About an even as the sum or the difference of two primes

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Abstract

(MSC=11) The present algebraic development begins simply by an exposition of the data of the problem. Our calculus is supported by a reasoning which must lead to an impossibility. We define the primal radius : For all x an integer greater or equal to 3, we define a primal number r for which $x-r$ and $x+r$ are prime numbers. We see then that Goldbach conjecture would be verified because $2x = (x + r) + (x - r)$. We prove the existence of r for all $x \geq 3$. We prove also the existence, for all x' an integer, of a primal radius r' for which $x'+r'$ and $r'-x'$ are prime numbers strictly greater than 2. De Polignac conjecture would be quickly verified because $2x' = (x'+r')-(r'-x')$.

Introduction

Goldbach and de Polignac conjectures seem actually impossible to be solved. Everyone has remarked the similarity between them. One stipulates that each even is the sum of two primes when the other stipulates that it is always the difference of two primes and that there is an infinity of such couples. In fact, we have used that similarity to solve those very old problems, we have considered in this research the conjectures as two faces of the same problem. For this, we have defined a notion : the primal radius. It allowed finally to prove the conjectures.

The Goldbach conjecture

Goldbach conjecture, fruit of personal works and correspondences between the mathematicians of the XVIII century (Leonard Euler, who was born exactly 300 years before 2007, was one of them), stipulates that an even number is always equal to the sum of two prime numbers. Let us suppose there exists an integer x ; $x \geq 3$, for which, for all $p_1 \geq 3, p_2 \geq 3$ prime numbers $b \neq 0$. We can pose for p_1 and p_2 distinct prime numbers verifying $p_1 > x > p_2$.

$$2x = p_1 + p_2 + 2b$$

b depends of p_1 and p_2. Then

$$x = \frac{p_1 + p_2}{2} + b$$

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And for all x, p_1, p_2, exists y whose expression is

$$y = \frac{p_1 - p_2}{2} + b$$

We pose

$$\begin{cases} x_1 = p_1 + 2b \\ x_2 = p_2 - 2b \\ x_3 = p_2 + 2b \\ x_4 = p_1 - 2b \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{p_1+p_2}{2} + b = \frac{p_1+x_2}{2} + 2b = \frac{x_1+p_2}{2} = \frac{x_1+x_2}{2} + b \\ \quad = \frac{p_1+x_3}{2} = \frac{x_1+x_3}{2} - b = \frac{x_4+x_3}{2} + b = \frac{x_4+x_2}{2} + 3b \\ y = \frac{p_1-p_2}{2} + b = \frac{p_1-x_2}{2} = \frac{x_1-p_2}{2} = \frac{x_1-x_2}{2} - b \\ \quad = \frac{p_1-x_3}{2} + 2b = \frac{x_1-x_3}{2} + b = \frac{x_4-x_3}{2} + 3b = \frac{x_4-x_2}{2} + b \\ x_1 + x_2 = p_1 + p_2 \end{cases}$$

LEMMA 1   The following formula

$$\begin{cases} x = \frac{p_1+p_2}{2} + b = \frac{p_1+x_2}{2} + 2b = \frac{x_1+p_2}{2} = \frac{x_1+x_2}{2} + b \\ \quad = \frac{p_1+x_3}{2} = \frac{x_1+x_3}{2} - b = \frac{x_4+x_3}{2} + b = \frac{x_4+x_2}{2} + 3b \\ y = \frac{p_1-p_2}{2} + b = \frac{p_1-x_2}{2} = \frac{x_1-p_2}{2} = \frac{x_1-x_2}{2} - b \\ \quad = \frac{p_1-x_3}{2} + 2b = \frac{x_1-x_3}{2} + b = \frac{x_4-x_3}{2} + 3b = \frac{x_4-x_2}{2} + b \\ x_1 + x_2 = p_1 + p_2 \end{cases}$$

imply that $\exists p_1, p_2$ prime numbers which verify

$$b = 0$$

. In fact, our hypothesis ($b \neq 0$ for all p_1, p_2) can not imply $b = 0$ without any condition. If the calculus leads to $b = 0$, the hypothesis is false because the formal calculus can not lead to $b = 0$.

Proof of lemma 1   If x is a prime number, $2x = x + x$ is the sum of two primes, then

$$p_1 - p_2 \neq 0$$

we will suppose firstly that

$$(x_1 - x_2)(x_1 + x_3) \neq 0$$

let

$$\begin{cases} \frac{x_1-x_2}{p_1-p_2} = \frac{p_1-p_2+4b}{p_1-p_2} = 1 + \frac{4b}{p_1-p_2} \\ \frac{p_1-p_2}{x_1-x_2} = \frac{x_1-x_2-4b}{x_1-x_2} = 1 - \frac{4b}{x_1-x_2} \end{cases}$$

we pose

$$\begin{cases} k = \frac{2b}{p_1-p_2} \\ k' = -\frac{2b}{(p_1-p_2+4b)} \end{cases}$$

if

$$kk' = 0 \Rightarrow b = 0$$

we will suppose
$$kk' \neq 0$$

But $\forall x, y, \exists \phi$ verifying $x = \phi y$

$$x + y = (\phi + 1)y = x_1 \neq 0$$

$$x - y = (\phi - 1)y = p_2 \neq 0$$

and $\forall k, k', \exists \alpha$ verifying $k = \alpha k'$

$$\Rightarrow k = \frac{2b}{p_1 - p_2} = \alpha \frac{-2b}{x_1 - x_2}$$

$$\Rightarrow x_1 - x_2 = -\alpha(p_1 - p_2)$$

$$\Rightarrow x_1 - x_2 - p_1 + p_2 = 4b = -(\alpha + 1)(p_1 - p_2)$$

$$\Rightarrow b = \frac{-(\alpha + 1)}{4}(p_1 - p_2)$$

$$\Rightarrow x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\alpha + 1}{4}(p_1 - p_2)$$

$$\Rightarrow y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\alpha + 1}{4}(p_1 + p_2)$$

or
$$x = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = \frac{\phi}{\phi - 1}p_2$$

and
$$y = \frac{(1 - \alpha)(p_1 - p_2)}{4} = \frac{1}{\phi - 1}p_2$$

let
$$\begin{cases} \frac{x_1 + x_3}{p_1 + p_2} = \frac{p_1 + p_2 + 4b}{p_1 + p_2} = 1 + \frac{4b}{p_1 + p_2} \\ \frac{p_1 + p_2}{x_1 + x_3} = \frac{x_1 + x_2 - 4b}{x_1 + x_3} = 1 - \frac{4b}{x_1 + x_3} \end{cases}$$

we pose
$$\begin{cases} m = \frac{2b}{p_1 + p_2} \\ m' = -\frac{2b}{x_1 + x_3} \end{cases}$$

$$mm' \neq 0$$

and $\forall m, m', \exists \beta$ verifying $m = \beta m'$

$$\Rightarrow m = \frac{2b}{p_1 + p_2} = \beta \frac{-2b}{x_1 + x_3}$$

$$\Rightarrow x_1 + x_3 = -\beta(p_1 + p_2)$$

$$\Rightarrow x_1 + x_3 - p_1 - p_2 = 4b = -(\beta + 1)(p_1 + p_2)$$

$$\Rightarrow b = \frac{-(\beta + 1)}{4}(p_1 + p_2)$$

$$\Rightarrow x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{(\beta + 1)}{4}(p_1 + p_2)$$

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$$\Rightarrow y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{(\beta + 1)}{4}(p_1 + p_2)$$

thus

$$x = \frac{1 - \beta}{4}p_1 + \frac{1 - \beta}{4}p_2 = \frac{\phi}{\phi - 1}p_2$$

$$y = \frac{1 - \beta}{4}p_1 - \frac{\beta + 3}{4}p_2 = \frac{1}{\phi - 1}p_2$$

we resume

$$x = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = \frac{1 - \beta}{4}p_1 + \frac{1 - \beta}{4}p_2 = \frac{\phi}{\phi - 1}p_2$$

and

$$y = \frac{(1 - \alpha)(p_1 - p_2)}{4} = \frac{1 - \beta}{4}p_1 - \frac{\beta + 3}{4}p_2 = \frac{1}{\phi - 1}p_2$$

$$b = \frac{-(\alpha + 1)}{4}(p_1 - p_2) = \frac{-(\beta + 1)}{4}(p_1 + p_2)$$

$$\Rightarrow \frac{\beta - \alpha}{4}p_1 = \frac{-2 - \alpha - \beta}{4}p_2$$

$$\Rightarrow \frac{\beta - \alpha}{p_2} = \frac{-2 - \alpha - \beta}{p_1}$$

but

$$4x - 4y + 4b = 2p_1 + 2p_2 + 4b - 2p_1 + 2p_2 - 4b + 4b = 4p_2 + 4b = 4P_2$$

And

$$4x + 4y - 4b = 2p_1 + 2p_2 + 4b + 2p_1 - 2p_2 + 4b - 4b = 4p_1 + 4b = 4P_1$$

Thus

$$4P_1 = 4p_1 + 4b = 4p_1 - (\alpha + 1)(p_1 - p_2) = (3 - \alpha)p_1 + (1 + \alpha)p_2$$

$$= 4p_1 - (\beta + 1)(p_1 + p_2) = (3 - \beta)p_1 - (1 + \beta)p_2 = \frac{1}{2}(6 - \alpha - \beta)p_1 + (\alpha - \beta)p_2)$$

And

$$4P_2 = 4p_2 + 4b = 4p_2 - (\alpha + 1)(p_1 - p_2) = -(1 + \alpha)p_1 + (5 + \alpha)p_2$$

$$= 4p_2 - (\beta + 1)(p_1 + p_2) = -(1 + \beta)p_1 + (3 - \beta)p_2 = \frac{1}{2}(-(2 + \alpha + \beta)p_1 + (8 + \alpha - \beta)p_2)$$

But p_1 and p_2 are primes.

$$(\beta - \alpha)p_1 = (-\alpha - \beta - 2)p_2$$

$$= (\beta - \alpha)(P_1 - b) = (\beta - \alpha)(P_1 + (\frac{\alpha + 1}{4})(P_1 - P_2))$$

$$= (\beta - \alpha)((\frac{5 + \alpha}{4})P_1 - (\frac{1 + \alpha}{4})P_2)$$

$$= (-\alpha - \beta - 2)(P_2 - b) = (-\alpha - \beta - 2)(P_2 + (\frac{\alpha + 1}{4})(P_1 - P_2))$$

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$$= (-\alpha - \beta - 2)((\frac{1+\alpha}{4})P_1 + \frac{3-\alpha}{4}P_2)$$

Thus
$$((\beta - \alpha)(5 + \alpha) + (\alpha + \beta + 2)(1 + \alpha))P_1$$
$$= ((-\alpha - \beta - 2)(3 - \alpha) + (\beta - \alpha)(1 + \alpha))P_2$$

And
$$(\beta - \alpha)P_1 = (-\alpha - \beta - 2)P_2 + b(\beta - \alpha + \alpha + \beta + 2)$$
$$= (-\alpha - \beta - 2)P_2 - (\beta + 1)(\frac{\alpha+1}{2})(P_1 - P_2)$$

Hence
$$(2\beta - 2\alpha + (\beta + 1)(\alpha + 1))P_1$$
$$= (-2\alpha - 2\beta - 4 + (\beta + 1)(\alpha + 1))P_2$$

Thus
$$((\beta - \alpha)(3 + \alpha) + (\alpha + 1)^2)P_1$$
$$= ((-\alpha - \beta - 2)(1 - \alpha) - (\alpha + 1)^2)P_2$$
$$= ((\beta + 1)(3 + \alpha) + (\alpha + 1)(\alpha + 1 - 3 - \alpha))P_1$$
$$= (-(\beta + 1)(1 - \alpha) - (\alpha + 1)(\alpha + 1 + 1 - \alpha))P_2$$
$$= ((\beta + 1)(3 + \alpha) - 2\alpha - 2)P_1$$
$$= (-(\beta + 1)(1 - \alpha) - 2\alpha - 2)P_2$$

Hence
$$((\beta + 1)(3 + \alpha) - 2(\alpha + 1))(p_1 + p_2)P_1$$
$$= (-(\beta + 1)(1 - \alpha) - 2(\alpha + 1))(p_1 + p_2)P_2$$
$$= ((\alpha + 1)(p_1 - p_2)(3 + \alpha) - 2(\alpha + 1)(p_1 + p_2))P_1$$
$$= (-(\alpha + 1)(p_1 - p_2)(1 - \alpha) - 2(\alpha + 1)(p_1 + p_2))P_2$$
$$= (\alpha + 1)((-\alpha + 1)p_1 - (3\alpha + 5)p_2)P_1$$
$$= (\alpha + 1)(-(\alpha + 3)p_1 - (1 + 3\alpha)p_2)P_2$$

If
$$((1 - \alpha)p_1 - (5 + 3\alpha)p_2)P_1 + ((\alpha + 3)p_1 + (1 + 3\alpha)p_2)P_2 = 0$$
$$= ((1 - \alpha)p_1 - (5 + 3\alpha)p_2)(p_1 + b) + ((\alpha + 3)p_1 + (1 + 3\alpha)p_2)(p_2 + b)$$
$$= (1 - \alpha)p_1^2 + (1 + 3\alpha)p_2^2 - 2(\alpha + 1)p_1p_2 + 4b(p_1 - p_2) = 0$$
$$= -(1 + \alpha)p_1^2 + 2p_1^2 + 2(1 + \alpha)p_2^2 + (\alpha + 1)p_2^2 - 2p_2^2 - 2(\alpha + 1)p_1p_2 + 4b(p_1 - p_2)$$
$$= (\alpha + 1)(-p_1^2 + 3p_2^2 - 2p_1p_2) + 2(p_1^2 - p_2^2) + 4b(p_1 - p_2)$$
$$= (\alpha + 1)(p_2 - p_1)(3p_2 + p_1) + 2(p_1^2 - p_2^2) + 4b(p_1 - p_2)$$
$$= 4b(3p_2 + p_1) + 2(p_1^2 - p_2^2) + 4b(p_1 - p_2)$$
$$= 4b(2p_1 + 2p_2) + 2p_1^2 - 2p_2^2 = 0$$

It means
$$4b = p_2 - p_1 = -(\alpha + 1)(p_1 - p_2) = -(\beta + 1)(p_1 + p_2)$$

$$\Rightarrow \alpha = 0, \quad \beta = \frac{-2p_2}{p_1 + p_2}$$

But

$$4k^2k'^2 = (k + k')^2 = k^2 + k'^2 + 2kk'$$

Thus

$$(4k'^2 - 1)k^2 - 2k'k - k'^2 = 0$$

$$\delta = k'^2 + k'^2(4k'^2 - 1) = 4k'^4$$

$$k = \frac{k' - 2k'}{4k'^2 - 1} = \frac{-k'}{4k'^2 - 1} = \frac{-k'}{2k' + 1}$$

$$1 - 2k' = 1 \Rightarrow k' = 0 \Rightarrow k = 0 \Rightarrow 4b = p_2 - p_1 = 0$$

Impossible, we have supposed $x_1 - x_2 \neq 0$. And as

$$((-\alpha + 1)p_1 - (3\alpha + 5)p_2)P_1 \neq (-(\alpha + 3)p_1 - (1 + 3\alpha)p_2)P_2$$

We deduce that

$$\alpha + 1 = 0 \Rightarrow 4b = -(\alpha + 1)(p_1 - p_2) = 0$$

P_1 and P_2 are p_1 and p_2 and

$$2x = p_1 + p_2$$

Thus $\alpha = \beta = -1$. b can not be different of zero. If

$$(x_1 - x_2)(x_1 + x_3) = 0 \Rightarrow (x_4 + x_2)(x_4 - x_3) \neq 0$$

let

$$\begin{cases} \frac{x_4 + x_2}{p_1 + p_2} = \frac{p_1 + p_2 - 4b}{p_1 + p_2} = 1 - \frac{4b}{p_1 + p_2} \\ \frac{p_1 + p_2}{x_4 + x_2} = \frac{x_4 + x_2 + 4b}{x_4 + x_2} = 1 + \frac{4b}{x_4 + x_2} \end{cases}$$

we pose

$$\begin{cases} 2k + 1 = 1 - \frac{4b}{p_1 + p_2} \\ 2k' + 1 = 1 + \frac{4b}{x_4 + x_2} \end{cases}$$

let

$$\begin{cases} \frac{x_4 - x_3}{p_1 - p_2} = \frac{p_1 - p_2 - 4b}{p_1 - p_2} = 1 - \frac{4b}{p_1 - p_2} \\ \frac{p_1 - p_2}{x_4 - x_3} = \frac{x_4 - x_3 + 4b}{x_4 - x_3} = 1 + \frac{4b}{x_4 - x_3} \end{cases}$$

we pose

$$\begin{cases} 2m + 1 = 1 - \frac{4b}{p_1 - p_2} \\ 2m' + 1 = 1 + \frac{4b}{x_4 - x_3} \end{cases}$$

if

$$kk' = 0 \Rightarrow b = 0$$

we will suppose

$$kk' \neq 0$$

But $\forall x, y, \exists \phi$ verifying $x = \phi y$

$$x + y = (\phi + 1)y = x_1 \neq 0$$

$$x - y = (\phi - 1)y = p_2 \neq 0$$

and $\forall k, k', \exists \alpha$ verifying $k = \alpha k'$

$$\Rightarrow k = \frac{-2b}{p_1 + p_2} = \alpha \frac{2b}{x_4 + x_3}$$

$$\Rightarrow x_4 + x_3 = -\alpha(p_1 + p_2)$$

$$\Rightarrow x_4 + x_3 - p_1 - p_2 = 4b = -(\alpha + 1)(p_1 + p_2)$$

$$\Rightarrow b = \frac{-(\alpha + 1)}{4}(p_1 + p_2)$$

$$\Rightarrow x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\alpha + 1}{4}(p_1 + p_2)$$

$$\Rightarrow y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\alpha + 1}{4}(p_1 + p_2)$$

or

$$y = \frac{(1 - \alpha)p_1 - (3 + \alpha)p_2}{4} = \frac{1}{\phi - 1}p_2$$

and

$$x = \frac{(1 - \alpha)(p_1 + p_2)}{4} = \frac{\phi}{\phi - 1}p_2$$

let

$$\begin{cases} \frac{x_4 - x_3}{p_1 - p_2} = \frac{p_1 - p_2 - 4b}{p_1 - p_2} = 1 - \frac{4b}{p_1 + p_2} \\ \frac{p_1 - p_2}{x_4 - x_3} = \frac{x_4 - x_3 + 4b}{x_4 - x_3} = 1 + \frac{4b}{x_4 - x_3} \end{cases}$$

we pose

$$\begin{cases} m = -\frac{2b}{p_1 - p_2} \\ m' = \frac{2b}{x_4 - x_3} \end{cases}$$

$$mm' \neq 0$$

and $\forall m, m', \exists \beta$ verifying $m = \beta m'$

$$\Rightarrow m = -\frac{2b}{p_1 - p_2} = \beta \frac{2b}{x_4 - x_3}$$

$$\Rightarrow x_4 - x_3 = -\beta(p_1 - p_2)$$

$$\Rightarrow x_4 - x_3 - p_1 + p_2 = 4b = -(\beta + 1)(p_1 - p_2)$$

$$\Rightarrow b = \frac{-(\beta + 1)}{4}(p_1 - p_2)$$

$$\Rightarrow y = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{(\beta + 1)}{4}(p_1 - p_2)$$

$$\Rightarrow x = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{(\beta + 1)}{4}(p_1 - p_2)$$

thus

$$y = \frac{1 - \beta}{4}p_1 - \frac{1 - \beta}{4}p_2 = \frac{1}{\phi - 1}p_2$$

$$x = \frac{1 - \beta}{4}p_1 + \frac{\beta + 3}{4}p_2 = \frac{\phi}{\phi - 1}p_2$$

we resume

$$x = \frac{(1-\beta)p_1 + (3+\beta)p_2}{4} = \frac{1-\alpha}{4}p_1 + \frac{1-\alpha}{4}p_2 = \frac{\phi}{\phi-1}p_2$$

and

$$y = \frac{(1-\beta)(p_1 - p_2)}{4} = \frac{1-\alpha}{4}p_1 - \frac{\alpha+3}{4}p_2 = \frac{1}{\phi-1}p_2$$

$$b = \frac{-(\alpha+1)}{4}(p_1 + p_2) = \frac{-(\beta+1)}{4}(p_1 - p_2)$$

$$\Rightarrow \frac{\beta-\alpha}{4}p_1 = \frac{2+\alpha+\beta}{4}p_2$$

$$\Rightarrow \frac{\beta-\alpha}{p_2} = \frac{2+\alpha+\beta}{p_1}$$

but

$$4x - 4y + 4b = 2p_1 + 2p_2 + 4b - 2p_1 + 2p_2 - 4b + 4b = 4p_2 + 4b = 4P_2$$

And

$$4x + 4y - 4b = 2p_1 + 2p_2 + 4b + 2p_1 - 2p_2 + 4b - 4b = 4p_1 + 4b = 4P_1$$

Thus

$$4P_1 = 4p_1 + 4b = 4p_1 - (\alpha+1)(p_1 + p_2) = (3-\alpha)p_1 - (1+\alpha)p_2$$

$$= 4p_1 - (\beta+1)(p_1 - p_2) = (3-\beta)p_1 + (1+\beta)p_2 = \frac{1}{2}(6 - \alpha - \beta)p_1 - (\alpha - \beta)p_2)$$

And

$$4P_2 = 4p_2 + 4b = 4p_2 - (\alpha+1)(p_1 + p_2) = -(1+\alpha)p_1 + (3-\alpha)p_2$$

$$= 4p_2 - (\beta+1)(p_1 - p_2) = -(1+\beta)p_1 + (5+\beta)p_2 = \frac{1}{2}(-(2+\alpha+\beta)p_1 + (8-\alpha+\beta)p_2)$$

But p_1 and p_2 are primes.

$$(\beta - \alpha)p_1 = (\alpha + \beta + 2)p_2$$

$$= (\beta - \alpha)(P_1 - b) = (\beta - \alpha)(P_1 + (\frac{\beta+1}{4})(P_1 - P_2))$$

$$= (\beta - \alpha)((\frac{5+\beta}{4})P_1 - (\frac{1+\beta}{4})P_2)$$

$$= (\alpha + \beta + 2)(P_2 - b) = (\alpha + \beta + 2)(P_2 + (\frac{\beta+1}{4})(P_1 - P_2))$$

$$= (\alpha + \beta + 2)((\frac{1+\beta}{4})P_1 + \frac{3-\beta}{4}P_2)$$

Thus

$$((\beta - \alpha)(5 + \beta) - (\alpha + \beta + 2)(1 + \beta))P_1$$

$$= ((\alpha + \beta + 2)(3 - \beta) + (\beta - \alpha)(1 + \beta))P_2$$

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And
$$(\beta - \alpha)P_1 = (\alpha + \beta + 2)P_2 + b(-\beta + \alpha + \alpha + \beta + 2)$$
$$= (\alpha + \beta + 2)P_2 - (\beta + 1)(\frac{\alpha + 1}{2})(P_1 - P_2)$$

Hence
$$(2\beta - 2\alpha + (\beta + 1)(\alpha + 1))P_1$$
$$= (2\alpha + 2\beta + 4 + (\beta + 1)(\alpha + 1))P_2$$

Thus
$$((\beta - \alpha)(3 + \beta) - (2\alpha + \beta + 3)(\beta + 1))P_1$$
$$= ((\alpha + \beta + 2)(1 - \beta) - (\beta + 1)(\alpha + 1))P_2$$
$$= ((\alpha + 1)(-3 - \beta - 2 - 2\beta) + (\beta + 1)(\beta + 1 + 3 + \beta))P_1$$
$$= ((\beta + 1)(1 - \beta) - (\alpha + 1)(-\beta - 1 + 1 - \beta))P_2$$
$$= ((\alpha + 1)(-5 - 3\beta) + 2(\beta + 1)(\beta + 2)P_1$$
$$= ((\beta + 1)(1 - \beta) + 2(\alpha + 1)\beta)P_2$$

Hence
$$((\beta + 1)(-5 - 3\beta) + 2(\beta + 1)(\beta + 2))(p_1 - p_2)P_1$$
$$= ((\beta + 1)(1 - \beta) + 2(\alpha + 1)\beta)(p_1 + p_2)P_2$$
$$= ((\beta + 1)(p_1 - p_2)(-5 - 3\beta) + 2(\beta + 1)(\beta + 2)(p_1 - p_2))P_1$$
$$= ((\beta + 1)(p_1 + p_2)(1 - \beta) + 2(\beta + 1)\beta(p_1 - p_2))P_2$$
$$= (\beta + 1)((-\beta - 1)p_1 + (\beta + 1)p_2)P_1$$
$$= (\beta + 1)((\beta + 1)p_1 + (1 - 3\beta)p_2)P_2$$

If
$$((-1 - \beta)p_1 - (1 + \beta)p_2)P_1 - ((\beta + 1)p_1 + (1 - 3\beta)p_2)P_2 = 0$$
$$= ((-1 - \beta)p_1 - (1 + \beta)p_2)(p_1 + b) - ((\beta + 1)p_1 + (1 - 3\beta)p_2)(p_2 + b)$$
$$= (-1 - \beta)p_1^2 + (-1 + 3\beta)p_2^2 - 2(\beta + 1)p_1 p_2 + b((-2 - 2\beta)p_1 - (2 + 4\beta)p_2) = 0$$
$$= -(1 + \beta)p_1^2 + 3(1 + \beta)p_2^2 - 4p_2^2 - 2(\beta + 1)p_1 p_2 + b((-2 - 2\beta)p_1 - (2 + 4\beta)p_2)$$
$$= (\beta + 1)(-p_1^2 + 3p_2^2 - 2p_1 p_2) - 4p_2^2 + 2b((-1 - \beta)p_1 - (1 + 2\beta)p_2)$$
$$= (\beta + 1)(p_2 - p_1)(3p_2 + p_1) - 4p_2^2 + 2b(-p_1 - p_2 - \beta(p_1 + 2p_2))$$
$$= 4b(3p_2 + p_1) - 4p_2^2 + 2b(-p_1 - p_2 - \beta(p_1 + 2p_2))$$
$$= \frac{p_2 - p_1}{4}(2p_1 + 10p_2 - \beta(p_1 + 2p_2)) - 4p_2^2 = 0$$

Impossible. And as
$$((-\beta - 1)p_1 - (\beta + 1)p_2)P_1 \neq ((\beta + 1)p_1 - (1 - 3\beta)p_2)P_2$$

We deduce that
$$\beta + 1 = 0 \Rightarrow 4b = -(\beta + 1)(p_1 - p_2) = 0$$

P_1 and P_2 are p_1 and p_2 and
$$2x = p_1 + p_2$$

Thus $\alpha = \beta = -1$. And
$$b = 0$$

Theorem

$$\forall x \geq 3, \exists p_1 \geq 3, p_2 \geq 3, x = \frac{p_1 + p_2}{2}$$

We did not pose any condition on p_1 and p_2. As $p_2 < x < p_1$ there are not an infinity of such primes.

THEOREM OF THE PRIMAL RADIUS   There exists r a primal radius for which $x + r$, $x - r$ are prime numbers $\forall x \geq 3$.

Proof of theorem of primal radius   For all $x \geq 3$ exists $p_1 \geq 3$, $p_2 \geq 3$ prime numbers for which

$$x = \frac{p_1 + p_2}{2}$$

if

$$r = \frac{p_1 - p_2}{2}$$

then

$$x + r = p_1$$
$$x - r = p_2$$

Corollary   Between x and $2x$ exists always a prime number $x < p_1 = x + r < 2x = p_1 + p_2$. If $2z + 1$ is an integer stricly greater than 8, exists always a prime p_3, which can be 3, for which $2z + 1 = p_3 + 2x = p_3 + p_1 + p_2$.

de Polignac conjecture   de Polignac conjecture stipulates that an even number is always equal to the difference between two prime numbers and that there is an infinity of such prime numbers. Let x an integer and p_1, p_2 prime numbers strictly greater than 2.

$$2x = p_1 - p_2 + 2b_{p_1, p_2} = p_1 - p_2 + 2b$$

For commodity, we have suppressed the indexes, but b depends of p_1 and p_2.

$$x = \frac{p_1 - p_2}{2} + b$$

But for all x, p_1, p_2, b exists y whose expression is

$$y = \frac{p_1 + p_2}{2} + b$$

We pose

$$\begin{cases} x_1 = p_1 + 2b \\ x_2 = p_2 - 2b \\ x_3 = p_2 + 2b \\ x_4 = p_1 - 2b \end{cases}$$

$$\Rightarrow \begin{cases} y = \frac{p_1+p_2}{2} + b = \frac{p_1+x_2}{2} + 2b = \frac{x_1+p_2}{2} = \frac{x_1+x_2}{2} + b \\ \quad = \frac{p_1+x_3}{2} = \frac{x_1+x_3}{2} - b = \frac{x_4+x_3}{2} + b = \frac{x_4+x_2}{2} + 3b \\ x = \frac{p_1-p_2}{2} + b = \frac{p_1-x_2}{2} = \frac{x_1-p_2}{2} = \frac{x_1-x_2}{2} - b \\ \quad = \frac{p_1-x_3}{2} + 2b = \frac{x_1-x_3}{2} + b = \frac{x_4-x_3}{2} + 3b = \frac{x_4-x_2}{2} + b \\ x_1 + x_2 = p_1 + p_2 \end{cases}$$

LEMMA 2   The following formula

$$\begin{cases} y = \frac{p_1+p_2}{2} + b = \frac{p_1+x_2}{2} + 2b = \frac{x_1+p_2}{2} = \frac{x_1+x_2}{2} + b \\ \quad = \frac{p_1+x_3}{2} = \frac{x_1+x_3}{2} - b = \frac{x_4+x_3}{2} + b = \frac{x_4+x_2}{2} + 3b \\ x = \frac{p_1-p_2}{2} + b = \frac{p_1-x_2}{2} = \frac{x_1-p_2}{2} = \frac{x_1-x_2}{2} - b \\ \quad = \frac{p_1-x_3}{2} + 2b = \frac{x_1-x_3}{2} + b = \frac{x_4-x_3}{2} + 3b = \frac{x_4-x_2}{2} + b \\ x_1 + x_2 = p_1 + p_2 \end{cases}$$

imply that $\exists p_1, p_2$ an infinity of couples which verify

$$b = 0$$

Proof of lemma 2   If $x = 0$ is a prime number, $2x = p - p$ an infinity of couples of primes, and

$$p_1 - p_2 \neq 0$$

we will suppose firstly that

$$(x_1 - x_2)(x_1 + x_3) \neq 0$$

let

$$\begin{cases} \frac{x_1-x_2}{p_1-p_2} = \frac{p_1-p_2+4b}{p_1-p_2} = 1 + \frac{4b}{p_1-p_2} \\ \frac{p_1-p_2}{x_1-x_2} = \frac{x_1-x_2-4b}{x_1-x_2} = 1 - \frac{4b}{x_1-x_2} \end{cases}$$

we pose

$$\begin{cases} k = \frac{2b}{p_1-p_2} \\ k' = -\frac{2b}{(p_1-p_2+4b)} \end{cases}$$

if

$$kk' = 0 \Rightarrow b = 0$$

we will suppose

$$kk' \neq 0$$

But $\forall x, y, \exists \phi$ verifying $y = \phi x$

$$x + y = (\phi + 1)x = x_1 \neq 0$$

$$x - y = (\phi - 1)x = p_2 \neq 0$$

Thus $\forall k, k', \exists \alpha$ verifying $k = \alpha k'$

$$\Rightarrow k = \frac{2b}{p_1 - p_2} = \alpha \frac{-2b}{x_1 - x_2}$$

$$\Rightarrow x_1 - x_2 = -\alpha(p_1 - p_2)$$

$$\Rightarrow x_1 - x_2 - p_1 + p_2 = 4b = -(\alpha + 1)(p_1 - p_2)$$

$$\Rightarrow b = \frac{-(\alpha + 1)}{4}(p_1 - p_2)$$

$$\Rightarrow y = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{\alpha + 1}{4}(p_1 - p_2)$$

$$\Rightarrow x = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{\alpha + 1}{4}(p_1 + p_2)$$

or

$$y = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = \frac{\phi}{\phi - 1}p_2$$

and

$$x = \frac{(1 - \alpha)(p_1 - p_2)}{4} = \frac{1}{\phi - 1}p_2$$

let

$$\begin{cases} \frac{x_1 + x_3}{p_1 + p_2} = \frac{p_1 + p_2 + 4b}{p_1 + p_2} = 1 + \frac{4b}{p_1 + p_2} \\ \frac{p_1 + p_2}{x_1 + x_3} = \frac{x_1 + x_2 - 4b}{x_1 + x_3} = 1 - \frac{4b}{x_1 + x_2} \end{cases}$$

we pose

$$\begin{cases} m = \frac{2b}{p_1 + p_2} \\ m' = -\frac{2b}{x_1 + x_2} \end{cases}$$

$$mm' \neq 0$$

and $\forall m, m', \exists \beta$ verifying $m = \beta m'$

$$\Rightarrow m = \frac{2b}{p_1 + p_2} = \beta \frac{-2b}{x_1 + x_3}$$

$$\Rightarrow x_1 + x_3 = -\beta(p_1 + p_2)$$

$$\Rightarrow x_1 + x_3 - p_1 - p_2 = 4b = -(\beta + 1)(p_1 + p_2)$$

$$\Rightarrow b = \frac{-(\beta + 1)}{4}(p_1 + p_2)$$

$$\Rightarrow y = \frac{p_1 + p_2}{2} + b = \frac{p_1 + p_2}{2} - \frac{(\beta + 1)}{4}(p_1 + p_2)$$

$$\Rightarrow x = \frac{p_1 - p_2}{2} + b = \frac{p_1 - p_2}{2} - \frac{(\beta + 1)}{4}(p_1 + p_2)$$

we deduce

$$y = \frac{1 - \beta}{4}p_1 + \frac{1 - \beta}{4}p_2 = \frac{\phi}{\phi - 1}p_2$$

$$x = \frac{1 - \beta}{4}p_1 - \frac{\beta + 3}{4}p_2 = \frac{1}{\phi - 1}p_2$$

we resume

$$y = \frac{(1 - \alpha)p_1 + (3 + \alpha)p_2}{4} = \frac{1 - \beta}{4}p_1 + \frac{1 - \beta}{4}p_2 = \frac{\phi}{\phi - 1}p_2$$

and

$$x = \frac{(1 - \alpha)(p_1 - p_2)}{4} = \frac{1 - \beta}{4}p_1 - \frac{\beta + 3}{4}p_2 = \frac{1}{\phi - 1}p_2$$

$$b = \frac{-(\alpha + 1)}{4}(p_1 - p_2) = \frac{-(\beta + 1)}{4}(p_1 + p_2)$$

$$\Rightarrow \frac{\beta - \alpha}{4}p_1 = \frac{-2 - \alpha - \beta}{4}p_2$$

$$\Rightarrow \frac{\beta - \alpha}{p_2} = \frac{-2 - \alpha - \beta}{p_1}$$

but

$$4y - 4x + 4b = 2p_1 + 2p_2 + 4b - 2p_1 + 2p_2 - 4b + 4b = 4p_2 + 4b = 4P_2$$

And

$$4x + 4y - 4b = 2p_1 + 2p_2 + 4b + 2p_1 - 2p_2 + 4b - 4b = 4p_1 + 4b = 4P_1$$

Thus

$$4P_1 = 4p_1 + 4b = 4p_1 - (\alpha + 1)(p_1 - p_2) = (3 - \alpha)p_1 + (1 + \alpha)p_2$$

$$= 4p_1 - (\beta + 1)(p_1 + p_2) = (3 - \beta)p_1 - (1 + \beta)p_2 = \frac{1}{2}(6 - \alpha - \beta)p_1 + (\alpha - \beta)p_2)$$

And

$$4P_2 = 4p_2 + 4b = 4p_2 - (\alpha + 1)(p_1 - p_2) = -(1 + \alpha)p_1 + (5 + \alpha)p_2$$

$$= 4p_2 - (\beta + 1)(p_1 + p_2) = -(1 + \beta)p_1 + (3 - \beta)p_2 = \frac{1}{2}(-(2 + \alpha + \beta)p_1 + (8 + \alpha - \beta)p_2)$$

But p_1 and p_2 are primes.

$$(\beta - \alpha)p_1 = (-\alpha - \beta - 2)p_2$$

$$= (\beta - \alpha)(P_1 - b) = (\beta - \alpha)(P_1 + (\frac{\alpha + 1}{4})(P_1 - P_2))$$

$$= (\beta - \alpha)((\frac{5 + \alpha}{4})P_1 - (\frac{1 + \alpha}{4})P_2)$$

$$= (-\alpha - \beta - 2)(P_2 - b) = (-\alpha - \beta - 2)(P_2 + (\frac{\alpha + 1}{4})(P_1 - P_2))$$

$$= (-\alpha - \beta - 2)((\frac{1 + \alpha}{4})P_1 + \frac{3 - \alpha}{4}P_2)$$

Thus

$$((\beta - \alpha)(5 + \alpha) + (\alpha + \beta + 2)(1 + \alpha))P_1$$

$$= ((-\alpha - \beta - 2)(3 - \alpha) + (\beta - \alpha)(1 + \alpha))P_2$$

And

$$(\beta - \alpha)P_1 = (-\alpha - \beta - 2)P_2 + b(\beta - \alpha + \alpha + \beta + 2)$$

$$= (-\alpha - \beta - 2)P_2 - (\beta + 1)(\frac{\alpha + 1}{2})(P_1 - P_2)$$

Hence

$$(2\beta - 2\alpha + (\beta + 1)(\alpha + 1))P_1$$

$$= (-2\alpha - 2\beta - 4 + (\beta + 1)(\alpha + 1))P_2$$

Thus

$$((\beta - \alpha)(3 + \alpha) + (\alpha + 1)^2)P_1$$

$$= ((-\alpha - \beta - 2)(1 - \alpha) - (\alpha + 1)^2)P_2$$

$$= ((\beta + 1)(3 + \alpha) + (\alpha + 1)(\alpha + 1 - 3 - \alpha))P_1$$

$$= (-(\beta + 1)(1 - \alpha) - (\alpha + 1)(\alpha + 1 + 1 - \alpha))P_2$$

$$= ((\beta + 1)(3 + \alpha) - 2\alpha - 2)P_1$$

$$= (-(\beta + 1)(1 - \alpha) - 2\alpha - 2)P_2$$

$$((\beta + 1)(3 + \alpha) - 2(\alpha + 1))(p_1 + p_2)P_1$$

$$= (-(\beta + 1)(1 - \alpha) - 2(\alpha + 1))(p_1 + p_2)P_2$$

$$= ((\alpha + 1)(p_1 - p_2)(3 + \alpha) - 2(\alpha + 1)(p_1 + p_2))P_1$$

$$= (-(\alpha + 1)(p_1 - p_2)(1 - \alpha) - 2(\alpha + 1)(p_1 + p_2))P_2$$

$$= (\alpha + 1)((-\alpha + 1)p_1 - (3\alpha + 5)p_2)P_1$$

$$= (\alpha + 1)(-(\alpha + 3)p_1 - (1 + 3\alpha)p_2)P_2$$

And as

$$((-\alpha + 1)p_1 - (3\alpha + 5)p_2)P_1 \neq (-(\alpha + 3)p_1 - (1 + 3\alpha)p_2)P_2$$

We deduce that

$$\alpha + 1 = 0 \Rightarrow 4b = -(\alpha + 1)(p_1 - p_2) = 0$$

If

$$(x_1 - x_2)(x_1 + x_3) = 0 \Rightarrow (x_4 + x_2)(x_4 - x_3) \neq 0$$

let

$$\begin{cases} \frac{x_4 + x_2}{p_1 + p_2} = \frac{p_1 + p_2 - 4b}{p_1 + p_2} = 1 - \frac{4b}{p_1 + p_2} \\ \frac{p_1 + p_2}{x_4 + x_2} = \frac{x_4 + x_2 + 4b}{x_4 + x_2} = 1 + \frac{4b}{x_4 + x_2} \end{cases}$$

we pose

$$\begin{cases} 2k + 1 = 1 - \frac{4b}{p_1 + p_2} \\ 2k' + 1 = 1 + \frac{4b}{x_4 + x_2} \end{cases}$$

let

$$\begin{cases} \frac{x_4 - x_3}{p_1 - p_2} = \frac{p_1 - p_2 - 4b}{p_1 - p_2} = 1 - \frac{4b}{p_1 - p_2} \\ \frac{p_1 - p_2}{x_4 - x_3} = \frac{x_4 - x_3 + 4b}{x_4 - x_3} = 1 + \frac{4b}{x_4 - x_3} \end{cases}$$

we pose

$$\begin{cases} 2m + 1 = 1 - \frac{4b}{p_1 - p_2} \\ 2m' + 1 = 1 + \frac{4b}{x_4 - x_3} \end{cases}$$

$\Rightarrow y = \frac{p_1 + 3p_2}{4} = \frac{p_1 + p_2}{2}$ then $p_1 = p_2 = y$ or $\frac{p_1 + p_2}{2} = \frac{p_1 + p_2}{4} \Rightarrow p_1 = p_2 = 0$ and it is impossible, thus

$$(x_1 - x_2)(x_1 + x_3) \neq 0$$

and

$$b = 0$$

The primal radius r here is equal to

$$r = \frac{p_1 + p_2}{2}$$

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with

$$x = \frac{p_1 - p_2}{2}$$

then

$$x + r = p_1$$

and

$$r - x = p_2$$

Its existence is proved. p_1 and p_2 are an infinity of couples of primes, because we did not specify any condition on them.

Conclusion

The conclusion is that Goldbach conjecture and de Polignac conjecture, which seem so inaccessible, are in fact true. Because, we have defined the primal radius. The conjecture of the primal radius seems to be a consequence of Goldbach and de Polignac conjectures. In fact, if we had not its concept in mind, there would not be the present proof of those conjectures.

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