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# Toward Fast Transform Learning

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## Abstract

Dictionary learning is a matrix factorization problem. It aims at finding a dictionary of atoms that best represents an image according to a given objective. A usual objective consists of representing an image or a class of images sparsely which has led to many state-of-the-art algorithms in image processing. In practice, all algorithms performing dictionary learning iteratively estimate the dictionary and a sparse representation of the images using this dictionary. However, the numerical complexity of dictionary learning restricts its use to atoms with a small support since the computations using the constructed dictionaries require too much resources to be deployed for large scale applications.

In order to alleviate these issues, this paper introduces a new strategy to learn dictionaries composed of atoms obtained by translating the composition of  $K$  convolutions with  $S$ -sparse kernels of known support. The dictionary update step associated with this strategy is a non-convex optimization problem. The purpose of the present paper is to study this non-convex optimization problem. We first reformulate the problem to reduce the number of its irrelevant stationary points. A Gauss-Seidel type algorithm, referred to as Alternative Least Square Algorithm, is introduced for its resolution. The search space of the considered optimization problem is of dimension  $KS$ , which is typically smaller than the size of the target atom and is much smaller than the size of the image. Moreover, the complexity of the algorithm is linear with respect to the size of the image, allowing larger atoms to be learned (as opposed to small patches). The conducted experiments show that, when  $K$  is large (say  $K = 10$ ), we are able to approximate with a very good accuracy many atoms such as apodized modified discrete cosines, curvelets, sinc functions or cosines. We also argue empirically that surprisingly the algorithm generally converges to a global minimum for large values of  $K$  and  $S$ .

**Keywords:** dictionary learning, matrix factorization, fast transform, sparse representation, deconvolution, global optimization, Gauss-Seidel, wavelets, curvelets.

## 1 Introduction

### 1.1 Problem Formulation

We consider  $d \in \mathbb{N}$  and  $d$ -dimensional signals living in a domain  $\mathcal{P} \subset \mathbb{Z}^d$  (i.e.,  $d = 1$  for 1D signals,  $d = 2$  for 2D images,...). Typically,  $\mathcal{P} = \{0, \dots, N-1\}^d$ , where  $N \in \mathbb{N}$  is the number of “pixels” along

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each axis. We consider an ideal target atom  $H \in \mathbb{R}^{\mathcal{P}}$  which we want to recover. To fix ideas, one might think of the target atom as a curvelet in 2D or an apodized modified discrete cosine in 1D. A weighted sum  $u \in \mathbb{R}^{\mathcal{P}}$  of translations of the target atom corrupted by additive noise is observed. More precisely, we are interested in measurements defined by

$$u = \alpha * H + b, \quad (1)$$

where  $b \in \mathbb{R}^{\mathcal{P}}$  is an additive noise,  $*$  stands for the circular discrete convolution<sup>1</sup> in dimension  $d$  and  $\alpha \in \mathbb{R}^{\mathcal{P}}$  is a code of known coefficients. A simple example is obtained when  $\alpha$  is a Dirac delta function. In this situation  $u$  reduces to a noisy version of  $H$ . Another interesting situation is when  $\alpha$  is a sparse code. This situation turns out to be more favorable since, in that case,  $H$  is seen several times with different realizations of the noise. The typical framework we have in mind includes situations where  $\alpha$  is a sparse code, and where  $\alpha$  contains coefficients that have been estimated by dictionary learning (DL) strategies such as those described in Section 1.2. In such situations, a DL algorithm alternates an estimation of  $\alpha$  and an estimation of  $H$ . Of course,  $\alpha$  is only approximatively known and the stability of the proposed estimation of  $H$  with respect to the noise affecting  $\alpha$  is crucial. Note finally that no assumption or constraint about the code  $\alpha$  is required. However, the performance of an estimator of  $H$  from the data  $u$  defined in (1) clearly depends on the conditioning of the convolution with respect to the value of  $\alpha$ .

The problem addressed in this paper consists of both estimating the unknown target atom  $H$  and expressing it as a composition of convolutions of sparse kernels. More precisely, we consider an integer  $K \geq 2$  and  $K$  convolutions of sparse kernels  $(h^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$ . We assume that all these kernels have less than a fixed number  $S$  of non-zero elements (i.e., that they are at most  $S$ -sparse). Furthermore, we assume that the support of the kernels (i.e., the locations in  $\mathcal{P}$  of their non-zero elements) are known or pre-set. Similarly to the code  $\alpha$ , the location of the non-zero elements can be designed manually or can be estimated by some other means. For instance, the supports could be obtained by alternating support and kernel estimations.

In order to manipulate the kernel supports, we define, for all  $k \in \{1, \dots, K\}$ , an injective support mapping  $S^k \in \mathcal{P}^S$ . The range of the support mapping is defined by

$$\text{rg}(S^k) = \{S^k(1), \dots, S^k(S)\}.$$

The set of constraints on the support of  $h^k$  (denoted by  $\text{supp}(h^k)$ ) takes the form

$$\text{supp}(h^k) \subset \text{rg}(S^k), \quad \forall k \in \{1, \dots, K\}. \quad (2)$$

For 1D signals, examples of simple support mappings include  $S^k(s) = k(s-1)$ ,  $\forall s \in \{1, \dots, S\}$ . A similar support is displayed in Figure 1 for 2D images. In addition to the support constraint (2), the convolution of the  $K$  kernels  $\mathbf{h} = (h^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$ , should approximate the target atom  $H$ , i.e.,

$$h^1 * \dots * h^K \approx H.$$

The motivations for considering such a decomposition are detailed in Section 1.2. They are both to approximate a large target atom  $H$  with a model containing few degrees of freedom and to obtain target atoms whose manipulation is numerically efficient. As an illustration, we mention the approximation of a curvelet target atom by a composition of convolutions that will receive a specific attention in our experiments (see Section 4.3.1).

Therefore, we propose to solve the following optimization problem

$$(P_0): \quad \begin{cases} \text{argmin}_{\mathbf{h} \in (\mathbb{R}^{\mathcal{P}})^K} \|\alpha * h^1 * \dots * h^K - u\|_2^2, \\ \text{subject to} \quad \text{supp}(h^k) \subset \text{rg}(S^k), \quad \forall k \in \{1, \dots, K\}. \end{cases}$$

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<sup>1</sup> All the signals in  $\mathbb{R}^{\mathcal{P}}$  are extended by periodization to be defined at any point in  $\mathbb{Z}^d$ .

where  $\|\cdot\|_2$  stands for the usual Euclidean<sup>2</sup> norm in  $\mathbb{R}^{\mathcal{P}}$ . For instance, in the favorable case where the code  $\alpha$  is a Dirac delta function and  $b = 0$  (noiseless case), the solution of  $(P_0)$  approximates the target atom  $H$  by a composition of sparse convolutions. At the other extreme, when the convolution with  $\alpha$  is ill-conditioned and the noise is significant, the solution of  $(P_0)$  estimates the target atom  $H$  and regularizes it according to the composition of sparse convolution model.

The problem  $(P_0)$  is non convex. Thus, depending on the values of  $K \geq 2$ ,  $(S^k)_{1 \leq k \leq K} \in (\mathcal{P}^S)^K$ ,  $\alpha \in \mathbb{R}^{\mathcal{P}}$  and  $u \in \mathbb{R}^{\mathcal{P}}$ , it might be difficult or impossible to find a good approximation of a global minimizer of  $(P_0)$ . The main objective of this paper is to study if such a problem lends itself to global optimization. Another important objective is to assess empirically if the computed compositions of convolutions provide good approximations of some atoms usually encountered in applications. The current paper gives empirical answers to these questions. In order to do so, it contains the description of an algorithm for solving  $(P_0)$  and its performance analysis.

Before describing the proposed algorithm, we mention some links between the optimization problem  $(P_0)$  and some known issues in sparse representation.

## 1.2 Motivations

The primary motivation for considering the observation model (1) comes from DL, which was pioneered by [17, 24] and has received a growing attention during the last ten years. It can be viewed as a way of representing data using a sparse representation. We invite the reader to consult [8] for more details about sparse representations and DL. Given a set of  $L$  images<sup>3</sup>  $(u^l)_{1 \leq l \leq L} \in (\mathbb{R}^{\mathcal{P}})^L$ , the archetype of the DL strategy is to look for a dictionary as the solution of the following optimization problem

$$\operatorname{argmin}_{\mathbb{H}, (\alpha^l)_{1 \leq l \leq L}} \sum_{l=1}^L \|\mathbb{H}\alpha^l - u^l\|_2^2 + \lambda \|\alpha^l\|_*,$$

where  $\mathbb{H}$  is a matrix whose columns have a bounded norm and form the atoms of the dictionary,  $\lambda \geq 0$  is a regularization parameter and  $\|\cdot\|_*$  is a sparsity-inducing norm such as the counting function (or  $\ell_0$  pseudo-norm) or the usual  $\ell_1$  norm. The DL optimization problem is sometimes formulated by imposing a constraint on  $\|\alpha^l\|_*$ . The resulting non-convex problem can be solved (or approximatively solved) by many methods including the “method of optimal direction” (MOD) [9] and, in a different manner, by K-SVD [1]. To better reflect the distribution of images, it can also be useful to increase the number of images and to use an online strategy [21]. Finally, note that an alternative model has been presented for task driven DL in [20]. Algorithmically, all these approaches rely on alternatively updating the codes  $(\alpha^l)_{1 \leq l \leq L}$  and the dictionary  $\mathbb{H}$ .

The problem considered in the current paper mimics an update step of the dictionary. In this context,  $\alpha$  is fixed and the target atom  $H$  is a column of the dictionary  $\mathbb{H}$ . The dictionary  $\mathbb{H}$  is made of translations of the target atom  $H$ . Note that updating several target atoms iteratively is a straightforward extension of the present paper. The main novelty of the proposed approach is to impose the learned atoms to be a composition of convolutions of sparse kernels. The interest for such a constraint is that it provides numerically effective dictionaries and permits to consider larger atoms. Indeed, the reconstruction operator

$$\begin{aligned} \mathbb{R}^{\mathcal{P}} &\longrightarrow \mathbb{R}^{\mathcal{P}} \\ \alpha &\longmapsto \alpha * h^1 * \dots * h^K \end{aligned}$$

and its adjoint can be computed by  $K$  convolutions with kernels of size  $S$ . As a consequence, the computation of the reconstruction operator and its adjoint have a computational complexity of  $O(KS\#\mathcal{P})$ , where  $\#\mathcal{P}$  denotes the cardinality of the set  $\mathcal{P}$ . Depending on the support mappings  $(S^k)_{1 \leq k \leq K}$ , this

<sup>2</sup> $\mathbb{R}^{\mathcal{P}}$  and  $\mathbb{R}^S$  are endowed with the usual scalar product denoted  $\langle \cdot, \cdot \rangle$  and the usual Euclidean norm denoted  $\|\cdot\|_2$ . We use the same notation whatever the vector space. We expect that the notation will not be ambiguous, once in context.

<sup>3</sup>Usually, DL is applied to small images such as patches extracted from large images.

complexity can be much smaller than a convolution with a kernel filling the “reachable support”

$$\mathcal{S} = \left\{ p \in \mathcal{P}, \exists (p_k)_{1 \leq k \leq K} \in \text{rg}(S^1) \times \cdots \times \text{rg}(S^K), \sum_{k=1}^K p_k = p \right\}. \quad (3)$$

In the latter case, the computational complexity is indeed equal to  $O(\#\mathcal{S}\#\mathcal{P})$  or  $O(\#\mathcal{P} \log(\#\mathcal{P}))$  if the convolutions are computed using a Fast Fourier Transform (FFT). Moreover, when several atoms are considered, the convolutions of sparse kernels can be arranged according to a tree structure to save even more computing resources. The typical example of an existing dictionary having a similar structure is the dictionary made of undecimated wavelets [35] or undecimated wavelet packets.

To conclude with this subject, having a numerically effective scheme for using a dictionary is crucial since the computational complexity of most algorithms favoring sparsity is proportional to the computational complexity of the matrix-vector multiplications involving  $\mathbb{H}$  and its transpose. In particular, for the DL algorithms alternating a sparse coding step and a dictionary update step, the sparse coding steps require less computational resources. These resources are therefore available for the dictionary update.

### 1.3 Related Works

Before going ahead, it is interesting to describe the structures of the dictionaries that have been considered in DL. Structured and parametric dictionaries have recently been considered with increasing interest. Interested readers can find a concise bibliographical note on that subject in [32]. In particular, the structures studied so far include orthobases [6] and unions of orthobases [16], translation invariant dictionaries [19], concatenation of learned and fixed dictionaries [27], dictionaries composed of patches with multiple sizes [22], dictionaries divided into ordered pieces [37], structures induced by structured codes [13, 14], and tight frames [4]. Other interesting dictionaries are characterized by several layers. These dictionaries can be constructed as the composition of a fixed transform and learned dictionaries [33, 25]. Dictionaries made of two layers based on a sparsifying transform and a sampling matrix (both layers can be learned by the algorithm investigated in [7]) have also been considered. Another attempt requires two layers to build separable atoms [31]. To the best of our knowledge, there only exists a few attempts for building dictionaries involving an arbitrary number of layers. In a slightly different context, dictionaries structured by Kronecker products have been proposed in [36]. Interestingly, despite the non-convexity of the corresponding energy, it is possible to find some of its global minima [39]. Finally, dictionaries structured by wavelet-like trees (similar to one we are targeting in this paper) using a dictionary update based on a gradient descent have been studied in [34].

When compared to the dictionaries mentioned in this Section, the structure of the proposed dictionary aims at obtaining a numerically efficient translation invariant dictionary, whose elementary atoms  $H$  can have large supports. Moreover, the update of the proposed structured dictionary reduces to a global optimization problem. Surprisingly, the proposed algorithm provides interesting solutions for relatively large values of the number of layers  $K$ , e.g.,  $K = 10$  seems very reasonable.

### 1.4 Paper Organization

The paper is organized as follows. Section 1 formulates the proposed dictionary update and provides motivations with references to previous works. A more practical problem formulation is introduced in Section 2. Section 3 presents an algorithm for approximating a dictionary atom as a composition of convolutions, in order to build a fast transform. The algorithm is based on an alternating least squares strategy whose steps are detailed carefully. Simulation results illustrating the performance of the proposed algorithm and its convergence properties are provided in Sections 4 and 5. Conclusions and future work are reported in Section 6.

## 2 Reformulating $(P_0)$

The problem  $(P_0)$  is not very tractable because it has many stationary points. Denote as  $\mathbf{h} = (h^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$  the sequence of kernels and as  $E$  the objective function of  $(P_0)$

$$E(\mathbf{h}) = \|\alpha * h^1 * \dots * h^K - u\|_2^2.$$

For any  $k \in \{1, \dots, K\}$ , the partial differential of the energy function  $\frac{\partial E}{\partial h^k}$  can be calculated easily, leading to

$$\frac{\partial E}{\partial h^k}(\mathbf{h}) = 2\tilde{H}^k * (\alpha * h^1 * \dots * h^K - u), \quad (4)$$

where

$$H^k = \alpha * h^1 * \dots * h^{k-1} * h^{k+1} * \dots * h^K, \quad (5)$$

and where the operator  $\tilde{\cdot}$  is defined for any  $h \in \mathbb{R}^{\mathcal{P}}$  as

$$\tilde{h}_p = h_{-p}, \quad \forall p \in \mathcal{P}. \quad (6)$$

Note that the notation  $H^k$  has been used instead of  $H^k(\mathbf{h})$  to improve readability.

As soon as  $h^{k_1} = h^{k_2} = 0$  for two distinct values of  $k_1$  and  $k_2 \in \{1, \dots, K\}$ , we have  $H^k = 0$ , for all  $k \in \{1, \dots, K\}$ , and thus

$$\frac{\partial E}{\partial h^k}(\mathbf{h}) = 0 \quad \forall k \in \{1, \dots, K\}.$$

As a consequence, nothing prevents a minimization algorithm solving  $(P_0)$  to get stuck at one of these stationary points, although it is usually not a global minimizer of  $(P_0)$ .

Furthermore,  $\forall \mathbf{h} \in (\mathbb{R}^{\mathcal{P}})^K$  and  $\forall (\mu_k)_{1 \leq k \leq K} \in \mathbb{R}^K$  such that  $\prod_{k=1}^K \mu_k = 1$ , we have

$$E[(\mu_k h^k)_{1 \leq k \leq K}] = E(\mathbf{h}),$$

while, for any  $k \in \{1, \dots, K\}$ ,

$$\frac{\partial E}{\partial h^k}[(\mu_k h^k)_{1 \leq k \leq K}] = \frac{1}{\mu_k} \frac{\partial E}{\partial h^k}(\mathbf{h}).$$

This relation results in an unbalanced situation where the gradient strongly depends on quantities which are irrelevant with regard to the value of the objective function.

To address the two issues mentioned above and reduce the number of irrelevant stationary points, we propose to include an additional constraint for the norms of the kernels  $h^k \in \mathbb{R}^{\mathcal{P}}$ ,  $\forall k \in \{1, \dots, K\}$ . More precisely, we consider a norm-to-one constraint  $\|h^k\|_2 = 1$ ,  $\forall k \in \{1, \dots, K\}$  and introduce an additional scaling factor  $\lambda \geq 0$ , to scale the result according to the target atom. To simplify notations, we write

$$\mathcal{D} = \left\{ \mathbf{h} = (h^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K \mid \forall k \in \{1, \dots, K\}, \|h^k\|_2 = 1 \text{ and } \text{supp}(h^k) \subset \text{rg}(S^k) \right\}$$

and define the following optimization problem

$$(P_1): \quad \arg\min_{\lambda \geq 0, \mathbf{h} \in \mathcal{D}} \|\lambda \alpha * h^1 * \dots * h^K - u\|_2^2.$$

Let us now analyze the properties of the optimization problem  $(P_1)$ .

**Proposition 2.1** (Existence of a solution). *For any  $(u, \alpha, (S^k)_{1 \leq k \leq K}) \in (\mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}} \times (\mathcal{P}^S)^K)$ , if*

$$\forall \mathbf{h} \in \mathcal{D}, \quad \alpha * h^1 * \dots * h^K \neq 0, \quad (7)$$

*then the problem  $(P_1)$  has a minimizer.*

*Proof.* First notice that  $\mathcal{D}$  is a compact set. Moreover, when (7) holds, the objective function of  $(P_1)$  is coercive in  $\lambda$ . Thus, for any threshold  $\mu$ , it is possible to build a compact set such that the objective function evaluated at any  $(\lambda, \mathbf{h})$  outside this compact set is larger than  $\mu$ . As a consequence, we can extract a converging subsequence from any minimizing sequence. Since the objective function of  $(P_1)$  is continuous in a closed domain, any limit point of this subsequence is a minimizer of  $(P_1)$ .  $\square$

Note that there might be refined alternatives to the condition (7). However, the investigation of the tightest condition for the existence of a minimizer of  $(P_1)$  is clearly not the subject of this paper. Concerning the existence of a solution, note that the objective function of  $(P_1)$  is not necessarily coercive, e.g., it is not coercive if there exists  $\mathbf{h} \in (\mathbb{R}^{\mathcal{P}})^K$  such that  $\alpha * h^1 * \dots * h^K = 0$ . In this situation, a minimizing sequence might be such that  $\lambda \alpha * h^1 * \dots * h^K$  and  $(h^k)_{1 \leq k \leq K}$  have accumulation points whereas  $\alpha * h^1 * \dots * h^K$  and  $\lambda$  go towards 0 and infinity. Note finally that we typically expect the condition (7) to hold as soon as the supports  $(S^k)_{1 \leq k \leq K} \in (\mathcal{P}^S)^K$  and  $\text{supp}(\alpha)$  are sufficiently localized. In our experiments, we have never encountered a situation where  $\alpha * h^1 * \dots * h^K$  equals zero.

We also have:

**Proposition 2.2** ( $(P_1)$  is equivalent to  $(P_0)$ ). *Let  $(u, \alpha, (S^k)_{1 \leq k \leq K}) \in (\mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}} \times (\mathcal{P}^S)^K)$  be such that (7) holds. For any  $(\lambda, \mathbf{h}) \in \mathbb{R} \times (\mathbb{R}^{\mathcal{P}})^K$ , we consider the kernels  $\mathbf{g} = (g^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$  defined by*

$$g^1 = \lambda h^1 \text{ and } g^k = h^k, \quad \forall k \in \{2, \dots, K\}. \quad (8)$$

*The following statements hold:*

1. *if  $(\lambda, \mathbf{h}) \in \mathbb{R} \times (\mathbb{R}^{\mathcal{P}})^K$  is a stationary point of  $(P_1)$  and  $\lambda > 0$  then  $\mathbf{g}$  is a stationary point of  $(P_0)$ .*
2. *if  $(\lambda, \mathbf{h}) \in \mathbb{R} \times (\mathbb{R}^{\mathcal{P}})^K$  is a global minimizer of  $(P_1)$  then  $\mathbf{g}$  is a global minimizer of  $(P_0)$ .*

*Proof.* The proof of 1. hinges on formulating the expression of a stationary point of  $(P_1)$ , then showing that the Lagrange multipliers associated with the norm-to-one constraint for the  $(h^k)_{1 \leq k \leq K}$  are all equal to 0. First, considering the partial differential of the objective function of  $(P_1)$  with respect to  $\lambda$  and a Lagrange multiplier  $\gamma_\lambda \geq 0$  for the constraint  $\lambda \geq 0$ , we obtain

$$\lambda \|\alpha * h^1 * \dots * h^K\|_2^2 - \left\langle \alpha * h^1 * \dots * h^K, u \right\rangle = \frac{\gamma_\lambda}{2}, \quad (9)$$

and

$$\lambda \gamma_\lambda = 0. \quad (10)$$

Then, considering Lagrange multipliers  $\gamma_k \in \mathbb{R}$  associated with each constraint  $\|h^k\|_2 = 1$ , we have for each  $k$

$$\lambda \tilde{H}^k * (\lambda \alpha * h^1 * \dots * h^K - u) = \gamma_k h^k, \quad \forall k \in \{1, \dots, K\}, \quad (11)$$

where  $H^k$  is defined by (5). Taking the scalar product of (11) with  $h^k$  and using both  $\|h^k\|_2 = 1$  and (9), we obtain

$$\gamma_k = \lambda \frac{\gamma_\lambda}{2} = 0, \quad \forall k \in \{1, \dots, K\}.$$

Hence, (11) takes the form

$$\lambda \tilde{H}^k * (\lambda \alpha * h^1 * \dots * h^K - u) = 0, \quad \forall k \in \{1, \dots, K\}. \quad (12)$$

When  $\lambda > 0$ , this immediately implies that the kernels  $\mathbf{g}$  defined by (8) satisfy

$$\frac{\partial E}{\partial h^k}(\mathbf{h}) = 0, \quad \forall k \in \{1, \dots, K\},$$

i.e., the kernels  $\mathbf{g} \in (\mathbb{R}^{\mathcal{P}})^K$  form a stationary point of  $(P_0)$ .

The proof of the item 2 is straightforward since for any  $(f^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$  satisfying the constraints of  $(P_0)$ <sup>4</sup>, we have

$$\begin{aligned} \|\alpha * g^1 * \dots * g^K - u\|_2^2 &= \|\lambda \alpha * h^1 * \dots * h^K - u\|_2^2 \\ &\leq \left\| \left( \prod_{k=1}^K \|f^k\|_2 \right) \alpha * \frac{f^1}{\|f^1\|_2} * \dots * \frac{f^K}{\|f^K\|_2} - u \right\|_2^2 \\ &\leq \|\alpha * f^1 * \dots * f^K - u\|_2^2. \end{aligned}$$

As a consequence, the kernels  $(g^k)_{1 \leq k \leq K}$  defined by (8) form a solution of  $(P_0)$ .  $\square$

To conclude this part, it is interesting to mention some structural properties of problem  $(P_1)$ . The objective function of  $(P_1)$  is a polynomial of degree  $2K$ . Thus, it is infinitely differentiable and non-negative. The objective function of  $(P_1)$  is non-convex. However, for any  $k \in \{1, \dots, K\}$ , the objective function of  $(P_1)$  is marginally quadratic and convex with respect to  $h^k$ . Finally,  $\mathcal{D}$  is a smooth but non convex set. It is not difficult to check that the following mapping provides an orthogonal projection onto  $\mathcal{D}$ :

$$\begin{aligned} (\mathbb{R}^{\mathcal{P}})^K &\longrightarrow \mathcal{D} \\ (h^k)_{1 \leq k \leq K} &\longmapsto (\bar{h}^k)_{1 \leq k \leq K}, \end{aligned}$$

where

$$\bar{h}^k = \begin{cases} \frac{h^k \mathbb{1}_{\text{rg}(S^k)}}{\|h^k \mathbb{1}_{\text{rg}(S^k)}\|_2} & , \text{ if } \|h^k \mathbb{1}_{\text{rg}(S^k)}\|_2 \neq 0, \\ \frac{1}{\sqrt{S}} \mathbb{1}_{\text{rg}(S^k)} & , \text{ otherwise,} \end{cases}$$

where  $\mathbb{1}_{\text{rg}(S^k)}$  is the characteristic function of  $\text{rg}(S^k)$ .

### 3 The alternating least squares algorithm

#### 3.1 Principle of the algorithm

The objective function in  $(P_1)$  being non-convex, there is in general no guaranty to find neither a global nor a local minimum of  $(P_1)$ . However, it makes sense to build a method finding a stationary point of  $(P_1)$ . For that purpose, we propose to alternate minimizations with respect to the kernels  $h^k, \forall k \in \{1, \dots, K\}$ . The resulting algorithm is often called a ‘‘Gauss-Seidel’’ or a ‘‘block coordinate descent’’. Although our convergence analysis will not rely on these results let us mention that some convergence properties of these algorithms have been studied in [18, 12, 30, 3]. Furthermore, the algorithm we are designing in the present paper shares many similarities with the algorithm designed in [15] for the best rank-1 approximation of tensors.

More precisely, for any  $k \in \{1, \dots, K\}$ , we propose to (alternatively) solve the following least square (LS) problems

$$(P_k): \quad \begin{cases} \text{argmin}_{\lambda \geq 0, h \in \mathbb{R}^{\mathcal{P}}} \|\lambda \alpha * h^1 * \dots * h^{k-1} * h * h^{k+1} * \dots * h^K - u\|_2^2, \\ \text{subject to } \text{supp}(h) \subset \text{rg}(S^k) \text{ and } \|h\|_2 = 1. \end{cases}$$

where the kernels  $(h_p^{k'})_{p \in \mathcal{P}}$  are fixed  $\forall k' \neq k$ . The resulting alternating least square (ALS) algorithm is described in Algo. 1.

<sup>4</sup>We further assume that  $\|f^k\|_2 \neq 0$ , for all  $k \in \{1, \dots, K\}$ , since the inequality is otherwise trivial.



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**Algorithm 1:** ALS algorithm

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**Input:** $u$ : target measurements; $\alpha$ : known coefficients; $(S^k)_{1 \leq k \leq K}$ : supports of the kernels  $(h^k)_{1 \leq k \leq K}$ .**Output:** $\lambda$  and kernels  $(h^k)_{1 \leq k \leq K}$  such that  $\lambda h^1 * \dots * h^K \approx H$ .**begin**Initialize the kernels  $(h^k)_{1 \leq k \leq K}$ ;**while** not converged **do**  **for**  $k = 1, \dots, K$  **do**    Update  $\lambda$  and  $h^k$  with a minimizer of  $(P_k)$ .**end**

---

### 3.2 Resolution of $(P_k)$

Before studying the existence of a minimizer of  $(P_k)$ , let us rewrite the problem  $(P_k)$  in a simpler form. Since the embedding from  $\mathbb{R}^S$  in  $\text{rg}(S^k) \subset \mathbb{R}^{\mathcal{P}}$  and the operator

$$\begin{aligned} \mathbb{R}^{\mathcal{P}} &\longrightarrow \mathbb{R}^{\mathcal{P}} \\ h &\longmapsto \alpha * h^1 * \dots * h^{k-1} * h * h^{k+1} * \dots * h^K, \end{aligned}$$

are linear, their composition can be described by a matrix-vector product  $C_k h$ , where the vector  $h \in \mathbb{R}^S$  and  $C_k$  is a  $(\#\mathcal{P}) \times S$  matrix. (The matrix  $C_k$  will be detailed in Section 3.3.)

A solution of  $(P_k)$  can therefore be constructed by embedding in  $\text{rg}(S^k) \subset \mathbb{R}^{\mathcal{P}}$  a solution of the equivalent problem (still denoted  $(P_k)$ , for simplicity)

$$(P_k): \quad \begin{cases} \text{argmin}_{\lambda \geq 0, h \in \mathbb{R}^S} \|\lambda C_k h - u\|_2^2 \\ \text{subject to } \|h\|_2 = 1. \end{cases}$$

where we consider that  $u$  has been vectorized. In order to solve this problem, we define

$$(P'_k): \quad \text{argmin}_{h \in \mathbb{R}^S} \|C_k h - u\|_2^2.$$

The problem  $(P'_k)$  is a LS problem which has a minimizer  $h^* \in \mathbb{R}^S$ . Moreover, the gradient of its objective function is

$$C_k^T (C_k h - u).$$

Finally, by computing a stationary point of the problem  $(P'_k)$ , we obtain:

$$h^* = (C_k^T C_k)^\dagger C_k^T u, \tag{13}$$

where  $(C_k^T C_k)^\dagger$  is the pseudo-inverse of  $C_k^T C_k$ . Setting

$$\lambda = \|h^*\|_2 \quad \text{and} \quad h^k = \begin{cases} \frac{h^*}{\|h^*\|_2} & , \text{ if } \|h^*\|_2 \neq 0, \\ \frac{1}{\sqrt{S}} \mathbb{1} & , \text{ otherwise} \end{cases} \tag{14}$$

where  $\mathbb{1} \in \mathbb{R}^S$  is a vector of ones. It is easy to check that we always have  $h^* = \lambda h^k$ . One can also show that any  $(\mu, g) \in \mathbb{R} \times \mathbb{R}^S$  satisfying the constraints of  $(P_k)$  is such that:

$$\begin{aligned} \|\lambda C_k h^k - u\|_2^2 &= \|C_k h^* - u\|_2^2, \\ &\leq \|C_k(\mu g) - u\|_2^2 = \|\mu C_k g - u\|_2^2. \end{aligned}$$

As a consequence,  $(P_k)$  has a minimizer defined by (13) and (14). Moreover, note that if  $(\lambda', h')$  is a solution of  $(P_k)$ , we can easily check that  $\lambda' h'$  is a minimizer of  $(P'_k)$ . The latter being unique when  $C_k$  is full column rank, we know that the solution of  $(P_k)$  is unique under that same condition.

Altogether, we obtain the update rule by embedding in  $\text{rg}(S^k) \subset \mathbb{R}^{\mathcal{P}}$  the solution described by (13) and (14). In order to apply these formulas, the main computational difficulties are to compute  $C_k^T u$ ,  $C_k^T C_k$  and the pseudo-inverse of  $C_k^T C_k$ . These computations are the subject of the next paragraph.

### 3.3 Computing $C_k^T u$ and $C_k^T C_k$

Considering Dirac delta functions for  $h \in \mathbb{R}^S$  and the linearity of  $C_k$ , we obtain for any  $h \in \mathbb{R}^S$

$$(C_k h)_p = \sum_{s=1}^S H_{p-S^k(s)}^k h_s, \quad \forall p \in \mathcal{P}$$

where  $H^k$  is defined in (5). In other words, each column of  $C_k$  is a vectorization of  $(H_{p-S^k(s)}^k)_{p \in \mathcal{P}}$ . For any  $p' \in \mathcal{P}$ , denote as  $\tau_{p'}$  the translation operator such that  $(\tau_{p'} v)_p = v_{p-p'}$ ,  $\forall (v, p) \in \mathbb{R}^{\mathcal{P}} \times \mathcal{P}$ . Using this notation, the  $s$ th column of  $C_k$  is a vectorization of  $\tau_{S^k(s)} H^k$ . Therefore, the  $s$ th line of  $C_k^T$  is the transpose of a vectorization of  $\tau_{S^k(s)} H^k$ . We finally have

$$(C_k^T v)_s = \langle \tau_{S^k(s)} H^k, v \rangle, \quad \forall v \in \mathbb{R}^{\mathcal{P}}. \quad (15)$$

Note that the computational complexity for computing  $H^k$  is  $O((K-1)S\#\mathcal{P})$ . Once  $H^k$  has been computed, the cost for computing  $(C_k^T u)_s$  is  $O(\#\mathcal{P})$ ,  $\forall s \in \{1, \dots, S\}$ , and therefore the cost for computing  $C_k^T u$  is  $O(S\#\mathcal{P})$ . Altogether, we obtain a complexity  $O(KS\#\mathcal{P})$ .

We can immediately deduce the form of  $C_k^T C_k$ . Indeed, each of its column is obtained by applying (15) in which we replace  $v$  by the column vector  $\tau_{S^k(s')} H^k$ , for some  $s' \in \{1, \dots, S\}$ . Therefore the coefficient of  $C_k^T C_k$  at the location  $(s, s') \in \{1, \dots, S\}^2$  is

$$(C_k^T C_k)_{s,s'} = \langle \tau_{S^k(s)} H^k, \tau_{S^k(s')} H^k \rangle. \quad (16)$$

This Gram matrix is symmetric, positive semidefinite and of size  $S \times S$ . Once  $H^k$  has been computed, the computational complexity for computing  $C_k^T C_k$  is  $O(S^2\#\mathcal{P})$ . The computation of its pseudo-inverse is a well studied problem and is a step of the algorithm that can be optimized. An off-the-shelf implementation using a singular value decomposition (SVD) typically requires  $O(S^3)$  operations.

Algorithm 2 summarizes all the steps required for the proposed ALS algorithm. The overall computational complexity is typically  $O((K+S)KS\#\mathcal{P})$  per iteration of the while loop<sup>5</sup>. It can be reasonably applied in situations where  $KS(K+S)$  is not too large. The most demanding case considered in the experiments described in this paper corresponds to  $KS^2 = 6250$  (corresponding to  $K = 10$  and  $S = 25$ ). In order to choose the number of iterations in the while loop, we have used the relative difference between the values of the objective function of  $(P_k)$  for two consecutive iterations. When this difference is lower than  $10^{-4}$ , we consider that we have reached a stationary point, and the algorithm stops.

### 3.4 Convergence of the algorithm

Before stating the convergence result, let us give a few notations.

First, notice that the result of an iteration of the for loop in Algorithm 2 only depends on the initial kernels  $\mathbf{h} \in \mathcal{D}$  and not on the initial scaling factor  $\lambda$ . If we consider an initial condition  $\mathbf{h} \in \mathcal{D}$  of the for loop in Algorithm 2, we denote the initial condition of the  $k$ th iteration by  $T_k(\mathbf{h})$ . For instance, we have  $T_1(\mathbf{h}) = \mathbf{h}$ . We also denote the scaling factor and the kernels resulting from the whole for loop by  $T(\mathbf{h})$ . More precisely, denoting as  $(\lambda^n, \mathbf{h}^n)_{n \in \mathbb{N}}$  the sequence generated by Algorithm 2, we have for all  $n \in \mathbb{N}$

$$(\lambda^{n+1}, \mathbf{h}^{n+1}) = T(\mathbf{h}^n).$$

<sup>5</sup>In the practical situations we are interested in,  $\#\mathcal{P} \gg S$  and  $S^3$  can be neglected when compared to  $(K+S)S\#\mathcal{P}$ .

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**Algorithm 2:** Detailed ALS algorithm
 

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**Input:**
 $u$ : target measurements;

 $\alpha$ : known coefficients;

 $(S^k)_{1 \leq k \leq K}$ : supports of the kernels  $(h^k)_{1 \leq k \leq K}$ .

**Output:**
 $(h^k)_{1 \leq k \leq K}$ : convolution kernels such that  $h^1 * \dots * h^K \approx H$ .

**begin**

 Initialize the kernels  $((h_p^k)_{p \in \mathcal{P}})_{1 \leq k \leq K}$ ;

**while** not converged **do**
**for**  $k = 1, \dots, K$  **do**

 Compute  $H^k$  according to (5)

 $O((K-1)S\#\mathcal{P})$ 

 Compute  $C_k^T C_k$  and  $C_k^T u$  according to (16) and (15);

 $O((S+1)S\#\mathcal{P})$ 

 Compute  $h^*$  according to (13);

 $O(S^3)$ 

 Update  $h^k$  and  $\lambda$  according to (14);

 $O(S)$ 
**end**


---

**Proposition 3.1** (Convergence of Algorithm 2). *For any  $(u, \alpha, (S^k)_{1 \leq k \leq K}) \in (\mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}} \times (\mathcal{P}^S)^K)$ , if*

$$\alpha * h^1 * \dots * h^K \neq 0, \quad \forall \mathbf{h} \in \mathcal{D}, \quad (17)$$

*then the following statements hold:*

1. *The sequence generated by Algorithm 2 is bounded and its limit points are in  $\mathbb{R} \times \mathcal{D}$ . The value of the objective function is the same for all these limit points.*
2. *For any limit point  $(\lambda^*, \mathbf{h}^*) \in \mathbb{R} \times \mathcal{D}$ , if for all  $k \in \{1, \dots, K\}$ , the matrix  $C_k$  generated using  $T_k(\mathbf{h}^*)$  is full column rank and  $C_k^T u \neq 0$ , then  $(\lambda^*, \mathbf{h}^*) = T(\mathbf{h}^*)$  and  $(\lambda^*, \mathbf{h}^*)$  is a stationary point of the problem  $(P_1)$ .*

*Proof.* The first item of proposition 3.1 can be obtained directly since 1) the sequence of kernels generated by the algorithm belongs to  $\mathcal{D}$  and  $\mathcal{D}$  is compact, 2) the objective function of  $(P_1)$  is coercive with respect to  $\lambda$  when (17) holds, and 3) the objective function is continuous and decreases during the iterative process.

To prove the second item of proposition 3.1, we consider a limit point  $(\lambda^*, \mathbf{h}^*) \in \mathbb{R} \times \mathcal{D}$ . We denote by  $F$  the objective function of  $(P_1)$  and denote by  $(\lambda^o, \mathbf{h}^o)_{o \in \mathbb{N}}$  a subsequence of  $(\lambda^n, \mathbf{h}^n)_{n \in \mathbb{N}}$  which converges to  $(\lambda^*, \mathbf{h}^*)$ . The following statements are trivially true, since  $F$  is continuous and  $(F(\lambda^n, \mathbf{h}^n))_{n \in \mathbb{N}}$  decreases:

$$\lim_{o \rightarrow \infty} F(T(\mathbf{h}^o)) = \lim_{o \rightarrow \infty} F(\lambda^o, \mathbf{h}^o) = F(\lambda^*, \mathbf{h}^*) \quad (18)$$

However, if for any  $k$  inside  $\{1, \dots, K\}$ , we have  $C_k^T u \neq 0$  and the matrix  $C_k$  generated using  $T_k(\mathbf{h}^*)$  is full column rank, then there exist an open neighborhood of  $T_k(\mathbf{h}^*)$  such that these conditions remain true for the matrices  $C_k$  generated from kernels  $\mathbf{h}$  in this neighborhood. As a consequence, the  $k$ th iteration of the for loop is a continuous mapping on this neighborhood. Finally, we deduce that there is a neighborhood of  $\mathbf{h}^*$  in which  $T$  is continuous.

Since  $T$  is continuous in the vicinity of  $\mathbf{h}^*$  and  $(\mathbf{h}^o)_{o \in \mathbb{N}}$  converges to  $(\mathbf{h}^*)$ , the sequence  $(T(\mathbf{h}^o))_{o \in \mathbb{N}}$  converges to  $T(\mathbf{h}^*)$  and (18) guarantees that

$$F(T(\mathbf{h}^*)) = F(\lambda^*, \mathbf{h}^*).$$

As a consequence, denoting  $\mathbf{h}^* = (h^{*,k})_{1 \leq k \leq K}$ , for every  $k \in \{1, \dots, K\}$ ,  $F(\lambda^*, h^{*,k})$  is equal to the minimal value of  $(P_k)$ . Since  $C_k$  is full column rank, we know that this minimizer is unique (see the end of Section 3.2) and therefore  $(\lambda^*, h^{*,k})$  is the unique minimizer of  $(P_k)$ . We can then deduce that  $(\lambda^*, \mathbf{h}^*) = T(\mathbf{h}^*)$ .

Finally, we also know that  $(\lambda^*, \mathbf{h}^*)$  is a stationary point of  $(P_k)$ . Combining all the equations stating that, for any  $k$ ,  $(\lambda^*, \mathbf{h}^{*,k})$  is a stationary point of  $(P_k)$ , we can find that  $(\lambda^*, \mathbf{h}^*)$  is a stationary point of  $(P_1)$ . □

### 3.5 Initialization of the algorithm and restart

First, it is interesting to note that the ALS algorithm does not need any initialization for  $\lambda$ . Moreover, the initial kernel values  $(h^k)_{1 \leq k \leq K}$  must satisfy the constraints and therefore belong to  $\mathcal{D}$ . When the problem  $(P_1)$  has a global minimizer, we denote by  $\mathbb{I} \subset \mathcal{D}$  the non-empty convergence set such that the ALS algorithm converges to a global minimizer when it has been initialized with an element of  $\mathbb{I}$ . Surprisingly, after running intensively the ALS algorithm, it appears that in many situations  $\mathbb{I}$  is actually large. In order to illustrate this aspect, we have chosen a simple initialization. It consists of initializing our algorithm by drawing a random variable uniformly distributed in  $\mathcal{D}$ . This is easily achieved [23] by using<sup>6</sup>

$$h^k = \frac{h}{\|h\|_2}, \quad \text{with} \quad h \sim \mathcal{N}_S(0, Id),$$

where  $\mathcal{N}_S(0, Id)$  is the centered normal distribution in  $\mathbb{R}^S$ . Our experiments will show that  $\mathbb{P}(\mathbf{h} \notin \mathbb{I})$  is often significantly smaller than 1 when  $\mathbf{h}$  is uniformly distributed in  $\mathcal{D}$ . Moreover, an advantage of this random initialization is that we can use a “restart” strategy to explore  $\mathcal{D}$ . More precisely, we propose to run the ALS algorithm  $R$  times, for  $R \in \mathbb{N}$ , and to return the result for which the objective function is the smallest. The probability that such a strategy fails to provide a global minimizer is equal to the probability that none of the  $R$  independent initializations belong to  $\mathbb{I}$ , i.e.,

$$\mathbb{P}(\text{not global}) = [\mathbb{P}(\mathbf{h} \notin \mathbb{I})]^R$$

which decays rapidly to 0, when  $\mathbb{P}(\mathbf{h} \in \mathbb{I})$  is not negligible. For instance, to guarantee

$$\mathbb{P}(\text{not global}) \leq \varepsilon$$

for  $\varepsilon > 0$ , we must take

$$R \geq R_\varepsilon = \frac{\log(\varepsilon)}{\log(\mathbb{P}(\mathbf{h} \notin \mathbb{I}))}. \quad (19)$$

Note that the number of restarts does not increase significantly when  $\varepsilon$  decreases. However, when  $\mathbb{P}(\mathbf{h} \in \mathbb{I})$  is small (or negligible) we have

$$R_\varepsilon \sim \frac{-\log(\varepsilon)}{\mathbb{P}(\mathbf{h} \in \mathbb{I})}.$$

The proposed “restart” strategy is therefore only reasonable when  $\mathbb{P}(\mathbf{h} \in \mathbb{I})$  is not too small.

## 4 Approximation experiments

### 4.1 Simulation scenario

Our first goal is to empirically assess the ability of a composition of convolutions to approximate a given target atom  $H \in \mathbb{R}^{\mathcal{P}}$ . We are also interested in observing the influence of the number of kernels  $K$  and of the size of the kernels on the approximation error. In order to do so, this section presents

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<sup>6</sup>For simplicity, in the formula below, we do not mention the mapping of  $\mathbb{R}^S$  into  $\mathbb{R}^{\mathcal{P}}$  necessary to build  $h^k$ .

results obtained for several 1D and 2D target atoms  $H$  (i.e.,  $d = 1$  or  $2$ ) that have been selected from dictionaries commonly used in signal and image processing.

For all the experiments in Section 4, we consider a size  $N \in \mathbb{N}$ , a dimension  $d \in \{1, 2\}$  and take  $\mathcal{P} = \{0, \dots, N-1\}^d$ . We consider a target atom  $H \in \mathbb{R}^{\mathcal{P}}$ , a code  $\alpha \in \mathbb{R}^{\mathcal{P}}$  and a zero mean Gaussian noise  $b \in \mathbb{R}^{\mathcal{P}}$  of variance  $\sigma^2$ . Throughout these experiments, we explore parameters up to  $K = 11$  and  $S = 25$ . Moreover, for a dimension  $d \in \{1, 2\}$  and a size  $c \in \mathbb{N}$ , we always consider the support mappings  $(S^k)_{1 \leq k \leq K} \in (\mathcal{P}^S)^K$  such that for all  $k \in \{1, \dots, K\}$

$$\text{rg}(S^k) = k\{-c, \dots, 0, \dots, c\}^d. \quad (20)$$

Note that centering these support mappings on  $p = 0$  is possible because of the periodization of  $\mathbb{R}^{\mathcal{P}}$ . Figure 1 shows an example of support mapping obtained for  $K = 4$ ,  $d = 2$  and  $c = 1$ .

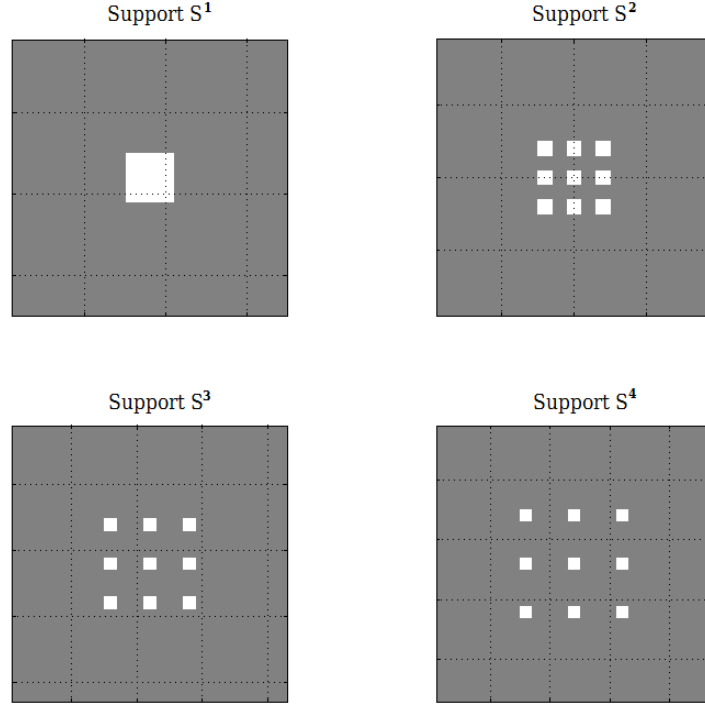


Figure 1: The supports  $\text{rg}(S^k)$  described by (20), for  $d = 2$ ,  $k \in \{1, 2, 3, 4\}$  and for  $c = 1$  (i.e.,  $S = 3 \times 3$ ). The representation is shifted so that the origin element of (20) is at the center of each image.

It is not difficult to show that the reachable support (3) associated with these support mappings is:

$$\begin{aligned} \mathcal{S} &= \left\{ \sum_{k=1}^K -ck, \dots, \sum_{k=1}^K ck \right\}^d \\ &= \left\{ -c \frac{K(K+1)}{2}, \dots, c \frac{K(K+1)}{2} \right\}^d. \end{aligned}$$

Therefore, the width of  $\mathcal{S}$  is given by  $K(K+1)c$  and its size (length or area) is  $(K(K+1)c)^d$ . Note that the size of  $\mathcal{S}$  is usually much smaller than the size of the search space, equal to  $K(2c+1)^d$ . The ratio between these two quantities corresponds to a “compression ratio” when describing the atom

with convolution kernels. This ratio behaves like  $\frac{K^{2d-1}}{2^d}$  when both  $c$  and  $K$  grow. Table 1 shows the compression ratio for a few values of  $(K, c)$  and  $d \in \{1, 2\}$ . The gain is clearly more interesting when increasing  $K$  compared to increasing  $c$ .

For most experiments, the support of  $H$  is contained into  $\mathcal{S}$ . When it is the case, we provide an indicator for the ability of the composition of convolutions to reduce the search space while filling the target atom's support. This indicator  $G$  is the ratio between the size of the effective support of  $H$  and the size of the actual search space using the  $K$   $\mathcal{S}$ -sparse kernels, i.e.,

$$G = \frac{\#\text{supp}_{\text{eff}}(H)}{K(2c+1)^d}$$

where

$$\text{supp}_{\text{eff}}(H) = \left\{ p \in \mathcal{P} \mid |H_p| \geq 10^{-4} (\max_{p \in \mathcal{P}} |H_p|) \right\}.$$

The role of the effective support is to realistically account for the energy localization in  $H$ . We will provide some values of  $G$  for the tests presented in this section.

Compression ratio		$K = 3$	$K = 4$	$K = 6$	$K = 10$
$d = 1$	$c = 1 (S = 3)$	0.67	1.00	1.67	3.00
	$c = 2 (S = 5)$	0.80	1.20	2.00	3.60
	$c = 3 (S = 7)$	0.86	1.29	2.14	3.86
$d = 2$	$c = 1 (S = 9)$	1.33	4.00	16.67	90.00
	$c = 2 (S = 25)$	1.92	5.76	24.00	129.60
	$c = 3 (S = 49)$	2.20	6.61	27.55	148.78

Table 1: Compression ratio  $\frac{(K(K+1)c)^d}{K(2c+1)^d}$  for various  $K$  and  $S$  in dimension  $d = 1$  and  $d = 2$ .

For each experiment, the quantities  $N, d, H, \alpha, \sigma, K, c$  and the number  $R$  of restarts are provided. Given these quantities, we compute  $u$  according to (1). Then, Algorithm 2 is run for a given number  $R$  of restarts and the result with the smallest objective function value is kept. The result of this process is denoted as  $(\lambda, (h^k)_{1 \leq k \leq K}) \in \mathbb{R} \times \mathcal{D}$  in what follows.

Given a result  $(\lambda, (h^k)_{1 \leq k \leq K}) \in \mathbb{R} \times \mathcal{D}$ , we evaluate the quality of the approximation of  $H$  by  $\lambda h^1 * \dots * h^K$  using the peak-signal-to-noise ratio (PSNR). Moreover, in order to consider that the size of the support of  $H$  can be much smaller than  $\#\mathcal{P}$ , the PSNR is normalized according to the size of the effective support of  $H$ . More precisely, it is defined by

$$\text{PSNR}_H = 10 \log_{10} \left( \frac{r^2}{\text{MSE}_H} \right)$$

where  $r = \max_{p \in \mathcal{P}} (H_p) - \min_{p \in \mathcal{P}} (H_p)$  is the dynamic range of the atom  $H$  and the mean-square-error (MSE) is defined by:

$$\text{MSE}_H = \frac{\|\lambda h^1 * \dots * h^K - H\|_2^2}{\#\text{supp}_{\text{eff}}(H)}. \quad (21)$$

The normalization defined in (21) is motivated by the nature of most atoms studied in this section: though their support may span over the whole set  $\mathcal{P}$ , most of their energy is concentrated in a small region.

Note that in noisy settings, PSNR values are provided in addition to the noise variance  $\sigma^2$ . These PSNR values inform us on the degradation between  $\alpha * H$  and  $u$ , and cannot be compared to the values of  $\text{PSNR}_H$ , which concern the reconstructed atom only. The only exception is the first experiment, of paragraph 4.2.1, where the code  $\alpha$  is a Dirac delta function.

We also provide a figure of merit reflecting both the quality of the convergence and the level of regularization induced by the composition of convolutions. It is defined as

$$\text{NRE} = \frac{\|\lambda \alpha * h^1 * \dots * h^K - u\|_2^2}{\|u\|_2^2}. \quad (22)$$

When NRE is large, either the convergence has not been reached or the values of  $K$  and  $S$  are too small to obtain a good approximation of  $H$ . When it is small, the algorithm has converged to a stationary point close to a global minimum and the values of  $K$  and  $S$  provide a good approximation of  $u$ . Note that this last property can be a problem when  $u$  is contaminated by a strong noise.

Finally, in order to assess the additional difficulty induced by the convolution with the code  $\alpha$ , we provide a measure of conditioning. Indeed, recovering  $H$  from  $u$  can be a badly conditioned problem (see (1)) yielding instabilities. For every experiment where  $\alpha$  is not a Dirac delta function, a histogram of the values of the modulus of its Fourier transform  $|\hat{\alpha}|$  is used to measure conditioning. The greater the range over which these values span, the worse the conditioning. The case of a sparse  $\alpha$  seems to be the best compromise between conditioning and redundancy, the latter being crucial to get a stable approximation of  $H$  in the presence of noise.

Note that NRE can be small whatever the conditioning because the value of the denominator in (22) depends on  $\alpha$ . For this reason,  $\text{PSNR}_H$  is still the most relevant indicator of the success of the algorithm.

## 4.2 1D targets

### 4.2.1 Apodized Modified Discrete Cosine

The modified discrete cosine transform (MDCT) has been successfully used in several signal processing applications such as audio coding [26]. The aim of the proposed experiment is to approximate an apodized modified discrete cosine (MDC) with a composition of convolutions. In order to do so, we apply the inverse MDCT to a Dirac delta function located at a given frequency, in a signal of size 512 (i.e.,  $d = 1$ ). We then apodize the MDC using the sine window  $(w_p)_{0 \leq p \leq 255}$  defined by:

$$w_p = \begin{cases} 0 & \text{if } p \in \{0, \dots, 127\} \cup \{384, \dots, 512\} \\ \sin\left[\pi \frac{(p-128)}{256}\right] & \text{if } p \in \{128, \dots, 383\}. \end{cases}$$

This type of window is, for instance, used in MDCT analysis for time-domain aliasing cancellation [28].

Figures 2 and 3 show examples of target atoms  $H$  obtained for frequencies 10Hz and 100Hz. The code  $\alpha$  used in this experiment is a Dirac delta function located at  $p = 256$ . In this simple, noiseless case,  $u$  equals  $H$ . As for all simulations conducted in this section, the kernel supports have been defined according to (20). We have used  $R = 50$  restarts because the simulation is very fast.

Moreover, we have considered  $5 \leq K \leq 11$  and  $5 \leq S = 2c + 1 \leq 11$ , corresponding to the values of  $\text{PSNR}_H$  reported in Table 2. One can see that the higher  $K$  and  $S$ , the better the approximation of  $H$ . This result is expected since increasing these parameters confers more flexibility to describe the target atom  $H$ , leading to a lower resulting objective function value (after algorithm convergence), which is inversely proportional to  $\text{PSNR}_H$ . Note that values of  $\text{PSNR}_H$  above 50 dB are obtained in many cases.

$\text{PSNR}_H(\text{dB})$	$K = 5$	$K = 7$	$K = 9$	$K = 11$
$c = 2$	14.43	17.32	23.81	38.26
$c = 3$	16.23	23.02	46.24	51.48
$c = 4$	18.45	34.84	54.32	54.33
$c = 5$	21.60	53.70	54.82	55.73

Table 2: MDC approximation for frequency 100Hz:  $\text{PSNR}_H$  for several values of  $K$  and  $c$ .

The approximations obtained for frequencies 10 and 100, both with  $K = 9$ ,  $S = 2c + 1 = 9$ , are depicted in Figures 2 and 3, respectively. More precisely, each figure shows the approximation  $\lambda h^1 * \dots * h^K$  and the atom  $H$ . Note that the resulting approximations are very accurate, and  $G = \frac{256}{9 \times 9} = 3.16$ .

The same experiment has been conducted for the frequency 100 with  $K = 9$ ,  $S = 2c + 1 = 9$  and  $R = 25$  restarts, with an additive white Gaussian noise of variance  $\sigma^2 \in [10^{-6}, 10^{-3}]$ . Figure 4 shows

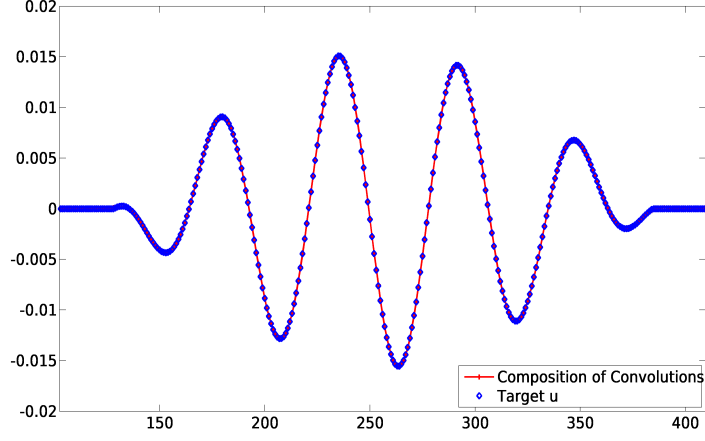


Figure 2: Approximation of an apodized frequency 10 MDC by the convolution of  $K = 9$  kernels of sparsity  $S = 9$  ( $\text{PSNR}_H = 58.88\text{dB}$ ).

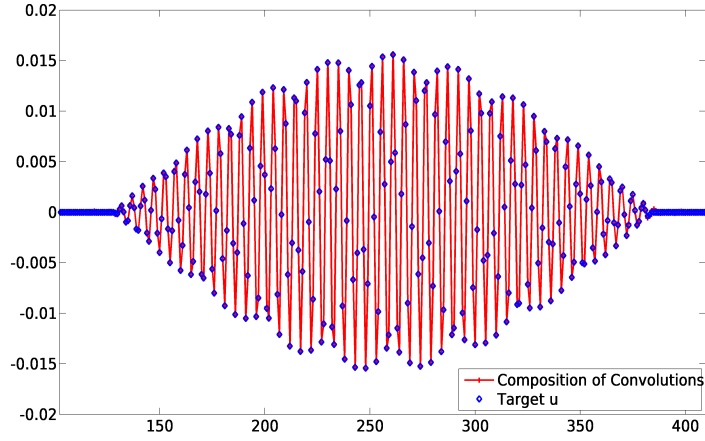


Figure 3: Approximation of an apodized frequency 100 MDC by the convolution of  $K = 9$  kernels of sparsity  $S = 9$  ( $\text{PSNR}_H = 54.32\text{dB}$ ).

$\text{PSNR}_H$  as a function of the noise variance. Note that  $\text{PSNR}_H$  is always higher than the PSNR between  $u$  and  $\alpha * H$ <sup>7</sup>. This means that the model ( $P_0$ ) reduces noise when  $u$  is a noisy apodized MDC. This denoising would be further improved with a sparse code  $\alpha$  containing several non-zero coefficients.

#### 4.2.2 Sinc function

This experiment aims at approximating the sinc function used to perform a linear zoom [38]. The sinc interpolation has been successfully approximated with splines [2]. Though the spline interpolation can be interpreted as a composition of convolutions, we use different kernel supports. The zoom factor is

<sup>7</sup>In this case the comparison is relevant, because  $\alpha$  is a Dirac delta function.



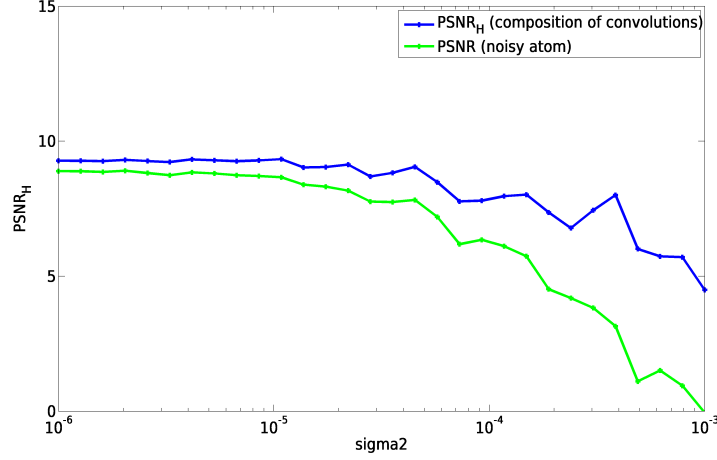


Figure 4:  $\text{PSNR}_H$  for the approximation of the apodized frequency 100 MDC by the convolution of  $K = 9$  kernels of sparsity  $S = 9$ , for  $10^{-6} \leq \sigma^2 \leq 10^{-3}$  (blue curve). The green curve is the PSNR between  $u$  and  $\alpha * H$ .

$Z = 3$  and the signal is of size 128. We therefore have  $d = 1$  and  $N = 3 \times 128 = 384$ . The target atom  $H$  is a sinc function obtained by computing the inverse Fourier transform of the characteristic function of a centered interval of length  $N/3$ . The signal to be zoomed corresponds to the first 128 values of the 128th column of the Barbara image.

The code  $\alpha$  has been built by upsampling this signal by a factor  $Z = 3$  (see Figure 5a). This upsampling has been performed by inserting 2 zeros between every couple of neighbors in the initial signal. We are obviously not in a case where  $\alpha$  is sparse. Moreover, the histogram of its Fourier transform displayed in Figure 5b shows that the convolution with  $\alpha$  is not very well conditioned. Indeed, the ratio between the highest Fourier coefficient and the lowest is 728.

The signal  $u$  has been constructed according to (1) for different noise levels. Moreover, as for all experiments in this section, kernel supports have been set according to (20).

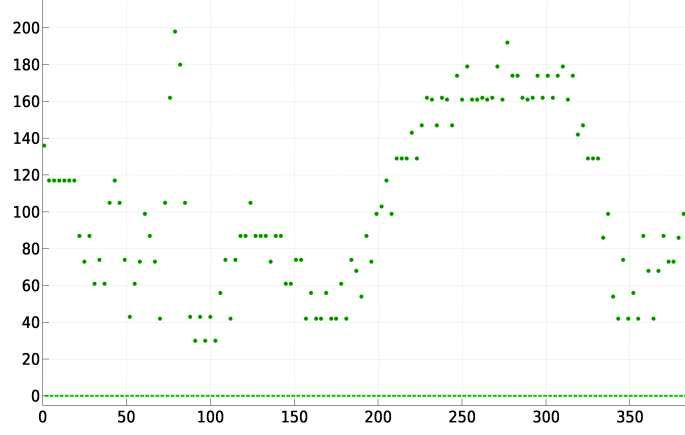
First, we have considered  $K = 9$  and  $c = 4$  (i.e.,  $S = 9$ ) and run  $R = 50$  restarts of Algorithm 2 for noiseless and noisy signals ( $\sigma^2 = 5$ ). Figure 6 shows the target sinc atom  $H$  and the approximation  $\lambda * h^1 * \dots * h^K$ , for the noisy and noiseless cases. In the noiseless case (Figure 6a), we see that the resulting composition of convolutions  $\lambda * h^1 * \dots * h^K$  is a good approximation of the sinc function. In the noisy case (Figure 6b), the approximation is less accurate, which is expected since there is no regularization and the convolution with  $\alpha$  is ill-conditioned.

The same experiment has been run for  $K \in \{3, 5, 7, 9\}$  and  $c \in \{1, 2, 3\}$  (i.e.  $S \in \{3, 5, 7\}$ ),  $R = 50$ , for both cases  $\sigma^2 = 0$  and  $\sigma^2 = 5$ . In the latter case, the PSNR between  $u$  and  $\alpha * H$  is 28.20 dB.

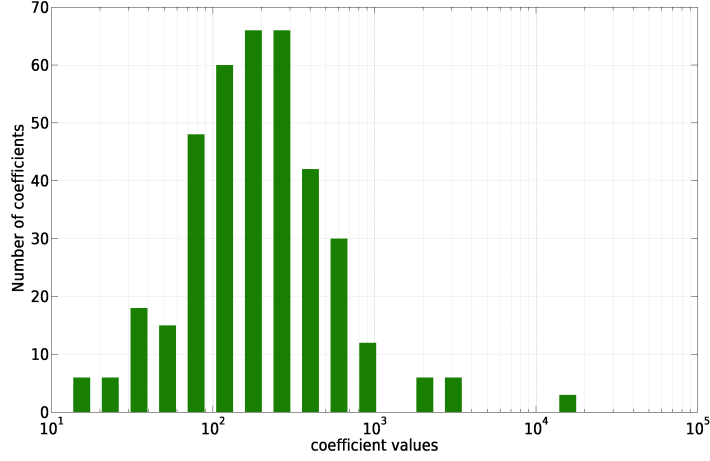
Table 3 contains the values of  $\text{PSNR}_H$  obtained for these parameters. In the noisy case (Table 3a),  $\text{PSNR}_H$  is only a little smaller than that of the noiseless case (Table 3b), which suggests that the method is robust to noise. To confirm this, a single case ( $K = 9$ ,  $c = 3$ ) is run for an increasing noise variance  $0 < \sigma^2 < 20$ . Figure 7 shows that when the noise variance increases (and PSNR between  $u$  and  $\alpha * H$  decreases),  $\text{PSNR}_H$  decreases at the same rate.

Moreover, in the presence of noise, increasing parameters  $K$  and  $S$  does not clearly improve  $\text{PSNR}_H$ . This is due to the lack of regularization, when  $K$  and  $c$  are large. We do not observe this phenomenon in Table 3c, which contains  $\text{PSNR}_H$  results for the noiseless case.

Tables 3b and 3d show, for the same experiments, the convergence criterion defined in (22). We observe that increasing  $K$  and  $c$  improves the criterion NRE, even in the noisy case, which is expected because of the conditioning of the convolution with  $\alpha$ . This simulation shows that it is possible to have a good reconstruction of the signal  $u$  with a poor approximation of the atom  $H$  when the convolution with  $\alpha$  is poorly conditioned.



(a) The code  $\alpha$ .



(b) Histogram of  $|\hat{\alpha}|$ , the modulus of the Fourier transform of the code.

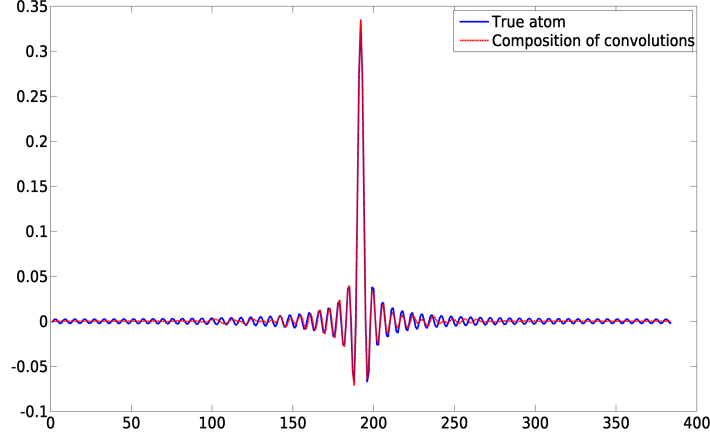
Figure 5: Code  $\alpha$  used in the approximation of a 1D sinc function.

Finally, it is interesting to test the stability of the proposed model to an imperfect knowledge of  $\alpha$ . For this purpose, a Gaussian noise  $b_\alpha \sim \mathcal{N}(0, \sigma_\alpha^2)$  has been added to the code  $\alpha$  used to solve  $(P_1)$  ( $u$  is still built with a noiseless  $\alpha$ ). We have set  $K = 9$  and  $c = 3$ , i.e.,  $S = 7$ , and have run the algorithm for several noise levels  $0 \leq \sigma_\alpha^2 \leq 15$ . Figure 8 shows that  $\text{PSNR}_H$  is stable with respect to  $\sigma_\alpha^2$ , even though NRE tends to increase with  $\sigma_\alpha^2$ . This suggests that the model is robust to an imperfect knowledge of  $\alpha$ .

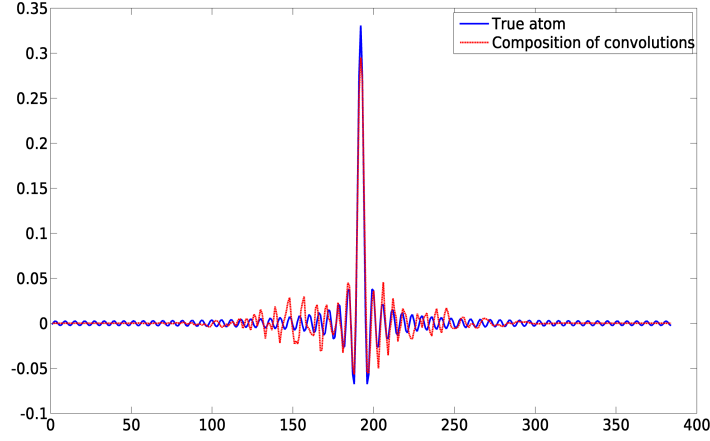
### 4.3 2D targets

#### 4.3.1 Curvelet

The aim of this experiment is to approximate a curvelet atom  $H$  in an image (i.e.,  $d = 2$ ) of size  $N \times N$  with  $N = 128$ . The curvelet is obtained by applying the inverse curvelet transform to a Dirac delta function, using the MCALAB toolbox [10]. The code  $\alpha$  corresponds to a Dirac delta function located



(a)  $\sigma^2 = 0$ ,  $\text{PSNR}_H = 44.47\text{dB}$ .



(b)  $\sigma^2 = 5$ ,  $\text{PSNR}_H = 35.68\text{dB}$ .

Figure 6: Approximation of a 1D sinc function with  $(K, c) = (9, 4)$  for different noise levels. The target sinc atom  $H$  and the composition of convolutions  $\lambda h^1 * \dots * h^K$ .

at the barycenter of the curvelet. Once again, the support mapping is the one described in (20), with either  $c = 1$  or  $c = 2$ . Note that this support mapping does not take the anisotropy of the curvelet into account. This is an unfavorable situation. All values of  $K$  satisfying  $3 \leq K \leq 11$  have been tested. We consider a noiseless case so that  $u$  is a simple translation of  $H$ . We have used  $R = 10$  restarts.

Figure 9 shows the target atom  $H$  and  $\lambda h^1 * \dots * h^K$ , for  $K = 7$  and  $c = 2$ . For these parameters, the size ratio between the effective support of the curvelet and the actual search space is  $G = 42.72$ . We observe that, although  $\text{PSNR}_H = 44.30$  dB, the accuracy of the approximation is not the same in different parts of the image. In particular, the tails of curvelet are not properly captured.

Table 4 contains the values of  $\text{PSNR}_H$  for various values of  $K$  and  $c$ . In this experiment, we were expecting that increasing  $K$  and  $S$  would improve the accuracy. It is not exactly what we observe in Table 4. For  $S = 5 \times 5$ , increasing  $K$  beyond a certain value actually makes  $\text{PSNR}_H$  decrease. This

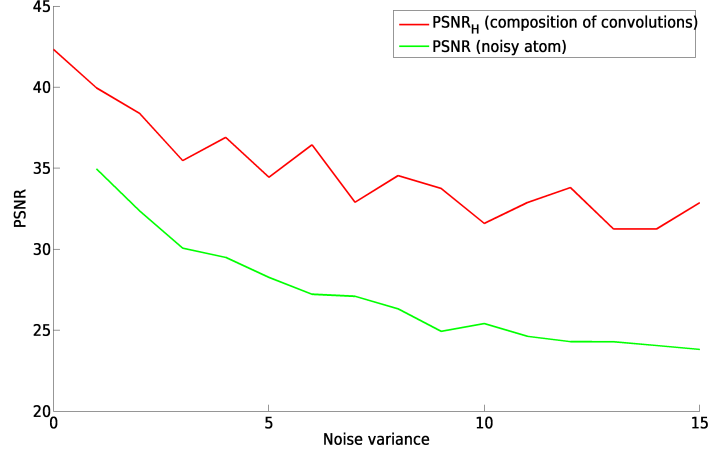


Figure 7: Evolution of  $\text{PSNR}_H$ , for the sinc target atom, with respect to the noise variance  $\sigma^2$ , for  $K = 9$ ,  $c = 3$ .

$\text{PSNR}_H(\text{dB})$	$K = 3$	$K = 5$	$K = 7$	$K = 9$	$\text{Conv} \times 10^{-3}$	$K = 3$	$K = 5$	$K = 7$	$K = 9$
$c = 1$	31.66	33.14	34.53	35.77	$c = 1$	5.0	4.6	4.1	4.1
$c = 2$	37.34	38.03	37.32	36.67	$c = 2$	3.5	3.5	3.4	3.4
$c = 3$	37.69	37.61	36.63	36.82	$c = 3$	3.5	3.4	3.3	3.2

(a) $\text{PSNR}_H$ for $\sigma^2 = 5$ .					(b) Convergence criterion Conv for $\sigma^2 = 5$ .				
$\text{PSNR}_H(\text{dB})$	$K = 3$	$K = 5$	$K = 7$	$K = 9$	$\text{Conv} \times 10^{-3}$	$K = 3$	$K = 5$	$K = 7$	$K = 9$
$c = 1$	31.46	33.91	32.89	33.79	$c = 1$	2.0	1.2	1.2	1.1
$c = 2$	37.59	38.95	39.29	39.49	$c = 2$	0.3	0.2	0.2	0.2
$c = 3$	39.14	41.86	41.93	42.07	$c = 3$	0.2	0.1	0.1	0.1

(c) $\text{PSNR}_H$ for $\sigma^2 = 0$ .					(d) Convergence criterion Conv for $\sigma^2 = 0$ .				
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Table 3: Accuracy and convergence criteria for the sinc approximation experiment, in the noise-free and noisy cases, with  $R = 50$  restarts.

$\text{PSNR}_H(\text{dB})$	$K = 3$	$K = 5$	$K = 7$	$K = 9$	$K = 11$
$S = 3 \times 3$	33.06	36.55	36.52	37.22	37.01
$S = 5 \times 5$	39.99	45.81	44.30	40.74	38.05

Table 4: Curvelet approximation:  $\text{PSNR}_H$  for several values of  $K$  and  $S$ .

NRE	$K = 3$	$K = 5$	$K = 7$	$K = 9$	$K = 11$
$c = 1$	1.99	0.89	0.90	0.76	0.80
$c = 2$	0.40	0.11	0.15	0.34	0.63

Table 5: Curvelet approximation: NRE for several values of  $K$  and  $c$ .

result can be explained by a lack of convergence, as is confirmed in Table 5. This problem could easily be corrected by an initialization exploiting the results obtained for smaller values of  $K$  and  $c$ .

Finally, Figure 10 shows the kernels  $(h^k)_{1 \leq k \leq K}$  computed for  $K = 7$  and  $S = 5 \times 5$ . We can observe that many kernel coefficients are close to zero, i.e., only the coefficients along the main direction of the curvelet have significant values. It is obvious that the simple isotropic dilation of the supports defined

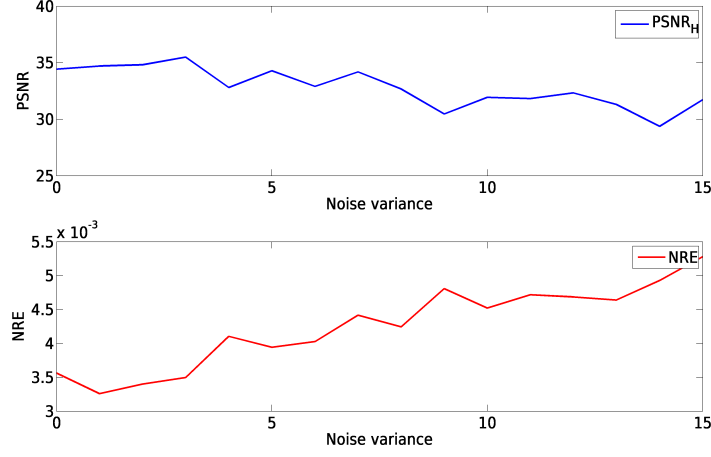


Figure 8: Evolution of  $\text{PSNR}_H$  and NRE, for the sinc target atom, with respect to the noise variance  $\sigma_\alpha$  on the code  $\alpha$ , for  $K = 9$ ,  $c = 3$ .



(a) Approximation  $\lambda h^1 * \dots * h^7$

(b) True curvelet  $H$

Figure 9: Curvelet approximation with  $K = 7$  and  $S = 5 \times 5$ . Comparison between  $\lambda h^1 * \dots * h^7$  and the target curvelet atom  $H$ . We have  $\text{PSNR}_H = 44.30$ .

by (20) is not appropriate for this curvelet. This raises the question of the adaptation of the support mappings  $(S^k)_{1 \leq k \leq K}$  to the atom's geometry.

#### 4.3.2 Cosine

The aim of this experiment is to approximate an atom representing a 2D cosine function in an image of size  $64 \times 64$  (i.e.,  $d = 2$  and  $N = 64$ ). In the context of image processing, such an atom can be seen as a large local cosine or a Fourier atom. Both are widely used in image processing. The interest of this atom is that it covers the whole image and is of a rather large support. Beside, patches of this size are

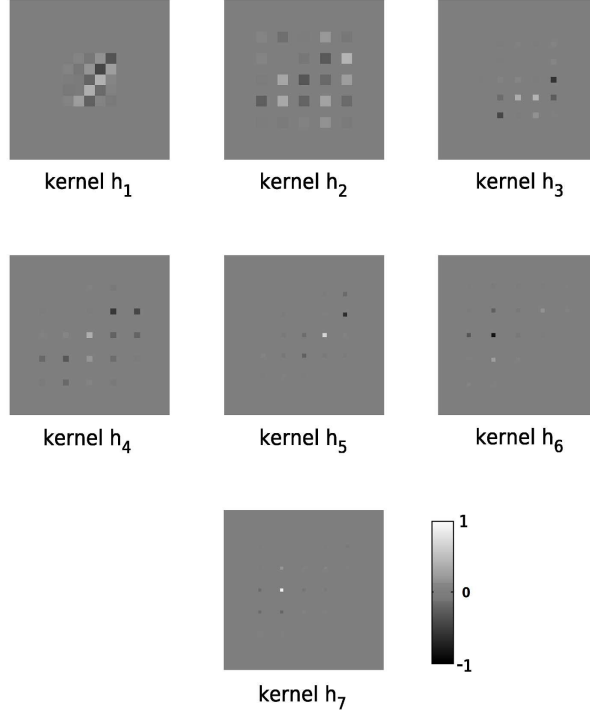


Figure 10: Curvelet approximation for  $K = 7$  and  $S = 5 \times 5$ . Zoom on the computed kernels  $(h^k)_{1 \leq k \leq 7}$ . The colormap is flattened around 0 to highlight the higher coefficients.

difficult to handle with existing dictionary learning strategies. The considered atom is given by

$$H_p = \cos \left( 2\pi \frac{\langle p, (2, 5) \rangle}{N} \right), \forall p \in \{0, \dots, 63\}^2.$$

The code  $\alpha$  is a sparse vector whose support elements are randomly chosen. More precisely, for all  $p \in \mathcal{P}$ , there is a probability  $10^{-1}$  that  $\alpha_p \neq 0$ . The values of the non-zero elements are then set according to the centered normal distribution  $\mathcal{N}(0, 1)$ . In other words, for a given  $p$ , the elements of code  $\alpha_p$  are assumed to be independent and identically distributed according to a Bernoulli-Gaussian distribution, that has been widely used in sparse signal and image deconvolution [5][29][11]. Therefore,  $u$  contains a few weighted translations of the cosine atom  $H^8$ , which should result in a better approximation of  $H$  using Algorithm 2. Figure 11 shows the code and the histogram of its Fourier transform. Note that the ratio between the largest and the smallest Fourier coefficients (in modulus) is 91, which corresponds to a reasonable conditioning. The target  $u$  is built with additive Gaussian noise of variance  $\sigma^2 = 0.5$ , which corresponds to a normalized PSNR $_H$  between  $\alpha * H$  and  $u$  of 22.08.

The support mapping is the same as for the previous experiment (see (20)) with, either  $c = 1$  ( $S = 3 \times 3$ ) or  $c = 2$  ( $S = 5 \times 5$ ). Different Values of  $K$  have been tested in the range  $3 \leq K \leq 11$ , each time with  $R = 15$  restarts.

Table 6 provides the PSNR $_H$  and NRE indicators in the studied range of parameters. In this experiment, we expect to obtain a somewhat regularized atom thanks to the repetitions induced by the sparse (and reasonably conditioned) code  $\alpha$ . We observe in Table 6 that PSNR $_H$  rises above 30 if parameters  $K$  and  $S$  are large enough. Even for  $K = 9$  and  $c = 2$ , the ratio between the number of variables describing the kernels and the size of the cosine is  $G = \frac{64 \times 64}{9 \times 5 \times 5} = 18.20$ .

<sup>8</sup>A sum of cosines of same frequency and different phases will yield a cosine of unchanged frequency.

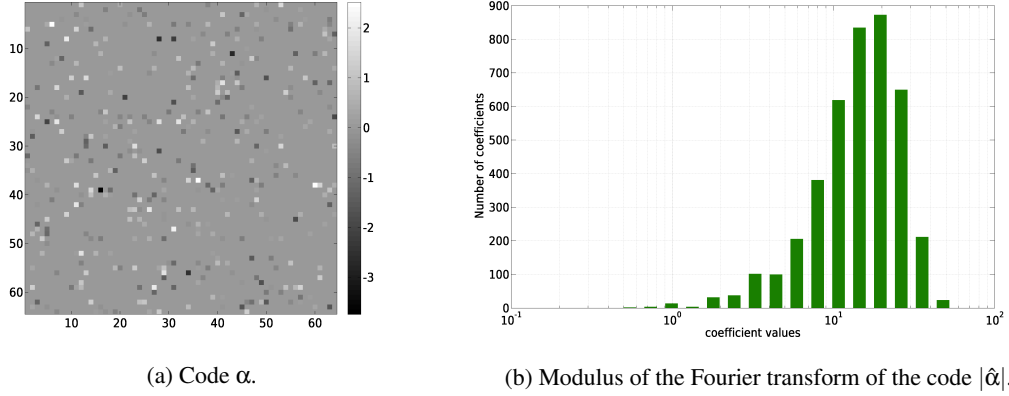


Figure 11: Cosine experiment: code  $\alpha$  and its conditioning.

PSNR <sub>H</sub> (dB)	$K = 3$	$K = 5$	$K = 7$	$K = 9$	$K = 11$
$c = 1$	11.79	12.27	13.81	25.15	30.09
$c = 2$	11.94	15.97	41.44	38.94	39.82

(a) 2D Cosine: PSNR<sub>H</sub>.

NRE	$K = 3$	$K = 5$	$K = 7$	$K = 9$	$K = 11$
$c = 1$	1.02	0.89	0.41	0.04	0.02
$c = 2$	0.96	0.24	0.01	0.01	0.01

(b) 2D Cosine: NRE.

Table 6: 2D Cosine approximation: PSNR<sub>H</sub> and NRE for several values of  $K$  and  $c$ .

Figure 12 shows the cosine image  $u$ , its approximation  $\lambda\alpha * h^1 * \dots * h^K$ , the actual atom  $H$  and  $\lambda h^1 * \dots * h^K$ , for  $K = 7$  and  $c = 2$ . The results obtained here are quite accurate even though the cosine image was corrupted by additive noise.

This experiment was also run with fixed  $K = 7$  and  $c = 2$  for an increasing noise variance, to test the robustness of the proposed model. Figure 13 shows the values of PSNR<sub>H</sub> associated with the reconstructed image, as a function of the noise variance. Note that PSNR<sub>H</sub> decreases at the same rate as the PSNR measuring the degradation between  $u$  and  $\alpha * H$ .

### 4.3.3 Wavelet decomposition

In this experiment, we consider a scenario reflecting the difficulties of DL. More precisely, we consider  $d = 2$ ,  $N = 512$ , a target atom  $H$  defined as a wavelet atom and a code  $\alpha$  resulting from the wavelet coefficients of a natural image. More precisely, the following operations have been conducted

- Select an image (here the Barbara image).
- Compute the wavelet transform of the image using the Daubechies wavelet db4 at level  $L$ . We used the official Matlab wavelet toolbox with  $L = 3$ .
- Select the set of coefficients associated with an orientation and a given decomposition level  $l$  such that  $1 \leq l \leq L$ . The low frequency at level  $l = L = 3$  was considered for the first experiment and the horizontal detail at level  $l = 3$  for the second experiment.
- Set the non selected wavelet coefficients to zero and compute the inverse wavelet transform. Add white Gaussian noise of variance  $\sigma^2 = 5$  to obtain  $u$ .
- Define  $\alpha$  as a zoom of factor  $2^l$  of the selected coefficients, where the zoom consists of interpolating with zeros.

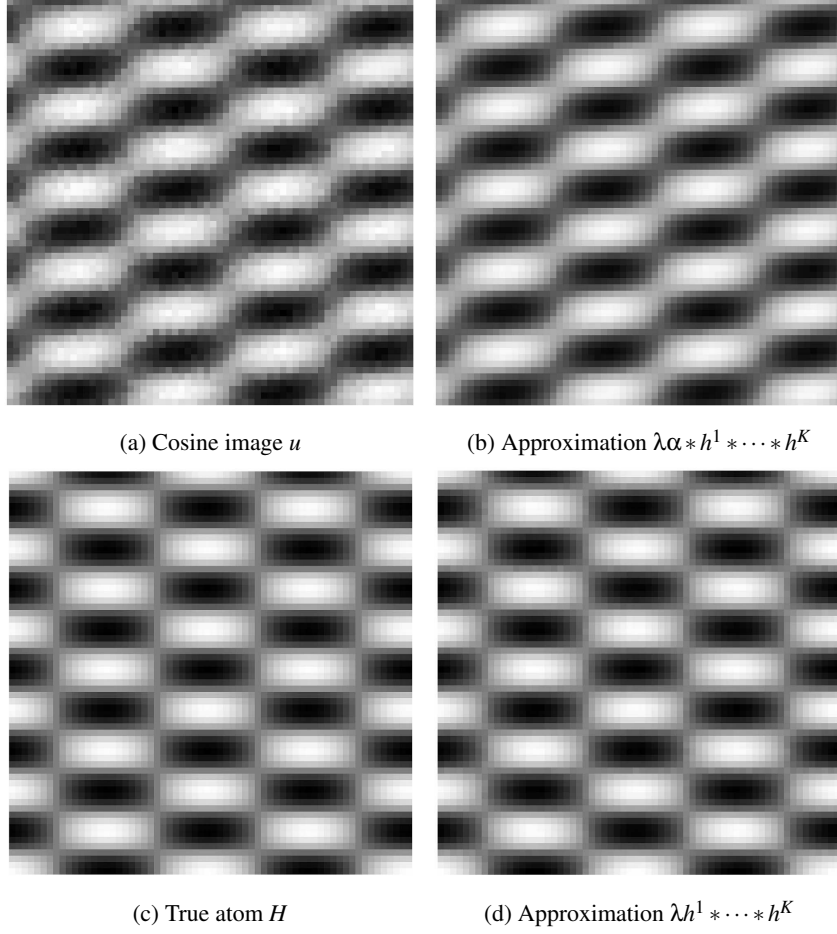


Figure 12: Cosine approximation with  $K = 7$ ,  $c = 2$ , and Gaussian noise of variance  $\sigma^2 = 0.5$ .



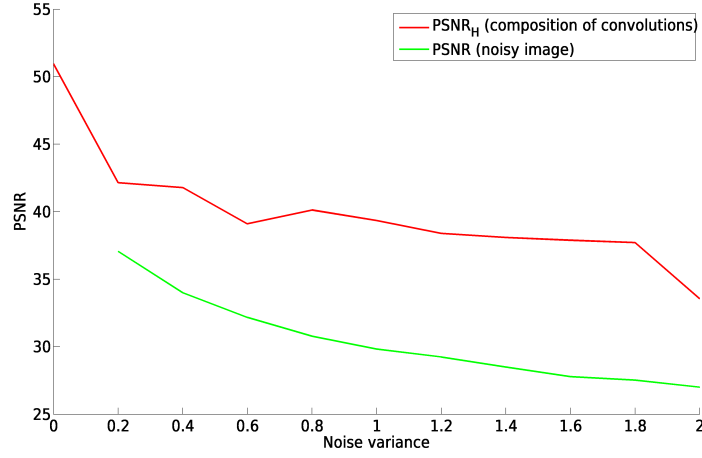


Figure 13: Evolution of  $\text{PSNR}_H$  of a reconstructed cosine atom when the noise variance  $\sigma^2$  varies in  $[0, 2]$ , for  $K = 6$  and  $c = 2(S = 5 \times 5)$ .

- Solve problem  $(P_k)$  with the code  $\alpha$ , the target atom  $u$ ,  $R = 1$ , with a support mapping defined by (20) for the parameters  $K = 6$  and  $S = 3 \times 3$  (i.e.  $c = 1$ ). Note that the knowledge of the supports associated with the composition of convolutions leading to the wavelet atom was not used in this experiment.

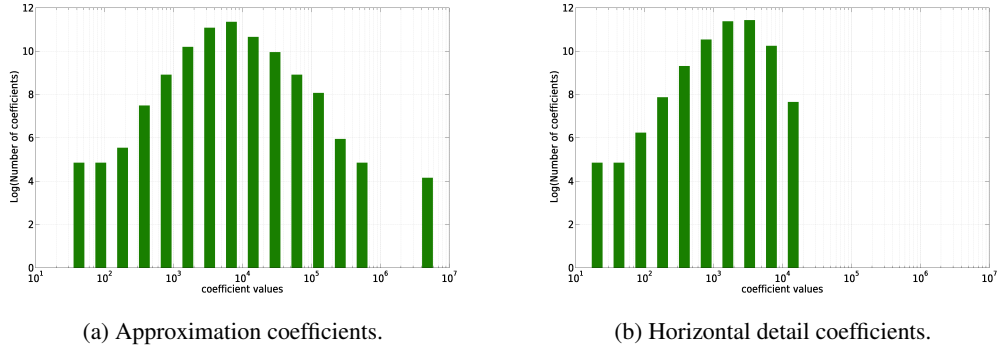


Figure 14: Logarithm of the histogram of  $|\hat{\alpha}|$ , the modulus of the Fourier transform of the code.

The results obtained for the low frequency wavelet atom at level 3 and the horizontal detail wavelet atom are shown in Figures 15 and 16. Note that the conditioning of the convolution with  $\alpha$  is less favorable for the estimation of the low frequency wavelet atom. Indeed,  $\alpha$  is sparser when selecting detail coefficients, which results in a better conditioned problem. Figure 14 shows histograms of  $|\hat{\alpha}|$  for both cases. Note that the ratio between the largest and the smallest Fourier coefficients (in modulus) is 532 for the horizontal detail case, and  $6.67 \times 10^4$  in the approximation case.

For both experiments (shown in Figures 15 and 16), the composition of convolutions is close to the corresponding wavelet atom. However,  $\text{PSNR}_H$  is larger for the horizontal detail case, which suggests that a good conditioning for  $\alpha$  is crucial. Unsurprisingly, convergence is better for the approximation coefficients case ( $\text{NRE} = 10^{-3}$ ) than for the horizontal detail case ( $\text{NRE} = 0.68$ ), though our primary

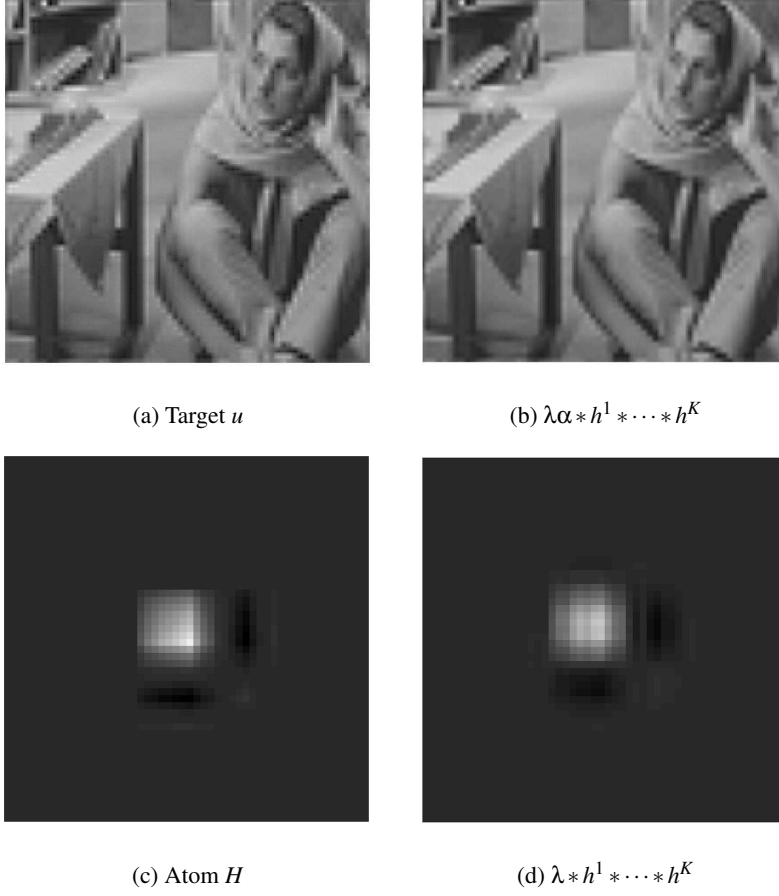


Figure 15: Estimation of the low frequency wavelet atom at level 3 (NRE =  $10^{-3}$  and  $\text{PSNR}_H = 29.94$ ).

concern remains accuracy on atom reconstruction. Note, however, that we did not run multiple restarts: a better convergence could still be achieved by increasing  $R$ .

Finally, we rerun both experiments with additive Gaussian noise of variance  $\sigma_\alpha^2 = 10$  corrupting the code  $\alpha$ . Note that this noise degrades the conditioning of the convolution with  $\alpha$ . For the horizontal detail coefficients, we obtain  $\frac{\max|\hat{\alpha}|}{\min|\hat{\alpha}|} = 4.99 \times 10^3$ , NRE = 1.62 and  $\text{PSNR}_H = 29.09$ . For the approximation coefficients, we obtain  $\frac{\max|\hat{\alpha}|}{\min|\hat{\alpha}|} = 8.47 \times 10^4$ , NRE = 0.002 and  $\text{PSNR}_H = 27.17$ . The horizontal detail case still gives a better result, as expected. Both approximations  $\lambda h^1 * \dots * h^K$  are shown in Figure 17.

## 5 Convergence Assessment

### 5.1 Simulation Scenario

This section evaluates

$$\mathbb{P}(\text{not global}) \quad \text{and} \quad R_\epsilon = \frac{\log(\epsilon)}{\log(1 - \mathbb{P}(\mathbf{h} \in \mathbb{I}))}$$

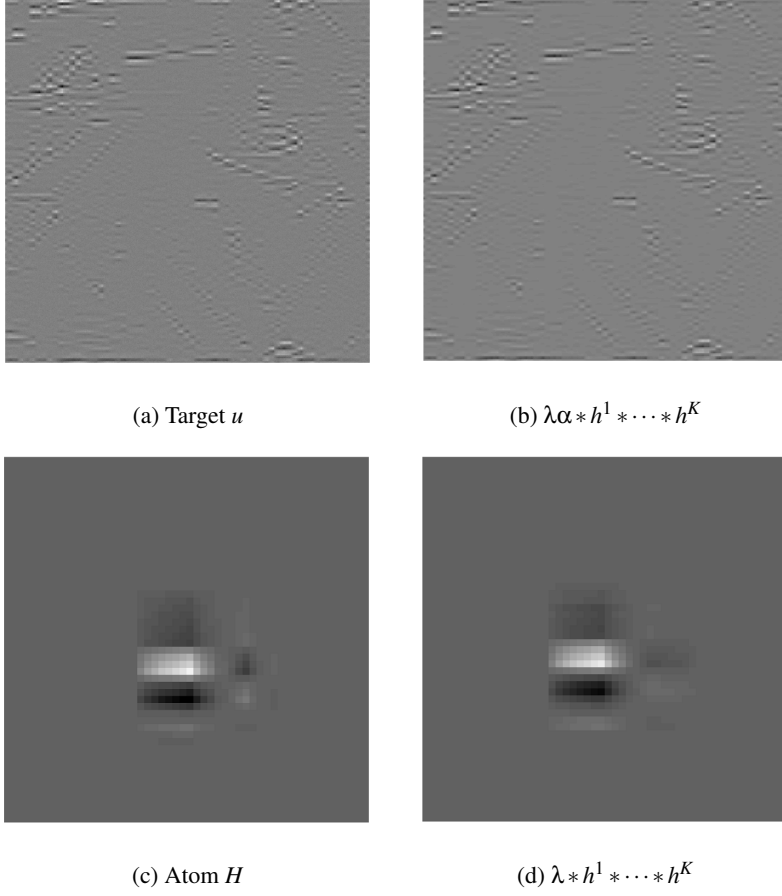


Figure 16: Estimation of the horizontal detail wavelet atom at level 3 (NRE = 0.68 and PSNR<sub>H</sub> = 36.61).

for various supports, kernels and noise levels. All the experiments have been conducted with one-dimensional signals of size  $\#P = 128$  and  $(K, S) \in \{2, \dots, 7\} \times \{2, \dots, 10\}$  and random support mappings  $\mathbf{S} = (S^k)_{1 \leq k \leq K}$ . For every  $k \in \{1, \dots, K\}$ , the support mapping  $S^k$  maps  $\{1, \dots, S\}$  into  $S$  distinct elements randomly drawn according to a uniform distribution in  $\{1, \dots, 10\}$ . Moreover, for any  $(k_1, k_2) \in \{1, \dots, K\}^2$ , with  $k_1 \neq k_2$ ,  $\text{rg}(S^{k_1})$  and  $\text{rg}(S^{k_2})$  are independent random vectors. We also consider  $K$  independent random kernels

$$h_p^k \begin{cases} \sim \mathcal{N}(0, 1) & , \text{ if } p \in \text{rg}(S^k) \\ = 0 & , \text{ otherwise.} \end{cases}$$

Finally, the code is set to  $\alpha = (1, 0, \dots, 0)$  (i.e., no translation) and the image  $u$  is obtained by convolving the kernels, i.e.,

$$u = \alpha * h^1 * \dots * h^K + b$$

where  $b \sim \mathcal{N}(0, \sigma^2 \mathbb{1}_S)$ ,  $\sigma^2$  is the noise variance and the set  $S$  is the “reachable support” defined in (3). Note that  $u$  is zero outside of the reachable support.

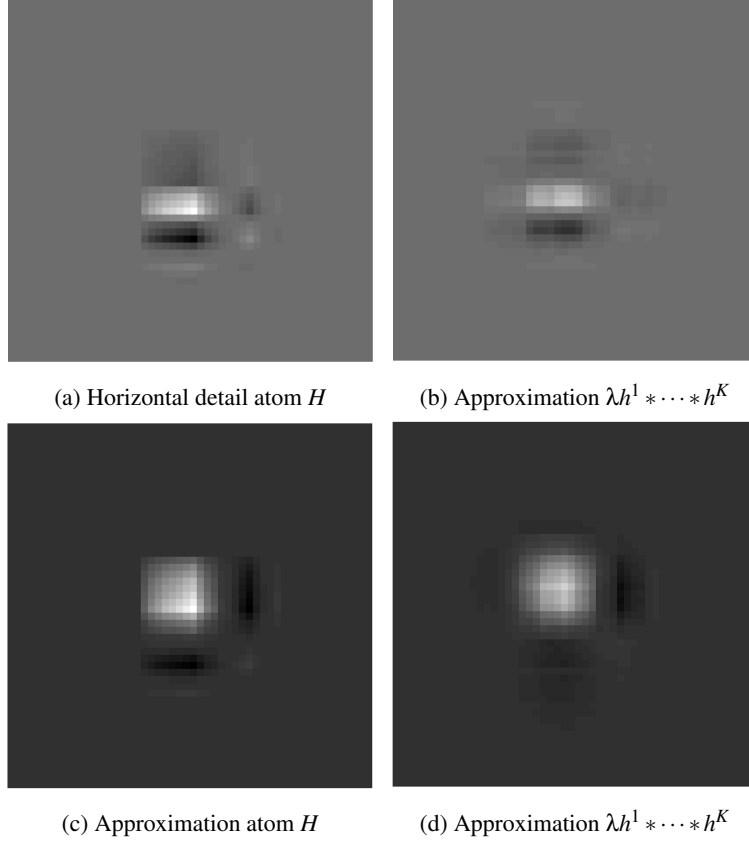


Figure 17: Approximations  $\lambda \alpha * h^1 * \dots * h^K$  obtained with a noisy code  $\alpha$  ( $\sigma_\alpha^2 = 10$ ). Horizontal detail atom (a) and its approximation (b) (PSNR<sub>H</sub> = 29.09, NRE = 1.62). Approximation atom (c) and its approximation (d) (PSNR<sub>H</sub> = 27.17, NRE = 2.10<sup>-3</sup>).

## 5.2 Performance measure

Given a problem defined by  $(u, \alpha, \mathbf{S})$ , a global minimizer  $\mathbf{h}^* = (h^{*,k})_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$  of  $(P_0)$  and a solution  $\bar{\mathbf{h}} = (\bar{h}^k)_{1 \leq k \leq K} \in (\mathbb{R}^{\mathcal{P}})^K$  provided by Algorithm 2, we denote the approximation error by

$$E_a(u, \alpha, \mathbf{S}) = \|\alpha * h^{*,1} * \dots * h^{*,K} - u\|_2^2.$$

For the problem constructed in the previous paragraph, we expect that

$$E_a(u, \alpha, \mathbf{S}) \leq \sigma^2 (\#\mathcal{S}),$$

where  $\sigma^2$  is the noise variance. Moreover, we know that  $E_a(u, \alpha, \mathbf{S}) = 0$  for  $\sigma = 0$ . We also denote the numerical error by

$$E_n(\bar{\mathbf{h}}, u, \alpha, \mathbf{S}) = \|\alpha * \bar{h}^1 * \dots * \bar{h}^K - u\|_2^2 - E_a(u, \alpha, \mathbf{S}).$$

The only quantity that we can actually observe is the sum of these two errors

$$\|\alpha * \bar{h}^1 * \dots * \bar{h}^K - u\|_2^2 = E_a(u, \alpha, \mathbf{S}) + E_n(\bar{\mathbf{h}}, u, \alpha, \mathbf{S}).$$

We therefore consider that Algorithm 2 has converged to a global minimum if

$$\|\alpha * \bar{h}^1 * \dots * \bar{h}^K - u\|_2^2 \leq \sigma^2 (\#\mathcal{S}) + 10^{-4} \|u\|_2^2. \quad (23)$$

Of course, this notion is not very accurate when  $\sigma^2$  is large.

### 5.3 Evaluation of $\mathbb{P}$ (not global)

For any fixed  $(K, S) \in \{2, \dots, 6\} \times \{2, \dots, 10\}$ , we have generated  $L = 50K^2$  signals. Each signal is labelled by an index  $l \in \{1, \dots, L\}$ . For every experiment, we consider  $R = 25$  random initializations according to a uniform distribution defined on the set of constraints associated with  $(P_1)$ , as described in Section 3.5. The corresponding outcome of Algorithm 2 is referred to as the  $r$ th result with  $r \in \{1, \dots, R\}$ . Finally, for any  $(l, r) \in \{1, \dots, L\} \times \{1, \dots, R\}$ , we introduce the following indicator function

$$\mathbb{1}(l, r) = \begin{cases} 1, & \text{if (23) holds for the } r\text{th result obtained from the } l\text{th input,} \\ 0, & \text{otherwise.} \end{cases}$$

The probability of reaching a global minimum of problem  $(P_1)$  is estimated as follows

$$\mathbb{P}(\text{global minimizer}) \simeq \frac{1}{LR} \sum_{l=1}^L \sum_{r=1}^R \mathbb{1}(l, r).$$

### 5.4 Results

Figures 18 and 19 show the results obtained in the noiseless ( $\sigma^2 = 0$ ) and noisy ( $\sigma^2 = 5 \times 10^{-2}$ ) cases respectively. In each figure, the curves show  $\mathbb{P}(\text{global minimizer})$  and the number of restarts needed to ensure a failure probability lower than  $\epsilon$ ,  $R_\epsilon = \frac{\log(\epsilon)}{\log(1 - \mathbb{P}((h^k)_{1 \leq k \leq K} \in \mathbb{I}))}$  for a given value of  $K$  whereas the  $x$  axis indicates the support size  $S$ . We can see that for very sparse kernels ( $S \leq 3$ ), the probability of success is quite high. However, this probability drops significantly when the support size increases. Surprisingly,  $\mathbb{P}(\text{global minimizer})$  increases when the support size increases. The more kernels we use (i.e., the larger  $K$ ), the steeper the decrease and increase. These results show that it is possible to obtain convergence to a global minimum with only a few restarts of the proposed algorithm even for relatively large values of  $K$ . The last experiments obtained in the noisy case show similar patterns. As a consequence, the described convergence properties seem to be robust to noise.

## 6 Conclusions

We introduced a new important problem whose purpose is to mitigate the computational issues encountered in most dictionary learning frameworks (which generally constrain the use of small patches). We proposed to consider atoms defined as a composition of convolutions with sparse kernels. The determination of these atoms required to solve a non-convex optimization problem. Using the sparsity of kernels to reduce the search space, we studied a computationally efficient algorithm based on alternate least squares minimizations. This algorithm has linear complexity with respect to the image size. It allows the learning of fast transforms by the dictionary update stage and permits to consider larger atoms. Our experiments showed that compositions of convolutions can approximate accurately many atom-like signals and images such as curvelets, cosines, wavelets, ... However, the full potential of these compositions of convolutions for approximation purposes still remains to be assessed.

Future work includes the definition of a tree structure for the proposed composition of kernel convolutions for dictionary learning applications. Designing efficient rules to learn the kernel supports also remains a large and unexplored issue which might have a huge impact on the performance of the proposed algorithm.

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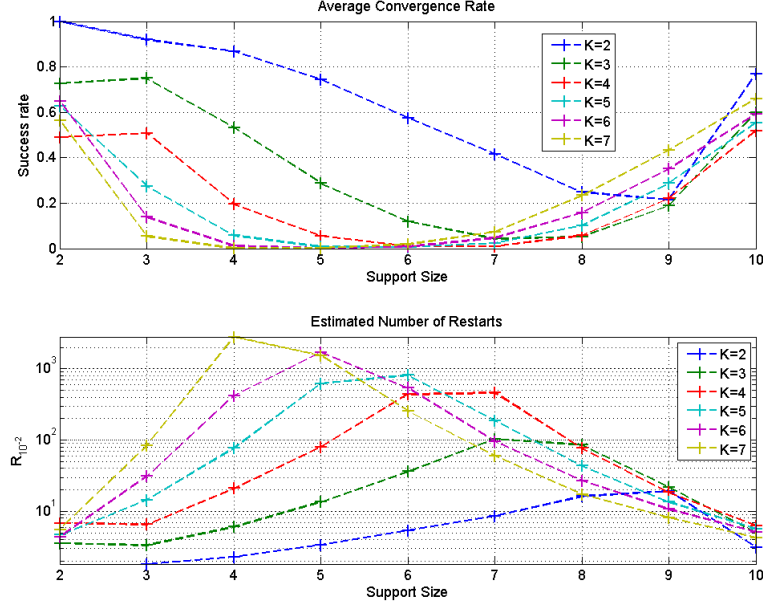


Figure 18: Convergence test for  $\sigma = 0$ : Estimated probability of reaching a global minimum (top) for every  $K \in \{2, \dots, 7\}$  and corresponding number of restarts  $R_e$  to guarantee  $\mathbb{P}(\text{global minimizer}) \geq 99\%$  (bottom). For every  $K \in \{2, \dots, 7\}$ , the results have been averaged over  $L = 50K^2$  inputs from which we have computed  $R = 25$  outputs.

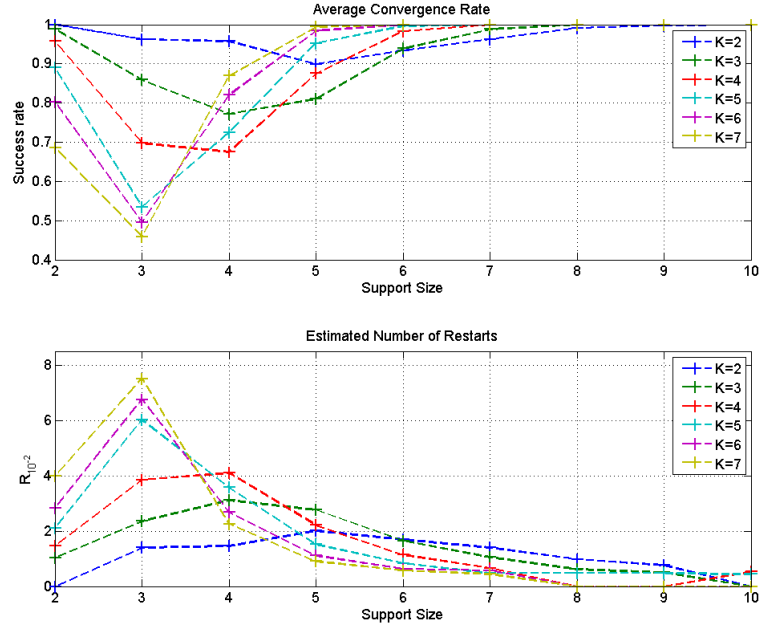


Figure 19: Convergence test for  $\sigma^2 = 5.10^{-2}$ : Estimated probability of reaching a  $L_2$  ball of radius  $\sigma\sqrt{\#S}$  around a global minimum (top) for every  $K \in \{2, \dots, 6\}$  and corresponding number of restarts  $R_e$  to guarantee  $\mathbb{P}(\text{global minimizer}) \geq 99\%$  (bottom). For every  $K \in \{2, \dots, 6\}$ , the results have been averaged over  $L = 50K^2$  inputs, from which we have computed  $R = 25$  outputs.

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