by the same letter in  $v$ .*
- (3) *Suppose  $v = v_1 \cdot v_2$  and let  $u$  be a word left-absorbed by  $v$  but not bitten from the right by  $v_1$ , then  $u$  and  $v_1$  commute and  $u$  is left absorbed by  $v_2$ .*

*Proof.* Assume  $v = t_1 \cdots t_n$ . Let  $r \subset t_i$  commute with  $t_1 \cdots t_{i-1}$  and  $s \subset t_j$  commute with  $t_1 \cdots t_{j-1}$ . Assume  $i \leq j$ . Then, either  $i = j$  or  $s$  commutes with  $t_i$ , which implies that  $s$  commutes with  $r$ . This yields both (1) and (2).

For (3), we apply induction on the length  $m$  of  $u = s_1 \cdots s_m$ . If  $m = 0$ , then there is nothing to prove. Otherwise, the subword  $u' = s_2 \cdots s_m$  is not bitten by  $v_1$  by assumption. Induction gives that  $u'$  commutes with  $v_1$  and is absorbed by  $v_2$ . The letter  $s_1$  cannot be absorbed by  $v_1$ , for otherwise  $s_1$  would also commute with  $u'$  and thus it would belong to the final segment of  $u$ . The word  $u$  would then be bitten by  $v_1$ . Since  $s_1$  is absorbed by  $v$  but not by  $v_1$ , it must commute with  $v_1$  and hence it is absorbed by  $v_2$  as well.  $\square$

Based on the the previous result, we introduce the following notions.

**Definition 5.10.** The *left stabiliser*  $\mathcal{S}_L(v)$  of a word  $v = t_1 \cdots t_n$  is the union of the sets

$$\mathcal{S}_L^j(v) = t_j \cap C(t_1 \cdots t_{j-1}).$$

The *right stabiliser*  $\mathcal{S}_R(v)$  is defined likewise or, alternatively, as  $\mathcal{S}_L(v^{-1})$

By Lemma 5.9(2), the sets  $\mathcal{S}_L^j(v)$  are either empty or intervals commuting with each other. Equivalent words have same stabilisers. In fact, if  $u \rightarrow v$  then  $\mathcal{S}_L(u) \subset \mathcal{S}_L(v)$ .

**Lemma 5.11.** *The letter  $s$  is absorbed by  $v$  if and only if  $s \subset \mathcal{S}_L(v)$ .*

Set

$$|s_1 \cdots s_m| = s_1 \cup \cdots \cup s_m.$$

Then  $u$  is absorbed by  $v$  if and only if  $|u| \subset \mathcal{S}_L(v)$ . Furthermore, the word  $v$  bites  $u$  from the right if and only if some element in the final segment of  $u$  is contained in  $\mathcal{S}_L(v)$ .

**Lemma 5.12.** *Given two words  $u$  and  $v$ , there is a unique decomposition  $u = u_1 \cdot u_2$  (up to commutation) such that:*

- $u_2$  is left-absorbed by  $v$ .
- $u_1$  is not bitten from the right by  $v$ .

*The decomposition of  $u$  depends only on the set  $\mathcal{S}_L(v)$ .*

*Proof.* We proceed by induction on the length of  $u$ . If  $u$  is not bitten by  $v$ , we set  $u_1 = u$  and  $u_2 = 1$ . Otherwise, up to permutation, we have  $u = u' \cdot s$ , where  $s$  is absorbed by  $v$ . Decompose  $u'$  as  $u'_1 \cdot u'_2$  and set  $u_1 = u'_1$  and  $u_2 = u'_2 \cdot s$ .

Uniqueness is proved in a similar fashion.  $\square$

We can now describe the general form of the product of two reduced words in  $\text{Cox}(N)$ .

**Theorem 5.13** (Decomposition Lemma). *Given two reduced words  $u$  and  $v$ , there are unique decompositions (up to permutation):*

$$u = u_1 \cdot u' \qquad v' \cdot v_1 = v,$$

such that:

- (a)  $u'$  is left-absorbed by  $v_1$ ,
- (b)  $v'$  is properly right-absorbed by  $u_1$ ,
- (c)  $u'$  and  $v'$  commute,
- (d)  $u_1 \cdot v_1$  is reduced.

It follows that  $u \cdot v \rightarrow u_1 \cdot v_1$ . We call such a decomposition *fine*.

*Proof.* We apply Lemma 5.12 to  $u$  and  $v$  to obtain a decomposition

$$u = u_1 \cdot u',$$

such that  $u'$  is left-absorbed by  $v$  and  $u_1$  is not bitten by  $v$  from the right. The same (in the other direction) with  $u_1$  and  $v$  yields

$$v' \cdot v_1 = v,$$

where  $v'$  is right-absorbed by  $u_1$  and  $v_1$  is not bitten from the left by  $u_1$ .

First, we show (c), that is, the words  $u'$  and  $v'$  commute. If not, let  $s$  the first element of  $u'$  which does not commute with  $v'$ . Since  $s$  is left-absorbed by  $v' \cdot v_1$ , it must be left-absorbed by  $v'$ . As  $u_1$  right-absorbs  $v'$ , it also right-absorbs  $s$ , which contradicts that  $u_1 \cdot u'$  is reduced. Lemma 5.9(3) gives that  $u'$  is absorbed by  $v_1$ , showing (a).

Let us now show (d): the product  $u_1 \cdot v_1$  is reduced. Otherwise, as  $v_1$  is not bitten from the left by  $u_1$ , it bites  $u_1$  from the right, i.e. it left-absorbs a letter  $s$  from the final segment of  $u_1$ . The Absorption Lemma 5.9, applied to  $u_1 = u_1^1 \cdot s$  and  $v'$ , which is right absorbed by  $u_1$ , gives (possibly after permutation) a decomposition  $v' = x \cdot y$ , where  $|x| \subset s$  and  $y$  commutes with  $s$ . There are two cases:

- (1) The word  $x = 1$ . Then  $s$  commutes with  $v'$  and is absorbed by  $v_1$ . This contradicts that  $u_1$  is not bitten by  $v_1$  from the right.
- (2) The word  $x$  is not trivial. As it is absorbed by  $s$  and  $s$  is right-absorbed by  $v_1$ , we have that  $x$  is right-absorbed by  $v_1$ . This contradicts that  $v' \cdot v_1$  is reduced.

The only point left to prove is that  $v'$  is properly right-absorbed by  $u_1$ . Otherwise, there is a letter  $t$  in  $v'$  which is absorbed but not properly absorbed by  $u_1$ . Then  $t$  occurs in the final segment of  $u_1$  and  $v' = t \cdot y$  up to commutation. In particular, the word  $u_1$  is bitten from the right by  $v'$  and thus by  $v$ , which contradicts our choice of  $u_1$ .

In order to show uniqueness, assume we are given another fine decomposition:

$$u = u_1 \cdot u' \qquad v' \cdot v_1 = v$$

We need only show the following four facts:

- (1) The word  $u'$  is left-absorbed by  $v$ : Since  $u'$  commutes with  $v'$  and is left-absorbed by  $v_1$ , then it is left-absorbed by  $v' \cdot v_1$  as well.

- (2) The word  $u_1$  is not bitten by  $v$  from the right: Suppose not and take a letter  $s$  in the final segment of  $u_1$  which is left-absorbed by  $v$ . Since  $u_1 \cdot v_1$  is reduced, the letter  $s$  must be left-absorbed by  $v'$ . Let  $t$  in  $v'$  containing  $s$ . However, the word  $t$  is right-absorbed by  $u_1$ . As  $u_1$  is reduced and  $s$  is in the final segment of  $u_1$ , the only possibility is that  $s = t$ . But then  $t$  is not properly left-absorbed by  $u_1$ , which is a contradiction.
- (3)  $v'$  is right-absorbed by  $u_1$ : By definition.
- (4)  $v_1$  is not bitten from the left by  $u_1$ : This clearly follows from the fact that  $u_1 \cdot v_1$  is reduced.

□

**Corollary 5.14.** *Let  $u$  and  $v$  be reduced words. Then  $v$  left-absorbs  $u$  if and only if  $uv = v$  in  $\text{Cox}(N)$ .*

Note that  $uv = v$  in  $\text{Cox}(N)$  if and only if  $u \cdot v \rightarrow v$ .

*Proof.* Clearly, if  $v$  left-absorbs  $u$ , then  $u \cdot v \rightarrow v$ . For the converse, apply the Decomposition Lemma 5.13 to  $u$  and  $v$  to obtain:

$$u = u_1 \cdot u' \qquad v' \cdot v_1 = v$$

such that  $u'$  is left-absorbed by  $v_1$ , the word  $v'$  is properly right-absorbed by  $u_1$ , the words  $u'$  and  $v'$  commute and  $u_1 \cdot v_1$  is reduced. By assumption, we have

$$u \cdot v \rightarrow u_1 \cdot v_1 \approx v = v' \cdot v_1.$$

Thus  $u_1 = v'$ . Since  $u_1$  must be properly right-absorb itself, this forces  $u_1$  to be trivial. Hence  $u = u'$  is left-absorbed by  $v$ . □

As in  $\text{Cox}(N)$  (or generally, in any semi-group), the identity  $uvx = uv$  holds if  $vx = v$ , we have the following.

**Corollary 5.15.** *Let  $u$  and  $v$  be reduced words and  $w$  the reduct of  $u \cdot v$ . Then  $\mathcal{S}_R(v) \subset \mathcal{S}_R(w)$ .*

**Definition 5.16.** The *wobbling* between two words is

$$\text{Wob}(u, v) = \mathcal{S}_R(u) \cap \mathcal{S}_L(v).$$

**Remark 5.17.** If  $u \cdot v$  is reduced, then every  $s \in \text{Wob}(u, v)$  is properly right-absorbed by  $u$  and properly left-absorbed by  $v$ .

*Proof.* If  $s$  is not properly right-absorbed by  $u$ , then  $s$  belongs to the final segment of  $u$ . Since  $s$  is left-absorbed by  $v$ , the product  $u \cdot v$  would not be reduced. □

**Lemma 5.18.** *Assume that  $v_1 \cdot v_2$  and  $u \cdot v_2$  are reduced. If  $v_1$  is right absorbed by  $u$ , then*

$$\text{Wob}(v_1 \cdot v_2, h) \subset \text{Wob}(u \cdot v_2, h).$$

*Proof.* The word  $u \cdot v_2$  is the reduct of  $u \cdot (v_1 \cdot v_2)$ . Corollary 5.15 yields that  $\mathcal{S}_R(v_1 \cdot v_2) \subset \mathcal{S}_R(u \cdot v_2)$ . □

We will now study the idempotents of  $\text{Cox}(N)$ .

**Definition 5.19.** A word is commuting if it consists of pairwise commuting letters.

The letters of the final segment of a word  $u$  form a commuting word, which we denote by  $\tilde{u}$  (up to equivalence).

Commuting words are automatically reduced. Since every subset of  $[0, N]$  can uniquely be written as the union of commuting intervals, a commuting word (up to equivalence) can be considered as just a set of numbers. The following is an easy observation:

**Lemma 5.20.** *Every word  $u$  is equivalent to a word  $x \cdot \tilde{u}$ , where  $\tilde{u}$  is the final segment of  $u$ .*

Note that no letter in the final segment of  $x$  commutes with  $\tilde{u}$ .

**Proposition 5.21.** *Let  $u$  and  $v$  reduced words such that  $v$  left-absorbs  $u$ . Then, up to permutation, there is a unique decomposition*

$$u = u' \cdot w \qquad w \cdot v' = v,$$

such that

- (1)  $u'$  is properly left-absorbed by  $v'$ ,
- (2)  $w$  commutes with  $u'$ ,
- (3)  $w$  is a commuting word.

*Proof.* Apply the Absorption Lemma 5.9 to  $v$  and  $u$ , which is completely left-absorbed by  $v$ . The letters of  $u$  which are not properly left-absorbed by  $v$  must commute with all other letters and form the word  $w$ .  $\square$

We obtain therefore the following consequence, which implies that a word is commuting if and only if it is an idempotents in  $\text{Cox}(N)$ .

**Corollary 5.22.** *A reduced word is commuting if and only if it absorbs itself (left, or equivalently, right).*

*Proof.* Clearly, if  $u$  is commuting, then  $|u| = \mathcal{S}_L(u)$ , so  $u$  absorbs itself. Suppose now that  $u$  left-absorbs itself. By the proposition applied to  $v = u$  we find  $u = w \cdot u' \approx w \cdot v'$  such that  $u'$  is properly left-absorbed by  $v'$  and  $w$  is a commuting word. It follows that  $u' = v'$  properly absorbs itself, i.e. the word  $u' = 1$ .  $\square$

We can now state a symmetric version of the Decomposition Theorem 5.13, combined with Proposition 5.21.

**Corollary 5.23** (Symmetric Decomposition Lemma). *Let  $u$  and  $v$  be two reduced words. Each can be uniquely decomposed (up to commutation) as:*

$$u = u_1 \cdot u' \cdot w \qquad w \cdot v' \cdot v_1 = v,$$

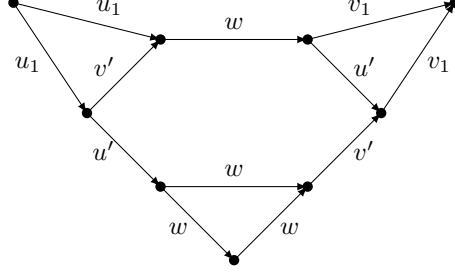
such that:

- (a)  $u'$  is properly left-absorbed by  $v_1$ ,
- (b)  $v'$  is properly right-absorbed by  $u_1$ ,
- (c)  $u'$ ,  $w$  and  $v'$  pairwise commute,
- (d)  $w$  is a commuting word,
- (e)  $u_1 \cdot w \cdot v_1$  is reduced.

In particular, we have  $u \cdot v \rightarrow u_1 \cdot w \cdot v_1$ .

*Proof.* Let

$$u = u_1 \cdot \bar{u}' \qquad v' \cdot \bar{v}_1 = v$$



be a fine decomposition as in Theorem 5.13. Apply Proposition 5.21 to  $\bar{u}'$  and  $\bar{v}_1$  to obtain

$$\bar{u}' = u' \cdot w \qquad w \cdot v_1 = \bar{v}_1,$$

where  $u'$  is properly left-absorbed by  $v_1$ ,  $w$  commutes with  $u'$  and  $w$  is a commuting word.

Uniqueness follows similarly.  $\square$

In order to describe canonical paths between elements (or rather, between flags) in the Fraïssé limit  $M_\infty^N$ , we require a stronger form of reduction, since applying twice the same operation  $\alpha_s$  does not necessarily yield a global application of  $\alpha_s$ , but rather a finite product of proper subletters.

**Definition 5.24.** The word  $u$  is *strongly reduced to*  $v$ , denoted by  $u \xrightarrow{*} v$ , if  $v$  is obtained from  $u$  by finitely many iterations of CANCELLATION, COMMUTATION, and

SPLITTING: Replace an occurrence of  $s \cdot s$  by a (possibly trivial) product  $t_1 \cdots t_n$  of letters  $t_i$ , each of which is properly contained in  $s$ .

If  $v$  is reduced, we call  $v$  a *strong reduct* of  $u$ .

As an example note that  $u \cdot u^{-1} \xrightarrow{*} 1$ .

Despite the possible confusion for the reader, we will not refer to reductions defined in 5.2 as *weak* reductions.

Related to the notion of strong reduction, we also consider the following partial ordering on words.

**Definition 5.25.** For words  $u$  and  $v$ , we define  $u \prec v$  if some permutation of  $u$  is obtained from  $v$  by replacing at least one letter  $s$  of  $v$  by a (possibly empty) product of proper subletters of  $s$ . By  $u \preceq v$ , we mean  $u \prec v$  or  $u \approx v$ .

**Lemma 5.26.**

- (1)  $\prec$  is transitive and well-founded.
- (2)  $u' \approx u \prec v \approx v'$  implies  $u' \prec v'$ .
- (3) If the strong reduction  $u \xrightarrow{*} v$  involves at least one cancellation or splitting, we have  $v \prec u$ .

Well-foundedness implies in particular that if  $u \prec v$ , then  $u \not\approx v$ . Furthermore, property (2) yields that  $\prec$  induces a partial order on  $\text{Cox}(N)$ , setting  $[u] \prec [v]$  if  $u \prec v$ , where both  $u$  and  $v$  are reduced. With this notation, the trivial word 1 becomes the smallest element.

*Proof.* To see that  $\prec$  is well-founded, we introduce an ordinal-valued rank function  $\text{ord}$ . For  $i$  in  $[0, N]$ , set  $\text{ord}_i(w)$  to be number of letters  $s$  in  $w$  with  $i + 1$  elements. Define now

$$\text{ord}(w) = \omega^N \text{ord}_N(w) + \omega^{N-1} \text{ord}_{N-1}(w) + \dots + \text{ord}_0(w).$$

Then  $u \prec v$  implies  $\text{ord}(u) < \text{ord}(v)$ .  $\square$

The semigroup  $\text{Cox}(N)$ , equipped with the order function as above, is an ordered semigroup in which left and right-cancellation are (almost) order-preserving.

**Lemma 5.27.** *Let  $w \cdot v$  be reduced and  $w \cdot v \preceq w \cdot v'$ . Then  $v \preceq v'$ .*

The condition that  $w \cdot v$  is reduced is needed, by taking  $v' = t \subsetneq s = w = v$  and  $w \cdot v \xrightarrow{*} 1$ .

*Proof.* By induction on the number of letters appearing in  $w$ , we need only consider the case where  $w = s$  for some interval  $s$ .

The assumption implies that  $s \cdot v$  is equivalent to a word  $u_s \cdot u'$  where  $u_s \preceq s$  and  $u' \preceq v'$ . The word  $u_s$  either equals  $s$  or is a product of proper subletters of  $s$ . If  $u_s = s$ , we have  $v \approx u' \preceq v'$  and are done. Otherwise, since  $s \cdot v$  is reduced, it follows that  $u_s = 1$ . This implies  $v \prec s \cdot v \approx u' \preceq v'$ .  $\square$

**Corollary 5.28.** *Given reduced words  $w \cdot v$  and  $v'$  such that  $w \cdot v$  is smaller than some strong reduct of  $w \cdot v'$ , then  $v \preceq v'$ .*

**Lemma 5.29.** *The partial order  $\preceq$  is compatible with the semigroup operation in  $\text{Cox}(N)$ .*

*Proof.* Given reduced words  $u, v$  and  $w$ , we have to show the following:

$$[u] \preceq [v] \Rightarrow [w][u] \preceq [w][v]$$

and

$$[u] \preceq [v] \Rightarrow [u][w] \preceq [v][w].$$

By symmetry, it is sufficient to show the first implication. By induction on  $|w|$ , it is enough to consider the case where  $w$  is a single letter  $s$ .

Suppose first that  $s$  is left-absorbed by  $v$ . By Corollary 5.14,

$$[s][v] = [v].$$

If  $s$  is also left-absorbed by  $u$ , we are clearly done. Otherwise, by Theorem 5.13, decompose  $u$  (up to permutation) as  $u = u' \cdot u_1$ , where  $s \cdot u_1$  is the reduct of  $s \cdot u$ . Also, write  $v = \bar{v} \cdot t \cdot v_1$  such that  $s \subset t$  and  $\bar{v}$  is in  $\mathbb{C}(s)$ . Now, the word  $u_1 \preceq u \preceq v$ , so write  $u_1 = \bar{u}_1 \cdot u_1^t \cdot \bar{u}_1^1$ , where  $\bar{u}_1 \preceq \bar{v}$ ,  $u_1^t \preceq t$  and  $u_1^1 \preceq v_1$ . Since  $s \cdot u_1$  is reduced, so is  $s \cdot \bar{u}_1 \cdot u_1^t = \bar{u}_1 \cdot s \cdot u_1^t$ .

This forces  $u_1^t$  to be either trivial or different from  $t$  (and  $s \neq t$  as well). In both cases, we have that  $s \cdot u_1^t \preceq t$ , which implies  $s \cdot u_1 \preceq v$ , so we are done.

If  $s$  is not left-absorbed by  $v$ , by Theorem 5.13, we can write (up to permutation)  $v = v' \cdot v_1$ , where  $v'$  is properly absorbed by  $s$  and  $s \cdot v_1$  is reduced. So  $[s][v] = [s \cdot v_1]$ . If  $s$  is left-absorbed by  $u$ , then

$$[s][u] = [u] \preceq [v' \cdot v_1] \prec [s \cdot v_1].$$



Otherwise, write  $u = \bar{u} \cdot u' \cdot u_1$  as above such that  $s \cdot u \rightarrow \bar{u} \cdot s \cdot u_1$ . Since  $\bar{u}$  and  $s$  commute, note that  $\bar{u} \cdot u_1$  is irreducible, since  $u$  is. Decompose  $\bar{u} \cdot u_1 = u'_1 \cdot u_{11}$  with  $u'_1 \preceq v'$  and  $u_{11} \preceq v_1$ . Since  $s \cdot \bar{u} \cdot u_1 = \bar{u} \cdot s \cdot u_1$  is reduced, the word  $u'_1$  must be trivial. Therefore  $s \cdot \bar{u} \cdot u_1 = s \cdot u_{11} \preceq s \cdot v_1$ .  $\square$

In particular, since  $1 \preceq v$  for any word  $v$ , we obtain the following result.

**Corollary 5.30.** *Let  $u$  be reduced. Given any word  $v$ , the reduction  $w$  of  $u \cdot v$  is  $\preceq$ -larger than  $u$ .*

In contrast to Proposition 5.3, uniqueness of strong reductions does no longer hold, e.g.  $s \cdot s \xrightarrow{*} s$  and  $s \cdot s \xrightarrow{*} 1$ . However, we get the following result, which allows us to permute the steps of the strong reduction:

**Proposition 5.31** (Commutation Lemma). *If  $x$  is a strong reduct of  $u \cdot v \cdot w$ , then there is a strong reduct  $y$  of  $v$  such that  $u \cdot y \cdot w \xrightarrow{*} x$ .*

*Proof.* Consider first the case where  $u = t$  has length 1, the word  $v$  has length 2 and  $w$  is empty. Suppose furthermore that in the first step of the reduction  $t \cdot v \xrightarrow{*} x$ , the letter  $t$  is deleted. It is easy to check that setting  $y$  as the reduct of  $v$ , the results follows, except if  $v = s \cdot s$ , the letter  $t$  is contained in  $s$  and the strong reduction is  $t \cdot (s \cdot s) \xrightarrow{*} s \cdot s \xrightarrow{*} x$ , where  $x$  is a product of letters which are properly contained in  $s$ . Then:

- If  $t = s$ , set  $y = s$ .
- If  $t \cdot x \xrightarrow{*} x$ , set  $y = x$ .
- Otherwise, apply Theorem 5.13 to  $x$  and  $t$  and decompose  $x = x' \cdot x_1$  such that  $|x'|$  is properly contained in  $t$  and  $t \cdot x_1$  is reduced. Set  $y = t \cdot x_1$ .

In all three cases, the strong reductions hold:

$$t \cdot (s \cdot s) \xrightarrow{*} t \cdot y \xrightarrow{*} x.$$

In order to show the proposition for the general case, motivated by the proof of 5.3, let us introduce the following rule:

**GENERALISED SPLITTING:** Given a word  $s_1 \cdots s_n$  and a pair of indices  $i \neq j$  such that  $s_i = s_j$  and  $s_i$  commutes with all  $s_k$ 's with  $k$  between  $i$  and  $j$ , delete  $s_j$  and replace  $s_i$  by a product of letters which are properly contained in  $s$ .

Note that a strong reduction consists of finitely many generalised cancellations and generalised splittings, followed by commutation (if needed).

If  $v$  is reduced, set  $y = v$ . Otherwise, we will apply induction on the  $\prec$ -order type of  $v$ . Suppose therefore that the assertion holds for all  $v' \prec v$  and consider  $x$  a strong reduct of  $u \cdot v \cdot w$ . If  $2 < |v|$ , then (after permutation) write  $v = v_1 \cdot a \cdot v_2$ , where  $a$  is a non-reduced word of length 2. Note that by assumption, the subword  $a \prec v$ , so there is a strong reduct  $b$  of  $a$  such that  $u \cdot v_1 \cdot b \cdot v_2 \cdot w \xrightarrow{*} x$ . Since  $a$  is not reduced, we have  $b \prec a$  and thus  $v_1 \cdot b \cdot v_2 \prec v$ . Induction yields the existence of a strong reduct  $y$  of  $v_1 \cdot b \cdot v_2$  such that

$$u \cdot y \cdot w \xrightarrow{*} x.$$

Note that  $v = v_1 \cdot a \cdot v_2 \xrightarrow{*} v_1 \cdot b \cdot v_2 \xrightarrow{*} y$ . Therefore, we may assume that  $v$  has length 2 and it is non-reduced. By the above discussion, the first step in the strong reduction

$$u \cdot v \cdot w \xrightarrow{*} x.$$

is either a generalised cancellation or a generalised splitting. If it involves only letters from  $v$ , its strong reduction is  $\preceq$ -smaller and one step shorter to the output  $x$ , so we are done by induction on the number of steps in the strong reduction. Likewise if the letters involved are in  $u \cdot w$ . Thus, we may assume that there are two letters  $t$  and  $r$  witnessing the reduction in the first step and, say, the letter  $t$  occurs in  $u$  and  $r$  in  $v$ .

We have two cases:

- The letter  $t$  is absorbed by  $v$ . In particular, the letter lies in the final segment  $\tilde{u}$ . Write  $u = u_1 \cdot t$ . If it was a generalised splitting, the result  $v' \prec v$  and  $u_1 \cdot v' \cdot w \xrightarrow{*} x$ . Induction gives a strong reduct  $x'$  of  $v'$  such that  $u_1 \cdot x' \cdot w \xrightarrow{*} x$ . In particular, we are now in the case  $t \cdot v \xrightarrow{*} x'$  and thus, by the discussion at the beginning of the proof, there exists a strong reduction  $y$  of  $v$  such that  $t \cdot y \xrightarrow{*} x'$ . Note that

$$u \cdot v \cdot w = u_1 \cdot (t \cdot v) \cdot w \xrightarrow{*} u_1(t \cdot y) \cdot w \xrightarrow{*} u_1 \cdot x' \cdot w \xrightarrow{*} x,$$

so we are done.

If the first step was a generalised cancellation, the word  $v$  does not change and now  $u_1 \cdot v \cdot w \xrightarrow{*} x$  in one step less. We obtain a strong reduct  $x'$  of  $v$  with  $u_1 \cdot x' \cdot w \xrightarrow{*} x$ . Again, note that  $t \cdot v \xrightarrow{*} v \xrightarrow{*} x'$  so, again by the previous discussion, there is a strong reduct  $y$  of  $v$  which does the job.

- Otherwise, the occurrence  $r$  in  $v$  is deleted. If  $r = t$ , we are in the previous case. Suppose hence  $r \subsetneq t$  and write  $u = u_1 \cdot t \cdot u_2$ , where  $u_2$  commutes with  $r$ . We may assume that  $v = r \cdot s$ . Note that  $r$  and  $s$  are comparable, since  $v$  is not reduced. If  $r \subseteq s$ , then set  $y = s$ , which is a strong reduct of  $v$ . We have that  $u \cdot y \cdot w \xrightarrow{*} x$ .

If  $s \subsetneq r$ , then  $s$  and  $u_2$  commute as well. Note that  $u_1 \cdot (t \cdot s) \cdot u_2 \cdot w = u \cdot s \cdot w \xrightarrow{*} x$  in one step less. We have that  $u_1 \cdot t \cdot u_2 \cdot w \xrightarrow{*} x$  and setting  $y = r$  does the job.

□

Despite the apparent arbitrariness of the strong reductions, they are orthogonal to the reduction without splitting, as the following result shows.

**Proposition 5.32.** *Let  $u$  and  $v$  be reduced words and consider  $x$  the reduct of  $u \cdot v$  and  $x^*$  some strong reduct of  $u \cdot v$ , where splitting occurs. Then  $x^* \prec x$ .*

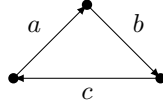
Note that that this is not true for the product of three reduced words:  $s \cdot s \cdot s$  can be strongly reduced to  $s$  by one splitting operation.

*Proof.* Remark first that, if  $w = s_1 \cdots s_n$  is a commuting word and  $y^*$  is a strong reduct of  $w \cdot w$ , then  $y^* = t_1 \cdots t_n$ , where each  $t_i$  is a strong reduct of  $s_i \cdot s_i$ . If splitting ever occurred in the reduction, then  $y^* \prec w$ .

To prove the proposition, choose decompositions  $u = u_1 \cdot u' \cdot w$  and  $w \cdot v' \cdot v_1 = v$ , as in Corollary 5.23. A general cancellation applied to  $u_1 \cdot u' \cdot w \cdot w \cdot v' \cdot v_1$  does the following: either the last letter of (a permutation of)  $u'$  is deleted, the first letter of  $v'$  is deleted or one letter in one of the copies of  $w$  is deleted. Hence, after finitely many generalised cancellations, the end result has the form  $z = u_1 \cdot u'' \cdot w' \cdot w' \cdot v'' \cdot v_1$ , where  $u''$  is a left end of  $u'$ , the subword  $v''$  is a right end of  $v'$  and  $w'$  is a

subword of  $w$ . A generalised splitting for  $z$  can only happen inside  $w' \cdot w'$ . So we obtain a word  $z' = u_1 \cdot u'' \cdot a \cdot v'' \cdot v_1$ , where  $a$  is obtain from  $w \cdot w$  by the splitting operation. If we apply the Commutation Lemma 5.31 to  $(u_1 \cdot v') \cdot a \cdot (u' \cdot v_1) \approx z'$ , we obtain a strong reduct  $b$  of  $a$  such that  $u_1 \cdot b \cdot v_1 \xrightarrow{*} x^*$ . The above observation gives that  $b \prec w$  and thus  $x^* \preceq u_1 \cdot b \cdot v_1 \prec u_1 \cdot w \cdot v_1 \approx x$ .  $\square$

Inspired by the following picture:



we deduce strong reductions from a given one, as long as products are involved.

**Proposition 5.33** (Triangle Lemma). *Let  $a$ ,  $b$  and  $c$  be reduced words. Then  $a \cdot b \xrightarrow{*} c^{-1}$  implies  $c \cdot a \xrightarrow{*} b^{-1}$  and  $b \cdot c \xrightarrow{*} a^{-1}$ .*

*Proof.* By symmetry, it is enough to show that  $a \cdot b \xrightarrow{*} c^{-1}$  implies  $c \cdot a \xrightarrow{*} b^{-1}$ . Suppose hence that  $a \cdot b \xrightarrow{*} c^{-1}$ . We apply induction on the  $\prec$ -type of  $a$  and  $b$ .

If  $a \cdot b$  is reduced, then  $c = b^{-1} \cdot a^{-1}$  and so  $c \cdot a = b^{-1} \cdot a^{-1} \cdot a \xrightarrow{*} b^{-1}$ . Thus, assume  $a \cdot b$  is not reduced. We distinguish the following cases (up to permutation):

- $a = a_1 \cdot s$ , where  $s$  is properly left-absorbed by  $b$ . Since  $b$  is the only strong reduct of  $s \cdot b$ , the Commutation Lemma 5.31 gives that

$$a \cdot b = a_1 \cdot (s \cdot b) \rightarrow a_1 \cdot b \xrightarrow{*} c^{-1}.$$

Since  $a_1 \prec a$ , induction gives that  $c \cdot a_1 \xrightarrow{*} b^{-1}$ , which implies that

$$c \cdot a = (c \cdot a_1) \cdot s \xrightarrow{*} b^{-1} \cdot s \rightarrow b^{-1}.$$

- $b = s \cdot b_1$ , where  $s$  is properly right-absorbed by  $a$ . Again  $a \cdot b = a \cdot (s \cdot b_1) \rightarrow a \cdot b_1 \xrightarrow{*} c^{-1}$ , so by induction  $c \cdot a \xrightarrow{*} b_1^{-1}$ . Thus

$$c \cdot (a \cdot s) \xrightarrow{*} b_1^{-1} \cdot s = b^{-1}.$$

Since  $a$  is the only strong reduct of  $a \cdot s$ , again Proposition 5.31 gives that  $c \cdot a \xrightarrow{*} b^{-1}$ .

- $a = a_1 \cdot s$  and  $b = s \cdot b_1$ . Since  $a_1 \cdot (s \cdot s) \cdot b_1 \xrightarrow{*} c^{-1}$ , Proposition 5.31 provides a strong reduct  $x$  of  $s \cdot s$  such that  $a_1 \cdot x \cdot b_1 \xrightarrow{*} c^{-1}$ . The word  $x$  is either  $s$  or a product of proper subletters of  $x$  and hence  $\prec$ -smaller than  $s$ . Since  $b = s \cdot b_1$  is reduced, apply Theorem 5.13 to decompose  $x = x_1 \cdot x'$ , where  $x'$  is properly left absorbed by  $b_1$  and  $x_1 \cdot b_1$  is reduced (If  $x = s$ , then  $x_1 = s$  and  $x' = 1$ ). Since  $x' \cdot b_1 \xrightarrow{*} b_1$ , the reduction  $(a_1 \cdot x_1) \cdot (x' \cdot b_1) \xrightarrow{*} c^{-1}$  implies  $a_1 \cdot x_1 \cdot b_1 \xrightarrow{*} c^{-1}$ . Since  $a_1 \prec a$  and  $x_1 \cdot b_1 \preceq b$ , induction gives that

$$c \cdot a_1 \xrightarrow{*} b_1^{-1} \cdot x_1^{-1}.$$

In particular,

$$c \cdot a = c \cdot a_1 \cdot s \xrightarrow{*} (b_1^{-1} \cdot x_1^{-1}) \cdot s \rightarrow b_1^{-1} \cdot s \rightarrow b^{-1}.$$

$\square$

We can now easily conclude the following:

**Corollary 5.34.** *If  $u$  and  $v$  are both reduced and  $u \cdot v \xrightarrow{*} 1$ , then  $v \approx u^{-1}$ .*

*Proof.* The Triangle Lemma (Proposition 5.33) yields  $1 \cdot u \xrightarrow{*} v^{-1}$  and  $v \cdot 1 \xrightarrow{*} u^{-1}$ . That is,  $u^{-1} \xrightarrow{*} v$  and  $v \xrightarrow{*} u^{-1}$ . Thus

$$u^{-1} \preceq v \preceq u^{-1},$$

and therefore  $v \approx u^{-1}$ .  $\square$

Recall by Corollary 5.14 that if  $u$  is the reduct of  $u \cdot v$ , then  $v$  is right-absorbed by  $u$ . This is no longer true for strong reductions: take for example

$$(s \cdot t) \cdot (t \cdot s \cdot t) = s \cdot (t \cdot t) \cdot (s \cdot t) \xrightarrow{*} s \cdot (s \cdot t) \xrightarrow{*} s \cdot t.$$

However, in certain situations we are still able to conclude the same for strong reductions as for reductions with no splitting.

**Lemma 5.35.** *Let  $u$  and  $v$  be reduced. If every letter in  $v$  which is right-absorbed by  $u$  is properly absorbed and  $u \cdot v \xrightarrow{*} u$ , then  $u \cdot v \rightarrow u$ .*

*Proof.* Apply Theorem 5.13 to obtain fine decompositions  $u = u_1 \cdot u'$  and  $v' \cdot v_1 = v$  such that  $u'$  is properly left-absorbed by  $v_1$ , the word  $v'$  is right-absorbed by  $u_1$ , the words  $u'$  and  $v'$  commute and  $u_1 \cdot v_1$  is reduced.

By hypothesis, the word  $v'$  is properly right-absorbed by  $u_1$ . The Commutation Lemma 5.31 applied to  $(u_1 \cdot v') \cdot (u' \cdot v_1) \xrightarrow{*} u$  gives

$$(u_1 \cdot v') \cdot (u' \cdot v_1) \rightarrow u_1 \cdot v_1 \xrightarrow{*} u.$$

Since  $u_1 \cdot v_1$  is reduced, we have  $u_1 \cdot v_1 = u$ . So  $v_1 = u'$  must properly absorb itself, which is a contradiction unless  $v_1 = 1$  and thus  $u \cdot v \rightarrow u$ .  $\square$

Let us conclude by giving a criteria for when a word wobbles inside two other. This will be useful for determining all possible paths between two given flags.

**Proposition 5.36.** *Let  $u \cdot v$  and  $w$  be reduced. If  $u \cdot w \xrightarrow{*} u$  and  $w^{-1} \cdot v \xrightarrow{*} v$ , then  $|w| \subset \text{Wob}(u, v)$ .*

*Proof.* By Remark 5.17, it is enough to prove that  $w$  is properly right-absorbed by  $u$  (and likewise for  $v$ ). We proceed by induction on the length of  $|v|$ .

If  $v = 1$ , then  $w^{-1} \cdot 1 \xrightarrow{*} 1$  implies  $w^{-1} = 1$ , since  $w$  is reduced.

Suppose now that  $v = s \cdot v_1$ . Set  $u \cdot s = u_1$ , which is again reduced. So is  $u_1 \cdot v_1 = u \cdot v$ .

The condition  $w^{-1} \cdot v \xrightarrow{*} v$  implies  $v^{-1} \cdot w \xrightarrow{*} v^{-1}$  by Proposition 5.33. This implies

$$v_1^{-1} \cdot (s \cdot w \cdot s) \xrightarrow{*} (v_1^{-1} \cdot s) \cdot s \xrightarrow{*} v_1^{-1}.$$

By the Commutation Lemma (Proposition 5.31), there is a strong reduct  $w_1$  of  $s \cdot w \cdot s$  with  $v_1^{-1} \cdot w_1 \xrightarrow{*} v_1^{-1}$ , or equivalently,  $w_1^{-1} \cdot v_1 \xrightarrow{*} v_1$ .

The Triangle Lemma 5.33 gives that  $s \cdot (w \cdot s) \xrightarrow{*} w_1$  implies  $w_1^{-1} \cdot s \xrightarrow{*} s \cdot w^{-1}$ , that is,  $s \cdot w_1 \xrightarrow{*} w \cdot s$ .

In particular, we have that  $u_1 \cdot w_1 = u \cdot (s \cdot w_1) \xrightarrow{*} u \cdot (w \cdot s) \xrightarrow{*} u \cdot s = u_1$ .

By the induction hypothesis applied to  $u_1, v_1$  and  $w_1$ , we have that  $w_1$  is properly right-absorbed by  $u_1 = u \cdot s$ . By Lemma 5.9 (3), write  $w_1$  as  $w_s \cdot w_u$  where  $w_s$  is properly absorbed by  $s$  and  $w_u$  is properly right-absorbed by  $u$  and commutes with  $s$ . Note that  $s \cdot w_u$  is the only strong reduct of  $s \cdot w_1$ . Proposition 5.31 yields that the strong reduction  $(s \cdot w_1) \cdot s \xrightarrow{*} w \cdot s \cdot s \xrightarrow{*} w$  factors through  $s \cdot w_u \cdot s \xrightarrow{*} w$ .

Since  $s \cdot w_u \cdot s$  is equivalent to  $s \cdot s \cdot w_u$ , there is strong reduct  $x$  of  $s \cdot s$  such that  $x \cdot w_u \xrightarrow{*} w$ . However, the product  $x \cdot w_u$  is already reduced and so  $x \cdot w_u = w$ . The reduct  $x$  is either  $s$  or consists of proper subletters of  $s$ . Suppose that  $x = s$ . Then  $u \cdot w = u \cdot s \cdot w_u = u \cdot s$ , since  $w_u$  is properly right-absorbed by  $u$  and commutes with  $s$ . This contradicts with  $u \cdot w \xrightarrow{*} u$ . Hence, the word  $x$  consists of proper subletters of  $s$ . By Theorem 5.13, since  $u \cdot s$  is reduced, decompose  $x$  into  $x' \cdot x_1$ , where  $x'$  is properly right-absorbed by  $u$  and  $u \cdot x_1$  is reduced. Then  $u \cdot x_1$  is the only strong reduct of  $u \cdot w = u \cdot x' \cdot x_1 \cdot w_u$ . We conclude that  $u \cdot x_1 = u$  and thus  $x_1 = 1$  by Corollary 5.14. Hence, the word  $w = x' \cdot w_u$  is properly right-absorbed by  $u$ .  $\square$

## 6. FLAGS AND PATHS

Let  $M$  be any colored  $N$ -space. As in Definition 4.16, recall that a *flag*  $F$  in  $M$  is a path  $a_0 - \dots - a_N$  of length  $N$ , where each  $a_i$  belongs to  $\mathcal{A}_i(M)$ . We call  $a_i$  the  *$i$ -vertex* of the flag  $F$ .

**Definition 6.1.** Given flags  $F$  and  $G$ , we say that  $G$  is obtained from  $F$  by the *weak operation*  $\alpha_s$  if  $s$  consists of the indexes where the vertices of  $F$  and  $G$  differ. A weak path of flags  $P$  is a sequence of flags  $F_0, \dots, F_n$ , where each  $F_i$  is obtained from  $F_{i-1}$  by a weak operation  $\alpha_{s_i}$ . We call  $s_1 \cdots s_n$  the *word* of  $P$ .

More generally, we define:

**Definition 6.2.** Let  $A$  be a subset of  $[0, N]$ . Two flags are *equivalent modulo*  $A$  if they have the same vertices in all levels outside  $A$ . We write  $F/A$  for the equivalence class of  $F$  modulo  $A$ .

Note that  $F/A$  is interdefinable with the set of vertices of  $F$  with levels outside  $A$ . For  $i$  in  $[0, N]$  and  $A = [0, N] \setminus \{i\}$ , the equivalence class  $F/A_i$  is interdefinable with the vertex  $f_i$ . We can say that  $F/A_i$  and  $F'/A_j$ , for  $i$  and  $j$  immediate successors, are connected in case they belong to a class of a common flag  $G$ . This induces a structure bi-interpretable with  $PS_N$ .

Any two flags can be connected by a weak flag path: decompose the set  $I$  of indices where the vertices of  $F$  and  $G$  differ as the disjoint union  $s_1 \cup \dots \cup s_n$  of intervals, such that  $s_i$  and  $s_j$  commute for  $i \neq j$ . Then  $F$  and  $G$  are connected by a weak path with word  $s_1 \cdots s_n$ . In particular, we obtain the following.

**Lemma 6.3.** *Two flags  $F$  and  $G$  are equivalent modulo  $A$  if and only if they can be connected by a weak path whose word consists of letters contained in  $A$ . Furthermore, there is such a path whose word is commuting.*

*In particular, any two flags are connected by a weak path, by taking  $A = [0, N]$ .*

Commuting letters in a path induces another path whose word is a permutation of the previous one.

**Lemma 6.4.** *Let  $s$  and  $t$  be commuting letters and assume that  $F$  and  $G$  are connected by a weak flag path with word  $s \cdot t$ . Then there is a unique weak flag path from  $F$  to  $G$  with word  $t \cdot s$ .*

*Proof.* Given the path  $F - H - G$  with word  $s \cdot t$ , define a new flag  $H'$  by replacing the  $s$ -part of  $H$  by the  $s$ -part of  $F$  and its  $t$ -part by the  $t$ -part of  $G$ . By construction, the weak path  $F - H' - G$  has word  $t \cdot s$ .

Uniqueness is clear since the  $s$ -part and the  $t$ -part of  $H'$  are determined by those of  $F$  and  $G$ .  $\square$

Iterating the previous result, since any permutation can be achieved by a sequence of transpositions of adjacent commuting letters, given a weak path  $P_u$  from  $F$  to  $G$  with word  $u$ , if  $v$  is a permutation of  $u$ , we can connect  $F$  and  $G$  by a weak path  $P_v$  with word  $v$ . Note that  $P_v$  does not depend on the sequence of transpositions and the collection of vertices of flags occurring in  $P_u$  agrees with the one of flags in  $P$ . We call the path  $P_v$  a *permutation of  $P_u$* .

We will now link the words appearing in weak paths with their distance as in Lemma 4.13.

**Lemma 6.5.** *Let  $t = (l, r)$  and  $F$  and  $G$  be equivalent modulo  $t$ . Let  $a_l$  and  $a_r$  the vertices of  $F$  (and  $G$ ) of level  $l$  and  $r$ , respectively. Given a subletter  $s \subset t$ , the following are equivalent:*

- a) *The flags  $F$  and  $G$  have finite  $s$ -distance in  $M_{a_l}^{a_r}$ .*
- b) *The flag  $F$  and  $G$  are connected by a weak flag path whose letters are contained in  $t$  but do not contain  $s$ .*

*Proof.* (a)  $\rightarrow$  (b): Consider a path  $b_0, \dots, b_n$  in  $\mathcal{A}_s(M_{a_l}^{a_r})$  connecting two vertices of  $F$  and  $G$ . For every  $i$  in  $\{1, \dots, n-1\}$ , pick a flag  $F_i$  containing  $b_i$  and  $b_{i+1}$  which agrees with  $F$  and  $G$  outside the levels in  $t$ . Set  $F_0 = F$  and  $F_n = G$ . If  $b_{i+1}$  has level  $j_i$ , then  $F_i$  and  $F_{i+1}$  are equivalent modulo  $t \setminus \{j_i\}$ . They are thus connected by a weak flag path whose letters are contained in  $t \setminus \{j_i\}$  and therefore none contains  $s$ . The concatenation of these flag paths gives the result.

(b)  $\rightarrow$  a): Let  $F = F_0 - \dots - F_n = G$  be a weak flag path whose letters are in  $t$  but do not contain  $s$ . For every  $i$  in  $\{0, n-1\}$ , the flags  $F_i$  and  $F_{i+1}$  have a common vertex in  $\mathcal{A}_s(M_{a_l}^{a_r})$ . Thus, we can connect  $F$  and  $G$  by a path whose vertices lie in  $\mathcal{A}_s(F_0) \cup \dots \cup \mathcal{A}_s(F_n)$  and hence, between  $a_l$  and  $a_r$ .  $\square$

In order to distinguish between weak operations between flags and global applications of  $\alpha_s$  to nice sets, as in Lemma 4.21, we introduce the following definition, at the level of flags.

**Definition 6.6.** For  $s = (l, r)$ , the flag  $G$  is obtained by a *global application of  $\alpha_s$*  from  $F$  if  $G$  is obtained by a weak application of  $\alpha_s$  from  $F$  and its new vertices have infinite distance in  $M_{a_l}^{a_r}$  from  $F$ , where  $a_l$  and  $a_r$  are the vertices of  $F$  (and  $G$ ) of level  $l$  and  $r$ , respectively.

Since a flag is in particular a nice set, these two definitions agree, by applying Lemma 6.5 to the case  $t = s$ :

**Corollary 6.7.** *Given an interval  $s$  and flags  $F$  and  $G$ , the following are equivalent:*

- a) *The flag  $G$  is obtained from  $F$  by a global application of  $\alpha_s$ , as in Lemma 4.21.*
- b) *The flag  $G$  is obtained from  $F$  by the weak operation  $\alpha_s$  and there is no weak flag path connecting them whose word consists of proper subletters of  $s$ .*

**Definition 6.8.** A *flag path* is a weak flag path where each flag is obtained from its predecessor by a global operation. If  $F$  and  $G$  are connected with a flag path with word  $u$ , we write

$$F \xrightarrow[u]{} G.$$

A flag path is *reduced* if its word is reduced.

**Lemma 6.9.** *If there is a weak path from  $F$  to  $G$  with word  $u$ , we have  $F \xrightarrow[v]{} G$  for some  $v$  with  $v \preceq u$ .*

*Proof.* By Lemma 6.3, choose a weak path  $F = F_0 - \dots - F_n = G$  whose word  $v = s_1 \cdots s_n$  is  $\preceq$ -smaller to  $u$  and minimal such. We need only show that this path is a flag path. Otherwise, some operation  $\alpha_{s_i}$  is not global and, by Corollary 6.7, we can connect  $F_{i-1}$  and  $F_i$  with a weak path whose word consists of proper subletters of  $s_i$ . The resulting word is  $\prec$ -smaller than  $v$ , contradicting its minimality.  $\square$

Combining the previous result and Corollary 6.7, we obtain the following:

**Corollary 6.10.** *If  $F$  and  $G$  are equivalent modulo  $t$ , then either  $F \xrightarrow[t]{} G$  or  $F \xrightarrow[x]{} G$ , for some product  $x$  whose factors are proper subletters of  $t$ .*

*Proof.* By Lemma 6.3, the flag  $G$  is obtained from  $F$  by a weak path  $P$  whose word  $x$  either equals  $t$  or consists of letters properly contained in  $t$ . By Lemma 6.9, we may assume that  $P$  is a flag path.  $\square$

We can now compose flag paths, using the results of the previous section.

**Lemma 6.11.** *Assume  $F \xrightarrow[s]{} G \xrightarrow[t]{} H$ .*

- (1) *If  $s$  and  $t$  commute, there is a unique  $G'$  with  $F \xrightarrow[t]{} G' \xrightarrow[s]{} H$ .*
- (2) *If  $s$  is a proper subset of  $t$ , then  $F \xrightarrow[t]{} H$ . Similarly, if  $t$  is a proper subset of  $s$ , then  $F \xrightarrow[s]{} H$ .*
- (3) *If  $s = t$ , then either  $F \xrightarrow[t]{} H$  or  $F \xrightarrow[x]{} H$ , for some product  $x$  whose factors are proper subletters of  $t$ .*

*In particular, a permutation of a flag path yields again a flag path, by (1).*

*Proof.* Property (1) follows easily from Lemma 6.4, since the permutation of a reduced word remains reduced.

For (2), assume  $s \subsetneq t$ . Then  $H$  is equivalent to  $F$  modulo  $t$ . So by Corollary 6.10, either  $F \xrightarrow[t]{} H$  or  $F \xrightarrow[x]{} H$ , where  $x$  consists of proper subletters of  $t$ . The latter implies that  $G \xrightarrow[s \cdot x]{} H$ , which contradicts the assumption  $G \xrightarrow[t]{} H$ . The proof is similar if  $t$  is a proper subset of  $s$ .

Property (3) clearly follows from Corollary 6.10, as  $F$  and  $H$  are equivalent modulo  $t$ .  $\square$

Lemma 6.9 yields the following.

**Corollary 6.12.** *Let  $F$  and  $G$  be two flags.*

- (1) *If  $F \xrightarrow[u]{} G$ , then  $F \xrightarrow[v]{} G$  for some strong reduct  $v$  of  $u$ .*
- (2) *If  $u$  is  $\prec$ -minimal with  $F \xrightarrow[u]{} G$ , then  $u$  is reduced.*

**Definition 6.13.** Let  $A$  be a subset of  $M$  and two vertices  $a_l$  and  $a_r$  in  $A$  such that  $a_l$  lies below  $a_r$  in  $A$ . The pair  $(a_l, a_r)$  is called *open* in  $A$  if there are vertices  $b$  and  $c$  in  $A_{a_l}^{a_r}$  whose distance in  $M_{a_l}^{a_r}$  is infinite.

A pair as before which is not open is called *closed*.

**Lemma 6.14.** *Let  $s = (l, r)$  be an interval and  $M$  be simply connected. Take a nice subset  $A$  of  $M$  with two distinguished vertices  $a_l$  and  $a_r$  of levels  $l$  and  $r$ , respectively. Given a flag  $F$  in  $A$  containing  $a_l$  and  $a_r$ , assume that  $F \xrightarrow[s]{} G$  for some flag  $G$  in  $M$ . Set  $B = A \cup G$ . If the pair  $(a_l, a_r)$  is closed in  $A$ , we have that:*

- (1) *The set  $B$  is obtained from  $A$  by a global application of  $\alpha_s$  on  $(a_l, a_r)$ .*

(2) The open pairs in  $B$  are exactly the open pairs of  $A$  together with  $(a_l, a_r)$ .

*Proof.* For the first assertion, by Lemma 4.21, we need only check that

$$d^{M_{a_l}^{a_r}}(d, A) = \infty,$$

where  $d$  is one of the new vertices of  $G$ .

Pick any  $b$  in  $A_{a_l}^{a_r}$  and choose some vertex  $c$  in  $F$  between  $a_l$  and  $a_r$ . Since  $(a_l, a_r)$  is closed in  $A$ , we have that  $d^{M_{a_l}^{a_r}}(b, c) < \infty$ . Since  $F \xrightarrow{s} G$ , Lemma 6.5 shows that  $d^{M_{a_l}^{a_r}}(c, d) = \infty$ . In particular,

$$d^{M_{a_l}^{a_r}}(b, d) = \infty,$$

which gives the desired result.

For the second assertion, clearly  $(a_l, a_r)$  is now open in  $B$ . We need only show there are no new open pairs in  $B$ . Consider an open pair  $(x, y)$ . If  $x$  is one of the new elements of  $G$ , then either  $y$  is either also in  $B \setminus A$  or in  $A$  and either equal to  $a_r$  or above of it. If both  $x$  and  $y$  lie in  $B \setminus A$ , they form a closed pair. If  $y = a_r$ , all vertices between  $x$  and  $y$  lie on  $B \setminus A$ , and thus the pair  $(x, y)$  is closed. If  $y$  lies above  $a_r$  in  $A$ , then all vertices between  $x$  and  $y$  are connected with  $a_r$  and thus their distance is finite, so  $(x, y)$  is closed.

Hence, we conclude that both  $x$  and  $y$  lie in  $A$ . Suppose  $(x, y)$  is not  $(a_l, a_r)$ . Either it was already open in  $A$  or there is a vertex  $d$  in  $B \setminus A$  whose distance to some  $b$  in  $A$  is infinite in  $M_x^y$ . In particular, the vertex  $x$  lies below  $a_l$  and  $y$  lies above  $a_r$ . Since  $(x, y)$  is closed in  $A$ , the distance between  $b$  and  $a_l$  in  $M_x^y$  is finite and thus  $b$  and  $d$  have finite distance in  $M_x^y$ , which is a contradiction.  $\square$

Flag paths provide scaffolds which are nice sets, as the following Lemma shows.

**Lemma 6.15.** *Let  $M$  be simply connected and  $F_0 \xrightarrow{s_1} F_1 \xrightarrow{s_2} \dots \xrightarrow{s_n} F_n$  be a reduced flag path in  $M$ . The following hold:*

- (1) *The set  $A_n = F_0 \cup F_1 \cup \dots \cup F_n$  is nice in  $M$ .*
- (2) *If  $a_0 - \dots - a_N$  are the vertices of  $F_n$ , then  $(a_l, a_r)$  is open in  $A_n$  if and only if the letter  $(l, r)$  belongs to final segment of  $s_1 s_2 \dots s_n$ .*

*Proof.* We prove it by induction on  $n$ . Let  $s_i = (l_i, r_i)$  and  $w_i = s_1 s_2 \dots s_i$ . If  $n = 0$ , there is nothing to prove, since any flag is nice and the word  $w_0$  is trivial.

Suppose hence that  $n > 0$  and let  $F_n = a_0 - \dots - a_N$ . Since  $w_n$  is reduced by assumption, the letter  $s_n$  does not belong to the final segment of  $w_{n-1}$ . Therefore, the pair  $(a_{l_n}, a_{r_n})$  appeared already in  $F_{n-1}$  and, by induction, it is closed in  $A_{n-1}$ , which is nice. Lemma 6.14 gives that so is  $A_n$ .

Furthermore, Lemma 6.14 also implies that  $(a_l, a_r)$  is open in  $A_n$  if and only if  $(a_l, a_r) = (a_{l_n}, a_{r_n})$  or it belongs to  $A_{n-1}$  and was already open in  $A_{n-1}$ . In particular, the pair  $(a_l, a_r)$  belongs to  $A_{n-1}$  if and only if either  $(l, r)$  commutes with  $s_n$  or  $(l, r)$  contains  $s_n$ . Since  $s_n$  is not contained in the final segment of  $w_{n-1}$ , induction gives that  $(a_l, a_r)$  is open in  $A_n$  iff  $(l, r) = s_n$  or  $(l, r)$  commutes with  $s_n$  and belongs to the final segment of  $w_{n-1}$ , which means that  $(l, r)$  belongs to the final segment of  $w_n$ .  $\square$

If the space is simply connected, we shall prove that there are no flag loops, unless they are not reduced.





*Proof.* Write  $u = s_1 \cdots s_i$  and  $v = s_{i+1} \cdots s_n$ . Suppose we are given flags  $H_i$  and  $H'_i$  as in the previous picture. Hence

$$F \xrightarrow[u]{\quad} H_i \xrightarrow[v]{\quad} G \qquad F \xrightarrow[u]{\quad} H'_i \xrightarrow[v]{\quad} G.$$

Let  $w$  be some reduced word with  $H_i \xrightarrow[w]{\quad} H'_i$ . By Corollary 6.12 and Proposition 6.17, the word  $u$  is a strong reduct of  $u \cdot w$ . Likewise, the word  $v$  is a strong reduct of  $w^{-1} \cdot v$ . Proposition 5.36 gives that  $|w| \subset \text{Wob}(u, v)$ , which yields the result.  $\square$

We finish this section by observing that nice sets are flag-connected.

**Proposition 6.20.** *Let  $M$  be simply connected and  $A$  some union of flags from  $M$ . The set  $A$  is nice if and only if any two flags in  $A$  can be connected by a reduced flag path which belongs to  $A$ .*

*Proof.* Clearly, any union of flags satisfies that  $A_a^b = A \cap M_a^b$ .

Suppose it is nice. Consider two flags  $F$  and  $G$  in  $A$  and connect them in  $M$  by some weak path. Since  $A$  is nice, we can find a weak path  $P$  belonging to  $A$  which is reduced *in the sense of  $A$* . In order to show that  $P$  is a flag path (in the sense of  $M$ ), we need only show that if  $G$  is obtained from  $F$  by a global application of  $\alpha_s$  in  $A$ , then it remains a global application of  $\alpha_s$  in  $M$ . Equivalently, for any  $b$  in  $G \setminus F$ , if  $d_s^A(b, F) = \infty$  then  $d_s^M(b, F) = \infty$ . This is exactly the definition of niceness.

Assume now that every two flags in  $A$  are connected in  $A$  by a reduced flag path. Consider two vertices  $b$  and  $c$  in  $\mathcal{A}_s(A)$  with finite  $s$ -distance in  $M$  and choose two flags  $F$  and  $G$  in  $A$  containing  $b$  and  $c$ , respectively. Lemma 6.5 (with  $t = [0, N]$ ) and Lemma 6.9 imply that we can connect  $F$  and  $G$  by a reduced path  $P$  with word  $u$  whose letters do not contain  $s$ . By assumption, there is a reduced flag path  $P'$  in  $A$  connecting  $F$  and  $G$  as well. Thus, the word of  $P'$  is a permutation of  $u$  by Proposition 6.17. So, again by Lemma 6.5, the points  $b$  and  $c$  are  $s$ -connected in  $A$  and hence  $A$  is nice.  $\square$

## 7. FORKING IN THE FREE PSEUDOSPACE

In this section we provide a detailed description of nonforking over nice sets and canonical bases. In particular, we obtain weak elimination of imaginaries. The theory  $\text{PS}_N$  has trivial forking and is totally trivial, as in [2].

We will work inside a sufficiently saturated model  $M$ . We start with an easy observation which follows immediately from Theorem 4.22.

**Proposition 7.1.** *The theory  $\text{PS}_N$  is  $\omega$ -stable.*

*Proof.* Work over a countable subset  $A$ , which we may assume to be nice. Theorem 4.22 shows that every 1-type over  $A$  lies in some nice set  $B$ , obtained from  $A$  by a finite number of applications  $\alpha_s$ . In particular, there are countably many quantifier-free types of such  $B$ 's over  $A$  and thus countably many types by Corollary 4.29. The theory  $\text{PS}_N$  is therefore  $\omega$ -stable.  $\square$

The following result will allow us to determine the type of a flag over a nice set.

**Proposition 7.2.** *Let  $X$  be a nice set and  $F$  a flag which is connected to a flag  $G$  in  $X$  by a reduced flag path  $P$  with word  $u$ . The following are equivalent:*

- (a) *Let  $v$  be a reduced word connecting  $G$  to another flag  $G'$  in  $X$ . Then  $F$  is connected to  $G'$  by the reduct of  $u \cdot v$ .*

- (b)  $u$  is the  $\preceq$ -smallest word connecting  $F$  to a flag in  $X$ .
- (c)  $u$  is  $\preceq$ -minimal among words connecting  $F$  to a flag in  $X$ .

*Proof.* (a)→(b) follows from Corollary 5.30.

(b)→(c) is trivial.

(c)→(a): Let  $G'$  be any flag in  $X$ . Then  $G$  is connected to  $G'$  by a flag path  $P$  with word  $v$ . By Proposition 6.20, we may assume that  $P$  in  $X$ . Choose a decomposition  $u = u_1 \cdot u' \cdot w$  and  $w \cdot v' \cdot v_1 = v$  as in Corollary 5.23, with corresponding paths

$$F \xrightarrow[u_1 \cdot u']{ } F^* \xrightarrow[w]{ } G \xrightarrow[w]{ } G^* \xrightarrow[v' \cdot v_1]{ } G',$$

where  $G^*$  is a flag in  $X$ .

Let  $b$  be a strong reduct of  $w \cdot w$  connecting  $F^*$  to  $G^*$ . If  $b \not\approx w$ , consider the reduced word  $c$  which connects  $F$  with  $G^*$ . Since  $c$  is a strong reduct of  $u_1 \cdot u' \cdot b$ , we have  $c \preceq u_1 \cdot u' \cdot b \prec u$ , a contradiction. So  $b$  is equivalent to  $w$ . We obtain a path from  $F$  to  $G'$  with word  $u_1 \cdot u' \cdot w \cdot v' \cdot v_1$ . Up to permutation, its only possible strong reduct is  $u_1 \cdot w \cdot v_1$ . So  $F$  connects to  $G'$  by word  $u_1 \cdot w \cdot v_1$ , which is the reduct of  $u \cdot v$ . □

**Definition 7.3.** Given a nice set  $X$ . We call a flag  $G$  in  $X$  a *base-point* of  $F$  over  $X$  if the conditions of Proposition 7.2 hold: The word connecting  $F$  to  $G$  is  $\preceq$ -minimal among words which connect  $F$  with flags in  $X$ .

**Lemma 7.4.** *Let  $X$  be a nice set and  $F_0 \xrightarrow{s_1} \dots \xrightarrow{s_n} F_n$  be a reduced flag path with  $F_n \in X$ . Then  $F_n$  is a basepoint of  $F_0$  over  $X$  if and only if the flag  $F_{i-1}$  is obtained from  $F_i \cup \dots \cup F_n \cup X$  by a global application of  $\alpha_{s_i}$  for all  $i \geq 1$ .*

*In particular, if  $F_n$  is a basepoint of  $F_0$  over  $X$ , then  $F_0 \cup \dots \cup F_n \cup X$  is nice.*

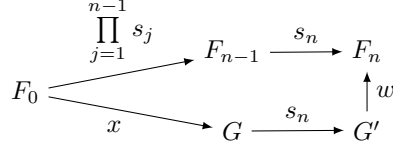
*Proof.* The equivalence for  $n = 1$  is clear, since  $F_0$  is obtained by an global application of  $\alpha_{s_1}$  from  $F_1 \cup X = X$  if and only if there is no connection of  $F_0$  to  $X$  by a product of proper subletters of  $s$  by Lemma 6.5.

Proceed now by induction over  $n$  and assume first that each  $F_{i-1}$  is obtained from  $F_i \cup \dots \cup F_n \cup X$  by a global application of  $\alpha_{s_i}$ . Lemma 4.21 implies that  $Y = F_1 \cup \dots \cup F_n \cup X$  is nice. Furthermore, the flag  $F_1$  is a basepoint of  $F_0$  over  $Y$ . We will show that property 7.2 (a) holds for  $F_0$  and  $F_n$  over  $X$ . Let  $G$  be a flag in  $X$ . Choose reduced words  $x, y$  and  $v$  with

$$F_0 \xrightarrow[x]{ } G, F_1 \xrightarrow[y]{ } G \text{ and } F_n \xrightarrow[v]{ } G.$$

Then  $x$  is the reduct of  $s_1 \cdot y$  and, by induction, the word  $y$  is the reduct of  $s_2 \cdots s_n \cdot v$ . So  $x$  is the reduct of  $s_1 \cdots s_n \cdot v$ . Therefore, the flag  $F_n$  is a basepoint of  $F_0$  over  $X$ .

For the other direction, note first that  $F_{n-1}$  is obtained from  $F_n \cup X = X$  by a global application of  $\alpha_{s_n}$ . So  $Y = F_{n-1} \cup F_n \cup X$  is nice. If we can show that  $F_{n-1}$  is a basepoint of  $F_0$  over  $Y$ , we can conclude by induction. For that, we will verify 7.2(b). Consider any flag  $G$  in  $Y$  and let  $x$  be the reduced word which connects  $F_0$  to  $G$ . If  $G$  belongs to  $X$ , we have  $s_1 \cdots s_{n-1} \prec s_1 \cdots s_n \preceq x$ . Otherwise, there are a flag  $G'$  in  $X$  and a word  $w$  commuting with  $s_n$  such the following diagram holds:



The reduced word  $x'$  connecting  $F_0$  with  $G'$  is a strong reduct of  $x \cdot s_n$ . Minimality of  $u = s_1 \cdots s_n$  yields that  $u \preceq x'$ . Corollary 5.28 gives that  $s_1 \cdots s_{n-1} \preceq x$ .  $\square$

**Corollary 7.5.** *Let  $G$  be a flag in a nice set  $X$ . Given a reduced word  $u$ , there is a flag  $F$  a path  $P$  from  $F$  to  $G$  with word  $u$  such that  $G$  is the basepoint of  $F$  over  $X$ . The set  $X \cup P$  is nice. The type of  $F$  over  $G$  (and thus, over  $X$ ) is uniquely determined.*

Denote these types by

$$p_u(G) \text{ and } p_u(G)|X.$$

In order to describe the regular types and the dimensions of  $\text{PS}_N$ , we will need a characterisation of nonforking over nice sets in terms of the reduction of the corresponding words connecting the paths.

**Lemma 7.6.** *Let  $F$  and  $G$  be flags, where  $G$  lies in a nice set  $X$ . The independence  $F \downarrow_G X$  holds if and only if  $G$  is a basepoint of  $F$  over  $X$ .*

*Proof.* Let  $u$  be the reduced word which connects  $F$  to  $G$ . Then the type  $p_u(G)$  of  $F$  over  $G$  has a canonical extension  $p_u(G)|Y$  to every nice set  $Y$  which contains  $G$ . Since  $\text{PS}_N$  is stable, it follows that  $p_u(G)|X$  is the only non-forking extension of  $p_u(G)$  to  $X$ .  $\square$

**Proposition 7.7.** *Given three flags with reduced paths  $F \xrightarrow{u} G$ ,  $G \xrightarrow{v} H$  and  $F \xrightarrow{w} H$ , we have that  $F \downarrow_G H$  if and only if  $u \cdot v \rightarrow w$ .*

*Proof.* If  $F \downarrow_G H$ , there is a nice set  $X$  containing  $G$  and  $H$  such that  $F \downarrow_G X$ . But then  $G$  is a basepoint of  $F$  over  $X$  and  $u \cdot v \rightarrow w$  follows.

Assume now  $u \cdot v \rightarrow w$ . Take  $P$  the reduced path from  $G$  to  $H$  with word  $v$ . The set  $P$  is nice. Enough to show  $F \downarrow_G P$  by verifying 7.2(a). Given any flag  $G'$  in  $P$ , by Lemma 6.18, we may assume that  $G'$  occurs in  $P$ . Thus, write  $v_1 \cdot v_2 = v$  with  $G \xrightarrow{v_1} G' \xrightarrow{v_2} H$ . If  $x$  is reduced with  $F \xrightarrow{x} G'$ , then

$$u \cdot v = (u \cdot v_1) \cdot v_2 \xrightarrow{*} x \cdot v_2 \xrightarrow{*} w.$$

By assumption  $u \cdot v \rightarrow w$ , so Proposition 5.32 yields that no splitting occurs in the strong reductions above. This implies that  $u \cdot v_1 \rightarrow x$ , which completes the proof.  $\square$

Note that the previous proof also yields  $x \cdot v_2 \rightarrow w$ , which will be used in the proof of Lemma 7.19. Furthermore, we have the following:

**Corollary 7.8.** *Given flags  $F$ ,  $G$  and  $H$  with  $F \downarrow_G H$ , then*

$$F \downarrow_G P,$$

where  $P$  is the reduced flag path connecting  $G$  to  $H$ .

We will now compute the Morley rank  $\text{MR}(p)$  and Lascar rank  $\text{U}(p)$  of certain types in  $\text{PS}_N$ .

**Definition 7.9.** Given reduced words  $u$  and  $v$ , we say that  $u$  is a proper left-divisor of  $v$  if  $u \not\approx v$  and there is a reduced  $w$  such that  $uw = v$  in  $\text{Cox}(N)$ .

Note that  $uw = v$  in  $\text{Cox}(N)$  is equivalent to  $u \cdot w \rightarrow v$ .

If  $u$  is a proper left-divisor of  $v$ , it follows by Corollary 5.30 that  $u \prec v$ . In particular, Lemma 5.26 yields that being a proper left-divisor is well-founded. Let  $\text{R}_{\text{div}}$  be its foundation rank and likewise let  $\text{R}_{\prec}$  denote the foundation rank with respect to  $\prec$ .

**Lemma 7.10.** *For every flag  $G$  and every reduced word  $u$ ,*

$$\text{U}(\mathfrak{p}_u(G)) = \text{R}_{\text{div}}(u).$$

*Proof.* Show  $\text{U}(\mathfrak{p}_u(G)) \leq \text{R}_{\text{div}}(u)$  by induction on  $\text{R}_{\text{div}}(u)$ . Assume that  $\alpha < \text{U}(\mathfrak{p}_u(G))$ . Then there is a nice extension  $X$  of  $G$  and a realisation  $F$  of  $\mathfrak{p}_u(G)$  such that  $\alpha \leq \text{U}(F/X)$ . Since  $F \not\downarrow_G X$ , the type of  $F$  over  $X$  is of the form  $\mathfrak{p}_v(H)|X$  for a reduced word  $v$  and some flag  $H$  in  $X$ . Proposition 7.2 (a) and Lemma 7.6 imply that  $v$  is a proper left-divisor of  $u$ . By induction, we have

$$\alpha \leq \text{U}(F/X) = \text{U}(\mathfrak{p}_v(H)) = \text{R}_{\text{div}}(v) < \text{R}_{\text{div}}(u),$$

which proves  $\text{U}(\mathfrak{p}_u(G)) \leq \text{R}_{\text{div}}(u)$ .

For the other direction, assume  $\alpha < \text{R}_{\text{div}}(u)$ . Then there is a proper left-divisor  $v$  of  $u$  such that  $\alpha \leq \text{R}_{\text{div}}(v)$ . Choose a reduced word  $w$  such that  $v \cdot w \rightarrow u$ . It is easy to construct a flag  $H$  with

$$F \underset{v}{\rightarrow} H \underset{w}{\rightarrow} G.$$

Actually, such an  $H$  exists whenever  $v \cdot w \xrightarrow{*} u$ . By Proposition 7.7 we have  $F \downarrow_H G$ . Let  $P$  be a path from  $H$  to  $G$  with associated word  $w$ . Seen as a collection of points, the path  $P$  is nice by Lemma 6.15. Corollary 7.8 gives that  $F \downarrow_H P$ , so  $\text{tp}(F/P) = \mathfrak{p}_v(H)|P$  and thus  $F \not\downarrow_G P$ . By induction,

$$\alpha \leq \text{R}_{\text{div}}(v) = \text{U}(\mathfrak{p}_v(H)) < \text{U}(\mathfrak{p}_u(G)).$$

□

**Lemma 7.11.** *For every flag  $G$  and reduced word  $u$ , we have that*

$$\text{MR}(\mathfrak{p}_u(G)) \leq \text{R}_{\prec}(u).$$

*Proof.* Extend  $\mathfrak{p}_u(G)$  to  $p = \mathfrak{p}_u(G)|X$ , where  $X$  is an  $\omega$ -saturated model containing  $G$ . The type  $p$  contains a formula  $\varphi(x)$  stating that there is a weak path connecting the flag  $x$  to  $G$  with word  $u$ . If  $F$  realizes  $\varphi$ , then either  $F$  realizes  $p$  or there is a path connecting  $F$  to  $X$  with word  $\prec$ -smaller than  $u$ . For the latter, induction gives that the Morley rank of  $F$  over  $X$  is strictly smaller than  $\text{R}_{\prec}(u)$ . Since  $X$  is  $\omega$ -saturated, this implies that  $\text{MR}(p) \leq \text{R}_{\prec}(u)$ . □

**Lemma 7.12.** *If  $u = s_1 \cdots s_n$  is reduced and  $|s_i| \geq |s_{i+1}|$  for  $i = 1, \dots, n-1$ , then*

$$\text{R}_{\text{div}}(u) = \text{R}_{\prec}(u) = \omega^{|s_1|-1} + \dots + \omega^{|s_n|-1}.$$

*Proof.* Let  $\text{ord}$  be the function introduced in the proof of Lemma 5.26. Recall that for any reduced word  $w$

$$R_{\text{div}}(w) \leq R_{\prec}(w) \leq \text{ord}(w).$$

If  $u$  satisfies the above hypotheses, then  $\text{ord}(u) = \omega^{|s_1|-1} + \dots + \omega^{|s_n|-1}$ . Hence, we need only that  $\text{ord}(u) \leq R_{\text{div}}(u)$ . By induction, it is enough to find, for every  $\alpha < \text{ord}(u)$ , a proper left-divisor  $u'$  of  $u$  satisfying the hypotheses of the Lemma such that  $\alpha \leq \text{ord}(u')$ .

There are two cases: If  $|s_n| = 1$ , set  $u' = s_1 \cdots s_{n-1}$ . If  $|s_n| > 1$ , let  $k$  be large enough such that

$$\alpha \leq \omega^{|s_1|-1} + \dots + \omega^{|s_{n-1}|-1} + \omega^{|s_n|-2} \cdot k$$

Then choose an appropriate sequence  $t_1 \cdots t_k$  of subletters of  $s_n$ , each of size  $|s_n| - 1$ , such that  $u' = s_1 \cdots s_{n-1} \cdot t_1 \cdots t_k$  is reduced.  $\square$

**Corollary 7.13.** *For every flag  $G$  and every reduced word  $u = s_1 \cdots s_n$  with  $|s_i| \geq |s_{i+1}|$  for  $i = 1, \dots, n-1$ ,*

$$U(\mathfrak{p}_u(G)) = \text{MR}(\mathfrak{p}_u(G)) = \omega^{|s_1|-1} + \dots + \omega^{|s_n|-1}.$$

However, Lascar and Morley rank may differ in general, as the following example shows.

**Remark 7.14.** Consider the word  $u = [0, 1][1, 3]$ . It is easy to see that  $R_{\text{div}}(u) = \omega^2$  and  $R_{\prec}(u) = \omega^2 + \omega$ , since the inversion antiautomorphism  $u \rightarrow u^{-1}$  preserves  $\prec$ . In particular, the Lascar rank of  $\mathfrak{p}_u(G)$  is  $\omega^2$ . To compute the Morley rank of  $\mathfrak{p}_u(G)$ , consider the following sequence of words

$$u_k = \underbrace{[1][0] \cdots [1][0]}_k [1, 3].$$

The Morley rank of  $u_k$  is at least  $R_{\text{div}}(u_k) = \omega^2$ . Since  $\mathfrak{p}_u(G)$  is the limit of the types  $\mathfrak{p}_{u_k}(G)$ , its Morley rank of  $\mathfrak{p}_u(G)$  is at least  $\omega^2 + 1$ . Actually, it is easy to show that  $\text{MR}(\mathfrak{p}_u(G)) = \omega^2 + 1$ .

The non-orthogonality classes of regular types over a nice set in  $\text{PS}_N$  are given by global operations of  $\alpha_s$  for  $s$  varying among all intervals. These types have trivial forking and therefore so does  $\text{PS}_N$ .

**Theorem 7.15.** *The theory  $\text{PS}_N$  is  $\omega$ -stable of rank  $\omega^N$ . Every type over a nice set  $X$  is non-orthogonal to some type  $\mathfrak{p}_s(G)|X$ , where  $G$  lies in  $X$ . Forking is trivial, that is, any three pairwise independent tuples are independent (as a set).*

*Proof.* By Lemma 7.11, the Morley rank of a flag cannot exceed  $R_{\prec}([0, N]) = \omega^N = U(\mathfrak{p}_{[0, N]}(G)) = \text{MR}(\mathfrak{p}_{[0, N]}(G))$ , by Corollary 7.13. Thus, the Lascar and Morley rank of a flag over the emptyset are both  $\omega^N$ . Let  $a$  be a vertex of  $F$ . Lascar inequalities imply that  $U(F/a) + U(a) \leq U(F)$ . Since  $U(a) > 0$ , this implies that  $U(a) = \omega^N$ , and therefore  $\text{MR}(a) = \omega^N$ .

Given a type  $p$  over  $X$ , we may assume it is the type of a flag  $F$  and thus determined by some reduced word  $u$  connecting  $F$  a basepoint  $G$  over  $X$ . In particular, take any  $s$  in the final segment of  $u$ . The type  $p$  is hence non-orthogonal to the type  $\mathfrak{p}_s(G)|X$ , since the connecting word of  $F$  over the nice set consisting of  $G$  together with a realisation of  $\mathfrak{p}_s(G)|X$  is  $\prec$ -smaller than  $u$ .

Since the type  $p_s(G)$  has monomial Lascar rank, it is regular. A different way to see this is by taking a non-forking realisation  $F$  of  $p_s(G)|X$  and a forking realisation  $F'$  to  $X$ . Now, since  $F'$  forks with  $X$  over  $G$ , Proposition 7.2(b) gives a flag  $G'$  in  $X$  such that the word connecting  $F'$  to  $G'$  is a finite product  $x$  of proper subletters of  $s$ . Since the reduction  $s \cdot x \xrightarrow{*} s$  involves no splitting, the flags  $F$  and  $F'$  are independent over  $G$  by Proposition 7.7. The type  $p_s(G)$  is regular, and so is  $p_s(G)|X$ .

Note that the geometry on every type  $p_s(G)$  is trivial: given three pairwise independent realisations  $F_1, F_2$  and  $F_3$  of  $p_s(G)$ , note that any flag in  $G \cup F_2 \cup F_3$  must be either  $G, F_2$  or  $F_3$ , for there are no new  $s$ -connections between them. Hence,

$$F_1 \downarrow_G F_2 \cup F_3$$

and forking is trivial on each  $p_s(G)|X$ . Since the theory is superstable, forking is trivial [6, Proposition 2].  $\square$

Nice sets are algebraically closed in  $\text{PS}_N^{\text{eq}}$ .

**Remark 7.16.** Let  $X$  be nice and  $F$  be a flag with  $F/A \in \text{acl}^{\text{eq}}(X)$  for some set  $A \subset [0, N]$ . Then, the class  $F/A$  lies in  $X^{\text{eq}}$ . That is, all vertices of  $F$  with level outside  $A$  belong to  $X$ .

Since  $X$  is nice, this is equivalent to  $F/A = G/A$  for some  $G$  in  $X$ .

*Proof.* Let  $u$  be the reduced word connecting  $F$  to a basepoint  $G$  over  $X$ . By taking a sufficiently large initial segment of a sequence of  $X$ -independent realisations of  $\text{tp}(F/X)$ , since the class  $F/A$  is algebraic, we may find another realisation  $F'$  with  $F \downarrow_G F'$  and  $F/A = F'/A$ . By Lemmata 6.3 and 6.9, there is a path connecting  $F$  and  $F'$  whose reduced word  $v$  satisfies  $|v| \subset A$ . Proposition 7.7 and the independence  $F \downarrow_G F'$  imply that  $v$  is the reduct of  $u \cdot u^{-1}$ . Thus  $|u| = |u \cdot u^{-1}| = |v| \subset A$ . In particular, the flags  $F$  and  $G$  are equivalent modulo  $A$ .  $\square$

Let us now explicitly describe canonical bases of types over nice sets. They are interdefinable with finite sets of real elements and hence  $\text{PS}_N$  has weak elimination of imaginaries (*cf.* Corollary 7.24).

**Theorem 7.17.** *Let  $u$  be a reduced word and  $G$  a flag. Then the canonical base of  $p_u(G)$  is interdefinable with  $G/\mathcal{S}_R(u)$ .*

Observe that  $G/\mathcal{S}_R(u)$  is interdefinable with a finite set by Definition 6.2.

*Proof.* We have to show that  $p_u(G)$  and  $p_u(G')$  have a common nonforking extension if and only if  $G$  and  $G'$  are equivalent modulo  $\mathcal{S}_R(u)$ . Or, in other words, given a nice set  $X$ , if  $F$  is a realisation of  $p_u(G)|X$ , then  $G' \in X$  is a basepoint of  $F$  over  $X$  if and only if  $G/\mathcal{S}_R(u) = G'/\mathcal{S}_R(u)$ .

If  $v$  is a reduced word connecting  $G$  and  $G'$ , then  $G/\mathcal{S}_R(u) = G'/\mathcal{S}_R(u)$  means that  $|v| \subset \mathcal{S}_R(u)$ , or equivalently by Lemma 5.11, that  $v$  is right-absorbed by  $u$ . Let  $w$  be the reduced word connecting  $F$  to  $G'$ . Then  $w$  is the reduct of  $u \cdot v$  by Proposition 7.2(a). The flag  $G'$  is a basepoint of  $F$  if and only if  $w \approx u$ . By Corollary 5.14, this is equivalent to  $v$  being right-absorbed by  $u$ .  $\square$

The following result will be useful in order to prove that the theory  $\text{PS}_N$  is not  $(N + 1)$ -ample.

**Lemma 7.18** (Basepoint Lemma). *Let  $X$  be a nice set and  $F$  connected by a reduced word  $u$  to its basepoint  $G$  in  $X$ . Assume  $u = w \cdot v$  and pick a flag  $H$  with*

$$F \xrightarrow{w} H \xrightarrow{v} G.$$

*If  $H/A \in X$  for some set  $A \subset [0, N]$ , then  $|v|$  is a subset of  $A$ .*

*Proof.* By Remark 7.16 and Corollary 6.12, there is a flag  $G'$  in  $X$  connected to  $H$  by a reduced word  $|v'| \subset A$ . The flag  $G$  is a basepoint of  $H$  over  $X$  by Lemma 7.4. Proposition 7.2 (b) gives that  $v \preceq v'$  and therefore  $|v| \subset |v'| \subset A$ .  $\square$

We finish the section with a strengthening of triviality, called *totally trivial* [6], that is, given any set of parameters  $X$  and tuples  $a$ ,  $b$  and  $c$  such that  $a$  is both independent from  $b$  and  $c$  over  $X$ , then it is independent from  $\{b, c\}$  over  $X$ . For theories of finite U-rank, both notions agree [6, Proposition 5].

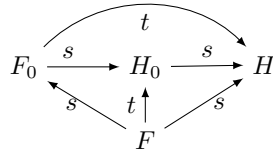
By Lemma 7.6, recall that, given a nice set  $X$  and a distinguished flag  $F_0$  in  $X$ , the following are equivalent for any flag  $F$ ,

- $F \downarrow_{F_0} X$
- $F \downarrow_{F_0} H$  for every flag  $H$  in  $X$
- $F_0$  is a basepoint of  $F$  over  $X$ .

Whilst considering flag paths, there is a simpler version of transitivity of non-forking, due to the nature of the reduction with non splitting.

**Lemma 7.19.** *Given flags  $H$ ,  $F$ ,  $H_0$  and  $F_0$ , then  $F \downarrow_{F_0} H_0$  and  $F \downarrow_{H_0} H$  imply  $F \downarrow_{F_0} H$ . If there is a reduced path  $F_0 \xrightarrow{v} H_0 \xrightarrow{w} H$ , the converse also holds:  $F \downarrow_{F_0} H$  implies  $F \downarrow_{F_0} H_0$  and  $F \downarrow_{H_0} H$ .*

Observe that the condition on the path being reduced is needed for the converse, as the following example shows, where  $t \subsetneq s$ :



Although  $F \downarrow_{F_0} H$ , since no splitting occurs when reducing  $s \cdot t$  to  $s$ , we have that  $F \not\downarrow_{F_0} H_0$ , as  $t$  is not the reduct of  $s \cdot s$ .

*Proof.* We will use throughout the proof the characterisation of independence between flags given by Proposition 7.7. It actually follows from the proof of Proposition 7.7 that the above converse holds, by taking  $F$ ,  $G$ ,  $G'$ ,  $H$  instead of  $H$ ,  $F$ ,  $H_0$ ,  $F_0$  in the proof. Alternatively, we may argue as follows: as  $H_0$  occurs in a reduced path  $P$  from  $F_0$  to  $H$ , the proof of Proposition 7.7 shows that  $F \downarrow_{F_0} P$ . This implies  $F \downarrow_{F_0} H_0$ . Since  $F_0 \xrightarrow{v} H_0 \xrightarrow{w} H$ , we have that  $F_0 \downarrow_{H_0} H$  by Proposition 7.7. This, together with  $F \downarrow_{F_0} H$ , the first part of the lemma and forking symmetry implies  $F \downarrow_{H_0} H$ .

Assume now  $F \downarrow_{F_0} H_0$  and  $F \downarrow_{H_0} H$ . Choose reduced paths  $F \xrightarrow{u} F_0$ ,  $F_0 \xrightarrow{v} H_0$ ,  $H_0 \xrightarrow{w} H$  and  $F_0 \xrightarrow{x} H$ . The word  $a$  which connects  $F$  to  $H_0$  is the reduct of



$u \cdot v$ . Also, the word  $b$  connecting  $F$  to  $H$  is the reduct of  $u' \cdot w$ . Hence, the word  $b$  is the reduct of  $u \cdot v \cdot w$ . If  $x$  were the reduct of  $v \cdot w$ , then  $b$  is the reduct of  $u \cdot x$ , so we are done. Therefore, suppose that splitting occurs in  $v \cdot w \xrightarrow{*} x$ . Treat first the case  $v = w = s$ . Then  $x$  is a product of proper subintervals of  $s$ . By the Decomposition Lemma 5.13, either  $s$  is right absorbed by  $u$ , or  $u = u_1 \cdot u'$ , where  $u'$  is properly absorbed by  $s$  and  $u_1 \cdot s$  is reduced. In the first case, the word  $x$  is properly absorbed by  $u$ , hence  $F \downarrow_{F_0} H$ .

For the second case, decompose  $u = u_1 \cdot u'$  as above. Then  $b$  (the word connecting  $F$  and  $H$ ) equals  $u_1 \cdot s$ . This cannot be a strong reduct of  $u_1 \cdot u' \cdot x$ , since the latter is  $\prec$ -smaller, contradicting Proposition 5.32.

For the general case, as in the proof of Proposition 5.32, we may assume that the splitting in  $v \cdot w \xrightarrow{*} x$  happens at the first step of the reduction. Write hence  $v = v' \cdot s$  and  $w = s \cdot w'$ , where

$$F_0 \xrightarrow{v'} K_1 \xrightarrow{s} H_0 \xrightarrow{s} K_2 \xrightarrow{w'} H.$$

The word  $y$  connecting  $K_1$  and  $K_2$  consists of proper subletters of  $s$ . By the first part of the proof, since  $F \downarrow_{F_0} H_0$ , we have that  $F \downarrow_{F_0} K_1$  and  $F \downarrow_{K_1} H_0$ . Similarly, we obtain  $F \downarrow_{H_0} K_2$  and  $F \downarrow_{K_2} H$ . By the previous discussion, we have that  $F \downarrow_{K_1} K_2$ . This, together with  $F \downarrow_{F_0} K_1$ , yields  $F \downarrow_{F_0} K_2$ , by induction on the length of  $v$ . Now, the word connecting  $F_0 \rightarrow K_2$  is a strong reduction of  $v' \cdot y$ , so  $\prec$ -smaller than  $v$ . Induction on the complexity of  $v$  together with  $F \downarrow_{K_2} H$  gives  $F \downarrow_{F_0} H$ , as desired.  $\square$

In order to prove the total triviality of  $\text{PS}_N$ , we will use the following lemma, a stronger form of which follows already from total triviality, without the assumption  $F_0 \downarrow_A B$ , since if

$$A \xrightarrow{s} B \xrightarrow{t} C,$$

where  $s$  and  $t$  commute with each other, then  $B$  is definable in  $A \cup C$ , by Lemma 6.19.

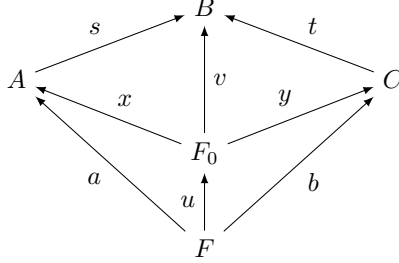
**Lemma 7.20.** *Let  $A, B, C, F, F_0$  be flags and  $s$  and  $t$  two commuting letters, such that  $A \xrightarrow{s} B \xrightarrow{t} C$ . If the following independencies hold:*

$$F \downarrow_{F_0} A, F \downarrow_{F_0} C \text{ and } F_0 \downarrow_A B,$$

then  $F \downarrow_{F_0} B$ .

*Proof.* In order to show that  $F \downarrow_{F_0} B$ , since  $F \downarrow_{F_0} A$ , by Lemma 7.19, we need only show  $F \downarrow_A B$ . Thus, consider a reduced word  $z$  with  $F \xrightarrow{z} B$  and connect the above flags by reduced paths as in the diagram below.

Assume for a contradiction that  $F \not\downarrow_A B$ . Then  $z$ , which is a strong reduct of  $a \cdot s$ , is not the reduct of  $a \cdot s$ . This has two consequences: first, the letter  $s$  does not occur in the final segment of  $z$ . Secondly, up to permutation, the path  $F \xrightarrow{a} A$  ends with a flag  $A' \xrightarrow{s} A$ , such that  $A'$  is connected to  $B$  by a word consisting of proper subletters of  $s$ . Since  $F_0 \downarrow_A B$ , such a flag  $A'$  cannot occur in any permutation of  $x$ . Thus, as  $a$  is a reduct of  $u \cdot x$ , it follows that  $s$  commutes with  $x$  and is in the final segment of  $u$ . In particular, the word  $x \cdot s$  is reduced, which implies that  $v$  is (up to permutation) the word  $x \cdot s$ .



On the other hand, the word  $v = x \cdot s$  is a strong reduct of  $y \cdot t$ . It is easy to see that this can only be possible if (after permutation)  $y$  has the form  $y' \cdot s$ , where  $y'$  and  $s$  commute. The independence  $F \downarrow_{F_0} C$  implies that  $b$  is the reduct of  $u \cdot y$ . Hence  $s$  still belongs to the final segment of  $b$ . Finally, since  $z$  is a strong reduct of  $b \cdot t$ , the word  $s$  must belong to the final segment of  $z$ , which contradicts that  $F \not\downarrow_A B$ .  $\square$

In order to ensure the independence of a flag with respect to a whole flag path over a nice set, it is enough to check the independence with respect to the set itself and the end flag of the path.

**Lemma 7.21.** *Let  $A$  be a nice set and a reduced path  $P$  connecting a flag  $H$  to a basepoint in  $A$ . Given a flag  $F_0$  in  $A$  and a flag  $F$ , we have that  $F \downarrow_{F_0} A \cup P$  if and only if  $F \downarrow_{F_0} A$  and  $F \downarrow_{F_0} H$ .*

*Proof.* Left-to-right is clear. Assume now that  $F \downarrow_{F_0} A$  and  $F \downarrow_{F_0} H$ . Since  $A \cup P$  is nice by Lemma 7.4, in order to check that  $F \downarrow_{F_0} A \cup P$ , we need to check that  $F \downarrow_{F_0} H'$  for any flag  $H'$  in  $A \cup P$  by the remark above Lemma 7.19. This is clear for flags in  $A$ , so let  $H'$  be in  $A \cup P$  but not in  $A$ .

We treat first the case where  $H'$  is in  $P$ . Let  $H_0$  be the base-point of  $H$  in  $A$ . We have then that  $F_0 \downarrow_{H_0} H$  and  $F \downarrow_{F_0} H$  by assumption, which implies  $F \downarrow_{H_0} H$  by Lemma 7.19. Since the path  $P$  is reduced, Lemma 7.19 gives  $F \downarrow_{H_0} H'$ , which together with  $F \downarrow_{F_0} H_0$  implies  $F \downarrow_{F_0} H'$ .

For the general case, we will proceed by induction on the length of  $P$ , based on the above paragraph. Thus, it suffices to consider the case where  $P$  has length 1 and let  $s$  be its letter:

$$H_0 \xrightarrow{s} H.$$

If  $H'$  is a flag in  $A \cup P$  not completely contained in  $A$ , it differs from  $H$  only on the indices outside  $s$ . As in the proof of Lemma 6.18, we can find a reduced word  $w$  commuting with  $s$  such that  $H' \xrightarrow{w} H$ . Furthermore, there is some flag  $H'_0$  in  $A$  with  $H'_0 \xrightarrow{w} H_0$  and  $H'_0 \xrightarrow{s} H'$ .

Note that  $H'_0$  is again a basepoint of  $H'$  over  $A$ , so in particular  $F_0 \downarrow_{H'_0} H'$ . By induction on the length of  $w$ , we may assume that  $w$  is a letter  $t$ . Setting  $A = H'_0$ ,  $B = H'$  and  $C = H$ , the hypotheses of Lemma 7.20 are satisfied. We conclude that  $F \downarrow_{F_0} H'$ , which gives the desired result.  $\square$

We now have all the ingredients to prove total triviality of forking.

**Proposition 7.22.** *The theory  $\text{PS}_N$  is totally trivial, that is, given any set of parameters  $X$  and tuples  $a$ ,  $b$  and  $c$  such that  $a$  is both independent from  $b$  and  $c$  over  $X$ , then it is independent from  $\{b, c\}$  over  $X$ . In particular, the canonical base of a tuple is the union of the canonical bases of each singleton.*

*Proof.* We may assume that our parameter set  $X$  is nice, by choosing a small model containing it independent from  $a, b, c$ .

Suppose first that the tuples  $a$ ,  $b$  and  $c$  consists of singletons: By transitivity, choose flags  $H_1$  and  $H_2$  independently from  $a$  over  $X$  containing  $b$  and  $c$  respectively. Choose now a flag  $F$  containing  $a$  independently from  $H_1$  and from  $H_2$  over  $X$ . We need only to show that

$$F \downarrow_X H_1 \cup H_2.$$

Let  $F_0$  and  $H_0$  be basepoints of  $F$  and  $H_1$  respectively over  $X$ . Since  $F \downarrow_{F_0} X$  and  $F \downarrow_X H_1$ , we have that  $F \downarrow_{F_0} X \cup P_1$  by Lemma 7.21, where  $P_1$  denotes the reduced flag path (connecting  $H_1$  to  $H_0$ ) determined by  $H_1$  over  $X$ . The set  $X \cup P_1$  is again nice by Lemma 7.4. Work now over  $X \cup P_1$  in order to show that  $F \downarrow_{F_0} X \cup P_1 \cup P_2$ , where  $P_2$  is the flag path given by  $H_2$  over  $X \cup P_1$ . Lemma 7.21 gives that  $F$  is independent from  $H_1 \cup H_2$  over  $X$ .

Transitivity of forking allows us to work with finite tuples by choosing accordingly nonforking extensions for each coordinate. The result now follows by local character.  $\square$

Since  $\text{PS}_N$  is superstable, [6, Proposition 7] allows to conclude the following.

**Corollary 7.23.** *The theory  $\text{PS}_N$  is perfectly trivial, that is, given any set of parameters  $X$  and tuples  $a$ ,  $b$  and  $c$  such that  $a$  and  $b$  are both independent over  $X$ , then so are they over  $X \cup \{c\}$ .*

**Corollary 7.24.** *The theory  $\text{PS}_N$  has weak elimination of imaginaries.*

*Proof.* By Proposition 7.22, in order to study the canonical base of a real tuple  $\bar{a}$  over an algebraically closed set  $B$  (in  $\text{PS}_N^{\text{eq}}$ ), we may assume that  $\bar{a}$  is an enumeration of a flag  $F$ . Furthermore, we may suppose that  $B$  is nice. By Theorem 7.17, the canonical base is interdefinable with a finite set, thus we get weak elimination of imaginaries.  $\square$

Although the theory  $\text{PS}_N$  is not 1-based, being  $N$ -ample by Proposition 8.1, it is 2-based, i.e. the canonical base of a type is determined by two independent realisations.

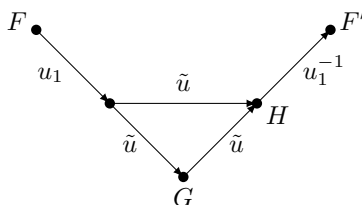
**Proposition 7.25.** *Let  $u$  be a reduced word and  $X$  a nice set. The canonical base of  $\text{p}_u(G)|X$  is algebraic over two independent realisations.*

*Proof.* Let  $F$  and  $F'$  be realisations of  $\text{p}_u(G)|X$ , which are  $X$ -independent. Since the base-point is only determined up to  $\mathcal{S}_R(u)$ -equivalence, pick a common base-point  $G$  in  $X$  for both  $F$  and  $F'$ .

As  $F \downarrow_X F'$  and  $F \downarrow_G X$ , combining Lemmas 7.19 and 7.21, we conclude that  $F \downarrow_G F'$ . Therefore, the word connecting  $F$  and  $F'$  is the reduction of  $u \cdot u^{-1}$ . Write  $u = u_1 \tilde{u}$ , where  $\tilde{u}$  is the final segment of  $u$ . Hence,

$$u \cdot u^{-1} \rightarrow u_1 \cdot \tilde{u} \cdot u_1^{-1},$$

as the diagram shows:



Note that  $G$  and  $H$  are equivalent modulo  $|w| \subset \mathcal{S}_R(u)$ . By Lemma 6.19, the flag  $H$  is determined by  $F$  and  $F'$  modulo  $\mathcal{S}_R(u) \cap \mathcal{S}_L(u_1^{-1})$  and thus, modulo  $\mathcal{S}_R(u)$ . In particular, the canonical base  $G/\mathcal{S}_R(u)$  is algebraic over  $F, F'$ .  $\square$

### 8. AMPLE YET NOT WIDE AMPLE

This last section shows that the ample hierarchy defined in 2.2 is proper, since the theory of the free  $N$ -dimensional pseudospace  $\text{PS}_N$  is  $N$ -ample but not  $(N+1)$ -ample. We will furthermore show that it is  $N$ -tight with respect to the family  $\Sigma$  of Lascar rank 1 types, if  $N \geq 2$ .

The proof that  $\text{PS}_N$  is  $N$ -ample is a direct translation of the proof exhibited in [2], which we nonetheless include for the sake of the presentation.

**Proposition 8.1.** *Consider a flag  $a_0 - \dots - a_N$ . We have the following:*

- (a)  $\text{acl}^{\text{eq}}(a_0, \dots, a_i) \cap \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1}, a_{i+1}) = \text{acl}^{\text{eq}}(a_0, \dots, a_{i-1})$  for every  $0 \leq i < N$ .
- (b)  $a_{i+1} \not\downarrow_{a_i} a_0, \dots, a_{i-1}$  for every  $1 \leq i < N$ .
- (c)  $a_N \not\downarrow a_0$ .

*In particular, the theory  $\text{PS}_N$  is  $N$ -ample.*

*Proof.* In order to prove (a), fix some  $i < N$  and choose parameters  $b_i, \dots, b_N$  independently from  $a_i, a_{i+1}$  such that

$$a_0 - \dots - a_{i-1} - b_i - \dots - b_N$$

is a flag. Set  $X = \{a_0, \dots, a_{i-1}, b_i, \dots, b_N\}$ , which is nice.

By Fact 2.1, assume for a contradiction that there is an element  $e$  in

$$\text{acl}^{\text{eq}}(X, a_i) \cap \text{acl}^{\text{eq}}(X, a_{i+1}) \setminus \text{acl}^{\text{eq}}(X).$$

Choose now  $a'_i$  realising  $\text{tp}(a_i/X, e)$ . Since the element  $e$  lies also in  $\text{acl}^{\text{eq}}(X, a'_i)$ , then  $a_i \not\downarrow_X a'_i$ . As the  $\leq$ -minimal word connecting  $a_i$  (or rather, the flag  $a_0 - \dots - a_N$ ) to  $X$  is  $[i, N]$ , it follows from Lemma 7.6 that  $a_i$  and  $a'_i$  (or rather, generic flags containing them) are connected through a finite product of proper intervals of  $[i, N]$ . Compactness (and Lemma 6.5) implies that there exists a natural number  $n$  such that

$$\text{tp}(a_i/X, e) \models d_{[i, N]}(x, a_i) \leq n.$$

Let  $m$  be such that  $2m > n$ . Consider the reduced word

$$u = \underbrace{[i+1, N] \cdot i \cdots [i+1, N] \cdot i}_{2m}.$$

Corollary 7.5 provides us with a flag  $F$  and a path  $P$  from  $G = a_0 - \cdots - a_N$  to  $F$  with word  $u$

$$F = F_0 \xrightarrow{[i+1, N]} F'_0 \xrightarrow{i} F_1 \xrightarrow{[i+1, N]} \cdots \xrightarrow{[i+1, N]} F'_{m-1} \xrightarrow{i} F_m = G$$

such that  $G$  is the basepoint of  $F$  over the nice set  $G$ . Since the  $F_i$  and  $F'_i$  are connected by the word  $[i, N]$  to  $G$ , they have all the same type over  $X$ . Denote

$$\begin{aligned} F_r &= a_0 - \cdots - a_{i-1} - a_i^r - a_{i+1}^r - \cdots - a_N^r \\ F'_r &= a_0 - \cdots - a_{i-1} - a_i^r - a_{i+1}^{r+1} - \cdots - a_N^{r+1}. \end{aligned}$$

Since  $F_0$  and  $F'_0$  have the same type over  $X$ , they have also the same type over  $Xa_i^0$  and therefore over  $Xe$ . This implies that  $e$  belongs to  $\text{acl}^{\text{eq}}(Xa_{i+1}^1)$ . Similarly, the flags  $F'_0$  and  $F_1$  have the same type over  $Xa_{i+1}^1$  and therefore over  $Xe$ , which implies that  $e$  belongs to  $\text{acl}^{\text{eq}}(Xa_i^1)$ . Iterating, we see that  $a_i^m$  has the same type over  $Xe$  as  $a_i$ . This implies that  $d_{[i, N]}(a_i^m, a_i) \leq n$ , which gives a contradiction since the shortest path between  $a_i$  and  $a_i^m$  in  $\mathcal{A}_{[0, N]}$  is

$$a_i^0 - a_{i+1}^1 - a_i^1 - \cdots - a_{i+1}^m - a_i^m,$$

of length  $2m$ .

For (b), chose generic flags  $F$  containing  $a_{i+1}$  and  $G$  containing  $a_0, \dots, a_i$ . The canonical base  $\text{Cb}(a_{i+1}/a_0, \dots, a_i)$  equals  $\text{Cb}(F/G)$ . On the other hand, the flags  $F$  and  $G$  are connected by the reduced word  $u = [0, i][i+1, N]$ . So

$$\text{Cb}(F/G) = G/\mathcal{S}_R(u) = G/([0, i-1] \cup [i+1, N]) = a_i$$

by Theorem 7.17, which gives the desired independence.

For (c), choose a generic flag  $F$  which contains  $a_N$  and a generic flag  $G$  which contains  $a_0$ . Then  $\text{Cb}(a_N/a_0)$  equals  $\text{Cb}(F/G)$ . On the other hand the reduced word connecting  $F$  to  $G$  is  $u = [0, N-1][1, N]$ , So

$$\text{Cb}(F/G) = G/\mathcal{S}_R(u) = G/[1, N] = a_0,$$

which is clearly not algebraic over  $a_1$ . Thus,

$$a_N \not\downarrow a_0.$$

□

Before the proof that  $\text{PS}_N$  is not  $(N+1)$ -ample, we need some auxiliary results on the nature of the reduced words arising from the hypothesis on ampleness.

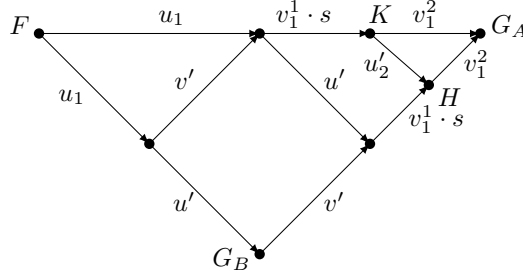
**Lemma 8.2.** *Consider nice sets  $A$  and  $B$  and a flag  $F$  such that  $\text{acl}^{\text{eq}}(AB) \cap \text{acl}^{\text{eq}}(A, F) = \text{acl}^{\text{eq}}(A)$  and  $F \downarrow_B A$ . Let  $u = u_B$  (resp.  $u_A$ ) be the  $\preceq$ -minimal word connecting  $F$  to a flag  $G_B$  in  $B$  (resp.  $G_A$  in  $A$ ) and let  $v$  be the reduced word connecting  $G_B$  to  $G_A$ . If*

$$u = u_1 \cdot u', \quad v' \cdot v_1 = v$$

*is the fine decomposition as in Theorem 5.13, then  $v_1$  is commuting.*

*Proof.* By hypothesis,  $F \downarrow_{G_B} G_A$ , so the product  $u_1 \cdot v_1$  is equivalent to  $u_A$ . Suppose for a contradiction that  $v_1$  is not commuting. Hence, we may decompose  $v_1 = v_1^1 \cdot s \cdot v_1^2$ , where  $v_1^1$  is the final segment of  $v_1$  and  $s$  does not commute with  $v_1^2$ .

By Lemma 5.9, we can write  $u' = u'_2 \cdot u'_1$ , where  $u'_1$  is left-absorbed by  $v_1^1 \cdot s$ , the word  $u'_2$  commutes with  $v_1^1 \cdot s$  and is left-absorbed by  $v_1^2$ . We have the following diagram:



where the path connecting  $K$  and  $H$  is given by  $u'_2$ . So the flags  $H$  and  $K$  are equivalent modulo  $|u'_2|$ .

Lemma 5.18 gives that  $\text{Wob}(v' \cdot v_1^1 \cdot s, v_1^2)$ , the wobbling of  $v$  at  $H$ , is contained in  $W = \text{Wob}(u_1 \cdot v_1^1 \cdot s, v_1^2)$ . In particular, by Lemma 6.19, the class  $H/W$  lies in  $\text{acl}^{\text{eq}}(AB)$ . So does  $K/(|u'_2| \cup W)$ , which also lies  $\text{acl}^{\text{eq}}(AF)$ . By assumption,  $K/(|u'_2| \cup W)$  lies in  $\text{acl}^{\text{eq}}(A)$  since  $\text{acl}^{\text{eq}}(AB) \cap \text{acl}^{\text{eq}}(AF) = \text{acl}^{\text{eq}}(A)$ , and therefore in  $A$  by Remark 7.16. Since  $u_A$  is  $\preceq$ -minimal connecting  $F$  to a flag in  $A$ , Lemma 7.18 implies

$$|v_1^2| \subset |u'_2| \cup W.$$

Observe that  $u'_2$  centralises  $s$  and  $W$  is contained in  $s \cup C(s)$ . Hence, so does  $|v_1^2|$ . Since  $v_1$  is reduced and  $v_1^2$  is commuting, no letter of  $v_1^2$  is contained in  $s$ . So  $v_1^2$  must commute with  $s$ , which contradicts the definition of  $v_1^2$ .  $\square$

**Proposition 8.3.** *Consider nice sets  $A$  and  $B$  and a flag  $F$  such that  $\text{acl}^{\text{eq}}(AB) \cap \text{acl}^{\text{eq}}(A, F) = \text{acl}^{\text{eq}}(A)$  and  $F \downarrow_B A$ . Let  $u = u_B$  (resp.  $u_A$ ) be the minimal word connecting  $F$  to a flag  $G_B$  in  $B$  (resp.  $G_A$  in  $A$ ) (These are the same hypotheses as in Lemma 8.2). Then, either  $F \downarrow_{A \cap B} AB$  or  $u$  is nontrivial and its final segment  $\tilde{u}$ , as a set of indices, is strictly contained in  $\tilde{u}_A$ , the final segment of  $u_A$ .*

*In particular, consider the reduced word  $v$  which connects  $G_B$  to  $G_A$  and the associated fine decomposition*

$$u = u_1 \cdot u', \quad v' \cdot v_1 = v,$$

as in Theorem 5.13. If

$$F \not\downarrow_{A \cap B} A,$$

then  $\tilde{u}$  is nontrivial and

$$|v'| \not\subseteq |\tilde{u}| \subsetneq |\tilde{u}_A|.$$

*Proof.* Since  $F \downarrow_B A$  and  $v$  is reduced connecting  $G_B$  to  $G_A$ , the word  $u \cdot v$  reduces to  $u_A$ . If

$$u = u_1 \cdot u' \qquad v' \cdot v_1 = v.$$

is the fine decomposition (cf. Theorem 5.13) applied to  $u$  and  $v$ , we may thus assume that  $u_A = u_1 \cdot v_1$ .

Let  $H$  be the flag in the path  $G_B \xrightarrow{v} G_A$  between  $v'$  and  $v_1$ . Likewise, let  $K$  be the flag in the path  $F \xrightarrow{u_A} G_A$  between  $u_1$  and  $v_1$ . Note that  $H$  and  $K$  are connected through  $u'$ . Furthermore, Lemma 5.18 gives that  $\text{Wob}(v', v_1)$  is contained in  $W = \text{Wob}(u_1, v_1)$ . Since  $H$  and  $K$  are equivalent modulo  $|u'|$  and  $H/\text{Wob}(v', v_1)$  lies in  $\text{acl}^{\text{eq}}(AB)$  by Lemma 6.19, it follows that  $K/(W \cup |u'|)$  lies in  $\text{acl}^{\text{eq}}(AB) \cap \text{acl}^{\text{eq}}(AF) = \text{acl}^{\text{eq}}(A)$  and whence in  $A$  by Remark 7.16. Lemma 7.18 gives now

$$|v_1| \subset |u'| \cup W.$$

Decompose the final segment of  $u$  as

$$\tilde{u} = w_1 \cdot w_2,$$

where  $w_2$  is the final segment of  $u'$  and  $w_1$  is a subword of the final segment of  $u_1$ . In particular  $u' = u'' \cdot w_2$  and  $w_1$  and  $u''$  commute. We show first that  $w_1$  and  $v_1$  commute: since  $u' \subset C(w_1)$  and  $W \subset \mathcal{S}_R(u_1) \subset |w_1| \cup C(w_1)$ , we have  $v_1 \subset |w_1| \cup C(w_1)$ . A letter  $s$  of  $v_1$  cannot be contained in  $|w_1|$ , since  $u_1 \cdot v_1$  is reduced. So  $s$  belongs to  $C(w_1)$ , which gives the desired result. Recall that  $v_1$  is commuting by Lemma 8.2. Thus, the final segment of  $u_A = u_1 \cdot v_1$  is

$$\tilde{u}_A = w_1 \cdot v_1,$$

which clearly contains  $\tilde{u}$ , as  $|w_2|$  is a subset of  $|v_1|$ .

Suppose the inclusion is not strict. Hence, we have  $|w_2| = |v_1|$ . Then  $|v_1| \subset \mathcal{S}_R(u)$  and hence  $|v| \subset \mathcal{S}_R(u)$ . So  $G_B$  and  $G_A$  are equivalent modulo  $\mathcal{S}_R(u)$ . In particular, the canonical base  $\text{Cb}(F/B)$  lies in  $A$  and thus

$$F \downarrow_{A \cap B} B.$$

Since  $F \downarrow_B A$ , transitivity of non-forking implies that  $F \downarrow_{A \cap B} AB$ .

Finally, assume that  $\tilde{u} = 1$ , which forces  $u = 1$  and thus  $v' = 1$ . In particular, since  $|v_1| \subset |\tilde{u}_A| \subset \mathcal{S}_R(u_A)$  and  $G_A$  and  $G_B$  are equivalent modulo  $v = v_1$ , they are equivalent modulo  $\mathcal{S}_R(u_A)$ , so  $\text{Cb}(F/A) = G_A/\mathcal{S}_R(u_A)$  lies in  $B$  and hence  $F \downarrow_{A \cap B} A$ .

Similarly, if  $|v'| \subset |\tilde{u}| \subset |\tilde{u}_A| \subset \mathcal{S}_R(u_A)$ , we conclude as before that  $\text{Cb}(F/A) = G_A/\mathcal{S}_R(u_A)$  lies in  $B$  and thus  $F \downarrow_{A \cap B} A$ . □

We can now state and prove the desired result.

**Theorem 8.4.** *The theory  $\text{PS}_N$  is not  $(N+1)$ -ample and is  $N$ -tight with respect to the family of Lascar rank 1 types.*

*Proof.* By Remark 2.5, we need only show that given tuples  $b_0, \dots, b_{N+1}$  with:

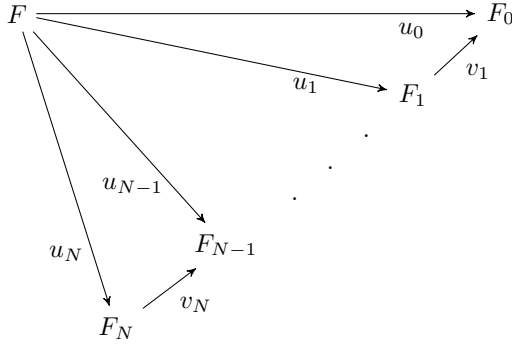
- (a)  $\text{acl}^{\text{eq}}(b_i, b_{i+1}) \cap \text{acl}^{\text{eq}}(b_i, b_{N+1}) = \text{acl}^{\text{eq}}(b_i)$  for every  $0 \leq i < N$ .
- (b)  $b_{N+1} \downarrow_{b_i} b_{i-1}$  for every  $1 \leq i \leq N$ ,

then there is some  $i$  in  $\{0, \dots, N-1\}$  such that

$$b_{N+1} \quad \downarrow \quad b_i \\ \text{acl}^{\text{eq}}(b_i) \cap \text{acl}^{\text{eq}}(b_{i+1})$$

By Fact 2.1, it suffices to prove this for tuples  $b_0, \dots, b_N$  which enumerate small models  $B_0, \dots, B_N$ , although for the proof, we only require that each  $B_i$  is nice. Total triviality (*cf.* Proposition 7.22) allows us to assume that  $b_{N+1}$  consists of a single flag  $F$ .

Choose for every  $i \leq N$  a basepoint  $F_i$  for  $F$  over  $B_i$ . Note that we obtain the following configuration:



such that  $u_i \cdot v_i$  reduces to  $u_{i-1}$ , for every  $i$  in  $\{1, \dots, N\}$ , due to (b). Proposition 8.3 implies that either, for some  $i < N$ ,

$$F \quad \downarrow \quad B_i \\ B_i \cap B_{i+1}$$

or the final segment  $\tilde{u}_{i+1}$  of  $u_{i+1}$  is non-trivial and strictly contained in  $\tilde{u}_i$  for all  $i < N$ .

The second possibility for every  $i < N$  delivers a strictly increasing sequence of length  $N+1$  of non-empty subsets of  $\{0, \dots, N\}$ , which implies that  $\tilde{u}_0$  equals  $[0, N]$  and thus  $u_0 = [0, N]$ . Hence

$$F \downarrow B_0,$$

and thus

$$F \quad \downarrow \quad B_0 \\ \text{acl}^{\text{eq}}(B_0) \cap \text{acl}^{\text{eq}}(B_1)$$

The first possibility implies

$$F \quad \downarrow \quad B_i \\ \text{acl}^{\text{eq}}(B_i) \cap \text{acl}^{\text{eq}}(B_{i+1})$$

as desired. This proves that  $\text{PS}_N$  is not  $(N+1)$ -ample.

Suppose now that  $N \geq 2$ . In order to show that  $\text{PS}_N$  is  $N$ -tight with respect to  $\Sigma$ , where  $\Sigma$  denotes the collection of all Lascar rank 1 types, assume we are given tuples  $b_0, \dots, b_N$  witnessing the following conditions:

- (a)  $\text{acl}^{\text{eq}}(b_0, \dots, b_i) \cap \text{acl}^{\text{eq}}(b_0, \dots, b_{i-1}, b_{i+1}) = \text{acl}^{\text{eq}}(b_0, \dots, b_{i-1})$  for every  $0 \leq i < N$ .



(b)  $b_{i+1} \downarrow_{b_i} b_0, \dots, b_{i-1}$  for every  $1 \leq i < N$ .

As in Remark 2.5, it follows that:

(c)  $\text{acl}^{\text{eq}}(b_{i+1}) \cap \text{acl}^{\text{eq}}(b_i) \subset \text{acl}^{\text{eq}}(b_0)$  for every  $1 \leq i < N$ .

(d)  $b_N \downarrow_{b_i} b_{i-1}$  for every  $1 \leq i < N$ .

(e)  $\text{acl}^{\text{eq}}(b_i, b_{i+1}) \cap \text{acl}^{\text{eq}}(b_i, b_N) = \text{acl}^{\text{eq}}(b_i)$  for every  $0 \leq i < N - 1$ .

Note that (almost) internality is preserved under taking nonforking restrictions. Furthermore, if a tuple  $d$  is (almost) internal over  $C$  and  $e$  is algebraic over  $Cd$ , then so is  $e$  (almost) internal over  $C$ . Thus, we may as before replace every  $b_i$  by a nice set  $B_i$  by Fact 2.1 and assume that  $b_N$  is a flag  $F$  by total triviality (*cf.* Proposition 7.22). In particular, we need to prove that  $\text{Cb}(F/B_0)$  is almost  $\Sigma$ -internal over  $B_1$ .

As before, let  $u_i$  be  $\preceq$ -minimal connecting  $F$  to a flag  $F_i$  of  $F$  in  $B_i$  for  $i < N$ . Since  $N \geq 2$ , there is (at least) one triangle to apply Proposition 8.3 and thus, either for some  $0 \leq i < N - 1$  we have that

$$F \downarrow_{B_i \cap B_{i+1}} B_i,$$

or the final segment  $\tilde{u}_{i+1}$  of  $u_{i+1}$  is non-trivial and strictly contained in  $\tilde{u}_i$  for every  $i < N$ . The independence

$$F \downarrow_{B_i \cap B_{i+1}} B_i$$

implies by properties (b) and (c) that  $F \downarrow_{\text{acl}^{\text{eq}}(B_0) \cap \text{acl}^{\text{eq}}(B_1)} B_0$ . So  $\text{Cb}(F/B_0)$  is algebraic over  $B_1$ , and hence internal over  $B_1$ .

Otherwise, if

$$F \not\downarrow_{B_i \cap B_{i+1}} B_i$$

for every  $i < N$ , then the final segment  $\tilde{u}_0$  must have length  $N$ . Consider the fine decomposition  $u_1 = u_1^1 \cdot u_1'$  and  $v_1^1 = v_1$  from Theorem 5.13. Proposition 8.3 implies that  $|v_1^1|$  is not fully contained in  $\tilde{u}_1$ , which must then have non-trivial centraliser. Since  $\tilde{u}_1$  has size  $N - 1$ , it must be either  $[2, N]$  or  $[0, N - 2]$ . Let us consider the first case. The canonical base  $\text{Cb}(b_N/B_0)$  is  $F_0$  modulo  $\mathcal{S}_R(u_0) = [1, N]$ , which is the 0-vertex  $f_0$  of  $F_0$ . Furthermore, since  $v_1 = [0] \cdot [1, N]$ , the vertex  $f_0$  is directly connected to  $B_1$  and, by theorem 7.15, it has rank 1 over  $B_1$ , so the canonical base  $\text{Cb}(F/B_0)$  is  $\Sigma$ -internal over  $B_1$ , which concludes the proof.  $\square$

## REFERENCES

- [1] A. Baudisch, A. Martin-Pizarro, M. Ziegler, *An alternative approach to the free pseudospace*, in preparation, (2013).
- [2] A. Baudisch, A. Pillay, *A free pseudospace*, J. Symb. Logic, **65**, 443–460, (2000).
- [3] T. Blossier, A. Martin-Pizarro, F.O. Wagner, *On variants of CM-triviality*, Fund. Mathematicae, **219**, 253–262, (2012).
- [4] D. Evans, *Ample dividing*, J. Symb. Logic **68**, 1385–1402, (2003).
- [5] R. Fraïssé, *Sur quelques classifications des systèmes de relations*, Publ. Scient. de l'Université d'Alger, 35–182 (1954).
- [6] J. B. Goode, *Some trivial considerations*, J. Symb. Logic **56**, 624–631 (1991).
- [7] E. Hrushovski, *A new strongly minimal set*, Annals of Pure and Applied Logic, **62**, 147–166 (1993).
- [8] P. Kowalski, A. Pillay, *Quantifier elimination for algebraic D-groups*, Trans. Amer. Math. Soc. **358**, 167–181 (2006).
- [9] M. Morley, *Categoricity in power*, Trans. Amer. Math. Soc., **114**, 514–538, (1965).

- [10] A. Pillay, *The geometry of forking and groups of finite Morley rank*, J. Symb. Logic **60**, 1251–1259 (1995).
- [11] A. Pillay, *A note on CM-triviality and the geometry of forking*, J. Symbolic Logic, **65**, 474–480, (2000).
- [12] K. Tent, *The free pseudospace is  $n$ -ample, but not  $(n + 1)$ -ample*, preprint, <http://arxiv.org/abs/1109.6608>, (2011).
- [13] K. Tent, M. Ziegler, *A course in Model Theory*, Cambridge University Press (2012).
- [14] F. O. Wagner, *CM-triviality and stable groups*, J. Symb. Logic **63**, 1473–1495 (1998).
- [15] M. Ziegler, *Strong Fraïssé limits*, preprint, [http://home.mathematik.uni-freiburg.de/ziegler/preprints/starker\\_fraisse.pdf](http://home.mathematik.uni-freiburg.de/ziegler/preprints/starker_fraisse.pdf), (2011).

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