Orthogonality and Dimensionality

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September 23, 2013

Abstract

In this article, we present what we believe to be a simple way to motivate the use of Hilbert spaces in quantum mechanics. To achieve this, we study the way the notion of dimension can, at a very primitive level, be defined as the cardinality of a maximal collection of mutually orthogonal elements (which, for instance, can be seen as spatial directions). Following this idea, we develop a formalism based on two basic ingredients, namely an orthogonality relation and matroids which are a very generic algebraic structure permitting to define a notion of dimension.

Having obtained what we call *orthomatroids*, we then show that, in high enough dimension, the basic ingredients of orthomatroids (more precisely the simple and irreducible ones) are isomorphic to generalized Hilbert lattices, so that the latter are a direct consequence of an orthogonality-based characterization of dimension.

1 Introduction

In axiomatic formulations of quantum mechanics, and especially those based on orthodox quantum logic, a central element, stated in a rather *ad hoc* manner by Mackey as its seventh axioms [8], is the fact that

> "The partially ordered set (or poset) of all questions in quantum mechanics is isomorphic to the poset of all closed subspaces of a [] Hilbert space."

This requirement can be divided into two properties: first that the poset of all questions is an orthomodular lattice, and that this orthomodular lattice is indeed isomorphic to the lattice of all closed subspaces of a Hilbert space.

Both properties have already been studied extensively. For instance, regarding the necessity of having an orthomodular lattice, Beltrametti et Cassinelli [1] suggest that orthomodularity corresponds to the survival of a notion of the logical conditional, which takes the place of the classical implication associated with Boolean algebra. Other attempts to justify the orthomodularity of the poset of questions can be found in [5] where the key idea, inspired from [12], is that

"There is a maximum amount of relevant information that can be extracted from a system.",

or in [2] where it is assumed that the considered lattice allows the definition of sufficiently many "points of view".

Regarding the second requirement, Piron's theorem provides conditions for an orthomodular lattice to be isomorphic to the lattice of all closed subspaces of a Hilbert space (or, more precisely, to a *generalized* Hilbert space, that is where it is not required that the used field is a "classical" one) [10, 11, 13].

In this article, we propose a different and simple approach to justify the use of a lattice isomorphic to the closed subspaces of a (generalized) Hilbert space, by considered two basic elements: the notion of dimension, and that of orthogonality. The notion of dimension, or similarly of degree of freedom, is central in physics and is a starting point of many axiomatic approaches of quantum mechanics. For instance, in [6], the assumption that one can define a *number of degrees of freedom* K, is made prior to the expression of the *five reasonable axioms* of the article. In [12], Postulate 1, which says:

"Postulate 1 (Limited information). There is a maximum amount of relevant information that can be extracted from a system."

can be interpreted as the existence of the dimension of a system.

The most general and abstract mathematical framework for dealing with dimension is given by the *theory of matroids* [7, 9] which is based on a general notion of independence (such as the linear independence of a basis in linear algebra). There exists several (although equivalent) ways to define a matroid, by providing for instance the collection of bases, or of independent families. In the following, we will use a third formulation, relying on closure operators, which leads us to the second basic element of our approach, that of orthogonality.

This notion arises naturally in Hilbert spaces by considering pairs of vectors which inner product is equal to zero. However, when considering the orthogonality relation only, all the machinery of linear and bilinear is not necessary, and one can (and we will) simply consider an orthogonality relation as being a symmetric and anti-reflexive relation. And even with such a simple definition, it is possible to define a notion of dimension, as the cardinal of a maximal collection of mutually orthogonal elements (in the case of vectors, we consider, of course, non-zero vectors).

In the following, we will use as primitive notion an orthogonality relation which is simply defined as a symmetric and anti-reflexive relation (the latter expressing that an element cannot be orthogonal to itself) and use it to define a dependance relation by stating that an element $x \in E$ depends on a subset $F \in \wp(E)$ if and only if x belongs to the bi-orthogonal closure of F:

$$x \in F^{\perp\perp} \qquad \text{(or, written more compactly, } x \in F^{\perp\!\!\!\perp}\text{)},$$

where $F^\perp = \{x \in E \,|\, \forall\, y \in F,\, x \perp y\}$. Equivalently, an subset F is independent if, and only if

$$\forall\, x \in F,\, x \notin (F \setminus \{x\})^{\perp\!\perp}.$$

Matroid theory ensures, through the *MacLane–Steinitz exchange property*, that all the bases (that is, all the maximal independant sets) of a given matroid have the same cardinality and that common cardinality will provide our definition of dimension.

We have mentioned earlier bases made of mutually orthogonal elements. They constitute a rather interesting class of bases to consider, and will be called *orthobases* in the following. Moreover, we will ensure that such bases do exist, and will even demand the additional requirement that every family of mutually orthogonal elements can be completed into an orthobasis. If elements of E are seen as spatial directions, this can be interpreted as a direct consequence of isotropy.

The rest of this article will be divided as follows: first, we will provide a formal definition of the structure we have described, in order to obtain what we call an *orthomatroid*. Then, in a second part, we will focus on the lattice associated to an orthomatroid and we will present a representation theorem for orthomatroids, based on Piron's theorem.

2 Orthomatroids

We first define the basic elements of our approach.

Definition 1 *Given a set E, a binary relation $\perp$ on E is an* orthogonality relation *if, and only if it verifies*

$$\begin{aligned} &\forall\, a, b \in E,\, a \perp b \iff b \perp a && \textit{Symmetry} \\ &\forall\, a, b \in E,\, a \perp b \implies a \neq b && \textit{Anti-reflexivity} \end{aligned}$$

In the following, such a pair $(E, \perp)$ will be called an orthoset.

Definition 2 *Given an orthoset $(E, \perp)$ and a subset $F \in \wp(E)$, we define its* orthogonal complement $F^\perp$ *as*

$$F^\perp = \{x \in E \,|\, \forall\, y \in F,\, x \perp y\},$$

and its (bi-orthogonal) closure $F^{\perp\!\perp} = \left(F^\perp\right)^\perp$.

Proposition 1 *Given an orthoset $(E, \perp)$, we have*

$$\begin{aligned} &\forall\, F \in \wp(E),\, F \cap F^\perp = \emptyset \\ &\forall\, F, G \in \wp(E),\, G \subseteq F^\perp \iff F \subseteq G^\perp && \text{(GC)} \end{aligned}$$

Proof For $F \in \wp(E)$ and $x \in F$, having $x \in F^\perp$ would imply that $\forall\, y \in F,\ x \perp y$ and, in particular, $x \perp x$ which is not possible. As a consequence, $F \cap F^\perp = \emptyset$. Now, (GC) trivially follows from the symmetry of our orthogonality relation, since

$$G \subseteq F^\perp \iff \forall\, x \in F, \forall\, y \in G,\ x \perp y \iff F \subseteq G^\perp.$$

□

This property shows that there is an antitone Galois connection [3, 4] between E and itself, realized both ways by $\cdot^\perp : F \mapsto F^\perp$. A direct consequence of this is the following result.

Proposition 2 *The bi-orthogonal closure operation is a closure operator on E, that is it verifies*

$$\begin{aligned} \forall\, F \in \wp(E),\ & F \subseteq F^{\perp\perp} && \text{Extensivity} \\ \forall\, F, G \in \wp(E),\ & F \subseteq G \implies F^{\perp\perp} \subseteq G^{\perp\perp} && \text{Monotony} \\ \forall\, F \in \wp(E),\ & \left(F^{\perp\perp}\right)^{\perp\perp} = F^{\perp\perp} && \text{Idempotence} \end{aligned}$$

Proof As stated above, this is a direct consequence of having a Galois connection. However, the direct proof of those facts is rather easy. Extensivity trivially follows from (GC), since

$$F \subseteq F^{\perp\perp} \iff F^\perp \subseteq F^\perp.$$

Now, if $F \subseteq G$, then $F \subseteq G^{\perp\perp}$ and hence $G^\perp \subseteq F^\perp$. By applying this deduction once again, we obtain

$$F \subseteq G \implies G^\perp \subseteq F^\perp \implies F^{\perp\perp} \subseteq G^{\perp\perp}.$$

Finally, for idempotence, we only need to prove that $\left(F^{\perp\perp}\right)^{\perp\perp} \subseteq F^{\perp\perp}$ which is equivalent, because of (GC), to $F^\perp \subseteq \left((F^{\perp\perp})^{\perp\perp}\right)^\perp = \left((F^\perp)^{\perp\perp}\right)^{\perp\perp}$. □

Following our initial discussion, we now want to turn an orthoset $(E, \perp)$ into a matroid. Since we have defined a closure operation on E using our orthogonality relation, it is natural to use the closure operator-based definition of a matroid. In that case, we only need to demand that the Mac Lane–Steinitz Exchange Property is verified by $\cdot^{\perp\perp}$:

$$\forall\, F \in \wp(E), \forall\, x, y \in E,\ x \in (F + y)^{\perp\perp} \setminus F^{\perp\perp} \implies y \in (F + x)^{\perp\perp}.$$

Here, $F + x$ denotes the set $F \cup \{x\}$. With this definition, a subset F of E is independent if, and only if (where $F - x$ denotes $F \setminus \{x\}$)

$$\forall\, x \in F,\ x \notin (F - x)^{\perp\perp}.$$

Let us now focus on subsets made of mutually orthogonal elements.

Definition 3 *A subset F of E is said to be* orthoindependent *if, and only if*

$$\forall\, x, y \in F,\ x \neq y \implies x \perp y.$$

Obviously, every orthoindependent subset is independent. Moreover, orthoindependent subsets verify the nice following property:

Proposition 3 *If $\{F_i\}$ is a chain of orthoindependent subsets, then $\bigcup F_i$ is also an orthoindependent subset.*

By application of Zorn's Lemma, the previous proposition implies that there exists maximal orthoindependent subsets and even that every orthoindependent subset is included in a maximal one.

An interesting question regarding orthoindependent subsets is whether, given a closed subset $F^{\perp\perp}$, there exists an orthoindependent subset I of $F^{\perp\perp}$ such that $I^{\perp\perp} = F^{\perp\perp}$ (in which case we would call I an *orthobasis* of $F^{\perp\perp}$).

As explained in the introduction, we will demand that this is actually the case, that every closed subset $F^{\perp\perp}$ of E admits an orthobasis. We even want to demand the stronger condition that every orthoindependent subset I of $F^{\perp\perp}$ can be completed into an orthobasis O of $F^{\perp\perp}$, which we formally state as

> (OB) Given a subset F of E and an orthoindependent subset I of $F^{\perp\perp}$, there exists an orthoindependent subset J such that $I \subseteq J$ and $J^{\perp\perp} = F^{\perp\perp}$.

Proposition 4 *If $(E, \perp)$ verifies the MacLane – Steinitz Exchange Property, then axiom (OB) is equivalent to the* Straightening Property*, which we define as*

$$\forall\, F \in \wp(E),\ \forall\, x \in E,\ x \notin F^{\perp\perp} \implies \exists\, y \in F^{\perp} : x \in (F + y)^{\perp\perp}.$$

Proof Suppose first that (OB) holds, and let $F \in \wp(E)$ and $x \in E$ be such that $x \notin F^{\perp\perp}$. Moreover, let B be an orthobasis of $F^{\perp\perp}$ which we extend into an orthobasis B' of $(F + x)^{\perp\perp}$, using (OB). For $y \in B' \setminus B$, it is clear from the orthoindependence of B' that $y \in F^{\perp}$. Finally, $y \in (F+x)^{\perp\perp} \setminus F^{\perp\perp}$ so that using the Exchange Property, we get $x \in (F + y)^{\perp\perp}$.

Conversely, suppose that the Straightening Property is verified. It can be remarked that, because of the Exchange Property, it can be equivalently stated as

$$\forall\, F \in \wp(E),\ \forall\, x \in E,\ x \notin F^{\perp\perp} \implies \exists\, y \in F^{\perp} : (F + x)^{\perp\perp} = (F + y)^{\perp\perp}.$$

Now, given an orthoindependent subset I of $F^{\perp\perp}$, let J be a maximal orthoindependent subset of $F^{\perp\perp}$ such that $I \subseteq J$. If $J^{\perp\perp} \neq F^{\perp\perp}$, then there exists an element $x \in F^{\perp\perp} \setminus J^{\perp\perp}$ and, following from the Straightening Property, there exists an element $y \in J^{\perp}$ such that $(J + x)^{\perp\perp} = (J + y)^{\perp\perp}$. This implies in particular that $y \in F^{\perp\perp}$ and hence $J + y$ is also an orthoindependent subset of $F^{\perp\perp}$. But this is absurd, since J was supposed maximal. As a consequence, we have $J^{\perp\perp} = F^{\perp\perp}$. □

We now summarize all these properties into what we call an *orthomatroid*:

Definition 4 (Orthomatroid) *An* orthomatroid *is an orthoset* $(E, \perp)$ *which verifies the two following properties :*

1. *Exchange Property*

$$\forall F \in \wp(E),\, \forall x, y \in E,\, x \in (F + y)^{\perp\perp} \setminus F^{\perp\perp} \implies y \in (F + x)^{\perp\perp}$$

2. *Straightening Property*

$$\forall F \in \wp(E),\, \forall x \in E,\, x \notin F^{\perp\perp} \implies \exists y \in F^{\perp} : x \in (F + y)^{\perp\perp}$$

Moreover, we will say that two orthomatroids $(E_1, \perp_1)$ and $(E_2, \perp_2)$ are *orthoisomorphic* if there exists a bijection $\varphi : E_1 \to E_2$ such that

$$\forall x, y \in E_1,\, x \perp_1 y \iff \varphi(x) \perp_2 \varphi(y).$$

We believe that orthomatroids provide a reasonable answer to the initial objective of formalizing a notion of orthogonality-based dimension. Here, the Exchange Property ensures that a correct notion of dimension can be defined: given a independent subset I, every independent subset J verifying $I^{\perp\perp} = J^{\perp\perp}$ has the same cardinality as I (which we call the rank of the orthomatroid). Moreover, considering orthoindependent subsets (which can be seen some sort of "preferred" independent subsets), the Straightening Property ensures that every closed subset admits an orthobasis and even that every orthoindependent subset can be extended into an orthobasis.

In the next section, we will study some properties of the lattices associated to orthomatroids, and provide a representation theorem for orthomatroids.

3 The Lattice associated to an Orthomatroid

Let us first remark that $\emptyset^{\perp} = E$ and $E^{\perp} = \emptyset$ (due to the anti-reflexivity of $\perp$), so that $\emptyset$ is closed. Moreover, for $x, y \in E$, if $x \in \{y\}^{\perp\perp}$, then $\{x\}^{\perp\perp} = \{y\}^{\perp\perp}$. By defining the relation $x \equiv y$ if and only if $\{x\}^{\perp\perp} = \{y\}^{\perp\perp}$, one clearly has an equivalence relation on E compatible with $\perp$:

$$x \equiv y \implies \forall z \in E,\, x \perp z \iff y \perp z.$$

An orthomatroid $(E, \perp)$ is said to be *simple* if $\forall x \in E$, $\{x\}^{\perp\perp} = \{x\}$ (this corresponds to the usual notion of simplicity for matroids, that is there are no loops and no parallel elements). Every orthomatroid can be *simplified* into a simple orthomatroid by considering its quotient by the previous equivalence relation.

Definition 5 *Given an orthomatroid* $M = (E, \perp)$*, we define the associated lattice* $\mathcal{L}_M$ *as the set* $\{F^{\perp\perp} \mid F \in \wp(E)\}$ *of its subspaces ordered by inclusion.*

Given an orthomatroid $M = (E, \perp)$ and its simplification $M_{/\equiv} = (E_{/\equiv}, \perp_{/\equiv})$, the associated lattices $\mathcal{L}_M$ and $\mathcal{L}_{M_{/\equiv}}$ are isomorphic. As a consequence, we will, without loss of generality, suppose that any lattice associated to an orthomatroid is indeed associated to a simple orthomatroid.

Since $\mathcal{L}_M$ is defined as the set of closed elements of a closure operator, it is a complete lattice, with operations

$$P \wedge Q = P \cap Q \qquad P \vee Q = (P^\perp \cap Q^\perp)^\perp = (P \cup Q)^{\perp\!\perp}.$$

It is also clearly atomistic (meaning that every element is the join of its atoms), and an ortholattice with $\cdot^\perp$ as orthocomplementation.

We now present two important properties that are verified by $\mathcal{L}_M$.

Proposition 5 (Orthomodularity) *The ortholattice $\mathcal{L}_M$ is orthomodular, that is it verifies*

$$\forall\, P, Q \in \mathcal{L}_M,\ P \leq Q \implies P = Q \wedge (P \vee Q^\perp).$$

Proof Let P and Q be in $\mathcal{L}_M$ such that $P \leq Q$. Obviously, $P \leq Q \wedge (P \vee Q^\perp)$. Conversely, let x be in $Q \wedge (P \vee Q^\perp)$ and suppose that $x \notin P$. One can then define $y \in P^\perp$ such that $(P + y)^{\perp\!\perp} = (P + x)^{\perp\!\perp}$. In particular, $y \in Q$ so that $y \in Q \wedge P^\perp$ and $y \in (Q \wedge P^\perp) \vee Q^\perp = \big(Q \wedge (P \vee Q^\perp)\big)^\perp$. Since $x \in Q \wedge (P \vee Q^\perp)$, this implies that $y \perp x$. But then, from $y \in P^\perp$ and $y \perp x$, one can deduce that $y \in (P + x)^\perp$ which is absurd since $y \in (P + x)^{\perp\!\perp}$. As a consequence, we have shown that $\forall\, x \in E$, $x \in Q \wedge (P \vee Q^\perp) \implies x \in P$ and finally that $P = Q \wedge (P \vee Q^\perp)$. □

Proposition 6 (Atom-covering) *The ortholattice $\mathcal{L}_M$ verifies the atom-covering property, that is for $F \in \wp(E)$ and $x \in E$, if $x \notin F^{\perp\!\perp}$, then $(F + x)^{\perp\!\perp}$ covers $F^{\perp\!\perp}$.*

Proof Let G be such that $F^{\perp\!\perp} < G^{\perp\!\perp} \leq (F + x)^{\perp\!\perp}$ with $x \notin F^{\perp\!\perp}$. We have to show that $G^{\perp\!\perp} = (F + x)^{\perp\!\perp}$. But since $F^{\perp\!\perp} < G^{\perp\!\perp}$, we can define $y \in G^{\perp\!\perp} \setminus F^{\perp\!\perp}$. We then have

$$(F + y)^{\perp\!\perp} \leq G^{\perp\!\perp} \leq (F + x)^{\perp\!\perp}.$$

But then, $y \in (F + x)^{\perp\!\perp} \setminus F^{\perp\!\perp}$ which implies that $(F + x)^{\perp\!\perp} = (F + y)^{\perp\!\perp}$ and finally that $G^{\perp\!\perp} = (F + x)^{\perp\!\perp}$. □

This shows that if M is an orthomatroid, then its associated lattice $\mathcal{L}_M$ is a complete atomistic orthomodular lattice that satisfies the covering law or, following Piron's terminology [10, 11], $\mathcal{L}_M$ is a *propositional system*. This algebraic structure plays an important role in the study of the mathematical foundations of quantum mechanics, with the following central result:

Theorem 1 (Pirons Representation Theorem) *Every irreducible propositional system of rank at least 4 is ortho-isomorphic to the lattice of (biorthogonally) closed subspaces of a generalized Hilbert space.*

Let us recall that a generalized Hilbert space $(\mathcal{H}, \mathbf{K}, \cdot^\star, \langle \cdot \,|\, \cdot \rangle)$ consists in a vector space $\mathcal{H}$ over a field $\mathbf{K}$ with an involution anti-automorphism $\cdot^\star : \alpha \in \mathbf{K} \mapsto \alpha^\star$ and an orthomodular Hermitian form $\langle \cdot \,|\, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{K}$ satisfying

$$\begin{gathered}\forall\, x, y, z \in \mathcal{H},\ \forall\, \lambda \in \mathbf{K},\ \langle \lambda x + y | z \rangle = \lambda \langle x | z \rangle + \langle y | z \rangle, \\ \forall\, x, y \in \mathcal{H},\ \langle x | y \rangle = \langle y | x \rangle^\star, \\ \forall\, S \in \wp(\mathcal{H}),\ S^\perp \oplus S^{\perp\perp} = \mathcal{H}\end{gathered}$$

where, obviously, $S^\perp = \big\{ x \in \mathcal{H} \;\big|\; \forall\, y \in S,\ \langle x | y \rangle = 0 \big\}$. We invite the reader to consult [13] for more informations.

If we consider the set $A(\mathcal{H})$ of vector rays of $\mathcal{H}$, and define the "natural" orthogonality relation $\perp_{\mathcal{H}}$ on $A(\mathcal{H})$ by

$$\mathbf{K}u \perp_{\mathcal{H}} \mathbf{K}v \iff \langle u | v \rangle = 0,$$

we obtain an orthomatroid:

Proposition 7 *If $\big(\mathcal{H}, \mathbf{K}, \cdot^\star, \langle \cdot \,|\, \cdot \rangle\big)$ is a generalized Hilbert space, then $(A(\mathcal{H}), \perp_{\mathcal{H}})$ is an orthomatroid, where $A(\mathcal{H}) = \big\{ \mathbf{K}u \;\big|\; u \in \mathcal{H} \setminus \{0\} \big\}$ and $\mathbf{K}u \perp_{\mathcal{H}} \mathbf{K}v \iff \langle u | v \rangle = 0$.*

Proof All we have to do is prove that the Exchange and Straightening properties are verified.

Regarding the Straightening Property, let $F \in \wp\big(A(\mathcal{H})\big)$ and $x = \mathbf{K}u$ be in $A(\mathcal{H}) \setminus F^{\perp\perp}$. Since $F^{\perp\perp} \oplus F^\perp = \mathcal{H}$, one can write $u = v + w$ with $v \in F^{\perp\perp}$ and $w \in F^\perp$. Since $x \notin F^{\perp\perp}$, one has $w \neq 0$ so that one can define $y = \mathbf{K}w \in A(\mathcal{H})$. Clearly, $y \in (F + x)^{\perp\perp}$ and $(F + y)^{\perp\perp} \subseteq (F + x)^{\perp\perp}$. If $\mathbf{K}t \in (F + y)^\perp$, then $\langle v | t \rangle = \langle u | t \rangle = 0$ so that $x \perp \mathbf{K}t$, thus showing that $x \in (F + y)^{\perp\perp}$ and finally that $(F + x)^{\perp\perp} = (F + y)^{\perp\perp}$.

Now, let $x = \mathbf{K}u$ and $y = \mathbf{K}v$ be such that $x \in (F + y)^{\perp\perp} \setminus F^{\perp\perp}$ and suppose, without loss of generality, that both x and y lie in $F^\perp$. If we define $\alpha = \frac{\langle u | v \rangle}{\langle v | v \rangle}$ and $w = u - \alpha v$, one has $\langle w | v \rangle = 0$ and $w \in F^\perp$ so that $w \in (F + y)^\perp$. But since $x \in (F + y)^{\perp\perp}$, one has $\langle w | u \rangle = 0$. As a consequence,

$$\langle w | w \rangle = \langle w | w + \alpha v \rangle - \alpha \langle w | v \rangle = \langle w | u \rangle - \alpha \langle w | v \rangle = 0,$$

and hence $w = 0$ (since an orthomodular Hermitian form is necessarily anisotropic [13]). Finally, this means that $x = y$, and thus $y \in (F + x)^{\perp\perp}$. □

Finally, in order to translate Piron's Representation Theorem in terms of orthomatroids, we need to define the notion of irreducibility. Following [13]

again, given a propositional system $\mathcal{P}$, the binary relation $\sim$ defined on the set $A(\mathcal{P})$ of atoms of $\mathcal{P}$ by

$$\forall\, x, y \in A(\mathcal{P}),\ x \sim y \iff \big(x \neq y \implies \exists\, z \in A(\mathcal{P}) \setminus \{x, y\} : z \leq x \vee y\big)$$

is an equivalence relation on $A(\mathcal{P})$. The equivalence classes of $A(\mathcal{P})$ are then called irreducible components of $A(\mathcal{P})$, and $\mathcal{P}$ is said to be irreducible if it has a single irreducible component.

In terms of orthomatroids, given a simple orthomatroid $(E, \perp)$, the previous equivalence relation can be reexpressed as

$$\forall\, x, y \in E,\ x \sim y \iff \mathrm{Card}\{x, y\}^{\perp\!\!\!\perp} \neq 2.$$

If we denote the irreducible components of E by $\big\{(E_i, \perp_i)\big\}_{i \in \mathcal{I}}$ (with $\perp_i$ being the restriction of $\perp$ to E_i), then $(E, \perp)$ can be seen as the disjoint union of its irreducible components (up to orthoisomorphism):

$$E = \biguplus_{i \in \mathcal{I}} E_i = \bigcup_{i \in \mathcal{I}} \big\{(i, x) \;\big|\; x \in E_i\big\}$$
$$(i, x) \perp (j, y) \iff \big(i \neq j \text{ or } (i = j \text{ and } x \perp_i y)\big)$$

We are now able to translate Piron's representation theorem in terms of orthomatroids, and we finally obtain the following representation theorem:

Theorem 2 (Representation of Orthomatroids) *Every simple and irreducible orthomatroid $(E, \perp)$ of rank at least 4 is orthoisomorphic to the orthomatroid $(A(\mathcal{H}), \perp_{\mathcal{H}})$ associated to a generalized Hilbert space $(\mathcal{H}, \mathbf{K}, \cdot^{\star}, \langle\, \cdot \,|\, \cdot \,\rangle)$.*

4 Conclusion

In this article, we have presented a formalism based on the idea that dimension can, at a very primitive level, be defined as the cardinality of a maximal collection of mutually orthogonal elements (which, for instance, can be seen as spatial directions). This formalism is based on two basic ingredients, namely an orthogonality relation and matroids which are a very generic algebraic structure permitting to define a notion of dimension.

Having obtained what we call *orthomatroids*, we then have shown that, in high enough dimension, the basic ingredients of orthomatroids (more precisely the simple and irreducible ones) are isomorphic to generalized Hilbert lattices.

Let us insist on the fact that in this approach, the use of generalized Hilbert lattices is derived from extremely simple and generic geometric assumptions. We believe that this result shows that instead of seeing the use of Hilbert lattices in quantum physics as puzzling, they (or, at least, generalized Hilbert lattices) should be seen as the most general (if not natural) way to model situations where a dimension can be defined in terms of orthogonality, such as in quantum mechanics. To that respect, Mackey's seventh axioms appear to be much less *ad hoc*.

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