HAL open science

Law of iterated logarithm and invariance principle for one-parameter families of interval maps

Daniel Schnellmann

To cite this version:

Daniel Schnellmann. Law of iterated logarithm and invariance principle for one-parameter families of interval maps. 2013. hal-00861987

HAL Id: hal-00861987

https://hal.science/hal-00861987v1

Preprint submitted on 14 Sep 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LAW OF ITERATED LOGARITHM AND INVARIANCE PRINCIPLE FOR ONE-PARAMETER FAMILIES OF INTERVAL MAPS

DANIEL SCHNELLMANN

Abstract. We show that for almost every map in a transversal one-parameter family of piecewise expanding unimodal maps the Birkhoff sum of suitable observables along the forward orbit of the turning point satisfies the law of iterated logarithm. This result will follow from an almost sure invariance principle for the Birkhoff sum, as a function on the parameter space. Furthermore, we obtain a similar result for general one-parameter families of piecewise expanding maps on the interval.

1. Introduction

In this introduction we consider only piecewise expanding unimodal maps. However, all the following results can be extended to more general families of piecewise expanding interval maps (see Section 2). We call a map $T : [0,1] \to [0,1]$ a *piecewise expanding unimodal map* or *tent map* if it is continuous and if there exists a turning point $c \in (0,1)$ such that $T|_{[0,c]}$ and $T|_{[c,1]}$ are $C^{1+\alpha}$, $\|1/T'\|_\infty < 1$ and $\|T'\|_\infty < \infty$, and $T(1) = T(0) = 0$. We assume that T is *mixing*, i.e., it is topologically mixing in the interval $[T^2(c), T(c)]$. Let μ denote the unique (hence ergodic) absolutely continuous invariant probability measure (acip) for T. By Birkhoff's ergodic theorem, μ almost every (or in this case also Lebesgue almost every) point $x \in [0,1]$ is *typical* for μ, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(T^i(x)) = \int_0^1 \varphi d\mu \,, \qquad \forall \varphi \in C^0 \,. \tag{1}$$

A natural question is how fast this convergence takes place. In order to answer this question one has to take a smaller set of observables: By [24] and [11], for any sequence α_n such that $\lim_{n\to\infty} \alpha_n = \infty$, there is a dense G_δ set in C^0 such that for all φ in this set one has

$$\lim_{n\to\infty} \alpha_n \Big| \frac{1}{n} \sum_{i=1}^{n} \varphi(T^i(x)) - \int_0^1 \varphi d\mu \Big| = \infty \,.$$

A suitable set of observables for which the question about the speed of convergence makes sense is for example the set of Hölder continuous functions (or more generally

Date: September 14, 2013.

Key words and phrases. Almost sure invariance principle, law of iterated logarithm, one-parameter families, interval maps, transversality, tent maps.

The author is grateful to Viviane Baladi for encouraging him to work on this question. This work was accomplished at DMA, ENS, Paris. The author was supported by the Swiss National Science Foundation.

the set of functions of generalised bounded variation; see Definition 2.5 below). For φ Hölder, set

$$\sigma(\varphi)^2 := \int_0^1 \Big(\varphi - \int \varphi d\mu\Big)^2 d\mu + 2\sum_{i>0} \int_0^1 \Big(\varphi - \int \varphi d\mu\Big)\Big(\varphi - \int \varphi d\mu\Big) \circ T^i d\mu \,. \tag{2}$$

Since we have exponential decay of correlation (see, e.g., Proposition 4.3 below), $\sigma(\varphi)$ is finite and since we can write $\sigma(\varphi) = \lim_{n\to\infty} n^{-1} Var(S_n)$, where S_n is the n-th Birkhoff sum, we see that $\sigma(\varphi) \geq 0$. If $\sigma(\varphi) = 0$, then φ is a co-boundary and there exists an L^1 function ψ so that $\varphi = \psi \circ T - \psi$ almost surely. Henceforth, we exclude this (degenerate) case, i.e., we will always assume that $\sigma(\varphi) > 0$. Turning back to the question about the speed of convergence of (1), it is shown in [18] that if we restrict ourself to the set of Hölder continuous observables φ then the *law of iterated logarithm (LIL)* holds: For a.e. $x \in [0,1]$, we have

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \sum_{i=1}^n \Big(\varphi(T^i(x)) - \int \varphi d\mu\Big) = \sigma(\varphi) \,. \tag{3}$$

For tent maps T the turning point c is of particular dynamical interest. A lot of information about the dynamics of T is contained in the forward orbit of c, and it is natural to ask if (3) holds when we take $x = c$. For a recent work where the assumption that the turning point satisfies the LIL is crucial, see [4]. However, even if we know that (1) and (3) hold for a.e. point x, it is a very difficult question to say wether they hold for a particular point x. So instead of asking for the LIL for c for a single tent map T, we perturb this map by a one-parameter family of tent maps and ask if the LIL for c holds for almost every map in this family. Let T_a, $a \in [0,1]$, be a one-parameter family of piecewise expanding unimodal maps through $T = T_0$. We make some natural regularity assumptions on the parameter dependency as, e.g., the turning point c_a is Lipschitz continuous in a and if $J \subset [0,1]$ is an interval on which $x \neq c_a$, then $a \mapsto T_a(x)$ is $C^{1+\alpha}$ on J (for the precise conditions we refer to the beginning of Section 2). Of course in order that the question of this paragraph makes sense we have to exclude trivial one-parameter families as for example the constant one or families for which the turning point is eventually mapped to a periodic point for all parameters. The right condition here is *transversality* which is a common non-degeneracy condition for one-parameter families of interval maps (see, e.g., [33], [2], [26], [5], [16], [32], [3] for previous occurrences of this condition in the literature). We say that the family of tent maps T_a is *transversal* at T_0 if there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \left| \frac{\partial_a T_a^j(c_a)|_{a=0}}{(\partial_x T_0^{j-1})(T_0(c_0))} \right| \leq C \,, \quad \forall\, j \text{ large.} \tag{4}$$

(If c_0 is periodic for T_0, then we take one-sided derivatives.) The transversality condition says that the a-derivative along the postcritical orbit is comparable to its x-derivate. Since the x-derivative is growing exponentially fast, this implies that if we change the parameter then the dynamics of the corresponding map will change fast which makes it possible to study the generic behaviour of the postcritical orbit. If the family T_a is *transversal* at T_0, then it is shown in [32] that for a.e. parameter

a close to 0 the turning point c_a is typical for the acip μ_a (for related results see [8], [31], and [14]). Given almost sure typicality of the turning point we can now ask for the speed of convergence of (1) in this setting.

The main result of this paper can be stated as follows (see also Theorem 3.1 in Section 3 below). To the best of the authors knowledge, it is the first result which treats the question of a LIL for a *specific* point in a dynamical systems. Recall the notation σ in (2). We will use the notation σ_a when considering the map T_a.

Theorem 1.1. *Assume that T_0 is mixing, its turning point c_0 is not periodic, and the family T_a is transversal at T_0. If φ is Hölder and $\sigma_0(\varphi) > 0$, then there exists $\epsilon > 0$ such that for almost every $a \in [0, \epsilon]$ the turning point c_a satisfies the LIL for the function φ under the map T_a, i.e.,*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \sum_{i=1}^{n} \Big(\varphi(T_a^i(c_a)) - \int \varphi d\mu_a\Big) = \sigma_a(\varphi)\,.$$

In order to prove Theorem 1.1, we will show a stronger property, the so called *almost sure invariance principle (ASIP)*, for the turning point. We say that the functions $\xi_i : [0, \epsilon] \to \mathbb{R}$, $i \geq 1$, satisfy the ASIP with error exponent $\gamma < 1/2$ if there exists a probability space supporting a Brownian motion W and a sequence of variables η_i, $i \geq 1$, such that

(i) $\{\xi_i\}_{i\geq 1}$ and $\{\eta_i\}_{i\geq 1}$ have the same distribution;
(ii) almost surely as $n \to \infty$,

$$\left| W(n) - \sum_{i=1}^{n} \eta_i \right| = O(n^\gamma)\,.$$

The following corollary is shown, e.g., in [29]. For other implications of the ASIP we refer to [17].

Corollary 1.2. *If the functions ξ_i satisfy the ASIP then they satisfy also the LIL and the central limit theorem. More precisely, if σ^2 is the variance of the related Brownian motion, then*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \sum_{i=1}^{n} \xi_i(a) = \sigma\,, \qquad \text{for a.e. } a \in [0, \epsilon]\,,$$

and, for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} m\Big(\Big\{a \in [0, \epsilon] \mid \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} \xi(a) \leq t\Big\}\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds\,,$$

where m denotes the Lebesgue measure.

For φ Hölder and such that $\sigma_0(\varphi) > 0$, for $i \geq 1$ and a small, set

$$\varphi_a(x) := \frac{1}{\sigma_a(\varphi)} \Big(\varphi(x) - \int_0^1 \varphi d\mu_a\Big). \tag{5}$$

Lemma 4.5 below guarantees[1] that $a \mapsto \sigma_a(\varphi)$ is continuous at 0 and, hence, the function φ_a is well-defined for a sufficiently close to 0. Due to this normalisation

[1]In order that condition (II) in Lemma 4.5 is satisfied, we assume that T_0 is mixing and c_0 is not periodic (see proof of Theorem 3.2).

we have

$$\sigma_a(\varphi_a) = 1\,, \qquad \text{and} \qquad \int \varphi_a d\mu_a = 0\,, \tag{6}$$

for all a sufficiently close to 0. We are going to show an ASIP for the functions

$$\xi_i(a) := \varphi_a(T_a^i(x))\,, \qquad i \geq 1\,. \tag{7}$$

Theorem 1.3. *Assume that T_0 is mixing, c_0 is not periodic, and the family T_a is transversal at T_0. Then there exists $\epsilon > 0$ such that the functions $\xi_i : [0, \epsilon] \to \mathbb{R}$, $i \geq 1$, satisfy the ASIP for all error exponents $\gamma > 2/5$.*

Remark 1.4. Theorem 1.3 and, hence, Theorem 1.1 hold also if c_0 is periodic and T_0 has a sufficiently high expansion (see Theorem 3.1 in Section 3 below). Because of the normalisation in the definition of the ξ_i's, the variance of the Brownian motion in the ASIP is equal to 1. For a comment on the optimality of the error exponent γ see the beginning of Section 6.

Regarding the proof of Theorem 1.3 we go along a classical method from probability theory which consists in writing the Birkhoff sum approximatively as a sum of blocks of polynomial size, then in approximating these blocks by a martingale difference sequence, and finally in applying Skorokhod's representation theorem which provides a link between a martingale and a Brownian motion. This strategy is illustrated on many examples in Philipp and Stout [29]. More precisely, using arguments from ergodic theory and perturbation theory, we go along the approach in [29, Section 3]. Note that the "usual" applications of [29] in dynamical systems refer to [29, Section 7] (see, e.g., [18], [12], [13], and [28]). The key property in these applications is a strong mixing condition which we do not have in our setting since loosely speaking the ξ_i's are *not* iterations of a fixed map and, thus, there is no underlying invariant measure. However, we can more or less replace this strong mixing condition by *uniformity of constants* in the Lasota-Yorke inequality for the family T_a (see condition (II) in Section 2). By Keller-Liverani perturbation arguments [22], we have then uniformity of constants for the exponential decay of correlation (see Proposition 4.3). This in turn can be used to show a certain exponential decay of correlation for the maps ξ_i's (see, e.g., the proof of Proposition 5.1) from which we are able to deduce similar estimates as in [29, Section 3]. In the recent work [15], Gouëzel uses spectral methods to show an almost sure invariance principle. His method is very powerful and it provides very good error estimates. However, we didn't find an easy way to apply these spectral techniques to our setting.

We would like to highlight that the main technical novelty or difficulty of this paper is to treat processes which are (at least "locally") close to processes generated by a dynamical system but for which there is no underlying invariant measure. Hence, various tools from ergodic theory cannot be applied directly and one has to "zoom in" in order to profit from ergodic theoretical facts. This explains the rather technical nature of this paper. The following classical example by Erdös and Fortet (see [19], p. 646; cf. also [10]) shows how careful one should be when one wants to show an ASIP for a process which is not but very close to a process generated by a dynamical system: Let $\varphi(x) = \cos(2\pi x) + \cos(4\pi x)$ and consider the sequence $\xi_i(x) = \varphi(2^i x)$, $i \geq 1$. ξ_i is a process generated by the doubling map $x \mapsto 2x \bmod 1$. It is straightforward to check that $\sigma(\varphi) > 0$ and, for instance by the above cited "dynamical" paper [18], it follows that the process ξ_i satisfies the ASIP. However, if

we change the process just slightly and consider instead $\xi_i(x) = \varphi((2^i - 1)x)$ then, surprisingly, this new process does not satisfy anymore the central limit theorem (and, thus, not either the ASIP).

As mentioned in the beginning of this section, the above presented results for tent maps hold for more general piecewise expanding maps on the interval. First, it is not essential to take the turning point as the point of interest. Any other point works fine as long as the a- and x-derivatives along its forward orbit are comparable. If we consider other piecewise expanding maps on the interval than tent maps, then we have to add two more conditions. The first one is to have uniform constants in the Lasota-Yorke inequality (see condition (II) in Section 2.2). This is a natural condition when applying perturbation theory. The second condition (see condition (III) in Section 2.2) is a bit more technical but satisfied for many one-parameter families, as it is shown in Section 3. In Section 3 we mention also how to apply the main result of this paper, Theorem 2.6, to obtain almost sure typicality results similar to the ones in [32] but under alternative conditions (see Theorem 3.5).

The present paper deals exclusively with maps which are uniformly hyperbolic. It is a natural question if we can obtain a similar result in a non-uniformly hyperbolic setting. An interesting candidate for this question is the quadratic family $f_a(x) = ax(1-x)$ with parameter $a \in (0, 4]$. Does the critical point $c = 1/2$ satisfy the LIL for Lebesgue almost every Collet-Eckmann (CE) map for sufficiently smooth observables? Despite a vast variety of results about the quadratic family, this question is still unsolved. To start with one should maybe content oneself with finding a positive Lebesgue measure set of CE parameters such that the critical points of the corresponding CE maps satisfy the LIL. Almost sure typicality of the critical point is known: By Avila and Moreira [1], the critical point for Lebesgue almost every CE map f_a in the quadratic family is typical for its SRB measure μ_a. (For the subset of CE parameters considered by Benedicks and Carleson this result was shown in [6].) An important ingredient in an attempt to find a positive measure set for which the turning point satisfies the LIL should be uniformity of constants in the set of CE parameters which one considers. For this one could follow the "start-up procedure" in Benedicks and Carleson [6] which yields, in addition to uniformity of constants, at each step nice "Markov partitions" on the parameter space. On the partition elements of these Markov partitions, which are intervals, one should be able to define functions $\xi_i(a)$ as in (7). (Observe that at each step one excludes parameter intervals from the previous Markov partition and, finally, one ends up with a Cantor set of positive Lebesgue measure.) In [28] where the ASIP is shown for a fixed CE map, they use a tower construction to get an induced system with uniform hyperbolicity where more or less a straight forward application of [29, Section 7] implies an ASIP which projects then down to the ASIP for the original CE map. Since in the parameter space one has to exclude an open and dense set of regular parameters, the "start-up procedure" in [6] might provide a way to replace this tower construction in [28] when one deals only with one single CE map.

The paper is organised as follows. In Section 2, we formulate a general model and give the main notations for the one-parameter families of piecewise expanding maps considered in this paper. This is followed by the main statement, Theorem 2.6. Section 3 contains examples of one-parameter families, such as families of tent maps, to which the result of this paper applies. Section 4 deals with elementary facts as

distortion estimates, uniform exponential decay of correlations, and the regularity of $a \mapsto \sigma_a$. Section 5 and 6 are dedicated to the proof of the main Theorem 2.6, i.e., the proof of an almost sure invariance principle.

2. Main statement

We begin this section with an introduction of the basic notation and a formulation of a suitable model for one-parameter families of *piecewise expanding maps* of the unit interval. A map $T : [0,1] \to [0,1]$ will be called *piecewise* $C^{1+\alpha}$, $0 < \alpha \leq 1$, if there exists a partition $0 = b_0 < b_1 < ... < b_p = 1$ of the unit interval such that for each $1 \leq k \leq p$ the restriction of T to the open interval (b_{k-1}, b_k) is a $C^{1+\alpha}$ function. Let $T_a : [0,1] \to [0,1]$, $a \in [0,1]$, be a one-parameter family of piecewise $C^{1+\alpha}$ maps and let $0 = b_0(a) < b_1(a) < ... < b_{p(a)}(a) = 1$ be the partition of the unit interval associated to T_a. We assume that the Hölder constants are uniform in a, i.e., there exist $0 < \alpha \leq 1$ and a constant C so that

$$|T_a'(x) - T_a'(y)| \leq C|x-y|^\alpha\,, \qquad \forall x, y \in (b_{k-1}(a), b_k(a)) \;\text{ and }\; \forall a \in [0,1]\,. \tag{8}$$

We assume that the maps are uniformly *expanding*, i.e., we assume that there are real numbers $1 < \lambda \leq \Lambda < \infty$ such that for every $a \in [0,1]$,

$$\|1/T_a'\|_\infty \leq \lambda^{-1}\,, \qquad \text{and} \qquad \|T_a'\|_\infty \leq \Lambda\,. \tag{9}$$

Remark 2.1. Regarding the framework in [21] and [30], it would be natural to skip the assumption that $\|T_a'\|_\infty \leq \Lambda$ and to replace the requirement that $x \mapsto T_a'(x)$ is piecewise α-Hölder by the requirement that $x \mapsto 1/T_a'(x)$ is piecewise α-Hölder. However, since our analysis on the parameter space seems to require some specific distortion estimates (see Lemma 4.4 below), we do not know how to make this improvement in our setting. For instance having in mind one-parameter families of one-dimensional Lorenz maps, it might be interesting to investigate such a more general setting.

We make the following natural assumptions on the parameter dependence.

(i) The number of monotonicity intervals for the T_a's is constant, i.e., $p(a) \equiv p_0$, and the partition points $b_k(a)$, $0 \leq k \leq p_0$, are Lipschitz continuous on $[0,1]$. It follows that there is a constant $\delta_0 > 0$ such that

$$b_k(a) - b_{k-1}(a) \geq \delta_0,$$

for all $1 \leq k \leq p_0$ and $a \in [0,1]$.

(ii) If $x \in [0,1]$ and $J \subset [0,1]$ is a parameter interval such that $b_k(a) \neq x$, for all $a \in J$ and $0 \leq k \leq p_0$, then $a \mapsto T_a(x)$ is $C^{1+\alpha}$ and $a \mapsto \partial_x T_a(x)$ is α-Hölder where the Hölder constants are independent on x. Further, the maps $x \mapsto \partial_a T_a(x)$, $x \in (b_{k-1}(a), b_k(a))$, $1 \leq k \leq p_0$, are α-Hölder continuous (where the Hölder constants are uniform in a).

In order to obtain an acip, we refer to a paper by G. Keller [21] (see Theorems 3.3 and 3.5 therein) who extended the results in [25] on piecewise expanding C^2 maps to a broader class of maps containing also piecewise expanding $C^{1+\alpha}$ maps: For a fixed $a \in [0,1]$ there exists a finite number of ergodic acip for T_a. Further, by [27] combined with the remark in [36] after Definition 4 on page 514 (regarding property (III) therein cf. also [30, Proposition 5.1]), there exist at most $p_0 - 1$ ergodic acip and the support of an ergodic acip is a finite union of intervals. Since we are always interested in only one ergodic acip, we can without loss of

generality assume that for each T_a, $a \in [0,1]$, there is a unique (hence ergodic) acip which we denote by μ_a. Let $K(a) = \mathrm{supp}(\mu_a)$. We say that T_a is *mixing* if it is topologically mixing on $K(a)$. For $a \in [0,1]$, let $\{D_1(a), ..., D_{p_1(a)}(a)\}$ be the connected components of $K(a) \setminus \{b_0(a), ..., b_{p_0}(a)\}$, i.e., the $D_k(a)$'s are the monotonicity intervals for $T_a : K(a) \to K(a)$. We assume the following.

(iii) The number of $D_k(a)$'s is constant in a, i.e., $p_1(a) \equiv p_1$ for all $a \in [0,1]$. The boundary points of $D_k(a)$, $1 \le k \le p_1$, are α-Hölder continuous in a.

2.1. **Partitions.** For a fixed parameter value $a \in [0,1]$, we denote by $\mathcal{P}_j(a)$, $j \ge 1$, the partition on the dynamical interval consisting of the maximal open intervals of smooth monotonicity for the map $T_a^j : K(a) \to K(a)$. More precisely, $\mathcal{P}_j(a)$ denotes the set of open intervals $\omega \subset K(a)$ such that $T_a^j : \omega \to K(a)$ is $C^{1+\alpha}$ and ω is maximal, i.e., for every other open interval $\tilde{\omega} \subset K(a)$ with $\omega \subsetneq \tilde{\omega}$, $T_a^j : \tilde{\omega} \to K(a)$ is no longer $C^{1+\alpha}$. Clearly, the elements of $\mathcal{P}_1(a)$ are the interior of the intervals $D_k(a)$, $1 \le k \le p_1$.

We will define similar partitions on the parameter interval $[0,1]$. Let $x_0 : [0,1] \to [0,1]$ be a $C^{1+\alpha}$ map from the parameter interval $[0,1]$ into the dynamical interval $[0,1]$ where we assume that

$$x_0(a) \in K(a) \setminus \{b_0(a), ..., b_{p_0}(a)\}, \qquad \forall a \in (0,1). \tag{10}$$

The points $x_0(a)$, $a \in [0,1]$, are the points of interest in this paper, i.e., we are interested in the properties of the forward orbit of these points under T_a. The assumption (10) is only for convenience and it helps to make the partitions $\mathcal{P}_j$ below well-defined. (If a map x_0 does not satisfy (10), then combining the fact that Lebesgue a.e. point $x \in [0,1]$ is eventually mapped into $K(a)$ under T_a with the transversality condition (I) below, one can derive that (10) is satisfied for some iteration of x_0 restricted to some smaller intervals located around $a = 0$.) The forward orbit of a point $x_0(a)$ under the map T_a we denote as

$$x_j(a) := T_a^j(x_0(a)), \qquad j \ge 0.$$

Observe that by assumption $x_j(a) \in K(a)$, for all $j \ge 0$ and $a \in [0,1]$.

Remark 2.2. Since a lot of information for the dynamics of T_a is contained in the forward orbits of the partition points $b_k(a)$, $0 \le k \le p_0$, an interesting choice of the map x_0 is

$$x_0(a) = \lim_{x \to b_k(a)\pm} T_a(x).$$

For example, in the case of tent maps we choose $x_0(a) = T_a^{j_0}(c_a)$, for j_0 sufficiently large (see Theorem 3.1 below).

Let $J \subset [0,1]$ be an interval. By $\mathcal{P}_j|J$, $j \ge 1$, we denote the partition consisting of all open intervals ω in J such that for each $0 \le i < j$, $x_i(a) \in K(a) \setminus \{b_0(a), ..., b_{p_0}(a)\}$, for all $a \in \omega$, and such that ω is maximal, i.e., for every other open interval $\tilde{\omega} \subset J$ with $\omega \subsetneq \tilde{\omega}$, there exist $a \in \tilde{\omega}$ and $0 \le i < j$ such that $x_i(a) \in \{b_0(a), ..., b_{p_0}(a)\}$. Observe that this partition might be empty which is, e.g., the case when $x_1(a)$ is equal to a boundary point $b_k(a)$ for all $a \in [0,1]$. However, such trivial situations (around $a = 0$) are excluded by the transversality condition (I) formulated in the next Section 2.2. Knowing that condition (I) is satisfied, then the partition $\mathcal{P}_j|J$, $j \ge 1$, around $a = 0$ can be thought of as the set of the (maximal) intervals of smooth monotonicity for $x_j : J \to [0,1]$ (cf. Lemma 2.4 below). We set $\mathcal{P}_0|J = J$. Finally, in view of condition (I) below, observe that if a

parameter $a \in [0,1]$ is contained in an element of $\mathcal{P}_j|[0,1]$, $j \geq 1$, then also the point $x_0(a)$ is contained in an element of $\mathcal{P}_j(a)$ which implies that T_a^j is differentiable in $x_0(a)$.

2.2. **Main statement.** We put two conditions on our sequence of maps x_j, $j \geq 0$, around $a = 0$. The first one (see condition (I) below) is a common *transversality* condition for one-parameter families of interval maps which was already mentioned in the introduction. The second one (see condition (III) below) is more technical. It is used for controlling the measure of the set of partition elements with a too small image. Further, in order to apply perturbation results we require that we have uniform constants in the Lasota-Yorke inequalities for the different maps in the family (see condition (II) below). This condition does not depend on the choice of the map x_0. Even if condition (III) is quite technical, assuming that conditions (I) and (II) hold, it is satisfied by many important one-parameter families of piecewise expanding maps, see Section 3. (Even if we suspect so, it is not clear to us if in general the transversality condition (I), possibly together with condition (II) and/or some weaker other conditions, implies condition (III).)

The transversality condition (I) requires that the derivatives of x_j and T_a^j at $a = 0$ are comparable. This is the very basic assumption in this paper. It says that locally the behaviour of the maps x_j are comparable to the behaviour of the maps T_a^j. Since the LIL holds for the maps T_a^j one can therefore hope to obtain similar properties for the maps x_j. Of course, in order to have transversality the choice of the map $x_0 : [0,1] \to [0,1]$ plays an important role. If, e.g., for every parameter $a \in [0,1]$, $x_0(a)$ is a periodic point for the map T_a, then x_j will have bounded derivatives and the dynamics of x_j is completely different from the dynamics of T_a. Henceforth, we will use the notations

$$T_a'(x) = \partial_x T_a(x) \qquad \text{and} \qquad x_j'(a) = \partial_a x_j(a), \quad j \geq 1.$$

(I) The right-derivatives $x_j'(0+)$, $j \geq 1$, of x_j in 0 exist and there is a constant $C \geq 1$ so that

$$\frac{1}{C} \leq \left| \frac{x_j'(0+)}{(T_0^j)'(x_0(0+))} \right| \leq C\,, \qquad \forall j \geq 1\,.$$

Further, for each $j \geq 1$, there exists a neighbourhood $V \subset [0,1]$ of 0 so that for all $a \in V \setminus 0$ and all $0 \leq i < j$, we have $x_i(a) \notin \{b_0(a), ..., b_{p_0}(a)\}$.

Remark 2.3. Looking at the proof of the following Lemma 2.4, one can derive that condition (I) is satisfied if

$$|x_0'(0+)| \geq \frac{\sup_{a\in[0,1]} \sup_{x\in K(0)} |\partial_a T_a(x)|_{a=0}|}{\lambda - 1} + 2L + 1,$$

where L is the Lipschitz constant of the partition points $b_0(a), ..., b_{p_0}(a)$. In other words, as soon as the initial derivative is sufficiently large we have transversality which makes it easy to verify this property numerically.

The following lemma ensures that if condition (I) holds then we can compare the a- and the x-derivatives along the forward orbit of $x_0(a)$ on an entire, sufficiently small interval around 0. Its proof is given at the end of this section.

Lemma 2.4. *Assume that the family T_a satisfies condition (I). Then, there exists $\epsilon > 0$ and a constant $C \geq 1$ so that for $\omega \in \mathcal{P}_j|[0,\epsilon]$, $j \geq 1$, we have*

$$\frac{1}{C} \leq \left| \frac{x_j'(a)}{(T_a^j)'(x_0(a))} \right| \leq C\,, \qquad \forall a \in \omega\,.$$

Furthermore, for each $j \geq 1$, the number of $a \in [0,\epsilon]$ which are not contained in any element $\omega \in \mathcal{P}_j|[0,\epsilon]$ is finite.

Apart from transversality we require also to have uniform constants in the Lasota-Yorke inequality. Let $\mathcal{L}_a : L^1([0,1]) \to L^1([0,1])$ be the ordinary (Perron–Frobenius) transfer operator, i.e.,

$$\mathcal{L}_a\varphi(x) = \sum_{T_a(y)=x} \frac{\varphi(y)}{|T_a'(y)|}\,.$$

The appropriate space of observables V_α, $0 < \alpha \leq 1$, for which $\mathcal{L}_a$ has a spectral gap and which is convenient for our setting was introduced in [21] (see also [35], [18], and [30] which treats the higher dimensional case). V_α is the space of functions of *generalised bounded variation.*

Definition 2.5 (Banach space V_α)**.** For $\varphi \in L^1(m)$ and $\delta > 0$, we define

$$\operatorname{osc}(\varphi,\delta,x) = \operatorname{ess\,sup} \varphi|_{(x-\delta,x+\delta)} - \operatorname{ess\,inf} \varphi|_{(x-\delta,x+\delta)}\,,$$

and, for $0 < \alpha \leq 1$ and $A > 0$, set

$$|\varphi|_\alpha = \sup_{0<\delta\leq A} \frac{1}{\delta^\alpha} \int_0^1 \operatorname{osc}(\varphi,\delta,x)dx\,.$$

The space V_α consists of all $\varphi \in L^1(m)$ such that $|\varphi|_\alpha < \infty$. On V_α we define the norm

$$\|\varphi\|_\alpha = |\varphi|_\alpha + \|\varphi\|_{L^1}\,.$$

(Observe that the norm $\|\cdot\|_\alpha$ depends also on the constant A.)

It follows immediately that V_α contains all α-Hölder functions. Further, by [21, Theorem 1.13] and [30, Proposition 3.4], the space V_α together with the norm $\|.\|_\alpha$ is a Banach space and there exists a constant $C = C(\alpha)$ so that for all $\varphi_1, \varphi_2 \in V_\alpha$ we have

$$\|\varphi_1\|_\infty \leq C\|\varphi_1\|_\alpha\,, \tag{11}$$

and

$$\|\varphi_1\varphi_2\|_\alpha \leq C\|\varphi_1\|_\alpha\|\varphi_2\|_\alpha\,. \tag{12}$$

Having introduced our main Banach space V_α we can now state our second condition. This condition is independent on the choice of the map x_0.

(II) T_0 is mixing and there exist constants $\epsilon > 0$, $C \geq 1$ and $0 < \tilde{\rho} < 1$ such that for all $\varphi \in V_\alpha$

$$\|\mathcal{L}_a^n\varphi\|_\alpha \leq C\tilde{\rho}^n\|\varphi\|_\alpha + C\|\varphi\|_{L^1}\,. \tag{13}$$

As already mentioned above the last condition is a bit more technical. It is used to guarantee that images by x_n of "most" elements in $\mathcal{P}_n$ are not too small (see Lemma 4.1). For an alternative condition see Remark 4.2 below.

(III) There exists $\epsilon > 0$ such that for all $\delta_0 > 0$ there exists a constant C so that

$$\sum_{\omega \in \mathcal{P}_n|[0,\epsilon]} \frac{1}{\|x_n|'_\omega\|_\infty} \leq Ce^{n^{\delta_0}}, \qquad \forall n \geq 1\,. \tag{14}$$

We can now state the main result of this paper. By Corollary 1.2, this result immediately implies the law of iterated logarithm. Recall the definition of σ in (2) (where the observable φ is now in the space V_α).

Theorem 2.6. *Let $T_a : [0,1] \to [0,1]$, $a \in [0,1]$, be a piecewise expanding one-parameter family, satisfying properties (i)-(iii) and condition (II) for some $0 < \alpha \leq 1$. If for a $C^{1+\alpha}$ map $x_0 : [0,1] \to [0,1]$ property (10) and conditions (I) and (III) are satisfied, then for all $\varphi \in V_\alpha$ such that $\sigma_0(\varphi) > 0$ there exists $\epsilon > 0$ so that the process $\xi_i : [0,\epsilon] \to \mathbb{R}$, $i \geq 1$, defined by*

$$\xi_i(a) = \frac{1}{\sigma_a(\varphi)}\Big(\varphi(x_i(a)) - \int \varphi d\mu_a\Big)\,, \tag{15}$$

satisfy the almost sure invariance principle for any error exponent $\gamma > 2/5$.

We conclude this section with the proof of Lemma 2.4.

Proof of Lemma 2.4. Recall that the boundary points $b_0(a), ..., b_{p_0}(a)$ are Lipschitz continuous and let L be their Lipschitz constant. By condition (I), we can take $j_0 \geq 1$ be so large that $|x'_{j_0}(0+)| \geq \sup_{a\in[0,1]} \sup_{x\in K(a)} |\partial_a T_a(x)|/(\lambda - 1) + 2L + 1$. Recall that by condition (I) there exists a neighbourhood $V \subset [0,1]$, so that $x_i(a) \notin \{b_0(a), ..., b_{p_0}(a)\}$, for all $a \in V \setminus 0$ and $0 \leq i < j_0$. Hence, by continuity, we find $\epsilon > 0$ (where $[0,\epsilon) \subset V$) so that

$$|x'_{j_0}(a)| \geq \frac{\sup_{a\in[0,1]} \sup_{x\in K(a)} |\partial_a T_a(x)|}{\lambda - 1} + 2L\,, \qquad \forall a \in (0,\epsilon)\,. \tag{16}$$

Let $j \geq 1$ and assume in the following formulas that, for the parameter values $a \in [0,\epsilon]$ under consideration, x_j and T_a^j are differentiable in a and $x_0(a)$, respectively. For $0 \leq k < j$ we have

$$x'_j(a) = (T_a^{j-k})'(x_k(a))x'_k(a) + \sum_{i=k+1}^{j} (T_a^{j-i})'(x_i(a))(\partial_a T_a)(x_{i-1}(a)), \tag{17}$$

which implies

$$\frac{x'_j(a)}{(T_a^j)'(x_0(a))} = \frac{1}{(T_a^k)'(x_0(a))}\left(x'_k(a) + \sum_{i=k+1}^{j} \frac{(\partial_a T_a)(x_{i-1}(a))}{(T_a^{i-k})'(x_k(a))}\right). \tag{18}$$

For $j > j_0$, choosing $k = 0$ and $k = j_0$, respectively, we get the following upper and lower bounds:

$$\begin{aligned} \frac{2L}{|(T_a^{j_0})'(x_0(a))|} \leq \left|\frac{x'_j(a)}{(T_a^j)'(x_0(a))}\right| \\ \leq \sup_{a\in[0,\epsilon]} \left(|x'_0(a)| + \frac{\sup_{x\in K(a)} |\partial_a T_a(x)|}{\lambda - 1}\right), \end{aligned} \tag{19}$$

where for the lower bound we used the assumption (16). It is only left to show that for each $j \geq j_0$ the number of $a \in [0,\epsilon]$ which are not contained in any element $\omega \in \mathcal{P}_j|[0,\epsilon]$ is finite. This is easily done by induction over j. Observe

first that, by the assumption on x_0, $x_j(a) \in K(a)$ for all $j \geq 0$ and $a \in [0,1]$. So the only case that prevents a to be contained in any element of $\omega \in \mathcal{P}_j|[0,\epsilon]$ is when $x_i(a) \in \{b_0(a), ...b_{p_0}(a)\}$, for some $i < j$. By the choice of ϵ above inequality (16), only 0 and ϵ might not be contained in any element of $\omega \in \mathcal{P}_{j_0}|[0,\epsilon]$. Assume that $j \geq j_0$ and consider the partition $\mathcal{P}_{j+1}|[0,\epsilon]$. From the lower bound in (19), we derive that $|x_j'(a)| \geq \lambda^{j-j_0} 2L > L$ for all a contained in an element of $\mathcal{P}_j|[0,\epsilon]$. Since the boundary points $b_k(a)$ are $\mathrm{Lip}(L)$, we have that $x_j(a) \in K(a) \setminus \{b_1(a), ..., b_k(a)\}$ for all but finitely many $a \in [0,\epsilon]$. Hence, by the induction assumption we conclude that the number of $a \in [0,\epsilon]$ which are not contained in any element $\omega \in \mathcal{P}_{j+1}|[0,\epsilon]$ is finite. This concludes the proof of Lemma 2.4. □

3. Tent maps and other examples

In this section we give some examples of piecewise expanding one-parameter families to which Theorem 2.6 can be applied.

We start with a trivial example which provides a good insight regarding the technical condition (III). Let $T_0 : [0,1] \to [0,1]$ be a mixing piecewise expanding map admitting a unique acip μ_0 with support, say, $[0,1]$. Let $\varphi \in V_\alpha$ so that $\sigma_0(\varphi) > 0$. We will deduce the well-known fact that the functions $\varphi(T_0^j(x)) - \int \varphi d\mu_0$, $j \geq 1$, satisfy the ASIP (see, e.g., [18]) from Theorem 2.6: As the one parameter family we take the constant family $T_a \equiv T_0$, for all $a \in [0,1]$. The map x_0 is the identity, i.e., $x_0(a) = a$. Obviously the transversality condition (I) is satisfied. The Lasota-Yorke inequality for the map T_0 which we need follows from [21, Theorem 3.2]. In order to apply Theorem 2.6, the remaining condition to verify is condition (III). Let $h = d\mu_0/dm$ be the density of μ_0. By [20] and [23], there exists a constant C so that

$$\frac{1}{C} \leq h(x) \leq C\,, \qquad \text{for a.e. } x \in [0,1]\,.$$

Since h is a fixed point of the transfer operator $\mathcal{L}_0$, we derive that

$$\sum_{T_0^n(y)=x} \frac{1}{|(T_0^n)'(y)|} \leq C \sum_{T_0^n(y)=x} \frac{h(y)}{|(T_0^n)'(y)|} = Ch(x) \leq C^2\,, \tag{20}$$

for a.e. x. Recall that the elements $\mathcal{P}_1(0)$ are of the form (b_{i-1}, b_i), for $1 \leq i \leq p_0$, and observe that the set of boundary points $\{\partial T_0^n(\omega) \mid \omega \in \mathcal{P}_n(0)\}$ consists of maximally $2np_0$ points. This implies that we can find points $x_1, ..., x_k$, $k \leq 2np_0$, which lie close to this set, so that

$$\sum_{\omega \in \mathcal{P}_n(0)} \frac{1}{\|(T_0^n)'|_\omega\|_\infty} \leq C \sum_{i=1}^{k} \sum_{T_0^n(y)=x_i} \frac{1}{|(T_0^n)'(y)|} \leq 2C^3 p_0 n\,. \tag{21}$$

(In the first inequality we used also a standard distortion estimate for piecewise expanding maps; see, e.g., (32) below.) Since, by definition, $x_n'(a) = (T_0^n)'(a)$ and $\mathcal{P}_n|[0,1] = \mathcal{P}_n(0)$, this concludes the verification of condition (III). Observe that in this trivial setting the right hand side of (14) is only increasing linearly in n.

We continue by studying some non-trivial examples, first the tent maps which is the main purpose of this paper and then β-transformations and Markov partition preserving families. In the end of this section, we give an application of our results in order to obtain almost sure typicality results similar to the ones in [32].

3.1. **Tent maps.** Let $T_a : [0,1] \to [0,1]$, $a \in [0,1]$, be a one-parameter family of tent maps, i.e., there exist $1 < \lambda \leq \Lambda < \infty$ and $0 < \alpha \leq 1$ so that, for each $a \in [0,1]$, the map $T_a : [0,1] \to [0,1]$ is continuous and there exists a turning point $c_a \in (0,1)$ such that $T_a|_{[0,c_a]}$ and $T_a|_{[c_a,1]}$ are $C^{1+\alpha}$, $0 < \alpha \leq 1$, (where the Hölder constants, see (8), are uniform in a), $\lambda \leq |T_a'(x)| \leq \Lambda$, for all $x \neq c_a$, and $T_a(1) = T_a(0) = 0$. Regarding the parameter dependency we assume that properties (i) and (ii) in the beginning of Section 2 are satisfied. Recall the definition (4) of a transversal family of tent maps T_a.

Theorem 3.1. *Assume that the family T_a be is transversal at T_0. Further, assume that T_0 is mixing and that the turning point c_0 is either not periodic or if p is its period then*

$$\lambda^{\alpha p} > 2\,. \tag{22}$$

If $\varphi \in V_\alpha$ so that $\sigma_0(\varphi) > 0$, then there exists $\epsilon > 0$ such that for a.e. $a \in [0,\epsilon]$ the turning point c_a satisfies the LIL for the function φ under the map T_a.

Proof. In order to prove Theorem 3.1, we will verify conditions (I)–(III). Then we can apply Theorem 2.6 and Corollary 1.2 which concludes the proof. (We have also to make sure that property (iii) in Section 2 is satisfied. This will follow, as a by-product, from the second last paragraph in this proof.)

Regarding condition (I), we define the map $x_0 : [0,1] \to [0,1]$ as $x_0(a) = T_a^{j_0}(c_a)$, where $j_0 \geq 1$ is so large that (4) holds for all $j \geq j_0$. Observe that, since T_0 is piecewise expanding and by (4), we find a constant $\delta > 0$ so that $x_0(a) \notin \{0, c_a, 1\}$, for all $a \in (0,\delta]$ (otherwise, in a neighbourhood of $a = 0$, c_a would be pre-periodic and hence $|x_j'(0)|$ would be bounded in j contradicting the transversality (4)). Hence, property (10) is satisfied for x_0 on the interval $[0,\delta]$. As before, using once more (4), for each $j \geq 1$, we find a neighbourhood $V \subset [0,\delta]$ of 0 so that $x_j(a) \neq c_a$, for all $a \in V \setminus 0$ (otherwise $|x_j'(a)|$ would be bounded). We conclude that x_0 satisfies condition (I) (and we can assume that x_0 satisfies (10) on the interval $[0,1]$).

We continue with the verification of condition (II) which is a condition on the family and which does not involve the map x_0. The problem in verifying condition (II) is to get uniform constants in the Lasota-Yorke inequality. If c_0 is not periodic let p be so large so that also in this non-periodic case inequality (22) is satisfied. [21, Theorem 3.2] and its proof shows that for all $\delta > 0$ and all $a \in [0,1]$ we find a constant $C = C(\delta, a)$ and $A = A(\delta, a) > 0$ (recall that the norm $\|\cdot\|_\alpha$ depends also on A) so that, setting $\rho = (2+\delta)/\lambda^{\alpha p}$, we have

$$\|\mathcal{L}_a^p \varphi\|_\alpha \leq \rho \|\varphi\|_\alpha + C\|\varphi\|_{L^1}\,. \tag{23}$$

By (22), we can fix $\delta > 0$ so small that $\rho < 1$. Hence, if we show that in a neighbourhood of 0 we can choose the constants C and A uniformly in a, then (23) combined with the assumption that T_0 is mixing implies condition (II). In order to verify this uniformity of C and A, we have to show that the constants K and A in [21, Lemma 3.1] can chosen independently on a in an neighbourhood of 0. Set $M = \delta\lambda^{-\alpha}(\delta/(16+2\delta))^{1-\alpha}$. By continuity we find an $\epsilon > 0$ so that $T_a^i(c_a) \neq c_a$, for all $a \in [0,\epsilon]$ and all $1 \leq i \leq p-1$. Hence, we find a constant $\kappa > 0$ so that for all $a \in [0,\epsilon]$ the sizes of the intervals of monotonicity for $T_a^p : [0,1] \to [0,1]$ are larger than κ. This and the fact that $x \mapsto |(T_a^p)'(x)|^{-1}$ is α-Hölder continuous on these monotonicity intervals imply that there is an integer k and a constant $0 < \kappa' \leq \kappa$ so that, for each $a \in [0,\epsilon]$, there is a refinement $\{I_1(a), I_2(a), ..., I_k(a)\}$

of the partition of $[0,1]$ into monotonicity intervals of T_a^p so that, for all $1 \le j \le k$, we have $\kappa' \le |I_j(a)| \le 2\kappa'$ and

$$\sup_{\substack{b_0<b_1<\ldots<b_n \\ b_0,\ldots,b_n \in I_j(a)}} \sum_{i=1}^{n} \left(\left| |(T_a^p)'(b_{i+1})|^{-1} - |(T_a^p)'(b_i)|^{-1} \right|^{1/\alpha} \right)^{\alpha} < M\,.$$

By this choice of $\{I_1(a), ..., I_k(a)\}$, we easily see that properties (16) and (17) in the proof of [21, Lemma 3.1] are satisfied. Further, setting $A = \kappa'\delta/(16+2\delta)$ corresponds to (17) in [21]. The remaining part of the proof of [21, Lemma 3.1] immediately shows then that the constant K therein only depends on the constants M, δ, and κ' which are by construction independent on $a \in [0,\epsilon]$.

It is left to verify condition (III). Let $h_a = d\mu_a/dm$ denote the density of the acip for T_a. We show first that there is a positive lower bound of h_a on its support which is uniform in a close to 0, i.e., there exists a constant $H < \infty$ so that

$$\text{ess inf}_{x\in K(a)}\, h_a(x) \ge H^{-1}\,, \qquad \text{for all } a \text{ close to } 0. \tag{24}$$

We claim that there exist $\epsilon > 0$ and an integer $N \ge 1$ so that, for all $a \in [0,\epsilon]$, there is an interval $I \subset K(a)$ of length $1/N$ so that $\text{ess inf}_{x\in I}\, h_a(x) \ge 1/2$. We show this claim by contradiction. By condition (II) (see (44) below), we find constants $\epsilon > 0$ and C so that, for all $a \in [0,\epsilon]$, we have the bound $\|h_a\|_\alpha \le C$. For $N \ge 1$, divide the unit interval into N disjoint intervals $I_1, ..., I_N$ of length $1/N$. For $1 \le \ell \le N$, let $M_\ell(a)$ and $m_\ell(a)$ denote the essential supremum and the essential infimum of h_a on I_ℓ, respectively. Since $1 = \int_0^1 h_a(x)dx \le \sum_{\ell=1}^N M_\ell(a)/N$, we get $\sum_{\ell=1}^N M_\ell(a) \ge 1$. Now, if the claim was not true, we find $a \in [0,\epsilon]$ so that $m_\ell(a) \le 1/2$, for all $1 \le \ell \le N$. From this we deduce

$$1/2 = 1 - 1/2 \le \sum_{\ell=1}^{N} (M_\ell(a) - m_\ell(a))/N \le \int_0^1 \text{osc}(h, 1/N, x)dx \le \|h_a\|_\alpha/N^\alpha$$

Since the right hand side tends to zero, uniformly in a for $N \to \infty$, we get a contradiction. Henceforth, fix $\epsilon > 0$ and $N \ge 1$ so that the just proven claim holds and, for $a \in [0,\epsilon]$, let $I(a)$ be the interval of length $1/N$ so that $\text{ess inf}_{x\in I(a)}\, h_a \ge 1/2$. We turn to the proof of (24). By the expansion of T_a, it follows that there exists an integer $0 \le k_0 \le \ln N/\ln\lambda$ such that $c_a \in T_a^{k_0}(I(a))$. Let $0 < \epsilon' \le \epsilon$, be so that T_a is mixing for all $a \in [0,\epsilon']$ (this is possible by condition (II); see the beginning of the proof of Proposition 4.3). Note that T_a mixing implies that the support $K(a)$ of the acip is equal to $[T_a^2(c_a), T_a(c_a)]$. From this we derive that property (iii) in Section 2 is satisfied. By [34] and since T_a is mixing, we have that $T_a : K(a) \to K(a)$ is exact, i.e., for each set $S \subset K(a)$ of positive Lebesgue measure it follows that $\lim_{j\to\infty} |K(a) \setminus T_a^j(S)| = 0$. Observe that, since T_a is a tent map, if J is an interval of length close to $K(a)(= [T_a^2(c_a), T_a(c_a)])$ then we have $T_a^2(K(a)) = K(a)$. Thus, exactness implies that there is an integer k_1 such that $T_a^{k_1}([c_a - 1/2N, c_a]) = T_a^k([c_a, c_a + 1/2N]) = K(a)$. Since the image of an interval by T_a^j, $j \ge 1$, changes continuously in a we can choose the integer k_1 independently on $a \in [0,\epsilon]$. Hence, we conclude that $T_a^{k_0+k_1}(I(a)) = K(a)$, for all $a \in [0,\epsilon]$. Using the equality

$$h_a(x) = \sum_{T_a^{k_0+k_1}(x)=y} \frac{h_a(y)}{|(T_a^{k_1+k_2})'(y)|}\,, \qquad \text{for a.e. } x\,,$$

the desired property (24) follows.

Let $\epsilon > 0$ be the constant in Lemma 2.4. It is shown in [32, Section 6.3] that there exists $0 < \epsilon' \leq \epsilon$ so that without loss of generality (otherwise inverse the order) if $0 \leq a_1 \leq a_2 \leq \epsilon'$ then for all $\omega_1 \in \mathcal{P}_n(a_1)$, $n \geq 1$, there exists (exactly) one $\omega_2 \in \mathcal{P}_n(a_2)$ so that ω_1 and ω_2 have the same combinatorics up to the iteration $n-1$. In order to apply the distortion estimate (32) below, we divide the interval $[0, \epsilon']$ into smaller intervals. For $n \geq 1$, let $\mathcal{I}_n$ be a partition of $[0, \epsilon']$ into intervals I of length approximately equal to $\epsilon'/n^{1/\alpha}$. For $I \in \mathcal{I}$, let a_I denote the right boundary point of I. By the proof of Lemma 2.4, it immediately follows that each two disjoint elements in $\mathcal{P}_n|[0, \epsilon']$ have different combinatorics up to $n-1$. Hence, for $I \in \mathcal{I}$, there exists an injective map from $\mathcal{P}_n|I$ to $\mathcal{P}_n(a_I)$ which maps each element in $\mathcal{P}_n|I$ to the element in $\mathcal{P}_n(a_I)$ with the same combinatorics up to $n-1$. Using Lemma 2.4 and the distortion estimate (32) below, we derive

$$\sum_{\omega_1 \in \mathcal{P}_n|I} \frac{1}{\|x_n'|_{\omega_1}\|_\infty} \leq C \sum_{\omega_2 \in \mathcal{P}(a_I)} \frac{1}{\|(T_{a_I}^n)'|_{\omega_2}\|_\infty} \leq C^2 n\,,$$

where the last inequality follows by (24), (20), and (21) ((24) guarantees that the constant C does not depend on a). Now, we can sum over the intervals in $\mathcal{I}_n$ which concludes the verification of condition (III) (where the right hand side in (14) increases in this setting like $n^{1+1/\alpha}$). □

Instead of taking the turning points c_a as the points of interest we can choose arbitrary points $x_0(a) \in [0, 1]$, as long as the transversality condition (I) is satisfied. However, in order to verify condition (III), we will still assume that the family itself is transversal at T_0. (It is quite likely that with some more work this assumption can be dropped.)

Theorem 3.2. *Assume that the family T_a is transversal at T_0. Further, assume that T_0 is mixing and that the turning point c_0 is either not periodic or if p is its period then* (22) *is satisfied. Let $x_0 : [0, 1] \to [0, 1]$ be a $C^{1+\alpha}$ map so that the transversality condition (I) is satisfied. If $\varphi \in V_\alpha$ so that $\sigma_0(\varphi) > 0$, then there exists $\epsilon > 0$ such that for almost every $a \in [0, \epsilon]$ the point $x_0(a)$ satisfies the LIL for the function φ under the map T_a.*

Proof. Condition (II) for the family T_a is already verified in the proof of Theorem 3.1.

Observe that in Theorem 3.2 we do not assume that x_0 satisfies (10). However, we can make the following reasoning. Observe that, for all $a \in [0, 1]$, all points in $(0, 1)$ are mapped after a finite number of iteration into $[T_a^2(c_a), T_a(c_a)]$. As explained in the beginning of the proof of Proposition 4.3 below, the fact that condition (II) is satisfied gives a constant $0 < \epsilon' \leq \epsilon$ so that T_a is mixing for all $a \in [0, \epsilon']$. Hence, $K(a) = [T_a^2(c_a), T_a(c_a)]$, for all $a \in [0, \epsilon']$. Since condition (I) is satisfied, we find $0 < \epsilon'' \leq \epsilon'$ and an iteration $k \geq 0$ so that $x_k(a) \in [T_a^2(c_a), T_a(c_a)] \setminus \{0, c_a, 1\}$, for all $a \in [0, \epsilon'']$. Hence, renaming x_k by x_0 (and considering the smaller interval $[0, \epsilon'']$), without loss of generality, we can assume in the remaining part of this proof that x_0 satisfies (10).

Regarding condition (III) we note that property (24) also holds in the setting of Theorem 3.2. Then we can follow word by word the last paragraph in the proof of Theorem 3.1 which concludes the verification of condition (III). □

3.2. **Generalised β-transformations and Markov partition preserving families.** First we consider a generalised form of β-transformations. Let $T : [0,\infty) \to [0,1]$ be piecewise $C^{1+\alpha}$, $0 < \alpha \leq 1$, and $0 = b_0 < b_1 < ...$ be the associated partition, where $b_k \to \infty$ as $k \to \infty$. We assume that T is right continuous and $T(b_k) = 0$, for each $k \geq 0$. Further, for each $a > 1$, we have $\|T'(a\,\cdot)^{-1}\|_{L^\infty([0,1])} < 1$ and $\|T'(a\,\cdot)\|_{L^\infty([0,1])} < \infty$. For $a_0 > 1$, we define the one-parameter family $T_a : [0,1] \to [0,1]$, $a \in [0,1]$, by $T_a(x) = T((a_0 + a)x)$. It is shown in [32, Lemma 5.1] that each T_a admits a unique acip μ_a whose support $K(a)$ is an interval adjacent to 0. Further, the length of K(a) is an increasing, piecewise constant function in a where the discontinuities are isolated point. Let $\lambda(a) = \text{ess inf}_{x\in[0,1]} |T_a'(x)|$. Regarding the verification of condition (II), we make sure that a similar condition as in (22) is satisfied: We assume that $b_j/a_0 \neq 1$, for all $j \geq 0$, and there exists $p \geq 1$ such that

$$\lambda(a_0)^{\alpha p} > 2\,, \quad \text{and} \quad T^i(b_j-) \neq b_k/a_0\,. \tag{25}$$

for all $1 \leq i \leq p-1$ and $k \geq 1$. Furthermore, we assume that $|K(a)|$ is constant in a neighbourhood of $a = 0$.

Theorem 3.3. *Let $x_0 : [0,1] \to [0,1]$ be a $C^{1+\alpha}$ map satisfying condition (I). If $\varphi \in V_\alpha$ so that $\sigma_0(\varphi) > 0$, then there exists $\epsilon > 0$ such that for almost every $a \in [0,\epsilon]$ the turning point $x_0(a)$ satisfies the LIL for the function φ under the map T_a.*

We continue with one-parameter families preserving a Markov structure. Assume that we have a one-parameter family $T_a : [0,1] \to [0,1]$, $a \in [0,1]$, as described in the beginning of Section 2 with a partition $0 \equiv b_0(a) < b_1(a) < ... < b_{p_0}(a) \equiv 1$ and satisfying properties (i)-(iii). We require additionally that the family T_a fulfils the following Markov property. Set $B_k(a) = (b_{k-1}(a), b_k(a))$, $1 \leq k \leq p_0$.

(M) For each $1 \leq k \leq p_0$ the image $T_a(B_k(a))$, $a \in [0,1]$, is a union of monotonicity intervals $B_\ell(a)$, $1 \leq \ell \leq p_0$ (modulo a finite number of points).

Theorem 3.4. *Let T_a be a family satisfying the Markov property (M) and let $x_0 : [0,1] \to [0,1]$ be a $C^{1+\alpha}$ map satisfying condition (I). If $\varphi \in V_\alpha$ so that $\sigma_0(\varphi) > 0$, then there exists $\epsilon > 0$ such that for almost every $a \in [0,\epsilon]$ the turning point $x_0(a)$ satisfies the LIL for the function φ under the map T_a.*

Proof of Theorems 3.3 and 3.4. Due to the Markov structure, the proof of Theorem 3.4 is much easier than the proofs of Theorem 3.1, 3.2, and 3.3. We leave it as an exercise to the reader. The proof of Theorem 3.3 is very similar to the proof of Theorem 3.1. Regarding property (10) we can argue as in the proof of Theorem 3.2. The fact that T_0 is mixing is shown in the last paragraph in [32, Section 5.2]. Property (25), ensures that we can go word by word along the verification of condition (II) in the proof of Theorem 3.1. Knowing that condition (II) is satisfied ensures that $\sigma_a(\varphi) > 0$ in an neighbourhood of 0 (see Lemma 4.5 below). It remains to verify condition (III). Observe that, by the construction of the family T_a, if $0 \leq a_1 \leq a_2 \leq 1$ then for all $\omega_1 \in \mathcal{P}_n(a_1)$, $n \geq 1$, there exists $\omega_2 \in \mathcal{P}_n(a_2)$ so that ω_1 and ω_2 have the same combinatorics up to the iteration $n-1$. Hence, if we show that the densities are uniformly bounded below on their support (see (24)), we can follow the last paragraph in the proof of Theorem 3.1 which concludes the verification of condition (III). The only obstacle in showing (24) might be the case when $K(0)$ is smaller than $K(a)$ but this case is excluded by our assumption on

the family T_a. The proof of (24) in a neighbourhood of $a = 0$ is done in detail in [32, inequality (30)]. □

3.3. **Almost sure typicality.** Let $T_a : [0,1] \to [0,1]$, $a \in [0,1]$, be a one-parameter family of piecewise expanding maps as described in Section 2 and satisfying properties (i)-(iii) therein. Let $x_0 : [0,1] \to [0,1]$ be a $C^{1+\alpha}$ map satisfying (10). As above let h_a denote the density of μ_a. As a corollary of Theorem 2.6 we get the following typicality result. Recall the definition of typical in (1).

Theorem 3.5. *If conditions (I)-(III) are satisfied and if there exists $\epsilon > 0$ and a constant C so that*

$$\text{ess inf}_{x\in K(a)} h_a(x) \geq C^{-1}\,, \qquad \forall a \in [0,\epsilon]\,, \tag{26}$$

then there exists $0 < \epsilon' \leq \epsilon$ so that $x_0(a)$ is typical for μ_a for a.e. $a \in [0,\epsilon']$.

Proof. For $\kappa > 0$ small, let $\mathcal{B} = \{(q-r, q+r)\cap[0,1] \mid q \in \mathbb{Q},\, r \in \mathbb{Q}\cap[0,\kappa]\}$. Observe that in order to prove Theorem 3.5, it is sufficient to show that there exists an $\epsilon' > 0$ so that, for each $B \in \mathcal{B}$, $x_0(a)$ satisfies the LIL for χ_B under the map T_a, for a.e. $a \in [0,\epsilon']$. From the proof of Theorem 2.6, we see that the constant ϵ in the assertion of Theorem 2.6 does only depend on the constant ϵ' in Proposition 4.3 and the length of the interval of parameters a on which $\sigma_a(\varphi) > 0$. Since ϵ' in Proposition 4.3 does only depend on the family T_a and not on the observable φ, it is enough to show that there exists $\delta > 0$ so that $\sigma_a(\chi_B) > 0$, for all $B \in \mathcal{B}$ and all $a \in [0,\delta]$. By Proposition 4.3 and (44) below, and (11), we find $\delta > 0$, C, and $0 < \rho < 1$ so that, for all $a \in [0,\delta]$, we have $\|h_a\|_\infty \leq C\|h_a\|_\alpha/2 \leq C$ and, for all $B \in \mathcal{B}$ and $a \in [0,\delta]$, we have

$$\Big|\int \chi_B\chi_B \circ T_a^n d\mu_a - \Big(\int \chi_B d\mu_a\Big)^2\Big| \leq C\|\chi_B h_a\|_\alpha \|\chi_B\|_{L^1}\rho^n \leq C^2|B|\rho^n\,,$$

for all $n \geq 1$, where in the last inequality we used also (12). Altogether, for $a \in [0,\delta]$, we derive

$$\begin{aligned}\sigma_a(\chi_B)^2 &= \int \chi_B d\mu_a - \Big(\int \chi_B d\mu_a\Big)^2 + 2\sum_{n\geq 1}\int \chi_B\chi_B\circ T_a^n d\mu_a - \Big(\int \chi_B d\mu_a\Big)^2 \\ &\geq C^{-1}|B| - 2NC^2|B|^2 - 2\sum_{n\geq N} C^2|B|\rho^n\,, \qquad \forall N \geq 1\,.\end{aligned}$$

Now, by taking $\kappa > 0$ in the definition of $\mathcal{B}$ sufficiently small, we can choose N so that $\sigma_a(\chi_B)^2 \geq |B|/2C$, for all $B \in \mathcal{B}$ and all $a \in [0,\epsilon]$. This concludes the proof of Theorem 3.5. □

Remark 3.6. The question of typicality of a point $x_0(a)$ for almost every parameter a in a general setting, was already studied in [32] (see also [8], [14], and [31] for more specific cases). Theorem 3.5 provides some alternative conditions. The method in [32] is inspired by a technique developed in [6] (see also [7] for another application of this technique). This method is very different from the one used in the present paper.

4. Preliminaries regarding the proof of Theorem 2.6

In this section, we fix an $\epsilon > 0$ which is at least so small as in Lemma 2.4 and conditions (II) and (III). When the meaning is clear, we will write $\mathcal{P}_j$ instead of $\mathcal{P}_j|[0,\epsilon]$.

We start with an elementary but important statement about the size of exceptionally small partition elements. Since we are far away from having Markov partitions, the image $x_j(\omega)$ of a partition element ω in $\mathcal{P}_j$ might be very small (despite the expansion of the map $x_j : \omega \to [0,1]$). If this image is too small it contains not sufficient information in order to use it in our analysis. From condition (III) we can derive a good control of the total size of partition elements having too small images for our purpose.

Lemma 4.1. *Assume that condition (III) is satisfied. Let $d_j > 0$, $j \geq 1$, be a sequence decaying at least stretched exponentially fast, i.e., there exists $\delta > 0$ so that*

$$\lim_{j\to\infty} d_j/e^{-j^\delta} < \infty\,. \tag{27}$$

There exists a constant C such that, for all $j \geq 1$, the size of the exceptional set $E_j := \{\omega \in \mathcal{P}_j \mid |x_j(\omega)| \leq d_j\} \subset \mathcal{P}_j$, has the upper bound

$$\Big|\bigcup_{\omega\in E_j} \omega\Big| \leq C d_j^{1/2}\,.$$

Proof. Take δ_0 in condition (III) strictly less than a δ satisfying (27). By the distortion estimate (31) below, for $\omega \in \mathcal{P}_j$ such that $|x_j(\omega)| \leq d_j$, we have $|\omega| \leq C\frac{d_j}{\|x_j|'_\omega\|_\infty}$. We conclude that

$$\Big|\bigcup_{\omega\in E_j} \omega\Big| \leq C d_j \sum_{\omega\in\mathcal{P}_j} \frac{1}{\|x_j|'_\omega\|_\infty} \leq C^2 d_j e^{j^{\delta_0}} \leq C^3 d_j^{1/2}\,.$$

□

Remark 4.2. Lemma 4.1 is the only place where we need condition (III). As an alternative condition to (III) it would be sufficient to require the following:

(III)' For each $\delta > 0$ there are constants C and $\beta > 0$ so that

$$|\{\omega \in \mathcal{P}_j \mid |x_j(\omega)| \leq e^{-j^\delta}\}| \leq C e^{-j^\beta}\,.$$

We preferred to put the slightly stronger condition (III) in Section 2 since it is the condition which we actually verify in the examples considered in Section 3.

Since the sequence of maps x_j is not the iteration of a fixed dynamical system admitting an invariant measure, in order to gain information about this sequence we have to switch locally from x_j to $T_{a_0}^j$ for some fixed parameter value a_0. After having switched we can profit from the abundant existing results for such a fixed mixing piecewise expanding map T_{a_0}. Very frequently we will use the exponential decay of correlations of T_{a_0}. Since we can only switch locally, we need that the constants in the decay of correlation for different T_a in the family are uniform.

Proposition 4.3 (Uniform decay of correlations)**.** *Assume that the family T_a satisfies condition (II). Then, the family T_a has uniform exponential decay of correlations for a close to 0, i.e., there exist constants $0 < \epsilon' \leq \epsilon$, $C \geq 1$, and $0 < \rho < 1$ such that for all $a \in [0,\epsilon']$, for all functions $\varphi \in V_\alpha$, and all $\psi \in L^1$ we have*

$$\Big|\int_0^1 \varphi\psi \circ T_a^n dm - \int_0^1 \varphi dm \int_0^1 \psi d\mu_a\Big| \leq C\|\varphi\|_\alpha \|\psi\|_{L^1}\rho^n\,, \quad \forall n \geq 1\,.$$

Proof. The proof is a direct application of the perturbation results of Keller and Liverani [22] using the estimates in Keller [21] and Saussol [30]. Observe that $\|\mathcal{L}_a^n\varphi\|_{L^1} = \|\varphi\|_{L^1}$. At the end of this proof we will show that for all $\varphi \in V_\alpha$

$$\|(\mathcal{L}_a - \mathcal{L}_0)\varphi\|_{L^1}/\|\varphi\|_\alpha = O(|a|^\alpha). \tag{28}$$

Combined with (13) in condition (II) and since T_0 is mixing, by [22], we find $0 < \epsilon' \leq \epsilon$ so that, for all $a \in [0, \epsilon']$, T_a is mixing and $\mathcal{L}_a$ can be written as $\mathbb{P}_a + \mathcal{Q}_a$ where $\mathbb{P}_a\varphi = h_a \int \varphi dm$ is a one-dimensional projection and where there are constants $C \geq 1$ and $0 < \rho < 1$ (both independent on a) so that $\|\mathcal{Q}_a^n\varphi\|_\alpha \leq C\rho^n\|\varphi\|_\alpha$, for all $n \geq 1$. Furthermore, for the later use we note that by [22] we get a constant $\kappa > 0$ such that

$$\|h_0 - h_a\|_{L^1} = O(|a|^\kappa)\,, \qquad \forall a \in [0, \epsilon']\,. \tag{29}$$

For $\varphi \in V_\alpha$ and $\psi \in L^1$, we get

$$\int \varphi\psi \circ T_a^n dm = \int [(\mathbb{P}_a + \mathcal{Q}_a^n)\varphi]\psi dm = \int \varphi dm \int \psi d\mu_a + \int \psi \mathcal{Q}_a^n \varphi dm\,.$$

Hence, using (11), we derive

$$\Big| \int \varphi\psi \circ T_a^n dm - \int \varphi dm \int \psi d\mu_a \Big| \leq C^2 \|\varphi\|_\alpha \|\psi\|_{L^1} \rho^n\,.$$

It remains to show (28). Recall the notation $b_0, ..., b_{p_0}$ for the partition points (the b_i's depend on a and are Lipschitz in a, say with Lipschitz constant L). Observe that $\|(\mathcal{L}_a - \mathcal{L}_0)\varphi\|_{L^1}$ is bounded above by

$$\sum_{i=0}^{p_0-1} \int \left| \frac{\varphi \circ T_a|^{-1}_{[b_i,b_{i+1}]}}{|T_a' \circ T_a|^{-1}_{[b_i,b_{i+1}]}|} \chi_{T_a([b_i,b_{i+1}])} - \frac{\varphi \circ T_0|^{-1}_{[b_i,b_{i+1}]}}{|T_0' \circ T_0|^{-1}_{[b_i,b_{i+1}]}|} \chi_{T_0([b_i,b_{i+1}])} \right|\,.$$

Let J_i be the interval $T_0([b_i, b_{i+1}])$ from which we subtract at each boundary point an interval of length $\Lambda L|a|$. Since the partition points b_i are Lipschitz in a, it follows that if $y \in (T_0|_{[b_i,b_{i+1}]})^{-1}(J_i)$ then $y \in (b_i(a'), b_{i+1}(a'))$, for all $a' \in [0, a]$. Furthermore, we have that $(T_a|_{[b_i,b_{i+1}]})^{-1}(J_i) \subset (b_i(0), b_{i+1}(0))$. Recall property (ii) in the beginning of Section 2, in particular, recall that $a' \mapsto T_{a'}'(y)$ is α-Hölder. Restricting the integral above to the interval J_i, we apply the triangle inequality and we split the integral into two integrals where the first one is (recall (11))

$$\int_{J_i} \left| \varphi(T_a|^{-1}_{[b_i,b_{i+1}]}(x)) \left(\frac{1}{|T_a'(T_a|^{-1}_{[b_i,b_{i+1}]}(x))|} - \frac{1}{|T_0'(T_0|^{-1}_{[b_i,b_{i+1}]}(x))|} \right) \right| dx$$
$$\leq C\|\varphi\|_{L^\infty}|a|^\alpha \leq C^2\|\varphi\|_\alpha |a|^\alpha\,,$$

and the second one is

$$\int_{J_i} \frac{1}{|T_0'(T_0|^{-1}_{[b_i,b_{i+1}]}(x))|} \left| \varphi(T_a|^{-1}_{[b_i,b_{i+1}]}(x)) - \varphi(T_0|^{-1}_{[b_i,b_{i+1}]}(x)) \right| dx$$
$$\leq C \int_{J_i} \operatorname{osc}(\varphi, C|a|, y) dy \leq C^2|a|^\alpha|\varphi|_\alpha \leq C^2|a|^\alpha\|\varphi\|_\alpha\,,$$

where we used the first inequality in (39) below (therein set $x_{i+1}^1 = x_{i+1}^2 = x$). In order to derive (28), it remains only to consider the integrals over $T_a([b_i, b_{i+1}]) \setminus J_i$ and $T_0([b_i, b_{i+1}]) \setminus J_i$, respectively. However, one easily sees that the measures of

these sets are bounded by a constant times $|a|$. Using once more (11), this concludes the proof. □

The next lemma is a collection of various distortion estimates. Recall the notations of the partitions in Section 2.1. In particular, recall that $\mathcal{P}_j(a)$ is the partition in the phase space, while $\mathcal{P}_j(=\mathcal{P}_j|[0,\epsilon])$ denotes the partition in the parameter space.

Lemma 4.4 (Distortion)**.** *There exists a constant C such that the following holds.*

For $a_1, a_2 \in [0,\epsilon]$ and $k \geq 1$, if $x \in [0,1]$, has the same combinatorics under T_{a_1} and T_{a_2} up to the $(k-1)$-th iteration, then

$$|T^k_{a_1}(x) - T^k_{a_2}(x)| \leq C\Lambda^k |a_1 - a_2| \,. \tag{30}$$

Let $\tilde{\omega} \in \mathcal{P}_k$. If $\omega \subset \tilde{\omega}$ is an interval, then

$$\left|\frac{x'_k(a_1)}{x'_k(a_2)}\right| \leq (1 + C|x_k(\omega)|^\alpha) \,, \qquad \forall a_1, a_2 \in \omega \,. \tag{31}$$

Let $k \geq 1$ and $a_1, a_2 \in [0,\epsilon]$ so that $|a_1 - a_2| \leq 1/k^{1/\alpha}$. If $\omega_1 \in \mathcal{P}_k(a_1)$ and $\omega_2 \in \mathcal{P}_k(a_2)$ have the same combinatorics up to the $(k-1)$-th iteration then

$$\left|\frac{(T^k_{a_1})'(x_1)}{(T^k_{a_2})'(x_2)}\right| \leq C \,, \qquad \forall x_1 \in \omega_1 \text{ and } x_2 \in \omega_2 \,. \tag{32}$$

Let $1 \leq k \leq \ell$. For $\omega \in \mathcal{P}_\ell$ and $a \in \omega$, we have

$$C^{-1} \leq \left|\frac{x'_\ell(a)/x'_k(a)}{(T^{\ell-k}_a)'(x_k(a))}\right| \leq C \,. \tag{33}$$

Proof. Property (33) follows immediately from Lemma 2.4.

We next show property (30). Set $x^1_i = T^i_{a_1}(x)$ and $x^2_i = T^i_{a_2}(x)$, $0 \leq i \leq k-1$. We assume that the constant C in the assertion of Lemma 4.4 satisfies $C \gg \delta_0^{-1}$ where δ_0 is the constant in property (i) in Section 2. By this choice, regarding the proof of (30) the only non-trivial situation is when $|a_1 - a_2| \ll \delta_0$. Recall that the partition points $b_0(a) < ... < b_{p_0}(a)$ are Lipschitz, say with constant L. Let $0 \leq i \leq k-1$ and take $\ell = \ell(i)$ so that $x^1_i \in (b_{\ell-1}(a_1), b_\ell(a_1))$. Since $|a_1 - a_2| \ll \delta_0$, we find $y \in (0,1)$ so that $|x^1_i - y| < L|a_1 - a_2|/2$ and $y \in (b_{\ell-1}(a), b_\ell(a))$, for all $a \in [a_1, a_2]$. By property (ii) in Section 2, it follows then that $|T_{a_1}(y) - T_{a_2}(y)| \leq C|a_1 - a_2|$. Hence, we derive

$$\begin{aligned}
&|T_{a_1}(x^1_i) - T_{a_2}(x^2_i)| \\
&\qquad \leq |T_{a_1}(x^1_i) - T_{a_1}(y)| + |T_{a_1}(y) - T_{a_2}(y)| + |T_{a_2}(y) - T_{a_2}(x^2_i)| \\
&\qquad \leq \Lambda|x^1_i - y| + C|a_1 - a_2| + \Lambda|y - x^2_i| \\
&\qquad \leq \Lambda(L + C)|a_1 - a_2| + \Lambda|x^1_i - x^2_i| \,. \qquad (34)
\end{aligned}$$

This estimate immediately implies (30).

Regarding property (31) observe first that by (18) (when $k = 0$ therein) we get

$$\frac{x'_k(a)}{(T^k_a)'(x_0(a))} = x'_0(a) + \sum_{j=1}^{k} \frac{(\partial_a T_a)(x_{j-1}(a))}{(T^j_a)'(x_0(a))} \,. \tag{35}$$

As in proving (30), we can assume that $|\omega| \ll \delta_0$ (otherwise we can compensate by possibly increasing the constant C). We proceed similarly as in deriving (34).

Let $0 \le i \le k-1$ and take $\ell = \ell(i)$ so that $x_i(a_1) \in (b_{\ell-1}(a_1), b_\ell(a_1))$. Since $|a_1 - a_2| \ll \delta_0$, we find $y \in (0,1)$ so that $|x_i(a_1) - y| < L|a_1 - a_2|/2$ and $y \in (b_{\ell-1}(a), b_\ell(a))$, for all $a \in [a_1, a_2]$. By property (ii) in Section 2, it follows that $|T'_{a_1}(y) - T'_{a_2}(y)| \le C|a_1 - a_2|^\alpha$. Hence, by a similar calculation as in (34), we get

$$|T'_{a_1}(x_i(a_1)) - T'_{a_2}(x_i(a_2))| \le C|a_1 - a_2|^\alpha + C|x_i(\omega)|^\alpha \le C^2|x_i(\omega)|^\alpha \,. \tag{36}$$

Thus,

$$\left|\frac{(T^j_{a_1})'(x_0(a_1))}{(T^j_{a_2})'(x_0(a_2))}\right| = \prod_{i=0}^{j-1}\left|\frac{T'_{a_1}(x_i(a_1))}{T'_{a_2}(x_i(a_2))}\right| \le 1 + C\sum_{i=0}^{j-1}|x_i(\omega)|^\alpha \,, \tag{37}$$

from which follows that

$$\left|\frac{1}{(T^j_{a_1})'(x_0(a_1))} - \frac{1}{(T^j_{a_2})'(x_0(a_2))}\right| \le C\frac{|x_{j-1}(\omega)|^\alpha}{|(T^j_{a_1})'(x_0(a_1))|} \,.$$

Recall that, by property (ii) in Section 2, $a \mapsto \partial_a T_a(x)$ and $x \mapsto \partial_a T_a(x)$ are α-Hölder continuous. Hence, using a "help" point y as above, we get

$$|(\partial_a T_a)|_{a=a_1}(x_{j-1}(a_1)) - (\partial_a T_a)|_{a=a_2}(x_{j-1}(a_2))| \le C|x_{j-1}(\omega)|^\alpha \,.$$

Combined with the α-Hölder continuity of x'_0, by comparing each term on the right hand side of (35) for $a = a_1$ and $a = a_2$, it follows

$$\begin{aligned}\left|\frac{x'_k(a_1)}{(T^k_{a_1})'(x_0(a_1))}\right| &\le \left|\frac{x'_k(a_2)}{(T^k_{a_2})'(x_0(a_2))}\right| + C|a_1 - a_2|^\alpha + C\sum_{j=1}^{k}\lambda^{-j}|x_{j-1}(\omega)|^\alpha \\ &\le \left|\frac{x'_k(a_2)}{(T^k_{a_2})'(x_0(a_2))}\right| + C^2|x_k(\omega)|^\alpha \,.\end{aligned}$$

Altogether, we have

$$\left|\frac{x'_k(a_1)}{x'_k(a_2)}\right| \le (1 + C|x_k(\omega)|^\alpha)\left|\frac{x'_k(a_1)/(T^k_{a_1})'(x_0(a_1))}{x'_k(a_2)/(T^k_{a_2})'(x_0(a_2))}\right| \le 1 + C^4|x_k(\omega)|^\alpha \,,$$

where in the last inequality we use the fact that $|x'_k(a_2)/(T^k_{a_2})'(x_0(a_2))| \ge C^{-1}$, by Lemma 2.4.

It is left to prove the distortion estimate (32). Choose two points $x^1_0 \in \omega_1$ and $x^2_0 \in \omega_2$ and, for $1 \le i \le k$, let $x^1_i = T^i_{a_1}(x^1_0)$ and $x^2_i = T^i_{a_2}(x^2_0)$. We claim that there is a constant C so that

$$|x^1_i - x^2_i| \le C\frac{1}{k^{1/\alpha}} + \frac{1}{\lambda^{k-i}} \,, \qquad \forall 0 \le i \le k \,. \tag{38}$$

In order to show (38), we proceed similarly as in showing (34). Let $0 \le i \le k-1$ and take $\ell = \ell(i)$ so that $x^1_i \in (b_{\ell-1}(a_1), b_\ell(a_1))$. By possible increasing the constant C in the assertion of Lemma 4.4 we can assume that $|a_1 - a_2| \ll \delta_0$ and we find $y \in (0,1)$ so that $|x^1_i - y| < L|a_1 - a_2|/2$ and $y \in (b_{\ell-1}(a), b_\ell(a))$, for all $a \in [a_1, a_2]$. Since $|y - x^2_i| \le \lambda^{-1}|T_{a_2}(y) - T_{a_2}(x^2_i)|$, we obtain

$$|x^1_i - x^2_i| \le L|a_1 - a_2|/2 + \frac{1}{\lambda}|T_{a_2}(y) - T_{a_1}(y)| + \frac{1}{\lambda}|T_{a_1}(y) - x^2_{i+1}| \,.$$

As in (34), we have $|T_{a_2}(y) - T_{a_1}(y)| \le C|a_2 - a_1|$, and note that

$$|T_{a_1}(y) - x^2_{i+1}| \le |T_{a_1}(y) - x^1_{i+1}| + |x^1_{i+1} - x^2_{i+1}| \le \Lambda L|a_1 - a_2|/2 + |x^1_{i+1} - x^2_{i+1}| \,.$$

Altogether, recalling that $|a_1 - a_2| \le 1/k^{1/\alpha}$, we find a constant C so that

$$|x_i^1 - x_i^2| \le C|a_1 - a_2| + |x_{i+1}^1 - x_{i+1}^2|/\lambda \le C/k^{1/\alpha} + |x_{i+1}^1 - x_{i+1}^2|/\lambda\,. \tag{39}$$

From this estimate we easily deduce (38).

By (36) and (38), for all $0 \le i \le k-1$, we obtain

$$|T'_{a_1}(x_i^1) - T'_{a_2}(x_i^2)| \le C|a_1 - a_2|^\alpha + C|x_i^1 - x_i^2|^\alpha \le C^2/k + C^2/\lambda^{\alpha(k-i)}\,,$$

which implies

$$\left|\frac{(T_{a_1}^k)(x_0^1)}{(T_{a_2}^k)'(x_0^2)}\right| \le \prod_{i=0}^{k-1} \frac{|T'_{a_1}(x_i^1)|}{|T'_{a_2}(x_i^2)|} \le \prod_{i=0}^{k-1} \frac{|T'_{a_2}(x_i^2)| + C^2/k + C^2/\lambda^{\alpha(k-i)}}{|T'_{a_2}(x_i^2)|}\,.$$

Since the right hand side is bounded by a constant independent on k, a_1 and a_2, this concludes the proof of (32). □

Recall the definition of $\sigma_a(\varphi)$ in (2) (where φ here is in the space V_α). In order to ensure that the functions $\xi_j(a)$, $j \ge 1$, defined in (15) depend nicely on a, we have to investigate the a-dependence of σ_a.

Lemma 4.5 (Regularity of $a \mapsto \sigma_a$)**.** *Assume that the family T_a satisfies condition (II). Let $\epsilon' > 0$ be the constant in Proposition 4.3. For each $\varphi \in V_\alpha$ there exist constants C and $\kappa > 0$ such that*

$$|\sigma_a(\varphi) - \sigma_{a'}(\varphi)| \le C|a - a'|^\kappa\,, \quad \forall\ a, a' \in [0, \epsilon']\,. \tag{40}$$

Proof. For simplicity we assume that $a' = 0$ and $\int \varphi d\mu_0 = 0$. The general case is proven similarly (cf. the last paragraph in this proof). For a constant $\kappa' > 0$ to be determined later in the proof, let $k_0 = k_0(a, 0)$ be minimal such that, for $\bar a = 0$ and $\bar a = a$, we have

$$\Big|2\sum_{k>k_0} \int \Big(\varphi - \int \varphi d\mu_{\bar a}\Big)\Big(\varphi - \int \varphi d\mu_{\bar a}\Big) \circ T_{\bar a}^k d\mu_{\bar a}\Big| \le a^{\kappa'}\,.$$

By Proposition 4.3, the absolute value of the integral in the sum is bounded by a constant (independent on a) times ρ^k which implies that

$$k_0 \le \kappa''|\log a| + C\,, \qquad \text{where } \kappa'' \to 0 \text{ as } \kappa' \to 0\,. \tag{41}$$

Observe that, for all $k \ge 0$,

$$\int \Big(\varphi - \int \varphi d\mu_a\Big)\Big(\varphi - \int \varphi d\mu_a\Big) \circ T_a^k d\mu_a = \int \varphi\varphi \circ T_a^k d\mu_a - \Big(\int \varphi d\mu_a\Big)^2\,.$$

We get

$$\begin{aligned}\sigma_a(\varphi)^2 - \sigma_0(\varphi)^2 &= \int \varphi^2 (h_a - h_0) dm - \Big(\int \varphi d\mu_a\Big)^2 \\ &\quad + 2\sum_{k=1}^{k_0} \Big(\int \varphi\varphi \circ T_a^k d\mu_a - \int \varphi\varphi \circ T_0^k d\mu_0 - \Big(\int \varphi d\mu_a\Big)^2\Big) + O(a^{\kappa'})\,.\end{aligned}$$

By (29), we immediately get that the absolute value of the first two terms on the right hand side and of the last term in the sum are bounded above by a constant (depending only on $\|\varphi\|_\infty$) times a^κ. Regarding the remaining two integrals, again by (29), we have

$$\int \varphi\varphi \circ T_a^k d\mu_a - \int \varphi\varphi \circ T_0^k d\mu_0 = \int \varphi\left(\varphi \circ T_a^k - \varphi \circ T_0^k\right) d\mu_0 + O\left(\|\varphi\|_\infty^2 a^\kappa\right)\,.$$

In order to bound the integral on the right hand side, we need the following sublemma.

Sublemma 4.6. *For all $a, a' \in [0, \epsilon']$ and $k \geq 1$, there exists a set of intervals $\mathcal{P}_k(a, a')$ such that for each $J \in \mathcal{P}_k(a, a')$ there exist $\omega \in \mathcal{P}_k(a)$ and $\omega' \in \mathcal{P}_k(a')$ such that $J = \omega \cap \omega'$ and ω and ω' have the same combinatorics (up to iteration $k-1$). Furthermore,*

$$\Big| \bigcup_{J \in \mathcal{P}_k(a,a')} J \Big| \geq |\operatorname{supp} \mu_a| - C\Big(\frac{p_1 \Lambda}{\lambda}\Big)^k |a - a'|^\alpha \,. \tag{42}$$

(Recall that p_1 is the number of elements in $\mathcal{P}_1(a)$.)

Proof. We show inductively in k that

$$\Big| \bigcup_{J \in \mathcal{P}_k(a,a')} J \Big| \geq |\operatorname{supp} \mu_a| - C \sum_{j=1}^{k} p_1^j \lambda^{-(j-1)} \Lambda^{j-1} |a - a'|^\alpha \,. \tag{43}$$

This immediately implies (42).

Recall that the properties (i) and (iii) in the beginning of Section 2 asserts that the boundary points of the elements in $\mathcal{P}_1(a)$ are α-Hölder continuous and the partition points $b_j(a)$, $0 \leq j \leq p_0$, are Lipschitz continuous in a. This immediately shows (43) for $k = 1$. Let $k > 1$ and assume the assertion holds for $k-1$. For $J_0 \in \mathcal{P}_{k-1}(a, a')$, let $\omega \in \mathcal{P}_k(a)|J_0$ and $j = j(\omega)$ such that $T_a^{k-1}(\omega) \subset (b_{j-1}(a), b_j(a))$. By (30) in Lemma 4.4 and by the Lipschitz continuity of $b_{j-1}(a)$ and $b_j(a)$, we derive

$$\begin{aligned} |T_{a'}^{k-1}(\omega) \cap (b_{j-1}(a'), b_j(a'))| &\geq |T_{a'}^{k-1}(\omega)| - 2L|a - a'| - 2C\Lambda^{k-1}|a - a'| \\ &\geq |T_{a'}^{k-1}(\omega)| - 3C\Lambda^{k-1}|a - a'| \,. \end{aligned}$$

If the right hand side is positive, then we find $\omega' \in \mathcal{P}_k(a')|J_0$ with the same combinatorics as ω. Furthermore, by the distortion estimate (32) in Lemma 4.4 (where we set $a_1 = a_2 = a'$), we find a constant C (independent on ω) such that

$$|\omega \cap \omega'| \geq |\omega| - C\lambda^{-(k-1)}\Lambda^{k-1}|a - a'| \,.$$

Since there are maximal p_1^{k-1} elements in $\mathcal{P}_{k-1}(a, a')$ and maximal p_1 elements in $\mathcal{P}_k(a)|J_0$, we derive that

$$\Big| \bigcup_{J \in \mathcal{P}_k(a,a')} J \Big| \geq \Big| \bigcup_{J_0 \in \mathcal{P}_{k-1}(a,a')} J_0 \Big| - Cp_1^k \lambda^{-(k-1)} \Lambda^{k-1} |a - a'| \,.$$

By the induction assumption, this concludes the proof of (43). □

Observe that (41) implies that a is bounded above by a constant times $e^{-k_0/\kappa''}$. Combined with (30) in Lemma 4.4, if $\kappa'' \leq 1/2 \log \Lambda$, we derive that for all $J \in \mathcal{P}_k(0, a)$, $k \leq k_0$,

$$|T_a^k(x) - T_0^k(x)| \leq C\Lambda^k a \leq C^2 a^{1/2} \,, \qquad \forall x \in J \,.$$

If in addition $\kappa'' \le \min(\alpha/4\log(p_1), \alpha/2\log(p_1\Lambda))$, it follows, for $k \le k_0$,

$$\begin{aligned}\int_{\mathrm{supp}(\mu_0)} \big|\varphi\circ T_a^k - \varphi\circ T_0^k\big| dy \\ &\le \sum_{J\in\mathcal{P}_k(0,a)} \int_J \big|\varphi\circ T_a^k - \varphi\circ T_0^k\big|\, dy + Cp_1^{k_0}\Lambda^{k_0}a^\alpha \\ &\le \#\{J\in\mathcal{P}_k(0,a)\}\int \mathrm{osc}(\varphi, C^2a^{1/2}, y)dy + C^2a^{\alpha/2}\,.\end{aligned}$$

Since the right hand side is bounded above by a constant times $p_1^{k_0}a^{\alpha/2}$ which is in turn bounded above by a constant times $a^{\alpha/4}$, altogether we derive that, for κ' sufficiently small, there exists a constant C (depending on $\|\varphi\|_\infty$ and $\|\varphi\|_\alpha$) such that

$$\big|\sigma_a(\varphi)^2 - \sigma_0(\varphi)^2\big| \le Ca^\kappa + Ck_0\Big(\|h_0\|_\infty a^{\alpha/4} + a^{2\kappa}\Big) + Ca^{\kappa'} \le C^2a^\kappa\,,$$

where in the last inequality we possibly have to decrease $\kappa > 0$.

If $a' \ne 0$ observe that by (13) it follows $\|h_a\|_\alpha = \lim_{n\to\infty}\|\mathcal{L}_a^n h_a\|_\alpha \le C\|h_a\|_{L^1}$, and by (29) we conclude that

$$\sup_{a\in[0,\epsilon']}\|h_a\|_\alpha < \infty\,. \tag{44}$$

Combined with (11), this ensures that the constant C is uniform in a'. This concludes the proof of Lemma 4.5. □

5. Switching locally from the parameter to the phase space

The aim of this section is to prove the following Proposition 5.1 which is the main estimate needed in verifying a law of large numbers for the squares of the blocks defined in the following Section 6 (see Lemma 6.2 therein). Its proof is given in the end of this section. Recall that in Theorem 2.6, we assume $\sigma_0(\varphi) > 0$. Hence, by Lemma 4.5, we find a constant $\epsilon > 0$ so that $\sigma_a(\varphi) > 0$, for all $a \in [0, \epsilon]$. (Note that by the assumption in Lemma 4.5 the present constant ϵ is smaller than the constant ϵ' in Proposition 4.3 which ensures that we can apply this proposition in the following.) Let

$$\lambda_0 = \min(\lambda^{\min(\alpha/3,\kappa/2)}, \rho^{-1/2}) > 1\,, \tag{45}$$

where $\kappa > 0$ is so small as in (29) and Lemma 4.5. Fix $\eta > 0$ so small that

$$\Big(\frac{p_1\Lambda}{\lambda}\Big)^\eta \le \lambda_0\,. \tag{46}$$

The expectation $E(\xi)$ of a function $\xi(a)$ is the integral $\epsilon^{-1}\int_0^\epsilon \xi(a)da$.

Proposition 5.1. *There exists a constant C (depending essentially only on φ and the constants in the uniform exponential decay of correlation of the family T_a) such that*

$$\Big|E\Big(\sum_{k=m}^{m+n-1}\xi_k(a)\Big)^2 - n\Big| \le C\,, \quad \forall\ m \text{ and } 1 \le n \le \eta m/2\,. \tag{47}$$

Furthermore, for m and n as in (47) *and $v = m - m^{1/4}$, if $\omega \in \mathcal{P}_v$ such that $\lambda_0^{-m^{1/4}} \le |x_v(\omega)| \le n^{-3/\alpha}$ then*

$$\Big| E\Big[\Big(\sum_{k=m}^{m+n-1} \xi_k(a)\Big)^2 \mid \{a \in \omega\}\Big] - n\Big| \le C\,. \tag{48}$$

Remark 5.2. With some more effort inequalities (47) and (48) can be proven to hold for all $n \ge 1$ (the argument uses a similar construction as in (66) below; the upper bound of $|x_v(\omega)|$ in (48) can be replaced, e.g., by $2\lambda_0^{-m^{1/4}}$).

The following lemma provides us with a tool to switch locally from the parameter space to the phase space. This can then be used in the proof of Proposition 5.1 to gain informations about the sequence x_j on the parameter space by considering the iterations $T_{a_0}^j$ on the phase space for a fixed parameter value a_0. Recall the definition (5) of φ_a.

Lemma 5.3 (Switching locally from parameter to phase space)**.** *There exists a constant C such that the following holds. Let $v, v_1, ..., v_{\ell_0}$, $1 \le \ell_0 \le 4$, be integers satisfying*

$$1 \le v \le v_1, ..., v_{\ell_0} \le v + \eta v\,.$$

Let ω be an interval such that there exists $\tilde\omega \in \mathcal{P}_v$ with $\omega \subset \tilde\omega$ and $|x_v(\omega)| \ge \lambda_0^{-v}$. For all $a_0 \in \omega$ we have

$$\frac{1}{|x_v(\omega)|}\int_{x_v(\omega)} \Big|\prod_{\ell=1}^{\ell_0} \xi_{v_\ell}(x_v|_\omega^{-1}(y)) - \prod_{\ell=1}^{\ell_0} \varphi_{a_0}(T_{a_0}^{v_\ell - v}(y))\Big| dy \le C\lambda_0^{-v}\,. \tag{49}$$

Proof. In order to prove Lemma 5.3, we need an ingredient similar to the one provided by Sublemma 4.6. The difference here is that we compare the partitions on the parameter space with the partitions on the phase space.

Sublemma 5.4. *Let v, ω, and a_0 be as in the assertion of Lemma 5.3, and let $v \le \nu \le v + \eta v$. There exists a set of intervals $\mathcal{P}_{v,\nu}(\omega, a_0)$ such that for each $J \in \mathcal{P}_{v,\nu}(\omega, a_0)$ there exist $\omega_1 \in \mathcal{P}_\nu|\omega$ and $\omega_2 \in \mathcal{P}_{\nu-v}(a_0)|x_v(\omega)$ such that $J = x_v(\omega_1) \cap \omega_2$ and $x_v(\omega_1)$ and ω_2 have the same combinatorics, i.e., for $a \in \omega_1$ and $x \in \omega_2$, $x_{v+i}(a)$ and $T_{a_0}^i(x)$ have the same combinatorics for $0 \le i < \nu - v$. Furthermore,*

$$\Big|\bigcup_{J \in \mathcal{P}_{v,\nu}(\omega,a_0)} J\Big| \ge |x_v(\omega)|(1 - C\lambda_0^{-v})\,. \tag{50}$$

Proof. The proof is similar to the proof of Sublemma 4.6. For $\nu = v$ there is nothing to show (by definition $x_v(\omega) = \mathcal{P}_0|x_v(\omega)$). Henceforth, we assume $\nu > v$. Given v, we show inductively in $\nu > v$ that

$$\Big|\bigcup_{J \in \mathcal{P}_{v,\nu}(\omega,a_0)} J\Big| \ge |x_v(\omega)| - C\sum_{j=v+1}^{\nu} p_1^{j-v}\lambda^{-(j-v-1)}\Lambda^{j-v-1}|\omega|^\alpha\,. \tag{51}$$

If $\nu \le v + \eta v$, inequality (51) implies then

$$\begin{aligned}\Big|\bigcup_{J \in \mathcal{P}_{v,\nu}(\omega,a_0)} J\Big| &\ge |x_v(\omega)| - C^2 p_1^{\nu-v}\lambda^{-(\nu-v)}\Lambda^{\nu-v}|\omega|^\alpha \\ &\ge |x_v(\omega)| - C^2\Big(\frac{p_1\Lambda}{\lambda}\Big)^{\eta v}\lambda^{-\alpha v} \ge |x_v(\omega)|(1 - C^3\lambda_0^{-v})\,,\end{aligned}$$

where in the last inequality we used the condition (46) on η and the fact that $|x_v(\omega)| \geq \lambda_0^{-v}$ (we used also that $|\omega| \leq C\lambda^{-v}$ which follows from Lemma 2.4). This concludes the proof of Sublemma 5.4.

Since, by properties (i) and (iii) in Section 2, the boundary points of $\omega \in \mathcal{P}_1(a_0)|x_v(\omega)$ are α-Hölder continuous and the partition points $b_0(a), ..., b_{p_0}(a)$ are Lipschitz continuous in a, this immediately shows (51) for $\nu = v+1$. Let $\nu > v+1$ and assume the assertion holds for $\nu - 1$. For $J_0 \in \mathcal{P}_{v,\nu-1}(\omega, a_0)$, let $\omega_2 \in \mathcal{P}_{\nu-v}(a_0)|J_0$ and $j = j(\omega_2)$ such that $T_{a_0}^{\nu-v-1}(\omega_2) \subset (b_{j-1}(a_0), b_j(a_0))$. By (30) in Lemma 4.4 and by the Lipschitz continuity of $b_{j-1}(a)$ and $b_j(a)$, we derive

$$\begin{aligned}&|x_{\nu-1}(x_v|_\omega^{-1}(\omega_2)) \cap (b_{j-1}(a), b_j(a'))| \\ &\qquad \geq |x_{\nu-1}(x_v|_\omega^{-1}(\omega_2))| - 2L|\omega| - 2C\Lambda^{\nu-v-1}|\omega|\,, \quad \forall a, a' \in \omega\,.\end{aligned}$$

If the right hand side is positive for an appropriate choice of $a, a' \in \omega$, then we find $\omega_1 \in \mathcal{P}_\nu|\omega$, where $x_v(\omega_1)$ and ω_2 have the same combinatorics. Furthermore, by the distortion estimate (32) in Lemma 4.4, we find a constant C such that

$$|x_v(\omega_1) \cap \omega_2| \geq |\omega_2| - C\frac{|\omega_2|}{|x_{\nu-1}(x_v|_\omega^{-1}(\omega_2))|}\Lambda^{\nu-v-1}|\omega| \geq |\omega_2| - C^2\Big(\frac{\Lambda}{\lambda}\Big)^{\nu-v-1}|\omega|\,,$$

where in the last inequality we used (33). Since there are maximal $p_1^{\nu-v-1}$ elements in $\mathcal{P}_{v,\nu-1}(\omega, a_0)$ and maximal p_1 elements in $\mathcal{P}_{\nu-v}(a_0)|J_0$, we derive that

$$\Big|\bigcup_{J\in\mathcal{P}_{v,\nu}(\omega,a_0)} J\Big| \geq \Big|\bigcup_{J_0\in\mathcal{P}_{v,\nu-1}(\omega,a_0)} J_0\Big| - C^2 p_1^{\nu-v}\lambda^{-(\nu-v-1)}\Lambda^{\nu-v-1}|\omega|\,.$$

By the induction assumption, this concludes the proof of (43). □

Recall that $\xi_{v_1}(a) = \varphi_a(x_{v_1}(a))$ (see (15)). For $a = x_v|_\omega^{-1}(y)$ and $a_0 \in \omega$, we write

$$\begin{aligned}&\xi_{v_1}(a) - \varphi_{a_0}(T_{a_0}^{v_1-v}(x_v(a))) \\ &\qquad = \varphi_a(x_{v_1}(a)) - \varphi_{a_0}(x_{v_1}(a)) + \varphi_{a_0}(x_{v_1}(a)) - \varphi_{a_0}(T_{a_0}^{v_1-v}(x_v(a)))\,.\end{aligned}$$

By Lemma 4.5 and (29), we easily see that the difference of the first two terms on the right hand side is bounded from above by a constant times $|\omega|^\kappa$. To estimate the integral over the difference of the last two terms we use the partition given by Sublemma 5.4. First, observe that, by Lemma 4.5 and (11), we find a constant C only dependent on φ (and, in particular, not on a) so that

$$\max(\|\varphi_a\|_\alpha, \|\varphi_a\|_{L^1}, \|\varphi_a\|_\infty) \leq C\|\varphi\|_\alpha \leq C^2\,, \qquad \forall a \in [0, \epsilon]\,. \tag{52}$$

If $J \in \mathcal{P}_{v,v_1}(\omega, a_0)$ and $y \in J$, then by (30) in Lemma 4.4 we have

$$\big|x_{v_1}(x_v|_\omega^{-1}(y)) - T_{a_0}^{v_1-v}(y)\big| \leq C\Lambda^{v_1-v}|\omega|\,,$$

which implies

$$\begin{aligned}
&\int_{x_v(\omega)} \big|\varphi_{a_0}(x_{v_1}(x_v|_\omega^{-1}(y))) - \varphi_{a_0}(T_{a_0}^{v_1-v}(y))\big| dy \\
&\leq \sum_{J\in\mathcal{P}_{v,v_1}(\omega,a_0)} \int_J \operatorname{osc}(\varphi_{a_0}, C\Lambda^{v_1-v}|\omega|, T_{a_0}^{v_1-v}(y))dy + C|x_v(\omega)|\lambda_0^{-v} \\
&\leq \sum_{J\in\mathcal{P}_{v,v_1}(\omega,a_0)} C\lambda^{-(v_1-v)} \int_{T_{a_0}^{v_1-v}(J)} \operatorname{osc}(\varphi_{a_0}, C\Lambda^{v_1-v}|\omega|, z)dz + C|x_v(\omega)|\lambda_0^{-v} \\
&\leq C^3\Big(\frac{p_1\Lambda^\alpha}{\lambda}\Big)^{v_1-v}|\omega|^\alpha + C|x_v(\omega)|\lambda_0^{-v}\,.
\end{aligned}$$

Altogether, we obtain (recall (52))

$$\begin{aligned}
&\frac{1}{|x_v(\omega)|}\int_{x_v(\omega)} \Big|\xi_{v_1}(x_v|_\omega^{-1}(y)) - \varphi_{a_0}(T_{a_0}^{v_1-v}(y))\Big| \prod_{\ell=2}^{\ell_0} |\xi_{v_\ell}(x_v|_\omega^{-1}(y))|dy \\
&\qquad\qquad \leq C\Big(\Big(\frac{p_1\Lambda^\alpha}{\lambda}\Big)^{\eta v}\frac{|\omega|^\alpha}{|x_v(\omega)|} + \lambda_0^{-v} + \frac{|\omega|^\kappa}{|x_v(\omega)|}\Big) \leq C^2\lambda_0^{-v}\,,
\end{aligned}$$

where in the last inequality we used the assumption that $|x_v(\omega)| \geq \lambda_0^{-v}$, the definition (45) of λ_0, and the condition (46) on η. Then, similarly we derive

$$\begin{aligned}
&\frac{1}{|x_v(\omega)|}\int_{x_v(\omega)} |\varphi_{a_0}(T_{a_0}^{v_1-v}(y))|\Big|\xi_{v_2}(x_v|_\omega^{-1}(y)) - \varphi_{a_0}(T_{a_0}^{v_2-v}(y))\Big| \\
&\qquad\qquad \prod_{\ell=3}^{\ell_0} |\xi_{v_\ell}(x_v|_\omega^{-1}(y))|dy \leq C\lambda_0^{-v}\,,
\end{aligned}$$

and so on. This concludes the proof of Lemma 5.3. □

Corollary 5.5. *There exists a constant C such that the following holds. Let $v, v_1, ..., v_{\ell_0}$, $1 \leq \ell_0 \leq 4$, be positive integers as in the assertion of Lemma 5.3 and let ω be an interval such that there exists $\tilde\omega \in \mathcal{P}_v$ with $\omega \subset \tilde\omega$ and $|x_v(\omega)| \geq \lambda_0^{-v}$. For all $a_0 \in \omega$ we have*

$$\frac{1}{|\omega|}\int_\omega \prod_{\ell=1}^{\ell_0} \xi_{v_\ell}(a)da = \frac{1+O(|x_v(\omega)|^\alpha)}{|x_v(\omega)|}\int_{x_v(\omega)} \prod_{\ell=1}^{\ell_0} \varphi_{a_0}(T_{a_0}^{v_\ell-v}(y))dy + O(\lambda_0^{-v})\,.$$

Proof. Doing the change of variables $y = x_v(a)$, $a \in \omega$, by the distortion estimate (31) in Lemma 4.4, we derive

$$\frac{1}{|\omega|}\int_\omega \prod_{\ell=1}^{\ell_0} \xi_{v_\ell}(a)da = \frac{1+O(|x_v(\omega)|^\alpha)}{|x_v(\omega)|}\int_{x_v(\omega)} \prod_{\ell=1}^{\ell_0} \xi_{v_\ell}(x_v|_\omega^{-1}(y))dy\,. \tag{53}$$

Applying Lemma 5.3 concludes the proof. □

5.1. **Proof of Proposition 5.1.** We are going to show (48). Let ω be as in the assertion. We write

$$\frac{1}{|\omega|}\int_\omega \Big(\sum_{k=m}^{m+n-1}\xi_k\Big)^2 = \frac{1}{|\omega|}\sum_{k=m}^{m+n-1}\Big(\int_\omega \xi_k^2 + 2\sum_{\ell=k+1}^{m+n-1}\int_\omega \xi_k\xi_\ell\Big)\,.$$

Hence, in order to prove (48), it is sufficient to show that there is a constant C such that

$$\Big|1-\frac{1}{|\omega|}\int_\omega\Big(\xi_k^2+2\sum_{\ell=k+1}^{m+n-1}\xi_k\xi_l\Big)\Big|\le C(m+n-k)^{-2}\,. \tag{54}$$

Let $a_0\in\omega$. By Corollary 5.5, we have

$$\begin{aligned}&\frac{1}{|\omega|}\int_\omega\Big(\xi_k^2+2\sum_{\ell=k+1}^{m+n-1}\xi_k\xi_l\Big)\\&=\frac{1}{|x_v(\omega)|}\int_{x_v(\omega)}\Big(\varphi_{a_0}^2\circ T_{a_0}^{k-v}+2\sum_{\ell=k+1}^{m+n-1}\varphi_{a_0}\circ T_{a_0}^{k-v}\varphi_{a_0}\circ T_{a_0}^{\ell-v}\Big)\\&\quad+O\big((m+n-k)(\lambda_0^{-(m-m^{1/4})}+|x_v(\omega)|^\alpha)\big)\,.\end{aligned}$$

Proposition 4.3 gives (recall also (52)), for all $\ell\ge k$,

$$\int_{x_v(\omega)}\varphi_{a_0}\circ T_{a_0}^{k-v}\varphi_{a_0}\circ T_{a_0}^{\ell-v}dm\;=\;|x_v(\omega)|\int\varphi_{a_0}\varphi_{a_0}\circ T_{a_0}^{\ell-k}d\mu_{a_0}+O(\rho^{k-v})\,.$$

By the normalisation (6) and applying once more Proposition 4.3, we have

$$\int\Big(\varphi_{a_0}^2+2\sum_{\ell=k+1}^{m+n-1}\varphi_{a_0}\varphi_{a_0}\circ T_{a_0}^{\ell-k}\Big)d\mu_{a_0}=1+O(\rho^{m+n-k})\,.$$

Hence, we conclude

$$\begin{aligned}&\Big|1-\frac{1}{|\omega|}\int_\omega\Big(\xi_k^2+2\sum_{\ell=k+1}^{m+n-1}\xi_k\xi_l\Big)\Big|\\&\qquad\le C(m+n-k)\big(\lambda_0^{-(m-m^{1/4})}+|x_v(\omega)|^\alpha+\rho^{k-v}/|x_v(\omega)|\big)+C\rho^{m+n-k}\,.\end{aligned}$$

Regarding (54), the first and last term on the right hand side are fine, and also the second term since by assumption $|x_v(\omega)|\le n^{-3/\alpha}$. For the remaining term we use the lower bound $|x_v(\omega)|\ge\lambda_0^{-m^{1/4}}$ which gives (recall the definition of λ_0 in (45))

$$\rho^{k-v}/|x_v(\omega)|\le\rho^{m^{1/4}}\lambda_0^{m^{1/4}}\le\lambda_0^{-m^{1/4}}\,.$$

In order to prove Proposition 5.1, it is only left to prove (47) which follows now easily from (48) combined with Lemma 4.1 in which we take $d_v=Ce^{-v^{1/5}}$ (where C is taken so that $d_v\ge\lambda_0^{-m^{1/4}}$). Recall that by Lemma 4.1, for each $v\ge1$, there is an exceptional set $E_v\subset\mathcal{P}_v$ so that $|E_v|\le Cd_v^{1/2}$ and $|x_v(\omega)|\ge d_v$ for all $\omega\in\mathcal{P}_v\setminus E_v$. Let $\mathcal{P}_v^*$ be a refinement of the partition $\mathcal{P}_v\setminus E_v$ so that for $\omega\in\mathcal{P}_v^*$ we have $d_v\le|x_v(\omega)|\le2d_v$. Since $d_v\ge\lambda_0^{-m^{1/4}}$, by (48), we obtain

$$\begin{aligned}E\Big(\sum_{k=m}^{m+n-1}\xi_k(a)\Big)^2&=O(|E_v|)+\sum_{\omega\in\mathcal{P}_v^*}|\omega|(n+O(1))/\epsilon\\&=\frac{\epsilon-|E_v|}{\epsilon}n+O(1)=n+O(d_v^{1/2}n)+O(1)\,.\end{aligned}$$

Since $d_v^{1/2}n\le Ce^{-m^{1/5}/C}m=o(1)$, this concludes the proof of (47) and, thus, the proof of Proposition 5.1.

6. Proof of Theorem 2.6 via Skorokhod's representation theorem

As mentioned in the introduction, in order to prove Theorem 2.6, we go along the classical, probabilistic approach in [29]. It consists in rearranging the Birkhoff sum as a sum of blocks of polynomial size where we then approximate the blocks by a martingale and apply Skorokhod's representation theorem to it. The optimal power of the polynomial size of the blocks in our setting is $2/3$ which gives then an error exponent $\gamma > 2/5$ in the almost sure invariance principle in Theorem 2.6. Being familiar with the technique in [29], it is natural to ask if the error exponent could be decreased to $\gamma > 1/3$: If one considers a fixed dynamical system as, e.g., in [18], then one could take $1/2$ as the power of the polynomial size of the block and when separating these blocks by small blocks of logarithmic (or very small polynomial) size then this would lead to an error exponent $\gamma > 1/3$. However, in our setting the estimate (69) below is not good enough to be able to establish an error exponent $\gamma > 1/3$, and we don't know how to improve this estimate. In the recent work [15], Gouëzel uses spectral methods to show an almost sure invariance principle and he obtains remarkable error estimates which are independent on the dimension of the process. For example for the maps studied in [18] he gets the error exponent $\gamma > 1/4$. However, we didn't find an easy way to apply these spectral methods to our setting. The strategy via Skorokhod's representation theorem is also convenient here because of its simplicity. Nevertheless, since our setting is rather special, we have to go step by step through the method of building blocks and approximating by martingales. In particular, we cannot apply directly the main statement in [29, Chapter 7] since the functions ξ_i are maps on the parameter space where the concept of invariant measures does not make any sense and we are not able to verify nor to formulate an analog of a strong mixing condition (cf. [29, 7.1.2]) in our setting. However, they are statements in [29] which we can take over more or less one to one. This will keep this section of a reasonable length.

6.1. **Building the blocks.** Fix a constant $\epsilon > 0$ as in the beginning of Section 5. This ensures that we can apply all the results in Sections 4 and 5. Take $\delta > 0$ sufficiently small (to be determined later on; see, e.g., the proof of Lemma 6.4 below). We approximate the functions $\xi_i : [0, \epsilon] \to [0, 1]$, $i \geq 1$, by stepfunctions χ_i. In order to do that, we introduce the σ-fields $\mathcal{F}_i$ which are generated by the intervals in $\mathcal{P}_{r_i} (= \mathcal{P}_{r_i} | [0, \epsilon])$ where $r_i = i + [i^\delta]$. Observe that, by (33) and (9),

$$|x_i(\omega)| \leq C\lambda^{-i^\delta}, \qquad \forall \omega \in \mathcal{P}_{r_i}. \tag{55}$$

The stepfunctions χ_i are defined as $\chi_i = E(\xi_i \mid \mathcal{F}_i)$. Recall the constants ρ in Proposition 4.3, λ_0 in (45), and η in (46). We introduce a constant ρ_0 defined as

$$\rho_0 = \max(\rho^{\eta/(1+\eta)}, \lambda_0^{-1/(1+\eta)}) < 1. \tag{56}$$

We have the following basic properties.

Lemma 6.1. *For almost every $a \in [0, \epsilon]$, we have*

$$|\xi_i(a) - \chi_i(a)| \leq \lambda^{-\alpha i^\delta/8}, \qquad \text{for all but finitely many } i \geq 1. \tag{57}$$

Furthermore, there exists a constant C such that for all $i \geq 1$ and $j \geq 0$ there exists an exceptional set of intervals $E_{i,j}$ so that for a.e. $a \in [0, \epsilon] \setminus E_{i,j}$

$$|E(\xi_{i+j} \mid \mathcal{F}_i)(a)| = |E(\chi_{i+j} \mid \mathcal{F}_i)(a)| \leq C \min(1, \rho_0^{j-2i^\delta}), \tag{58}$$

where $E_{i,j} = \emptyset$, for $j \le 2i^\delta$, and $|E_{i,j}| \le C\rho_0^{(j-i^\delta)/2}$, otherwise.

Proof. We show first (57). Let $\omega \in \mathcal{P}_i$ and fix an arbitrary parameter a_ω in ω. For $\tilde\omega \in \mathcal{P}_{r_i}|\omega$ and $a_0 \in \tilde\omega$, by the definition of φ_a, (29), and Lemma 4.5, we have

$$\chi_i(a_0) = \frac{1}{|\tilde\omega|}\int_{\tilde\omega} \xi_i(a)da = \frac{1}{|\tilde\omega|}\int_{\tilde\omega} \varphi_{a_\omega}(x_i(a))da + O(|\omega|^\kappa)\,,$$

which implies that, for a.e. $a_0 \in \tilde\omega$,

$$|\varphi_{a_\omega}(x_i(a_0)) - \chi_i(a_0)| \le \text{ess sup}_{a\in\tilde\omega}\, \varphi_{a_\omega}(x_i(a)) - \text{ess inf}_{a\in\tilde\omega}\, \varphi_{a_\omega}(x_i(a)) + C|\omega|^\kappa$$

Recall the estimate (55). Let $E_i = \{\omega \in \mathcal{P}_i \mid |x_i(\omega)| \le \lambda^{-\alpha i^\delta/2}\}$. We get

$$\begin{aligned}
&|\{|\xi_i - \chi_i| \ge \lambda^{-\alpha i^\delta/8}\}| \\
&\le \lambda^{\alpha i^\delta/8}\Big[C|E_i| + \sum_{\omega\in\mathcal{P}_i\setminus E_i}\int_\omega |\xi_i(a) - \chi_i(a)|da\Big] \\
&\le \lambda^{\alpha i^\delta/8}\Big[C|E_i| + \sum_{\omega\in\mathcal{P}_i\setminus E_i}\frac{C|\omega|}{|x_i(\omega)|}\Big(\int_{x_i(\omega)} |\varphi_{a_\omega}(y) - \chi_i(x_i|_\omega^{-1}(y))|dy + C|\omega|^\kappa\Big)\Big] \\
&\le \lambda^{\alpha i^\delta/8}\Big[C|E_i| + \sum_{\omega\in\mathcal{P}_i\setminus E_i}\frac{C|\omega|}{|x_i(\omega)|}\Big(\int_0^1 \text{osc}(\varphi_{a_\omega}, C\lambda^{-i^\delta}, y)dy + 2C|\omega|^\kappa\Big)\Big]\,.
\end{aligned}$$

The integral is bounded by a constant times $\lambda^{-\alpha i^\delta}$ (recall (52)). By Lemma 4.1, we have $|E_i| \le C\lambda^{-\alpha i^\delta/4}$. It follows

$$|\{|\xi_i - \chi_i| \ge \lambda^{-\alpha i^\delta/8}\}| \le C\lambda^{-\alpha^\delta/8}\,. \tag{59}$$

By Borel-Cantelli this concludes the proof of (57).

We turn to the proof of (58). If $j \le 2i^\delta$, there is nothing to prove. If $j \ge 2i^\delta$, let $k = \max(r_i, (i+j)/(1+\eta))$. Denoting by $\widetilde{\mathcal{P}}_k$ the σ-field generated by the intervals in $\mathcal{P}_k$, observe that we have

$$|E(\xi_{i+j} \mid \mathcal{F}_i)(a)| = |E(E(\xi_{i+j} \mid \widetilde{\mathcal{P}}_k) \mid \mathcal{F}_i)(a)|\,.$$

Hence, in order to prove (58), it is sufficient to consider the terms

$$\frac{1}{|\omega|}\Big|\int_\omega \xi_{i+j}(a)da\Big|\,, \qquad \omega \in \mathcal{P}_k\,.$$

For $\omega \in \mathcal{P}_k$, we have, by (31),

$$\frac{1}{|\omega|}\Big|\int_\omega \xi_{i+j}(a)da\Big| \le \frac{C}{|x_k(\omega)|}\Big|\int_{x_k(\omega)} \xi_{i+j}(x_k|_\omega^{-1}(y))dy\Big|\,.$$

Regarding Lemma 5.3, we can only give a good estimate of the right hand side, if the image of ω under x_k is sufficiently large. Hence, we define the exceptional set

$$E_{i,j} = \{\omega \in \mathcal{P}_k \mid |x_k(\omega)| \le \rho_0^{j-i^\delta}\}\,.$$

By the definition of k, we derive that $\rho_0^{j-i^\delta}$ is smaller than $\rho_0^{k^\delta/2}$ for $j \ge 2i^\delta$ small and smaller than $\rho_0^{\eta k}$ for j large. In particular, this implies that $\rho_0^{j-i^\delta}$ is decaying stretched exponentially fast in k. Applying Lemma 4.1, we derive that $|E_{i,j}| \le C\rho_0^{(j-i^\delta)/2}$ for some constant C (since the constants in the above two upper bounds for $\rho_0^{j-i^\delta}$ are uniform, the proof of Lemma 4.1 easily shows that this constant C can be chosen uniformly in i and j). On the other hand, by the

definition of k, $\rho_0^{j-i^\delta}$ is greater than $\rho_0^{(1+\eta)k}$ which in turn is, by the definition (56) of ρ_0, greater than λ_0^{-k}. In other words $|x_k(\omega)| \geq \lambda_0^{-k}$, for $\omega \in \mathcal{P}_k \setminus E_{i,j}$, and we can apply (49) in Lemma 5.3 which gives (observe that by the definition of k we have $k \leq i+j \leq k+\eta k$)

$$\frac{1}{|x_k(\omega)|}\Big|\int_{x_k(\omega)} \xi_{i+j}(x_k|_\omega^{-1}(y)) - \varphi_{a_0}(T_{a_0}^{i+j-k}(y))dy\Big| \leq C\lambda_0^{-k} \leq C\rho_0^{j-i^\delta}\,.$$

By Proposition 4.3 and (52), we get

$$\frac{1}{|x_k(\omega)|}\Big|\int_{x_k(\omega)} \varphi_{a_0}(T_{a_0}^{i+j-k}(y))dy\Big| \leq C\rho^{i+j-k}\,.$$

Since $i+j-k \geq \eta(j-i^\delta)/(1+\eta)$ for all $j \geq 2i^\delta$ and for k as defined above, by the definition (56) of ρ_0, we get

$$\frac{1}{|\omega|}\Big|\int_\omega \xi_{i+j}(a)da\Big| \leq C(\rho^{\eta(j-i^\delta)/(1+\eta)} + \rho_0^{j-i^\delta}) \leq 2C\rho_0^{j-i^\delta}\,,$$

which concludes the proof of (58). □

We define blocks of integers I_j, $j \geq 1$, inductively where $I_1 = \{1\}$ and I_j contains $[j^{2/3}]$ consecutive integers and there are no gaps between the blocks. For $j \geq 1$, we set

$$y_j := \sum_{i \in I_j} \chi_i\,.$$

Let $M = M(N)$ denote the index of y_j containing χ_N. Observe that there exists a constant C so that

$$C^{-1}N^{3/5} \leq M \leq CN^{3/5}\,, \qquad \forall N \geq 1\,. \tag{60}$$

By (57), for a.e. $a \in [0,\epsilon]$, we find a constant $C(a)$ so that

$$\Big|\sum_{i=1}^N \xi_i(a) - \sum_{j=1}^M y_j(a)\Big| \leq \sum_{i=1}^N |\xi_i(a) - \chi_i(a)| + C|I_M| \leq C(a) + CN^{2/5}\,, \tag{61}$$

for all $N \geq 1$. Hence, in order to prove Theorem 2.6 it is sufficient to consider the sum $\sum_{j=1}^M y_j$.

6.2. **Law of large numbers for y_j^2.** In this section we will prove the following key lemma. It is the main technical ingredient in the proof of Theorem 2.6.

Lemma 6.2. *For a.e. $a \in [0,\epsilon]$, there exists a constant C such that*

$$\Big|N - \sum_{j=1}^M y_j^2(a)\Big| \leq CN^{2\gamma}\,, \qquad \forall N \geq 1, \tag{62}$$

(where $\gamma > 2/5$ is the error exponent in Theorem 2.6).

Before we start with the proof of Lemma 6.2, we recall a version of the strong law of large numbers by Gal and Koksma. Its proof is, e.g., given in [29, Theorem A.1].

Theorem 6.3 (Gal-Koksma's strong law of large numbers)**.** *Let z_j, $j \geq 1$, be zero-mean random variables and assume that there exist a real number $q \geq 1$ and a constant C such that*

$$E\Big(\sum_{j=m+1}^{m+n} z_j\Big)^2 \leq C((m+n)^q - m^q)\,, \quad \forall\ m \geq 0 \text{ and } n \geq 1\,.$$

Then for all $\iota > 0$, we have $\frac{1}{n^{q/2+\iota}} \sum_{j=1}^n z_j \to 0$ almost surely.

Proof of Lemma 6.2. In this proof we will mainly work with the original ξ_i instead of their approximations χ_i. Let

$$w_j = \sum_{i \in I_j} \xi_i\,.$$

Writing $y_j^2 - w_j^2 = (y_j + w_j)(y_j - w_j)$, by (57), we derive that $\sum_{j\geq 1} |y_j^2 - w_j^2|$ is almost surely finite. Hence, it is sufficient to prove (62) where y_j is replaced by w_j. Regarding the w_j's we claim that it is sufficient to show that for all $\iota > 0$ there is a constant C such that

$$E\Big(\sum_{j=m+1}^{m+n} w_j^2 - Ew_j^2\Big)^2 \leq C((m+n)^{8/3+\iota} - m^{8/3+\iota})\,, \quad \forall m \geq 0,\ n \geq 1\,. \tag{63}$$

Indeed, by the estimate (47) in Proposition 5.1, we have

$$\Big|N - \sum_{j=1}^M w_j^2\Big| \leq CM + \Big|\sum_{j=1}^M w_j^2 - Ew_j^2\Big|\,.$$

Hence, applying Theorem 6.3 to (63) and recalling (60), concludes the proof of (62) (where y_j is replaced by w_j).

In the following we will prove (63). Observe that $(Ew_j^2)^2 \leq Ew_j^4$. We have

$$E\Big(\sum_{j=m+1}^{m+n} w_j^2 - Ew_j^2\Big)^2 \leq 2\sum_{j=m+1}^{m+n}\Big(Ew_j^4 + \sum_{k=j+1}^{m+n} |Ew_j^2 w_k^2 - Ew_j^2 Ew_k^2|\Big)\,. \tag{64}$$

We consider first Ew_j^4. For $\iota > 0$ small, let $S = \{(v_1, v_2, v_3, v_4) \in I_j^4 \mid v_1 \leq v_2 \leq v_3 \leq v_4 \text{ and either } v_2 - v_1 \geq j^\iota \text{ or } v_4 - v_3 \geq j^\iota\}$. We have

$$\int w_j(a)^4 da \leq C \sum_{(v_1,v_2,v_3,v_4)\in S} \Big|\int \prod_{\ell=1}^4 \xi_{v_\ell}(a) da\Big| + Cj^{4/3+2\iota}\,. \tag{65}$$

Let $(v_1, ..., v_4) \in S$. We consider first the case when $v_4 - v_3 \geq j^\iota$. In order to apply Lemma 5.3 we have to get rid of partition elements with a too small image. Let $E_{v_3} = \{\omega \in \mathcal{P}_{v_3} \mid |x_{v_3}(\omega)| \geq \rho^{j^\iota/2}\}$. By (60) and Lemma 4.1, the measure of E_{v_3} is decaying stretched exponentially fast in j. For $\omega \in \mathcal{P}_{v_3} \setminus E_3$ and $a_0 \in \omega$, by equality (53) (for $\ell_0 = 4$) combined with Lemma 5.3 (for $\ell_0 = 1$), we derive

$$\begin{aligned}
&\frac{1}{|\omega|}\Big|\int_\omega \prod_{\ell=1}^4 \xi_\ell(a) da\Big| \\
&\qquad \leq \frac{C}{|x_{v_3}(\omega)|}\Big|\int_{x_{v_3}(\omega)} \Big(\prod_{\ell=1}^3 \xi_\ell(x_{v_3}|_\omega^{-1}(y))\Big)\varphi_{a_0}(T_{a_0}^{v_4-v_3}(y))dy\Big| + C\lambda_0^{-v_3}\,.
\end{aligned}$$

Let $L_\ell = x_{v_\ell} \circ x_{v_3}|_\omega^{-1}$, $1 \le \ell \le 3$. For $y \in x_{v_3}(\omega)$, we have $\xi_{v_\ell}(x_{v_3}|_\omega^{-1}(y)) = \varphi_a(L_\ell(y))$ where $a = x_{v_3}|_\omega^{-1}(y)$. Hence, by Lemma 4.5 and (29), we derive that $|\xi_{v_\ell}(x_{v_3}|_\omega^{-1}(y)) - \varphi_{a_0}(L_\ell(y))| \le C|\omega|^\kappa$. It follows

$$\Big|\int \prod_{\ell=1}^4 \xi_{v_\ell}(a)da\Big| \le C|E_{v_3}| + \sum_{\omega\in\mathcal{P}_{v_3}\setminus E_{v_3}} |\omega|\Big[\frac{C}{|x_{v_3}(\omega)|}\Big|\int_{x_{v_3}(\omega)} \Big(\prod_{\ell=1}^3 \varphi_{a_0}(L_\ell(y))\Big) \varphi_{a_0}(T_{a_0}^{v_4-v_3}(y))dy\Big| + \frac{C|\omega|^\kappa}{|x_{v_3}(\omega)|} + C\lambda_0^{-v_3}\Big]. \tag{66}$$

Observe that, by the distortion estimate (33), we have $|L_\ell'(y)| \le C\lambda^{-(v_3-v_1)}$, which implies $|\chi_{x_{v_3}(\omega)}\varphi_{a_0} \circ L_\ell|_\alpha \le C|\chi_{x_{v_\ell}(\omega)}\varphi_{a_0}|_\alpha$. Since $\|\chi_{x_{v_3}(\omega)}\varphi_{a_0} \circ L_\ell\|_\infty = \|\chi_{x_{v_\ell}(\omega)}\varphi_{a_0}\|_\infty$, by (11),

$$\|\chi_{x_{v_3}(\omega)}\varphi_{a_0} \circ L_\ell\|_\alpha \le C\|\chi_{x_{v_\ell}(\omega)}\varphi_{a_0}\|_\alpha \le C^2\|\varphi_{a_0}\|_\alpha\,,$$

where in the last inequality we used (12). Hence, by Proposition 4.3 and (52), the absolute value of the integral on the right hand side in (66) is bounded from above by a (uniform) constant times $\rho^{v_4-v_3} \le \rho^{j^\iota}$. By the definition of $|E_{v_3}|$, we get that $|x_{v_3}(\omega)|\rho^{j^\iota} \le \rho^{j^\iota/2}$, for all $\omega \in \mathcal{P}_{v_3} \setminus E_{v_3}$. The second and last term in the sum on the right hand side of (66) decays exponentially fast in v_3. Altogether, we conclude that $|\int \prod_{\ell=1}^4 \xi_{v_\ell} da|$ is decaying stretched exponentially fast in j whenever $(v_1, ..., v_4) \in S$ and $v_4 - v_3 \ge j^\iota$.

The case when $v_2 - v_1 \ge j^\iota$ is easier. Instead of considering the functions L_ℓ, we can apply directly Lemma 5.3 with $\ell_0 = 4$. Then, a similar reasoning gives also the stretched exponential decay of $|\int \prod_{\ell=1}^4 \xi_{v_\ell} da|$ in this case. Altogether, recalling (65) and observing that $|S|$ is growing only polynomially fast in j, for each $\iota > 0$ we find a constant C so that

$$Ew_j^4 \le Cj^{4/3+2\iota}\,, \qquad \forall j \ge 1\,. \tag{67}$$

Regarding the term $Ew_j^2 w_k^2$, we can assume that $k \ge j+2$ since for $k = j+1$ we just can apply Cauchy's inequality and (67) for estimating $Ew_j^2 w_{j+1}^2$ and Proposition 5.1 for estimating $Ew_j^2 Ew_{j+1}^2$ which yields the upper bound $j^{4/3+2\iota}$ for $|Ew_j^2 w_{j+1}^2 - Ew_j^2 Ew_{j+1}^2|$. (For the other terms we have to give a better bound otherwise the bound we get when summing over k is not good enough.) Henceforth, let $k \ge j+2$. We first give a good upper bound for $|Ey_j^2 w_k^2 - Ey_j^2 Ew_k^2|$. This is convenient, since y_j is constant on elements of the partition $\mathcal{P}_{m-m^{1/4}}$, where m denotes the smallest integer in I_k. Let $v = m - m^{1/4}$. Since $d_v := \lambda_0^{-m^{1/4}}$ is decaying stretched exponentially fast in v, we can apply Lemma 4.1 and we find an exceptional set $E_v \subset \mathcal{P}_v$ so that $|E_v| \le Cd_v^{1/2}$ and $|x_v(\omega)| \ge d_v$ for all $\omega \in \mathcal{P}_v \setminus E_v$. Let $\mathcal{P}_v^*$ be a refinement of the partition $\mathcal{P}_v \setminus E_v$ so that for $\omega \in \mathcal{P}_v^*$ we have $d_v = \lambda_0^{-m^{1/4}} \le |x_v(\omega)| \le |I_k|^{-3/\alpha}$. Applying the local estimate (48) in Proposition 5.1, we obtain

$$\Big|E\Big(w_k(a)^2 \mid \{a \in \omega\}\Big) - |I_k|\Big| \le C\,, \qquad \forall \omega \in \mathcal{P}_v^*\,.$$

Recall that y_j is constant on elements of $\mathcal{P}_v^*$. We get

$$Ey_j^2 w_k^2 = \sum_{\omega\in\mathcal{P}_v^*} y_j^2(\omega)|\omega| E\Big(w_k(a)^2 \mid \{a\in\omega\}\Big)/\epsilon + O(\lambda_0^{-m^{1/4}/2}\|y_j^2 w_k^2\|_\infty)$$
$$\le Ey_j^2(|I_k| + C) + O(\lambda_0^{-m^{1/4}/2}k^{8/3})\,.$$

On the other hand, by the global estimate (47) in Proposition 5.1, we have $Ey_j^2 Ew_k^2 \ge Ey_j^2(|I_k| - C)$. Altogether, we derive

$$|Ey_j^2 w_k^2 - Ey_j^2 Ew_k^2| \le CEy_j^2\,. \tag{68}$$

Writing $E|w_j^2 - y_j^2| = E|w_j + y_j||w_j - y_j| \le Cj^{2/3}E|w_j - y_j|$, by (57) and (59), we derive that $E|w_j^2 - y_j^2|$ is stretched exponentially decreasing in j. Hence, by (68) and once more by (47), it follows

$$|Ew_j^2 w_k^2 - Ew_j^2 Ew_k^2| \le C|I_j| \le Cj^{2/3}\,. \tag{69}$$

Recalling (64), we can now easily derive (63). This concludes the proof of Lemma 6.2. □

6.3. **Martingale representation and embedding procedure.** In this section we will follow closely Sections 3.4 and 3.5 in [29]. Let $\mathcal{L}_j$, $j \ge 1$, be the σ-field generated by $(y_1, y_2, ..., y_j)$, and set

$$u_j = \sum_{k\ge 0} E(y_{j+k} \mid \mathcal{L}_{j-1})\,.$$

Then $\{Y_j, \mathcal{L}_j\}$ defined by $Y_j = y_j + u_{j+1} - u_j$ is a martingale difference sequence. Recalling the definition of the σ-fields $\mathcal{F}_i$ in the beginning of Section 6.1, we see that $\mathcal{L}_{j-1} \subset \mathcal{F}_{i(j)}$ where $i(j) = \max\{i \in I_{j-1}\}$. Hence, we can write

$$u_j = \sum_{k\ge 1} E(E(\xi_{i(j)+k} \mid \mathcal{F}_{i(j)}) \mid \mathcal{L}_{j-1})\,.$$

Recall (58) in Lemma 6.1 and the to it related notations. Recall also that $i(j) \le Cj^{5/3}$ (see, e.g., (60)). Setting $E_j = \cup_{k\ge 0} E_{i(j),k}$, we have $|E_j| \le C\rho_0^{i(j)^\delta/2} \le C^2 e^{-j^{5\delta/3}/C}$, and there exists a constant C so that for a.e. $a \in [0,\epsilon] \setminus E_j$ we have

$$|u_j(a)| \le Cj^{5\delta/3}\,. \tag{70}$$

Further, for $\ell \ge 0$, we derive

$$|u_j(a)| \le \max(Cj^{5\delta/3}, C\ell)\,, \tag{71}$$

for a.e. $a \in E_{i(j),\ell} \setminus \cup_{k>\ell} E_{i(j),k}$. Since

$$|E_{i(j),\ell}| \le \begin{cases} C\rho_0^{(\ell - i(j)^\delta)/2} \le C\rho_0^{\ell/4}\,, & \text{if } \ell \ge 2i(j)^\delta \ge j^{5\delta/3}/C \\ 0\,, & \text{otherwise,} \end{cases} \tag{72}$$

for $\delta > 0$ sufficiently small, we see that $|\{|u_{M+1}| \ge N^\gamma\}|$ is summable over $N \ge 1$ (recall (60)). We conclude that, for a.e. $a \in [0,\epsilon]$, there exists a constant C so that

$$\Big|\sum_{j=1}^M y_j(a) - Y_j(a)\Big| = |u_{M+1} - u_1| \le CN^\gamma\,. \tag{73}$$

In other words, in the following we can work with the martingale difference sequence Y_j instead of y_j. The Y_j inherit the law of large numbers shown for y_j:

Lemma 6.4. *For a.e. $a \in [0,\epsilon]$, there exists a constant C so that*

$$\Big| N - \sum_{j=1}^{M} Y_j^2(a) \Big| \le CN^{2\gamma}\,, \qquad \forall N \ge 1, \tag{74}$$

(where $\gamma > 2/5$ is the error exponent in Theorem 2.6).

Proof. Put $v_j = u_j - u_{j+1}$. Since $Y_j^2 = y_j^2 - 2y_jv_j + v_j^2$, by Lemma 6.2 and Cauchy's inequality, it is sufficient to show that, for a.e. $a \in [0,\epsilon]$, there exists a constant C so that

$$\sum_{j=1}^{M} v_j^2 \le CN^{4\gamma-1}\,. \tag{75}$$

Observe that by (60) we have $N^{4\gamma-1}/M \ge CN^{4\gamma-8/5}$, and since $\gamma > 2/5$ we have $4\gamma - 8/5 > 0$. By (70) and (71), there exists a constant C so that for all $\delta > 0$ sufficiently small and all M sufficiently large we have, for $1 \le j \le M$,

$$v_j(a)^2 \le Cj^{10\delta/3} \le N^{4\gamma-1}/M\,, \qquad \text{for a.e. } a \in [0,\epsilon] \setminus (E_j \cup E_{j+1}), \tag{76}$$

and, for all $\ell \ge 0$ and $1 \le j \le M$, we have

$$v_j(a)^2 \le \max(Cj^{10\delta/3}, C\ell^2) \le \max(N^{4\gamma-1}/M, C\ell^2)\,, \tag{77}$$

for a.e. $a \in (E_{i(j),\ell} \cup E_{i(j+1),\ell}) \setminus (\cup_{k>\ell} E_{i(j),k} \cup E_{i(j+1),k})$. Combined with (72), we get

$$|\{a \in [0,\epsilon] \mid \sum_{j=1}^{M} v_j^2 \le N^{4\gamma-1}\}| \le N^{-(4\gamma-1)} \sum_{j=1}^{M} \sum_{\ell \ge \sqrt{N^{4\gamma-1}/CM}} C\ell^2 C\rho_0^{\ell/4}\,,$$

where the right hand side is summable in N. This concludes the proof of (75) and, thus, the proof of the lemma. □

Lemma 6.5. *For a.e. $a \in [0,\epsilon]$, there exists a constant C so that*

$$\Big| \sum_{j=1}^{M} E(Y_j^2 \mid \mathcal{L}_{j-1}) - Y_j^2(a) \Big| \le CN^{2\gamma}\,, \qquad \forall N \ge 1, \tag{78}$$

(where $\gamma > 2/5$ is the error exponent in Theorem 2.6).

Proof. Set $R_j = Y_j^2 - E(Y_j^2 \mid \mathcal{L}_{j-1})$ and observe that $\{R_j, \mathcal{L}_j\}$ is a martingale difference sequence. By the definition of Y_j and by Minkowski's inequality, we have

$$ER_j^2 \le 4EY_j^4 \le C(Ew_j^4 + E|w_j^4 - y_j^4| + Ev_j^4)\,.$$

By (67), for all $\iota > 0$ we find a constant C so that $Ew_j^4 \le Cj^{4/3+\iota}$. Since $w_j^4 - y_j^4 = (w_j^2 + y_j^2)(w_j + y_j)(w_j - y_j)$, we can apply (59) and we derive that $E|w_j^4 - y_j^4|$ is uniformly bounded in j. By (70) and (71), we derive that $Ev_j^4 \le Cj^{4\delta}$. Hence, for all $\iota > 0$, we have

$$\sum_{j \ge 1} j^{-7/3-\iota} ER_j^2 < \infty\,,$$

and by a martingale result (see, e.g., [9]) we get that $\sum_{j\geq 1} j^{-7/6-\iota} R_j$ converges almost surely. By Kronecker's Lemma we conclude that, for a.e. $a \in [0,\epsilon]$, there exists a constant C so that

$$\sum_{j=1}^{M} R_j \leq CM^{7/6+\iota} \leq C^2 N^{21/30+\iota}\,,$$

where we used (60) in the last inequality. Since $21/30 < 4/5 < 2\gamma$ this concludes the proof of the lemma. □

Now we apply the following martingale embedding result to the martingale difference sequence Y_j. For a proof see, e.g., [17, Theorem A.1].

Theorem 6.6 (Skorokhod's representation theorem)**.** *Let $\{\sum_{j=1}^{M} Y_j,\ \mathcal{L}_M\}$, $M \geq 1$, be a zero-mean and square-integrable martingale. Then there exists a probability space which supports a zero-mean and square-integrable martingale $\{\sum_{j=1}^{M} \widetilde{Y}_j,\ \widetilde{\mathcal{L}}_M\}$, $M \geq 1$, a Brownian motion W, and a sequence of nonnegative variables T_j, $j \geq 1$, such that*

- *$\{Y_j\}_{j\geq 1}$ and $\{\widetilde{Y}_j\}_{j\geq 1}$ have the same distribution;*
- *$\sum_{j=1}^{M} \widetilde{Y}_j = W(\sum_{j=1}^{M} T_j)$ almost surely;*
- *$E(T_j \mid \mathcal{G}_{j-1}) = E(\widetilde{Y}_j^2 \mid \mathcal{G}_{j-1})$ almost surely, where $\mathcal{G}_j$ is the σ-field generated by $\{W(t),\ 0 \leq t \leq \sum_{\ell\leq j} T_\ell\}$.*

We will keep the same notation, i.e., instead of writing $\widetilde{Y}_j$ and $\widetilde{\mathcal{L}}_j$, we keep writing Y_j and $\mathcal{L}_j$. Since $\mathcal{L}_j \subset \mathcal{G}_j$, for all $j \geq 1$, we have

$$E(T_j \mid \mathcal{G}_{j-1}) = E(Y_j^2 \mid \mathcal{G}_{j-1}) = E(Y_j^2 \mid \mathcal{L}_{j-1})\,, \tag{79}$$

almost surely. We can now show a strong law of large numbers for the sequence T_j.

Lemma 6.7. *For a.e. $a \in [0,\epsilon]$, there exists a constant C so that*

$$\Big| N - \sum_{j=1}^{M} T_j \Big| \leq CN^{2\gamma}\,, \qquad \forall N \geq 1, \tag{80}$$

(where $\gamma > 2/5$ is the error exponent in Theorem 2.6).

Proof. By (79) we get

$$N - \sum_{j=1}^{M} T_j = \Big[N - \sum_{j=1}^{M} Y_j^2\Big] + \sum_{j=1}^{M} [Y_j^2 - E(Y_j^2 \mid \mathcal{L}_{j-1})] + \sum_{j=1}^{M} [E(T_j \mid \mathcal{G}_{j-1}) - T_j]\,,$$

almost surely. By Lemma 6.4 and Lemma 6.5, the first two terms are almost surely bounded by a constant times $N^{2\gamma}$. Write $R_j = E(T_j \mid \mathcal{G}_{j-1}) - T_j$. By (79), $\{R_j, \mathcal{G}_j\}$ is a martingale difference sequence satisfying $ER_j^2 \leq 4EY_j^4$. Hence, we can go along the proof of Lemma 6.5 and we get the same upper bound for this term. □

Now we can go word by word along the proof of [29, Lemma 3.5.3] replacing $1/2 - \alpha/2 + \gamma$ and Lemma 3.5.1 therein by γ and Lemma 6.7 from our setting, respectively, and we obtain

$$\Big| \sum_{j=1}^{M} Y_j - W(N) \Big| = O(N^{\gamma})\,, \qquad \text{almost surely.}$$

Recalling (61) and (73), this concludes the proof of Theorem 2.6.

References

[1] A. Avila and C.G. Moreira. Statistical properties of unimodal maps: physical measures, periodic points and pathological laminations. *Publ. Math. Inst. Hautes Études Sci.*, 101:1–67, 2005.

[2] Artur Avila, Mikhail Lyubich, and Welington de Melo. Regular or stochastic dynamics in real analytic families of unimodal maps. *Invent. Math.*, 154(3):451–550, 2003.

[3] V. Baladi, M. Benedicks, and D. Schnellmann. Whitney–Hölder continuity of the SRB measure for transversal families of smooth unimodal maps. *arXiv:1302.3191v2*, 2013.

[4] V. Baladi, S. Marmi, and D. Sauzin. Natural boundary for the susceptibility function of generic piecewise expanding unimodal maps. *Ergodic Theory Dynam. Systems*, 2013. To appear.

[5] V. Baladi and D. Smania. Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps. *Ann. Sci. Éc. Norm. Supér.*, 45:861–926, 2012.

[6] M. Benedicks and L. Carleson. On iterations of $1 - ax^2$ on $(-1, 1)$. *Ann. of Math. (2)*, 122(1):1–25, 1985.

[7] M. Björklund and D. Schnellmann. Almost sure equidistribution in expansive families. *Indag. Math.*, 20(2):167–177, 2009.

[8] H. Bruin. For almost every tent-map the turning point is typical. *Fund. Math.*, 155(3):215–235, 1998.

[9] Y.S. Chow. Local convergence of martingales and the law of large number. *The Annals of Mathematical Statistics*, 36:552–558, 1965.

[10] J.-P. Conze and S. Le Borgne. Limit law for some modified ergodic sums. *Stoch. Dyn.*, 11(1):107–133, 2011.

[11] A. del Junco and J. Rosenblatt. Counterexamples in ergodic theory and number theory. *Math. Ann.*, 245:185–197, 1979.

[12] M. Denker and W. Philipp. Approximation by Brownian motion for Gibbs measures and flows under a function. *Ergodic Theory Dynam. Systems*, 4:541–552, 1984.

[13] C. Dupont. Bernoulli coding map and almost sure invariance principle for endomorphisms of $\mathbb{P}^k$. *Prob. Theory Relat. Fields*, 146:337–359, 2010.

[14] B. Faller and C.-E. Pfister. A point is normal for almost all maps $\beta x + \alpha \bmod 1$ or generalized β-maps. *Ergodic Theory Dynam. Systems*, 29:1529–1547, 2009.

[15] S. Gouëzel. Almost sure invariance principle for dynamical systems by spectral methods. *Ann. Probab.*, 38(4):1639–1671, 2010.

[16] J. Graczyk and G. Swiatek. Generic hyperbolicity in the logistic family. *Ann. of Math.*, 146:1–52, 1997.

[17] P. Hall and C.C. Heyde. *Martingale Limit Theory and its Application*. Academic Press, New York, 1980.

[18] F. Hofbauer and G. Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.*, 180:119–140, 1982.

[19] M. Kac. Probability methods in some problems of analysis and number theory. *Bull. Amer. Math. Soc.*, 55:641–665, 1949.

[20] G. Keller. Piecewise monotonic transformations and exactness. *Collection : Seminar on Probability, Rennes*, 1978. Exp. 6, 32p.

[21] G. Keller. Generalized bounded variation and application to piecewise monotonic transformations. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 69:461–478, 1985.

[22] G. Keller and C. Liverani. Stability of the spectrum for transfer operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)*, 28(1):141–152, 1999.

[23] Z.S. Kowalski. Invariant measure for piecewise monotonic transformations has a lower bound on its support. *Bull. Acad. Pol. Sci. Math.*, 27:53–57, 1979.

[24] U. Krengel. On the speed of convergence in the ergodic theorem. *Monatsh. Math.*, 86:3–6, 1978.

[25] A. Lasota and J.A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.*, 186:481–488, 1973.

[26] G. Levin. On an analytic approach to the Fatou conjecture. *Fund. Math.*, 171:177–196, 2002.

[27] T.-Y. Li and J.A. Yorke. Ergodic transformations from an interval into itself. *Trans. Amer. Math. Soc.*, 235:183–192, 1978.

[28] I. Melbourne and M. Nicol. Almost sure invariance principle for nonuniformly hyperbolic systems. *Comm. Math. Phys.*, 260:131–146, 2005.

[29] W. Philipp and W.F. Stout. Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc.*, 161, 1975.

[30] B. Saussol. Absolutely continuous invariant measures for multidimensional expanding maps. *Israel J. Math.*, 116:223–248, 2000.

[31] J. Schmeling. Symbolic dynamics for β-shifts and self-normal numbers. *Ergodic Theory Dynam. Systems*, 17(3):675–694, 1997.

[32] D. Schnellmann. Typical points for one-parameter families of piecewise expanding maps of the interval. *Discrete Contin. Dyn. Syst.*, 31(3):877–911, 2011.

[33] M. Tsujii. Positive Lyapunov exponents in families of one-dimensional dynamical systems. *Invent. Math.*, 111(1):113–137, 1993.

[34] G. Wagner. The ergodic behaviour of piecewise monotonic transformations. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 46:317–324, 1979.

[35] S. Wong. Some metric properties of piecewise monotonic mappings of the unit interval. *Trans. Amer. Math. Soc.*, 246:493–500, 1978.

[36] S. Wong. Hölder continuous derivatives and ergodic theory. *J. Lond. Math. Soc.*, 22:506–520, 1980.

DMA, UMR 8553, École Normale Supérieure, 75005 Paris, France
E-mail address: daniel.schnellmann@gmail.com