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Pascal Romon

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# ANNALES DE L'INSTITUT FOURIER

PASCAL ROMON

**A rigidity theorem for Riemann's minimal surfaces**

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## A RIGIDITY THEOREM FOR RIEMANN'S MINIMAL SURFACES

by Pascal ROMON

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### 1. Introduction.

We are interested here in properly embedded minimal surfaces in  $\mathbb{R}^3$ , which are invariant under a translation  $\tau$ . Much has been proven about complete embedded minimal surfaces in general (see [6], [7], [8] for example) and singly-periodic ones in particular ([1], [2], [5]). However the classification of examples satisfying certain geometric constraints, like fixed genus or number of ends, is still open in almost all cases.

The most famous of these surfaces are Riemann's Examples (see fig. 1), which are, with the catenoid, the only minimal surfaces fibered by circles. However, while the catenoid is fibered by circles only, Riemann's Examples' circles tend periodically to parallel lines. Two successive lines bound an annulus. Using Schwarz Reflection, we can extend this annulus by rotation around these lines and their images, in order to reconstruct the complete surface  $\mathcal{R}$ . The translation  $\tau$  induced by rotation around two successive lines leaves the surface invariant. In fact there is a one-parameter family of singly-periodic minimal surfaces. A more detailed construction of  $\mathcal{R}$  will be found in section 2 and in [3].

A natural question to ask is whether there are any non trivial  $\epsilon$ -deformations of Riemann's Examples (keeping some straight lines fixed), or if all minimal planar domains bounded by straight lines and lying in a slab belong in this family. The latter was proven in [3] to be true if the

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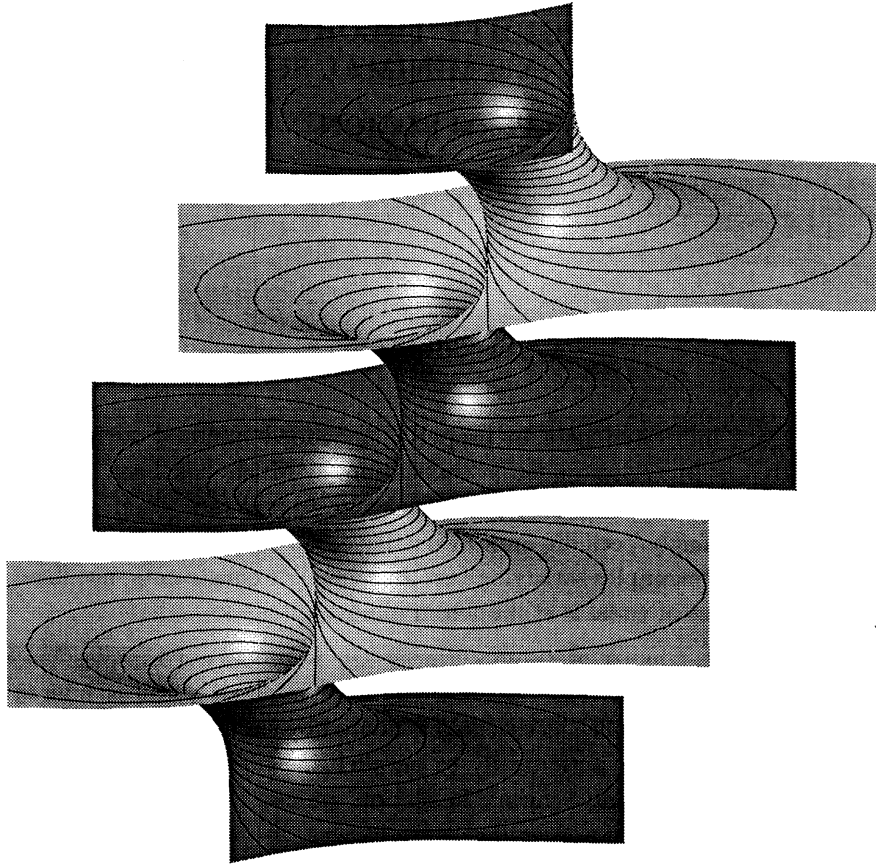


Fig. 1. Part of a Riemann Example with its level curves, which are circles.

lines are parallel and the total curvature of the domain is  $4\pi$  <sup>(1)</sup>. We will now generalize that result to the following :

**THEOREM 1.** — *If  $A$  is a properly embedded minimal annulus, punctured in a single point, that lies between two parallel planes, and is bounded by two parallel lines in these planes, then  $A$  is a piece of a Riemann Example.*

Equivalently, we can state a result about minimal surfaces properly embedded in  $\mathbb{R}^3$  modulo a translation, since they are in 1-1 correspondance with singly-periodic embedded minimal surfaces.

**THEOREM 2.** — *Any complete minimal torus embedded in some  $\mathbb{R}^3/\tau\mathbb{Z}$  with planar ends and two straight lines, and total curvature  $16\pi$  or less, is a covering of a Riemann Example.*

*Proof of Theorem 2.* — Let  $M$  be such a surface. After a rotation in  $\mathbb{R}^3$ , if necessary, we may assume that its ends are all horizontal. The number of ends is even because  $M$  is embedded and thus orientable (see [3]). Using the Callahan-Hoffman-Meeks formula (see [2])

$$\int_M |K| dA = 4\pi(r + \gamma - 1)$$

where  $\gamma$  is the genus of  $M$  and  $r$  the number of ends, we deduce the following : if the total curvature of  $M$  is  $8\pi$ , there are only two ends, which obviously correspond to each of the two lines. Furthermore, the degree of the Gauss map  $g$  is 2, by the Gauss-Bonnet Theorem. If, on the other hand, the total curvature is  $16\pi$ , then there are four ends and the degree of the Gauss map  $g$  is 4. Since the ends are planar, the Gauss map is branched at each one of them, and takes the values 0 and  $\infty$  alternatively. Thus their branching order is exactly 1.

Since  $\text{genus}(M)=1$ , any fundamental domain is planar, and so is its intersection with a horizontal slab. Namely, if there were topology in any subdomain, say genus  $\gamma$ , then the genus of  $M$  would be at least  $\gamma + 1$ . So if  $\mathcal{P}$  is a limit plane for an end, its intersection with  $M$  must be a simple infinite curve. The genus 1 assumption forbids any closed loop, and if there were another curve going to infinity, then the branching order of  $g$  would be strictly greater than 1, which is excluded. Recall that the order of the

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<sup>(1)</sup> It was also proven by Toubiana [11] and then by Pérez and Ros [9] that the lines have to be parallel, provided they lie in parallel planes.

Gauss map at a finite point equals the number of level curves crossing at that point, minus 1, and at a planar end, the order is equal to the number of curves going to that end divided by 2. Consider now the two horizontal lines : rotation around them generates the whole surface, so that they bound a fundamental piece of the surface, lying between two horizontal planes. Furthermore, it is a planar domain, punctured at most once. So we are back to Theorem 1.

I also want to thank Harold Rosenberg, David Hoffman and Fu Sheng Wei for their support and fruitful discussions.

## 2. Riemann's Examples.

### 2.1. Some notations.

We will first recall briefly the construction of Riemann's Examples. Let  $\mathbb{T}$  denote the rectangular torus whose vertices are  $(0, \mu, \mu + i, i)$  in the complex plane ( $\mu \in \mathbb{R}_+^*$ ). We can associate to this torus the lattice generated by  $(2\omega_1, 2\omega_3)$ , where  $(\omega_1, \omega_2, \omega_3)$  are defined as  $\left(\frac{\mu}{2}, \frac{\mu + i}{2}, \frac{i}{2}\right)$  as usual. Let  $P$  be the unique elliptic function having a double pole at 0, a double zero at  $\omega_2$  (and no other pole or zero), and two other branch points at  $\omega_1$  and  $\omega_3$ , whose respective values are  $\lambda, -1/\lambda$ , where  $\lambda$  is a positive constant characterizing the lattice. Then  $P$  has the following properties of symmetry :  $P(-z) = P(z)$  and  $P(\bar{z}) = \overline{P(z)}$ . Furthermore it satisfies the functional equation :  $P'(z)^2 = C_\lambda P(P - \lambda)(P + 1/\lambda)$ , for some positive real  $C_\lambda$  depending on the lattice. The function  $P$  will allow us to define the Weierstrass-Enneper representation of Riemann's Example on this torus. For more details about this function, see [3].

### 2.2. The Weierstrass-Enneper representation.

We define a minimal immersion  $R : \mathbb{T} \rightarrow \mathbb{R}^3$  up to translation by :

$$R(z) = \operatorname{Re} \int^z (1 - g^2, i(1 + g^2), 2g) \eta$$

where  $g = P$  is a degree two meromorphic map from  $\mathbb{T}$  to  $S^2$  corresponding to the stereographic projection of the Gauss map of the surface, and  $\eta = i dz/g$  is a holomorphic 1-form defined on  $\mathbb{T}$  minus the ends. The

ends will naturally be 0 and  $\omega_2$ , and then we define  $\tau$  to be the translation of vector

$$\operatorname{Re} \oint (1 - g^2, i(1 + g^2), 2g)\eta$$

the integration being taken along any "vertical" simple loop, that is homologous to the segment  $[0, i]$  oriented upward. Then  $R$  induces a proper finite total curvature embedding of the torus in  $\mathbb{R}^3/\tau$ . This embedding also has the following properties : each horizontal curve on  $\mathbb{T}$  is mapped to a horizontal circle, except for the heights 0 and  $1/2$ , where the curves are mapped to horizontal lines. There is a natural rotational symmetry around these lines (see fig. 1). Conversely, we can state the uniqueness theorem proven in [3].

**THEOREM 3.** — *Let  $A$  be a properly embedded minimal annulus bounded by two parallel lines; assume further that  $A$  lies between two planes with one line in each plane. Then there exists a real constant  $\mu$  such that  $A$  lies in the image of  $R$ , up to obvious isometries.*

### 3. General properties.

#### 3.1. Extension of the surface.

Let us now prove Theorem 1. We will start with two horizontal parallel lines  $L_0$  and  $L_1$  bounding a properly embedded minimal annulus  $A$  with one end. We also suppose that  $A$  lies between two horizontal planes, each containing  $L_0$  or  $L_1$  respectively. Applying Schwarz Reflection Principle, we can rotate by  $180^\circ$  around those lines in order to extend the surface. Repeating this procedure, we get a complete orientable minimal surface  $S$ , with parallel horizontal ends. If we define the translation  $\tau$  as the composition of the two basic rotations around  $L_0$  and around  $L_1$ , then the quotient  $M = S/\tau\mathbb{Z}$  is an *orientable, genus 1, embedded minimal surface, with four horizontal ends*. It is a general result in minimal surface theory ([8]) that any complete finite topology properly embedded minimal surface is conformally equivalent to a punctured Riemann surface. Our purpose will now be to prove that  $M$  is conformally a rectangular torus.

### 3.2. The first conformal representation.

Let  $\sigma$  denote the  $180^\circ$  rotation in space around  $L_1$ ; then  $B = A \cup \sigma(A)$  is a fundamental domain of  $M$  and is topologically an annulus bounded by two lines,  $L_0$  and  $L_2 = \sigma(L_0)$ , minus three points, namely  $l_1, l_2$  and  $\sigma(l_2)$ . Here  $l_1$  is the end at the height of  $L_1$ , and we will write abusively  $l_1 \in L_1$ ;  $l_2$  is the additional end in between the two original lines. Since  $B$  is minimal, it is conformally isomorphic to an annulus in the complex plane, that is  $\{z \in \mathbb{C}; 1 < |z| < r\}$ ; from now on, we will see  $B$  as that particular annulus. Furthermore the third coordinate  $X_3$  of the embedding is harmonic and constant on the boundary (here circles); thus it has to be something like

$$X_3(z) = a \log |z| + b.$$

An obvious consequence of this is that  $L_1$  is also a circle in this conformal representation, whose radius is the geometric mean of the radii of the boundary circles.

Considered as an isometry of  $\mathbb{R}^3$ ,  $\sigma$  is orientation-preserving; but by construction of  $M$ , we see that  $\sigma$  sends the normal vector at any point  $q \in M$  to the opposite of the normal at  $\sigma(q)$ :

$$(1) \quad \sigma_*(n(q)) = -n(\sigma(q)).$$

We can now conclude that  $\sigma$  is an orientation-reversing isometry of the surface. But the only anti-conformal map from  $B$  into itself that sends  $L_0$  to  $L_2$  and leaves the circle  $L_1$  pointwise invariant is the affine inversion  $z \mapsto k/\bar{z}$ , where  $k$  is some real constant uniquely determined by the position of  $L_1$ . So that  $M$  is conformally this annulus in the complex plane with boundary identified through  $\sigma$ . However we will use another representation.

### 3.3. Parameterization on the flat rectangular torus.

We now consider now the rectangle in  $\mathbb{C}$  whose vertices are  $0, \mu, \mu+i, i$  (as in the section 2, whose notations we will use). It is conformally equivalent to the annulus defined above, provided we identify the vertical edges. The equivalence mapping is  $z \mapsto e^{icz}$  for some real constant  $c$ . We see also that the lines  $L_0, L_1, L_2$  correspond to horizontal segments of height respectively equal to  $0, 1/2, 1$  (see figure 2). For the sake of simplicity, we will keep on calling these lines  $L_0, L_1$  and  $L_2$ .

Note that  $X_3$  is here an affine function of  $\text{Im}(z)$ , therefore  $X$  maps the horizontal lines to horizontal curves in  $\mathbb{R}^3$ .  $L_1$  is the middle horizontal



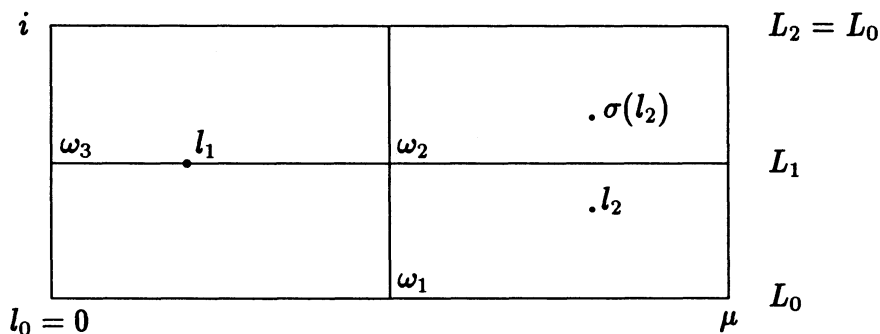


Fig. 2

line, and the rotation  $\sigma$  in space induces a line symmetry on the torus. It is easy to see that  $\sigma$  is here the usual reflection around  $L_1$ , which identifies any point of  $L_0$  with the point of  $L_2$  with the same real part. Therefore the conformal model of  $M$  is a *rectangular* torus. In that quotient,  $\sigma$  can be seen as the usual complex conjugation. Then we have to remove the four ends. We will keep on calling  $l_k$  the end that is “on” the line  $L_k$ , for  $k = 0, 1$  and we can always suppose  $l_0 = 0$ . Let  $l_2$  still be the end between  $L_0$  and  $L_1$  and, because of the symmetry,  $\bar{l}_2$  is the other one. So that  $M \simeq \mathbb{T} - \{l_0, l_1, l_2, \bar{l}_2\}$  conformally. We deduce from section 2 that in the case of the double covering of Riemann’s Example, the four ends are located at  $(0, \omega_3, \omega_2 - i/4, \omega_2 + i/4)$  (provided of course we locate  $l_0$  at 0). We will prove that these points are the only possible choices.

**3.4. The Weierstrass-Enneper representation.**

Now that we have a proper conformal model for  $M$ , we will try to determine the Weierstrass-Enneper parametrization of the immersion  $X : M \rightarrow \mathbb{R}^3$ . Let us sum up what we know about  $M$  : it is a finite topology, orientable, embedded minimal surface; its genus is equal to 1, with four ends. Then the Euler characteristic  $\chi(M) = -4$ ; the total curvature is finite and satisfies the Callahan-Hoffman-Meeks formula :

$$\int_M K dM = 2\pi(\chi(M) - 4) = -16\pi = -4\pi d(g).$$

Here  $g$  denotes the *extended* Gauss map, which is a meromorphic map from  $\mathbb{T}$ , the completion of  $M$  as a Riemann surface, to  $S^2$ , the extended complex plane. Thus the degree of  $g$  is 4. Using the orientability of  $M$ , we may suppose that the normal map at  $l_0$  and  $l_1$  is vertical in the upward

direction while at  $p$  and its conjugate it is vertical but goes downward. In terms of the Gauss map,  $l_0$  and  $l_1$  are poles, while  $l_2$  and  $\bar{l}_2$  are zeros.

Up to a scaling, we have  $X_3(z) \equiv \operatorname{Re} \int^z 2i \, dz \pmod{2}$ . Thus if  $\eta$  denotes the holomorphic 1-form associated to  $g$  in the Weierstrass representation, then  $g\eta = i \, dz$ .

$$\eta = \frac{i \, dz}{g}$$

Equivalently, we can write  $dh = g\eta = i \, dz$  where  $h$  is the “height function”. Furthermore, the ends are double points; namely, at a flat end,  $g$  must have branching order greater or equal to 1; but the degree of  $g$  is 4, so *all the ends of  $M$  have branching order 1 for  $g$* . We can now easily write the coordinates of the immersion :

$$\begin{cases} X_1(z) = \operatorname{Re} \int^z (1 - g^2)\eta = \operatorname{Re} \int^z (g^{-1} - g)i \, dz = \operatorname{Im} \int^z (g - g^{-1})dz \\ X_2(z) = \operatorname{Re} \int^z i(1 + g^2)\eta = -\operatorname{Re} \int^z (g^{-1} + g)dz \\ X_3(z) = \operatorname{Re} \int^z 2g\eta = \operatorname{Re} \int^z 2i \, dz = -2\operatorname{Im}(z - z_0). \end{cases}$$

### 3.5. Properties of the Gauss map $g$ .

We have seen that  $M$  has by construction a symmetry, namely the complex conjugation corresponding to the rotation around the lines in space. Whether we choose rotation around  $L_0$  or around  $L_1$  does not matter, since they yield the same isometry on the torus. The most important point is the behaviour of  $g$  under conjugation. If we choose the lines  $L_0$  and  $L_1$  to follow the  $X_2$  direction, the property stated in (1) in terms of the normal vector translates as

$$g(\bar{z}) = \overline{g(z)}.$$

We check easily that  $z = \bar{z}$  on the torus is equivalent to saying that  $z$  belongs to  $L_0$  or  $L_1$ , and in that case  $g$  has to be real, which means exactly that the normal map is orthogonal to the  $X_2$  direction.

Since  $g$  is an elliptic function of degree 4, we know it can be written as  $F(\wp(z)) + \wp'(z)G(\wp(z))$ , where  $F$  and  $G$  are rational functions, and  $\wp$  is the Weierstrass  $p$ -function associated with the lattice defining this torus. Rather than  $\wp$ , we will use the function  $P$  defined in 2.1, which similarly generates the field of elliptic functions, and has the advantage of having a double zero at  $\omega_2 = \frac{\mu + i}{2}$ . Then

PROPOSITION. — *The end  $l_1$  is one of the two branch points of  $P$  on  $L_1$ , namely  $\omega_2$  or  $\omega_3$ .  $F$  is a degree 2 fraction with simple poles at infinity and  $P(l_1)$ , while  $G$  is only a degree 1 fraction, with a pole at  $P(l_1)$  and a zero at infinity, unless it vanishes everywhere.*

COROLLARY 1. — *There exist a degree 2 polynomial  $E$ , which does not vanish at  $P(l_1)$ , and a complex constant  $\nu$  such that :*

$$g(z) = \frac{E(P(z)) + \nu P'(z)}{P(z) - P(l_1)}.$$

COROLLARY 2. — *Since  $g$  is invariant under complex conjugation, then  $\nu$  and all the coefficients in the expansion of  $E$  are real.*

*Proof of proposition.*

– By considering a neighbourhood of zero in the  $z$ -plane, we can see that  $F$  has a simple pole at infinity, whereas  $G$  vanishes there. Namely, if  $F(w) = cw^k + o(w^k)$  and  $G(w) = cw^h + o(w^h)$  when  $w \rightarrow \infty$  (here and afterwards,  $c$  will denote some non zero complex constant), then  $g(z) \sim cz^{-2k} + cz^{-3-2h}$ . This is compatible with the double pole requirement if and only if  $h \leq -1$  and  $k = 1$ .

– We will now prove by contradiction that  $l_1$  equals  $\omega_2$  or  $\omega_3$  (in other words  $2l_1 \equiv 0$  on the torus). Indeed let us suppose that  $l_1$  is not a branch point of  $P$ . We can find a function  $g$  having double poles at 0 and  $l_1$ , and double zeros at  $l_2$  and  $\bar{l}_2$ , provided these points satisfy  $2(l_1 - l_2 - \bar{l}_2) \equiv 0$  on the torus, by the Abel-Jacobi theorem. However, in order to eliminate the periods, the Gauss map must also satisfy the following criterion (proven in 4.1) : both  $g$  and  $g^{-1}$  have no residues. We will prove that under the hypothesis  $2l_1 \not\equiv 0$ , the function  $f = g^{-1}$  has non-vanishing residues at  $l_2$ . We proceed in technical lemmas.

LEMMA 1. —  $f'''(0) = 0$ .

*Proof.* — This condition is actually equivalent to  $\text{Res}(g, 0) = 0$ . Indeed, let us write the Taylor expansion of  $f = 1/g$  in a neighbourhood of the origin : if  $f(z) = az^2 + bz^3 + O(z^4)$  then :

$$g(z) = \frac{1}{f(z)} = \frac{1}{az^2} \left( 1 - \frac{bz}{a} + O(z^2) \right).$$

So  $b = -a^2 \operatorname{Res}(g, 0)$ , which vanishes by hypothesis. Hence  $f'''(0) = 0$ .

Q.E.D.

LEMMA 2. — Let  $\widehat{f}$  be the odd part of  $f$ ; then :

$$\widehat{f} = \frac{\alpha P'}{[(P - P(l_2))(P - P(\bar{l}_2))]^2}$$

where  $\alpha \in \mathbb{C}^*$  is a constant

*Proof.* — By hypothesis  $2l_1 \neq 0$ , hence  $l_2 + \bar{l}_2 \neq 0$ . Consequently,  $\pm l_2, \pm \bar{l}_2$  are four distinct point on  $\mathbb{T}$ .  $\widehat{f}$  is defined by  $\widehat{f}(z) = \frac{1}{2}(f(z) - f(-z))$ , and has double poles at each of these four points, and nowhere else : hence  $d(\widehat{f}) = 8$ . Being odd,  $\widehat{f}$  must vanish at  $0, \omega_1, \omega_2, \omega_3$ ; using lemma 1, we see that  $\widehat{f}$  has a zero of order at least 5 at the origin. We conclude that it has simple zeros at  $\omega_j, j = 1, 2, 3$ . Finally we consider  $\widehat{f}/P'$ , which has the same poles as  $\widehat{f}$  with the same multiplicity, and only one zero of order 8 at the origin. It can only be  $\alpha[(P - P(l_2))(P - P(\bar{l}_2))]^{-2}$  for some  $\alpha \in \mathbb{C}^*$ .

Q.E.D.

LEMMA 3. —  $\operatorname{Res}(\widehat{f}, l_2) = \frac{1}{2} \operatorname{Res}(f, l_2)$ .

*Proof.* — Since  $f(-z)$  is non singular at  $-l_2$ , then in some neighbourhood of  $l_2$ , one can write :

$$\widehat{f}(z) = \frac{1}{2}(f(z) - f(-z)) = \frac{1}{2}(f(z) + O(1)).$$

Q.E.D.

LEMMA 4. — The residues of  $\widehat{f}$  and  $f$  at  $l_2$  never vanish.

*Proof.* — Let us write the Taylor expansion of  $P$  in a neighbourhood of  $l_2$ ;  $P(l_2/\zeta) = a + b\zeta + c\zeta^2 + O(\zeta^3)$ . Notice that neither  $a$  nor  $b$  vanish. Then

$$(P - P(l_2))^{-2} = \frac{1}{b^2 \zeta^2} \left( 1 - \frac{2c}{b} \zeta + O(\zeta^2) \right)$$

$$(P - P(\bar{l}_2))^{-2} = \frac{1}{a'^2} \left( 1 - \frac{2b}{a'} \zeta + O(\zeta^2) \right)$$

where  $a' = a - \bar{a}(a' \neq 0$  because  $\bar{l}_2 \neq -l_2$ ).

$$\frac{P'}{[(P - P(l_2))(P - P(\bar{l}_2))]^2} = \frac{1}{ba'^2\zeta^2} \left( 1 + \frac{2c}{b}\zeta + O(\zeta^2) \right) \\ \left( 1 - 2 \left( \frac{b}{a'} + \frac{c}{b} \right) \zeta + O(\zeta^2) \right) = \frac{1}{ba'^2\zeta^2} \left( 1 - \frac{2b}{a'}\zeta + O(\zeta^2) \right).$$

The residue of  $\hat{f}$  is  $-2\alpha a'^{-3}$  which never vanishes under our hypotheses. Using lemma 3, we see that  $f$  itself has non vanishing residues, which contradicts section 4.1. We must exclude this case. Q.E.D.

Therefore we know that  $l_1 = \omega_2$  or  $l_1 = \omega_3$ ; equivalently,  $P(l_1) = 0$  or  $P(l_1) = -1/\lambda$ . One deduces that  $F$  has another simple pole at  $P(l_1)$ , while  $G$  can have at most a simple pole there.

- Let us finally look for the other possible poles of  $G$ . There must be at least one pole, since we have shown  $G$  vanishes at infinity (unless  $G \equiv 0$ ). If  $G$  has a pole at some  $P(z_0)$ , then  $z_0$  must be a pole for  $g$ . In fact, if it were not, then  $P'(z)G(P(z))$  would have to stay bounded near  $z_0$ ; this is never possible, whether  $z_0$  is a branch point of  $P$  or not. Thus  $P(z_0)$  is either infinity or  $P(l_1)$ . The first is excluded, so we can conclude that  $G$  has a simple pole at  $P(l_1)$ . The corollaries follow. Q.E.D.

### 4. The period constraints.

#### 4.1. The different types of periods.

In order for  $X$  to be well defined, we need to check that there are no periods for  $X$  on the torus. These real periods fall into two distinct categories : "analytical" periods coming from poles of a differential form, and "geometric" periods coming from the homology. For the latter, we will take into account only the real periods along any horizontal loop (by horizontal loop, we mean homologous to  $L_0$ ).

It is obvious already that  $dX_3 = -2\text{Im}dz$  has no period, either around the ends, or along a horizontal loop. However  $dX_1 = \text{Im}((g - g^{-1})dz)$  and  $dX_2 = -\text{Re}((g + g^{-1})dz)$  may have periods around the poles and zeros of  $g$ . Clearly their analytical periods vanish if and only if  $g$  and  $g^{-1}$  have no residues. There remains then to check the homological periods.

#### 4.2. The planar symmetry.

We will show that there has to be a plane of symmetry for the immersion  $X$ , and how this translates into analytical terms.

PROPOSITION. — *The function  $g$  is even; this implies that there is a plane of symmetry perpendicular to the  $X_2$  direction.*

We will need the following

LEMMA. — *The residue of  $g$  at 0 is  $-2\nu$ .*

*Proof of the lemma.* — We have found an expression of  $g$  as

$$g(z) = \frac{E(P(z)) + \nu P'(z)}{P(z) - P(l_1)}$$

where  $l_1$  is the pole of  $g$  on  $L_1$ , and is one of the two branch points of  $P$  on  $L_1$ . Obviously,  $\frac{E(P(z))}{P(z) - P(l_1)}$  is even, and thus has no residue at 0; then

$$\text{Res}(g, 0) = \nu \text{Res}(P'/(P - P(l_1)), 0).$$

Near zero, we know that

$$P'(z)/(P(z) - P(l_1)) \sim -2/z.$$

This proves the lemma.

*Proof of the proposition.* — According to the previous section, it is necessary for  $\nu$  to vanish; thus  $g(-z) = g(z)$ . One checks easily that it amounts to saying that there is a vertical plane of symmetry, orthogonal to the  $X_2$  direction. QED.

Finally, we deduce

COROLLARY. —  $P(l_2) \in \mathbb{R}$  and there is a real number  $\alpha$  such that :

$$g(z) = \alpha \frac{(P(z) - P(l_2))^2}{P(z) - P(l_1)}.$$

*Proof.* — Now that we have eliminated the odd part in the expansion of  $g$ , we can easily determine its accurate expression. In fact, let us compute

the derivative of  $g$  :

$$g'(z) = P'(z) \frac{E'(P(z))(P(z) - P(l_1)) - E(P(z))}{(P(z) - P(l_1))^2}$$

and apply the condition  $g(l_2) = g'(l_2) = 0$ . Since obviously  $P(l_2) \neq P(l_1)$  and  $P'(l_2) \neq 0$ ,  $P(l_2)$  has to be a double zero of  $E$ . Furthermore all parameters in the expansion of  $g$  must be real in order to satisfy the requirement that  $g(\bar{z}) = \overline{g(z)}$ . QED.

**4.3. The value of  $l_1$ .**

We have two possibilities for  $l_1$  : either  $\omega_2$  or  $\omega_3$ , whose values under  $P$  are respectively 0 and  $-1/\lambda < 0$ . We also know from section 3.5 that  $g(L_1)$  lies either in  $\overline{\mathbb{R}}_+^*$  or  $\overline{\mathbb{R}}_-^*$ . Namely the image of each of these lines lies in  $\overline{\mathbb{R}}$  but doesn't include zero, and reaches  $\infty$  only once (though with order 1). We will prove that the sign of  $g$  along both  $L_0$  and  $L_1$  must be the same, namely the sign of  $\alpha$ . As a corollary this will force  $l_1$  to be  $\omega_3$ .

*Proof.* — A necessary condition on  $g$  is that neither  $g$  nor  $g^{-1}$  have residues on  $\mathbb{T}$ . Thus

$$\int_{L_1} g^{-1} dz = \int_{L_0} g^{-1} dz$$

both lines having the same orientation (say increasing real part). It is clear then that  $g$  has the same sign on both of them. Suppose now that  $l_1 = \omega_2$ , or equivalently,  $P(l_1) = 0$ . Let us evaluate  $g$  in the neighbourhood of this pole. It is easy to compute that for small  $\zeta$ ,  $P(l_1 + \zeta) \sim -\frac{C_\lambda}{4} \zeta^2$ . Hence

$$g(l_1 + \zeta) \sim -\frac{4}{C_\lambda} \alpha P(l_2)^2 / \zeta^2. \text{ Meanwhile, near zero, on } L_0, g(z) \sim \frac{4}{C_\lambda} \alpha / z^2.$$

Both expressions have opposite signs, we get a contradiction (recall  $C_\lambda > 0$ ). We can now write :

$$g(z) = \alpha \frac{(P(z) - P(l_2))^2}{P(z) + 1/\lambda}.$$

**4.4. The period condition at the  $l_2$  end.**

Let us now consider periods around  $l_2$  (or  $\bar{l}_2$  which is equivalent). We need only check that the residue of  $g^{-1}$  vanishes. Therefore we write for

small  $u$ ,

$$P(l_2 + u) = P(l_2) + au + bu^2/2 + O(u^3)$$

then

$$(P(l_2 + u) - P(l_2))^2 = a^2u^2(1 + bu/a + O(u^2))$$

and

$$\begin{aligned} g^{-1}(l_2 + u) &= \frac{((P(l_2) + 1/\lambda) + au + O(u^2))(1 - bu/a + O(u^2))}{\alpha a^2 u^2} \\ &= \frac{(P(l_2) + 1/\lambda) + (a - b(P(l_2) + 1/\lambda)/a)u + O(u^2)}{\alpha a^2 u^2}. \end{aligned}$$

Therefore the residue is  $\frac{a^2 - b(P(l_2) + 1/\lambda)}{\alpha a^3}$ . Note that  $a = P'(l_2) \neq 0$ .

We will now show that only two possible choices of  $l_2$ , up to conjugation, make this residue vanish. The condition is :

$$a^2 = b(P(l_2) + 1/\lambda)$$

where  $a, b$  are respectively the first and second derivative of  $P$  at  $l_2$ . Let us first compute  $b$  in terms of the other quantities. The structural equation for  $P$  being  $P'^2 = C_\lambda P(P + 1/\lambda)(P - \lambda)$ , derivation yields :

$$2P'P'' = C_\lambda P'(P(P + 1/\lambda) + P(P - \lambda) + (P + 1/\lambda)(P - \lambda)),$$

therefore

$$\begin{cases} 2b = C_\lambda (P(l_2)(P(l_2) + 1/\lambda) + P(l_2)(P(l_2) - \lambda) \\ \quad + (P(l_2) + 1/\lambda)(P(l_2) - \lambda)) \\ a^2 = C_\lambda P(l_2)(P(l_2) + 1/\lambda)(P(l_2) - \lambda) \\ a^2 = b(P(l_2) + 1/\lambda). \end{cases}$$

This implies :

$$2P(l_2)(P(l_2) - \lambda) = (P(l_2)(P(l_2) + 1/\lambda) + P(l_2)(P(l_2) - \lambda) + (P(l_2) + 1/\lambda)(P(l_2) - \lambda))$$

namely

$$P(l_2)^2 + 2P(l_2)/\lambda - 1 = 0$$

whose solutions are  $P(l_2) = \frac{\pm \sqrt{\lambda^2 + 1} - 1}{\lambda}$ .



4.5. The homology period.

We have hencewith verified that no real period can arise from loops around  $l_2, \bar{l}_2, 0$  or  $l_1$ . This leaves us with two distinct families of solutions. However there remains to check the homological condition, which will eventually reduce the solution to the known Riemann's Example.

Suppose  $\gamma$  is a simple horizontal loop. We must have :

$$\begin{cases} \text{Im} \int_{\gamma} (g^{-1} - g) dz = 0 \\ \text{Re} \int_{\gamma} (g^{-1} + g) dz = 0. \end{cases}$$

We suppose of course that  $\gamma$  does not meet any zero or pole of  $g$ ; however, in order to evaluate the integral, we can split it and consider homologous curves. The first integral is always zero, because of the invariance of  $g$  under conjugation. Namely, the integrals of  $g$  and  $g^{-1}$  along  $\gamma$  are both real. The second equation can be written as

$$\alpha \int_{\gamma} \frac{(P(z) - P(l_2))^2}{P(z) + 1/\lambda} dz + \frac{1}{\alpha} \int_{\gamma} \frac{P(z) + 1/\lambda}{(P(z) - P(l_2))^2} dz = 0$$

where we must find a real  $\alpha$ , for one of the two possible  $P(l_2)$ . A necessary and sufficient condition is that both integrals have opposite signs (or both 0). But we see at once that  $\int_{\gamma} g^{-1} dz$  is positive if  $\alpha$  is, by integrating along  $L_0$ . Thus  $\int g dz$  and  $\alpha$  must have opposite signs. However this integral can be seen as a polynomial in  $P(l_2)$ ; in fact let us define :

$$\begin{aligned} \mathcal{P}(w) &= \frac{1}{\alpha} \int_{\gamma} g dz = \int \frac{(P(z) - w)^2}{P(z) + 1/\lambda} dz \\ &= \int \left[ (P(z) + 1/\lambda) - 2(w + 1/\lambda) + \frac{(w + 1/\lambda)^2}{P(z) + 1/\lambda} \right] dz \\ &= \left( -\mu(2w + 1/\lambda) + \int P(z) dz + (w + 1/\lambda)^2 \int \frac{dz}{P(z) + 1/\lambda} \right). \end{aligned}$$

The study of  $\mathcal{P}$  shows that :  $\mathcal{P}$  is a upward-oriented parabola which is positive at  $-1/\lambda$ , negative at 0, and attains its minimum *after*  $\lambda$ . Therefore  $\mathcal{P}\left(\frac{-\sqrt{\lambda^2 + 1} - 1}{\lambda}\right) > 0$ , which excludes the smallest possible value for  $P(l_2)$ . On the other hand,

$$\mathcal{P}\left(\frac{\sqrt{\lambda^2 + 1} - 1}{\lambda}\right) < 0,$$

which corresponds to the known Riemann's Example.

*Proof.* — Note first that the leading term in  $\mathcal{P}$  is positive. In fact we can integrate  $\int \frac{dz}{P(z) + 1/\lambda}$  along  $L_0$  instead of  $\gamma$ , where we have the following inequality :  $P(z) \geq \lambda$  (remember  $\lambda > 0$ ). So  $P(z) + 1/\lambda \geq \lambda + 1/\lambda > 0$ ; this yields the desired result but also a bound on the integral, namely :

$$\int_{L_0} \frac{dz}{P(z) + 1/\lambda} < \int_{L_0} (\lambda + 1/\lambda)^{-1} dz = \frac{\mu\lambda}{1 + \lambda^2}.$$

Let us now evaluate the position for the minimum of this parabola, which we will call  $w_m$  :

$$w_m = -1/\lambda + \frac{\mu}{\int \frac{dz}{P(z) + 1/\lambda}}.$$

It is very easy to verify that the previous inequality is equivalent to

$$w_m > \lambda.$$

Third and last point, we would like to have an estimate of  $\mathcal{P}$  at 0 and  $-1/\lambda$ ,

$$\mathcal{P}(0) = -\frac{\mu}{\lambda} + \int P(z) dz + \frac{1}{\lambda^2} \int \frac{dz}{P(z) + 1/\lambda}.$$

By integrating along  $L_1$ , it is obvious that  $\int P(z) dz < 0$ ; we know more, namely  $-\mu/\lambda < \int P(z) dz < 0$ . Therefore to prove that  $\mathcal{P}(0) < 0$  we need only show that

$$\frac{1}{\lambda^2} \int \frac{dz}{P(z) + 1/\lambda} < \frac{\mu}{\lambda}$$

that is

$$\int \frac{dz}{P(z) + 1/\lambda} < \mu\lambda.$$

Using the previous result, we know that

$$\int \frac{dz}{P(z) + 1/\lambda} < \frac{\mu\lambda}{1 + \lambda^2}$$

which is sufficient. And when  $w = -1/\lambda$ ,

$$\mathcal{P}(-1/\lambda) = \frac{\mu}{\lambda} + \int P(z)dz$$

which is positive, as seen previously.

QED.

#### 4.6. Conclusion.

The Riemann Example is only minimal surface which is conformally a once-punctured annulus lying between two parallel planes, bounded by two lines in those planes. Up to scaling, there is only a one-parameter family of such examples, parameterized by  $\mu$  (or  $\lambda$ ).

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Pascal ROMON,  
Mathematics Dept.  
University of Massachusetts  
Amherst, MA 01003 (USA).