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# Retraction obstruction to time-varying stabilization

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## Abstract

This paper address the problem of the global stabilization on a total space of a fiber bundle with a compact base space. We prove that, under mild assumptions (existence of a continuous section and forward unicity of solutions), no equilibrium of a continuous system defined on such a state space can be globally asymptotically uniformly stabilized using continuous time-varying feedback.

*Key words:* Retraction obstruction, Lyapunov stability, stabilization.

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## 1 Introduction

Topological obstruction to stabilization is a long standing problem of control theory detailed in the introduction of [1]. There are two main topological obstructions to continuous stabilization. First, the *Brockett obstruction* is a local obstruction related to the structure of the controlled systems involving the nature of feedback controls [2]. Then, the *retraction obstruction* is a global obstruction related to the structure of the underlying state space: if the state space of the system has the structure of a vector bundle over a compact manifold, no continuous *static* feedback can globally stabilize an equilibrium. This result has been proved and its consequences studied in detail in [3].

The Brockett condition, which is necessary in the case of continuous time-invariant feedback controls, does not remain necessary for driftless controllable systems with time-varying feedback. The existence of such feedbacks has been proved in [4], while [5] gave an explicit de-

sign under an additional condition on the Control Lie Algebra (see [5, Assumption 1]). A natural question is, hence, to wonder whether continuous time-varying feedback controls could also avoid the retraction obstruction as suggested in the introduction of [6].

In [3, Remark 1], the authors mention that their result also handle the case of dynamic feedback. Indeed a dynamic feedback is usually seen as a dynamic extension where the augmented state is stable. Estimation of parameters or observer-based control are in this scope. In that case, the result of [3] is applicable directly, with the method exposed in their remark. Nevertheless, a time-varying static feedback is also a dynamic extension using a timer  $\dot{\tau} = 1$ . However in this situation, there is no convergence to a single equilibrium point, but to a submanifold; the result of [3] is therefore not applicable in this context, following [3, Remark 1].

The aim of this paper is to prove that, in the second case, the obstruction still remains: a time-varying feedback control which is globally asymptotically uniformly stabilizing a system defined on a fiber bundle with a compact manifold as its base space does not exist.

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## 2 Retraction obstruction

By a *manifold* we mean a smooth, positive dimensional, connected manifold without boundary. The definition of a *fiber bundle* is given for instance in [7].

Our purpose is to link the topological property of contractibility of a manifold to the existence of a globally asymptotically stable equilibrium. Let us introduce the definitions we will be using and some useful properties.

**Definition 1** *Let  $E$  be a topological space and  $x_0 \in E$ . A retraction of  $E$  on  $x_0$  is a continuous mapping  $h : [0, 1] \times E \rightarrow E$  such that for all  $x \in E$ ,  $h(0, x) = x$  and  $h(1, x) = x_0$ . A topological space  $E$  is said to be contractible if there exists a retraction of  $E$ .*

**Proposition 2** [8, Section 2.4] *No compact manifold is contractible.*

Consider a controlled system defined by

$$\dot{y} = f_0(y, u) \quad y \in \mathcal{N}, u \in \mathcal{U}, \quad (1)$$

with  $\mathcal{N}$  a manifold and  $\mathcal{U}$  a set of admissible controls, and where  $f_0$  is a continuous vector field.

We wonder about the global stabilizability of the system via a time-varying feedback  $u(y, t)$ . To do that, we will set  $\dot{\tau} = 1$  and look at the partial asymptotic stability of the closed-loop system:

$$\begin{aligned} \dot{y} &= f_0(y, u(y, \tau)) \\ \dot{\tau} &= 1 \end{aligned} \quad (2)$$

Hence, let us introduce the definitions of partial stability we will be using. Those definitions are adapted from [9].

**Definition 3** *Let  $\mathcal{M} = \mathcal{N} \times \mathcal{T}$  be a product manifold. Consider  $f = (f_1, f_2)$  a forward complete continuous vector field on  $\mathcal{M}$  with the property of unicity of solutions in forward time. We denote by  $\Phi$  the semiflow of  $f$  and  $p_1$  the canonical projection on  $\mathcal{N}$ .*

- (1) *We say that  $y_\infty \in \mathcal{N}$  is a partial equilibrium if for all  $\tau \in \mathcal{T}$ , we have  $f_1(y_\infty, \tau) = 0$ .*
- (2) *A partial equilibrium  $y_\infty \in \mathcal{N}$  is said to be partially stable uniformly in  $\tau$  if for all  $U \subset \mathcal{N}$  neighborhood of  $y_\infty$  there exists  $V \subset \mathcal{N}$  a neighborhood of  $y_\infty$  such that for all  $y \in V$  and for all  $\tau \in \mathcal{T}$ ,  $p_1 \circ \Phi(t, (y, \tau)) \in U$  for all  $t \geq 0$ .*
- (3) *A partial equilibrium  $y_\infty \in \mathcal{N}$  is said to be partially globally asymptotically stable uniformly in  $\tau$  if it is partially stable uniformly in  $\tau$  and if for all  $(y, \tau) \in \mathcal{M}$  we have  $p_1 \circ \Phi(t, (y, \tau)) \rightarrow y_\infty$  when  $t \rightarrow +\infty$ .*

**Remark 4** *The last item of definition 3 is slightly different from the more standard ones. Indeed, to prove our result, we only need the stability to be uniform with respect to  $\tau$ . The uniformity with respect to  $\tau$  of the convergence, which is usually required, is not necessary here.*

The following definition, inspired by [10, Chapter 12] about time-varying stabilizability, is here given in the partial stability context.

**Definition 5** *The system (1) is said to be globally asymptotically uniformly stabilizable by means of a continuous generalized time-varying feedback if there exist a point  $y_\infty \in \mathcal{N}$ , a manifold  $\mathcal{T}$ , a continuous mapping  $f_2 : \mathcal{N} \times \mathcal{T} \rightarrow \mathbb{T}\mathcal{T}$  with  $f_2(y, \tau) \in \mathbb{T}_\tau\mathcal{T}$  for all  $y \in \mathcal{N}$  and all  $\tau \in \mathcal{T}$  and a continuous control law  $u(y, \tau)$  such that  $y_\infty$  is a partial equilibrium of the closed loop system  $(\dot{y}, \dot{\tau}) = (f_0(y, u(y, \tau)), f_2(y, \tau))$  and is partially globally asymptotically stable uniformly in  $\tau$ .*

**Remark 6** *Let us note that, taking  $\mathcal{T} = \mathbb{R}$  and  $f_2(y, \tau) = 1$ , this definition boils down to the definition of global asymptotic stabilization by means of a continuous time-varying feedback. In the generalized time-varying setting, the variable  $\tau$  can be stable or not, scalar or vector, bounded or not.*

In [3, Theorem 1], the authors use Proposition 2 to prove that if a manifold  $\mathcal{N}$  admits a structure of fiber bundle over a compact manifold, then no continuous vector field over  $\mathcal{N}$  can have a unique globally asymptotically stable equilibrium. Hence, they conclude that the system (1) cannot be globally asymptotically stabilized by means of a state feedback. Let us prove now that [3, Theorem 1] can be extended in the following way to the time-varying setting.

**Theorem 7** *Assume that  $\mathcal{N}$  is a manifold with a structure of fiber bundle over a compact manifold  $\mathcal{Q}$ . If there exists a continuous section of the bundle, then the system (1) is not globally asymptotically uniformly stabilizable by means of a continuous generalized time-varying feedback in such a way that the augmented vector field has the forward unicity of solutions property.*

**Proof.** *Ad absurdum*, assume that there exists a continuous dynamic feedback which globally asymptotically uniformly stabilizes the system (1) in such a way that the augmented vector field has the forward unicity of solutions. Let us denote by  $\tau$  the added variable, and  $\tau \in \mathcal{T}$ . We have  $\dot{\tau} = f_2(y, \tau)$ , and the system (1) can be rewritten in an extended form, with  $f_1(y, \tau) = f_0(y, u(y, \tau))$ :

$$\begin{pmatrix} \dot{y} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} f_1(y, \tau) \\ f_2(y, \tau) \end{pmatrix}. \quad (3)$$

The equation (3) defines a continuous vector field  $f$  on the manifold  $\mathcal{M} = \mathcal{N} \times \mathcal{T}$ , with the forward unicity

of solutions property. Moreover, there exists a partially globally asymptotically stable equilibrium uniformly in  $\tau$  denoted by  $y_\infty \in \mathcal{N}$ .

Let us denote  $\pi_0 : \mathcal{N} \rightarrow \mathcal{Q}$  the fiber bundle projection. Set  $p_1 : \mathcal{M} \rightarrow \mathcal{N}$  the first canonical projection and set  $\pi = \pi_0 \circ p_1$ . We denote  $q_\infty = \pi_0(y_\infty)$ . Similarly, fix  $\tau_0$  in  $\mathcal{T}$  and set  $\sigma(q) = (\sigma_0(q), \tau_0)$  where  $\sigma_0 : \mathcal{Q} \rightarrow \mathcal{N}$  is a continuous section of  $\pi_0$ . Clearly,  $\sigma$  is a continuous section of  $\pi$ . We also note that the manifold  $\mathcal{M}$  trivially inherits a structure of fiber bundle over  $\mathcal{Q}$  with projection  $\pi$ . The vector field  $f$  is continuous and has the forward unicity of solutions property. Therefore, it admits a semiflow  $\Phi$ [11]. Let us denote:

$$h : [0, 1] \times \mathcal{Q} \rightarrow \mathcal{Q}$$

$$(\lambda, q) \mapsto \begin{cases} \pi \circ \Phi \left( \ln \left( \frac{1}{1-\lambda} \right), \sigma(q) \right) & \text{if } \lambda \neq 1 \\ q_\infty & \text{if } \lambda = 1 \end{cases}$$

Since we clearly have  $h(0, q) = q$  and  $h(1, q) = q_\infty$ , let us prove the continuity of  $h$ . This mapping is obviously continuous on  $[0, 1) \times \mathcal{Q}$ .

Let us show the continuity at  $(1, q)$  for  $q \in \mathcal{Q}$ . Let  $(\lambda_n, q_n) \in [0, 1) \times \mathcal{Q}$  be a sequence of points converging to  $(1, q)$ . We set

$$t_n = \ln \left( \frac{1}{1-\lambda_n} \right), \quad x_n = \sigma(q_n), \quad x = \sigma(q).$$

We have  $t_n \rightarrow +\infty$  and, by continuity of the section,  $x_n \rightarrow x$ . Let  $U \subset \mathcal{Q}$  be a neighborhood of  $q_\infty$  and  $U_0 = \pi_0^{-1}(U) \subset \mathcal{N}$  the corresponding neighborhood of  $y_\infty$ . By partial stability uniformly in  $\tau$ , there exists  $V_0 \subset \mathcal{N}$  a neighborhood of  $y_\infty$  such that for all  $y \in V_0$ , all  $\tau \in \mathcal{T}$  and all  $t \geq 0$ , we have  $p_1 \circ \Phi(t, (y, \tau)) \in U_0$ .

On the other hand, the partial attractivity of  $y_\infty$  means that  $p_1 \circ \Phi(t, x) \rightarrow y_\infty$  when  $t \rightarrow \infty$ . Thus, there exists  $T > 0$  such that  $p_1 \circ \Phi(T, x) \in V_0$ . By continuity, there exists  $N_1 > 0$  such that for all  $n > N_1$  we have  $p_1 \circ \Phi(T, x_n) \in V_0$ . Therefore, for all  $t \geq T$ , we have  $p_1 \circ \Phi(t, x_n) \in U_0$ . But  $t_n \rightarrow +\infty$ , so there exists  $N_2 > 0$  such that for all  $n > N_2$ ,  $t_n > T$ . Thus, for all  $n > N = \max(N_1, N_2)$ , we have  $p_1 \circ \Phi(t_n, x_n) \in U_0$ . Hence, for all  $n > N$ ,  $h(\lambda_n, q_n) = \pi_0 \circ p_1 \circ \Phi(t_n, x_n) \in U$ . Since  $U$  is an arbitrary neighborhood of  $q_\infty$ , the mapping  $h$  is continuous.

However, the mapping  $h$  defines a retraction of the compact manifold  $\mathcal{Q}$  on  $q_\infty$ , which leads to the expected contradiction thanks to Proposition 2.

■

**Example 8** *Let us consider the following system, defined on the circle:*

$$\ddot{\theta} = u, \quad \theta \in \mathbb{S}^1. \quad (4)$$

*By using the angular velocity  $\omega = \dot{\theta}$  we can rewrite the system as*

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = u \end{cases}. \quad (5)$$

*Here the state space is the tangent space of the circle, denoted by  $\mathbb{TS}^1$ . The tangent space of a manifold has always a structure of vector bundle over that manifold, and therefore  $\mathbb{TS}^1$  has a structure of vector bundle over the compact manifold  $\mathbb{S}^1$  with projection  $\pi_0$  given by:*

$$\pi_0(\theta, \omega) = \theta.$$

*Moreover, being a vector bundle, the tangent bundle admits  $\sigma_0$ , the zero section, as a continuous section. One may wonder if it is possible to design a continuous feedback control  $u(\theta, \omega, t)$  globally stabilizing a state  $(\theta_0, 0)$  such that the closed-loop system has uniqueness of solution in forward time.*

*Since  $\mathbb{S}^1$  is compact, from Theorem 7 the system (5) cannot be globally asymptotically uniformly stabilized.*

Finally, taking into account Theorem 7 and [3, Theorem 1], we have the following result: consider the system (1) defined on a manifold with a structure of fiber bundle over a compact manifold. If there exists a continuous section of the bundle, then no continuous dynamic feedback can globally asymptotically uniformly stabilize the system in such a way that the closed loop system has the forward unicity of solutions.

### 3 Conclusion

This paper extends [3, Theorem 1] to the case of time-varying feedback control. We prove that under mild assumptions, no continuous time-varying feedback control can avoid the retraction obstruction; that is, no continuous time-varying feedback control can globally asymptotically uniformly stabilize an equilibrium on a state space which has a structure of fiber bundle over a compact manifold.

This topological obstruction on compact manifolds prevents us from having continuous globally asymptotically uniformly stabilizing feedback (neither static nor time-varying nor dynamic). Moreover it is proved in [12] that the obstruction still remains for discontinuous autonomous vector fields or differential inclusions. Finally the topological obstruction appears to be a strong constraint on stabilization, and few possibilities remain. First, since the uniformity property of the partial stability is indeed being used in our proof, the possibility of non-uniform global stabilizability still remains open. Second, hybrid feedbacks can be considered as suggested in [12]. Finally, other notions of solutions for discontinuous systems exist; some of them might not inherit the same obstruction to global stabilization.

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