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# GAUSSIAN FLUCTUATIONS FOR LINEAR SPECTRAL STATISTICS OF LARGE RANDOM COVARIANCE MATRICES

JAMAL NAJIM AND JIANFENG YAO

ABSTRACT. Consider a  $N \times n$  matrix  $\Sigma_n = \frac{1}{\sqrt{n}} R_n^{1/2} X_n$ , where  $R_n$  is a nonnegative definite Hermitian matrix and  $X_n$  is a random matrix with i.i.d. real or complex standardized entries. The fluctuations of the linear statistics of the eigenvalues:

$$\text{Trace } f(\Sigma_n \Sigma_n^*) = \sum_{i=1}^N f(\lambda_i), \quad (\lambda_i) \text{ eigenvalues of } \Sigma_n \Sigma_n^*,$$

are shown to be gaussian, in the regime where both dimensions of matrix  $\Sigma_n$  go to infinity at the same pace and in the case where  $f$  is of class  $C^3$ , i.e. has three continuous derivatives. The main improvements with respect to Bai and Silverstein's CLT [5] are twofold: First, we consider general entries with finite fourth moment, but whose fourth cumulant is non-null, i.e. whose fourth moment may differ from the moment of a (real or complex) Gaussian random variable. As a consequence, extra terms proportional to

$$|\mathcal{V}|^2 = |\mathbb{E}(X_{11}^n)^2|^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{11}^n|^4 - |\mathcal{V}|^2 - 2$$

appear in the limiting variance and in the limiting bias, which not only depend on the spectrum of matrix  $R_n$  but also on its eigenvectors. Second, we relax the analyticity assumption over  $f$  by representing the linear statistics with the help of Helffer-Sjöstrand's formula.

The CLT is expressed in terms of vanishing Lévy-Prohorov distance between the linear statistics' distribution and a Gaussian probability distribution, the mean and the variance of which depend upon  $N$  and  $n$  and may not converge.

**AMS 2000 subject classification:** Primary 15A52, Secondary 15A18, 60F15.

**Key words and phrases:** large random matrix, fluctuations, linear statistics of the eigenvalues, central limit theorem.

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## 1. INTRODUCTION

Empirical random covariance matrices, whose probabilistic study may be traced back to Wishart [57] in the late twenties, play an important role in applied mathematics. After Marčenko and Pastur's seminal contribution [43] in 1967, the large dimensional setting (where the dimension of the observations is of the same order as the size of the sample) has drawn a growing interest, and important theoretical contributions [5, 51, 36] found many applications in multivariate statistics, electrical engineering, mathematical finance, etc., cf. [4, 19, 41, 44]. The aim of this paper is to describe the fluctuations for linear spectral statistics of large empirical random covariance matrices. It will complete the picture already provided by Bai and Silverstein [5] and will hopefully provide a generic result of interest for practitioners.

**The model.** Consider a  $N \times n$  random matrix  $\Sigma_n = (\xi_{ij}^n)$  given by:

$$\Sigma_n = \frac{1}{\sqrt{n}} R_n^{1/2} X_n, \quad (1.1)$$

where  $N = N(n)$  and  $R_n$  is a  $N \times N$  nonnegative definite hermitian matrix with spectral norm uniformly bounded in  $N$ . The entries  $(X_{ij}^n; i \leq N, j \leq n, n \geq 1)$  of matrices  $(X_n)$  are real or complex, independent and identically distributed (i.i.d.) with mean 0 and variance 1. Matrix  $\Sigma_n \Sigma_n^*$  models a sample covariance matrix, formed from  $n$  samples of the random vector  $R_n^{1/2} X_1^n$ , with the population covariance matrix  $R_n$ . In the asymptotic regime where

$$N, n \rightarrow \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty, \quad (1.2)$$

(a condition that will be simply referred as  $N, n \rightarrow \infty$  in the sequel), we study the fluctuations of linear spectral statistics of the form:

$$\text{tr} f(\Sigma_n \Sigma_n^*) = \sum_{i=1}^N f(\lambda_i), \quad \text{as } N, n \rightarrow \infty \quad (1.3)$$

where  $\text{tr}(A)$  refers to the trace of  $A$  and the  $\lambda_i$ 's are the eigenvalues of  $\Sigma_n \Sigma_n^*$ . This subject has a rich history with contributions by Arharov [3], Girko (see [23, 24] and the references therein), Jonsson [37], Khorunzhiy et al. [40], Johansson [35], Sinai and Soshnikov [53, 54], Cabanal-Duvillard [16], Guionnet [26], Bai and Silverstein [5], Anderson and Zeitouni [2], Pan and Zhou [46], Chatterjee [18], Lytova and Pastur [42], Bai et al. [8], Shcherbina [50], etc. There are also more recent contributions for heavytailed entries (see for instance Benaych-Georges et al. [10]).

In their '04 article [5], Bai and Silverstein established a CLT for the linear spectral statistics (1.3) as the dimensions  $N$  and  $n$  grow to infinity at the same pace ( $N/n \rightarrow c \in (0, \infty)$ ) and under two important assumptions:

- (1) The entries  $(X_{ij}^n)$  are centered with unit variance and a finite fourth moment equal to the fourth moment of a (real or complex) gaussian standard variable.

- (2) Function  $f$  in (1.3) is analytic in a neighbourhood of the asymptotic spectrum of  $\Sigma_n \Sigma_n^*$ .

Such a result proved to be highly useful in probability theory, statistics and various other fields.

The purpose of this article is to establish a CLT for linear spectral statistics (1.3) for general entries  $X_{ij}^n$  with finite fourth moment and for non-analytic functions  $f$ , sufficiently regular, hence to relax both Assumptions (1) and (2) in [5].

It is well known since the paper by Khorunzhiy et al. [40] that if the fourth moment of the entries differs from the fourth moment of a Gaussian random variable, then a term appears in the variance of the trace of the resolvent, which is proportional to the fourth cumulant of the entries. This term does not appear if Assumption (1) holds true, because in this case, the fourth cumulant is zero.

In Pan and Zhou [46], Assumption (1) has been relaxed under an additional assumption on matrix  $R_n$ , which somehow enforces structural conditions on  $R_n$  (in particular, these conditions are satisfied if matrix  $R_n$  is diagonal). In Hachem et al. [39, 29], CLTs have been established for specific linear statistics of interest in information theory, with general entries and (possibly non-centered) covariance random matrices with a variance profile. In Bao et al. [9], the CLT is established for the white model (where  $R_n$  is equal to the identity matrix) with general entries with finite fourth moment, featuring terms in the covariance proportional to the square of the second non-absolute moment and to the fourth cumulant.

In Lytova and Pastur [42] and Shcherbina [50], both assumptions have been relaxed for the white model. In this case, it has been proved that mild integrability conditions over the Fourier transform of  $f$  was enough to establish the CLT. In Bai et al. [8], fluctuations for the white model are addressed as well, for non-analytic functions  $f$ . Following Shcherbina's ideas, Guédon et al. [25] establish a CLT for linear statistics of large covariance matrices with vectors with log-concave distribution. Following Lytova and Pastur, Yao [58] relaxes the analyticity assumption in [5] by using interpolation techniques and Fourier transforms. We follow here a different approach, inspired from Bordenave [12].

**Non-Gaussian entries.** The presence of matrix  $R_n$  yields interesting phenomena at the CLT level when considering entries with non-Gaussian fourth moment: terms proportionals to the fourth cumulant and to  $|\mathbb{E}(X_{11}^n)^2|^2$  appear in the asymptotic variance (described in Section 2.3); however their convergence is not granted under usual assumptions (roughly, under the convergence of  $R_n$ 's spectrum), mainly because these extra-terms also depend on the eigenvectors of  $R_n$ . As a consequence, such terms may not converge unless some very strong structural assumption over  $R_n$  (such as  $R_n$  diagonal) is made. This lack of convergence has consequences on the description of the fluctuations.

Denote by  $L_n(f)$  the (approximately) centered version of the linear statistics (1.3), to be properly defined below. Instead of expressing the CLT in the usual way, i.e. ( $\xrightarrow{\mathcal{D}}$  stands for the convergence in distribution):

$$L_n(f) \xrightarrow[N, n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(\mathcal{B}_\infty^f, \Theta_\infty^f), \quad (1.4)$$

for some well-defined parameters  $\mathcal{B}_\infty^f, \Theta_\infty^f$ , we prove that the distribution of the linear statistics  $L_n(f)$  becomes close to a family of Gaussian distributions, whose parameters (mean

and variance) may not converge. More precisely, we establish that there exists a family of Gaussian random variables  $\mathcal{N}(\mathcal{B}_n^f, \Theta_n^f)$ , such that

$$d_{\mathcal{L}P}(L_n(f), \mathcal{N}(\mathcal{B}_n^f, \Theta_n^f)) \xrightarrow{N, n \rightarrow \infty} 0, \quad (1.5)$$

where  $d_{\mathcal{L}P}$  denotes the Lévy-Prohorov distance (and in particular metrizes the convergence of laws). Details are provided in Section 2.5 and the fluctuation results are stated in Theorem 1 (for the resolvent  $f(\lambda) = (\lambda - z)^{-1}$ ) and Theorem 2 (for  $f$  of class  $C^3$ , the space of functions with third continuous derivative).

From a technical point of view, the analysis of the extra-term proportionnal to the fourth cumulant requires to cope with quadratic forms of the resolvent (counterpart of isotropic Marčenko-Pastur law). We provide the needed results in Section 5.

Expressing the CLT as in (1.5) makes it possible to avoid any cumbersome assumption related to the joint convergence of  $R_n$ 's eigenvectors and eigenvalues; the technical price to pay however is the need to get various uniform (in  $N, n$ ) controls over the sequence  $\mathcal{N}(\mathcal{B}_n, \Theta_n)$ . This is achieved by introducing a matrix meta-model in Section 2.6. The case where matrix  $R_n$  is diagonal is simpler and the fluctuations express in the usual way (1.4); it is handled in Section 3.4. Remarks on the white case ( $R_n = I_N$ ) are also provided in Sections 3.5 and 4.2.

This framework may also prove to be useful for other interesting models such as large dimensional information-plus-noise type matrices [20, 30] and more generally mixed models combining large dimensional deterministic and random matrices.

**Non-analytic functions.** In Section 3, we establish the CLT for the trace of the resolvent

$$\text{tr}(\Sigma_n \Sigma_n^* - z I_N)^{-1}.$$

In order to transfer the CLT from the resolvent to the linear statistics of the eigenvalues  $\text{tr} f(\Sigma_n \Sigma_n^*)$ , we will use (Dynkin-)Helffer-Sjöstrand's representation formula<sup>1</sup> for a function  $f$  of class  $C^{k+1}$  and with compact support [22, 34]. Denote by  $\Phi_k(f) : \mathbb{C}^+ \rightarrow \mathbb{C}$  the function:

$$\Phi_k(f)(x + iy) = \sum_{\ell=0}^k \frac{(iy)^\ell}{\ell!} f^{(\ell)}(x) \chi(y), \quad (1.6)$$

where  $\chi : \mathbb{R} \rightarrow \mathbb{R}^+$  is smooth, compactly supported, with value 1 in a neighbourhood of 0. Function  $\Phi_k(f)$  coincides with  $f$  on the real line and is an appropriate extension of  $f$  to the complex plane. Let  $\bar{\partial} = \partial_x + i\partial_y$ , then Helffer-Sjöstrand's formula writes:

$$\text{tr} f(\Sigma_n \Sigma_n^*) = \frac{1}{\pi} \text{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(f)(z) \text{tr}(\Sigma_n \Sigma_n^* - z I_N)^{-1} \ell_2(dz), \quad (1.7)$$

where  $\ell_2$  stands for the Lebesgue measure over  $\mathbb{C}^+$ . An elementary proof of Formula (1.7) can be found in [13, Chap. 5]. Closest to our work are the papers by Pizzo, O'Rourke, Renfrew and Soshnikov [45, 48] where the fluctuations of the entries of regular functions of Wigner and large covariance matrices are studied.

We believe that representation formula (1.7) provides a very streamlined way to handle non-analytic functions and in fact enables us to state the fluctuations for the linear statistics for functions of class  $C^3$ , a lower regularity requirement than in [8, 42, 50, 58].

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<sup>1</sup>In [33, Notes of chap. 8], it is written "This formula is due to Dynkin but was popularized by Helffer and Sjöstrand in the context of spectral theory, leading many authors to call it the Helffer-Sjöstrand formula".

**Bias in the CLT and asymptotic expansion for the linear spectral statistics.** Beside the fluctuations, a substantial part of this article is devoted to the study of the bias that we describe hereafter. In order to center the linear spectral statistics  $\text{tr} f(\Sigma_n \Sigma_n^*)$ , we consider the (first order) expansion of  $\frac{1}{N} \mathbb{E} \text{tr} f(\Sigma_n \Sigma_n^*)$ :

$$\frac{1}{N} \mathbb{E} \text{tr} f(\Sigma_n \Sigma_n^*) = \mathcal{E}_{0,n}(f) + \mathcal{O}\left(\frac{1}{N}\right),$$

where  $\mathcal{E}_{0,n}(f)$  is  $\mathcal{O}(1)$  and does not depend on the distribution of the entries of  $X_n$ , and define  $L_n(f)$  as:

$$L_n(f) = \text{tr} f(\Sigma_n \Sigma_n^*) - N \mathcal{E}_{0,n}(f).$$

A precise description of  $L_n(f)$  is provided in Section 2.4. In order to fully characterize the fluctuations of  $L_n(f)$ , we must study the second order expansion of  $\frac{1}{N} \mathbb{E} \text{tr} f(\Sigma_n \Sigma_n^*)$ :

$$\frac{1}{N} \mathbb{E} \text{tr} f(\Sigma_n \Sigma_n^*) = \mathcal{E}_{0,n}(f) + \frac{\mathcal{E}_{1,n}(f)}{N} + o\left(\frac{1}{N}\right),$$

which will naturally yield the bias of  $L_n(f)$ , as  $\mathbb{E} L_n(f) = \mathcal{E}_{1,n}(f) + o(1)$ . Asymptotic expansions for various matrix ensembles have already been studied, see for instance Pastur et al. [1], Bai and Silverstein [5], Haagerup and Thorbjørnsen [27, 28], Schultz [49], Capitaine and Donati-Martin [17], Vallet et al. [56], Hachem et al. [32], etc.

The asymptotic bias is expressed in Theorem 1 for the resolvent. In order to lift asymptotic expansions from the resolvent to smooth functions, we combine ideas from Haagerup and Thorbjørnsen [27] and Loubaton et al. [32, 56] together with some Gaussian interpolation and the use of Helffer-Sjöstrand's formula. For smooth functions, the statement is given in Theorem 3. Somehow surprisingly, the condition over function  $f$  is stronger for the asymptotic expansion to hold than for the CLT as function  $f$  needs to be of class  $C^{18}$  (cf. Remark 4.4).

**Outline of the paper.** In Section 2, we provide some general background; we describe the covariance of the normalized trace of the resolvent of  $\Sigma_n \Sigma_n^*$  in Section 2.3 and its bias in Section 2.4. In Section 3, we state the fluctuation theorem (Theorem 1) for the trace of the resolvent. In Section 4, we state the fluctuation theorem (Theorem 2) for general linear statistics and describe its bias in Theorem 3. Sections 5, 6 and 7 are respectively devoted to the proofs of Theorems 1, 2 and 3.

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## 2. GENERAL BACKGROUND - VARIANCE AND BIAS FORMULAS

**2.1. Assumptions.** Recall the asymptotic regime where  $N, n \rightarrow \infty$ , cf. (1.2), and denote by

$$c_n = \frac{N}{n}, \quad \ell^- = \liminf \frac{N}{n} \quad \text{and} \quad \ell^+ = \limsup \frac{N}{n}.$$

**Assumption A-1.** The random variables  $(X_{ij}^n ; 1 \leq i \leq N(n), 1 \leq j \leq n, n \geq 1)$  are independent and identically distributed. They satisfy:

$$\mathbb{E}X_{ij}^n = 0, \quad \mathbb{E}|X_{ij}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|X_{ij}^n|^4 < \infty.$$

**Assumption A-2.** Consider a sequence  $(R_n)$  of deterministic, nonnegative definite hermitian  $N \times N$  matrices, with  $N = N(n)$ . The sequence  $(R_n, n \geq 1)$  is bounded for the spectral norm as  $N, n \rightarrow \infty$ :

$$\sup_{n \geq 1} \|R_n\| < \infty.$$

In particular, we will have:

$$0 \leq \lambda_R^- \triangleq \liminf_{N, n \rightarrow \infty} \|R_n\| \leq \lambda_R^+ \triangleq \limsup_{N, n \rightarrow \infty} \|R_n\| < \infty.$$

**2.2. Resolvent, canonical equation and deterministic equivalents.** Denote by  $Q_n(z)$  (resp.  $\tilde{Q}_n$ ) the resolvent of matrix  $\Sigma_n \Sigma_n^*$  (resp. of  $\Sigma_n^* \Sigma_n$ ):

$$Q_n(z) = (\Sigma_n \Sigma_n^* - zI_N)^{-1}, \quad \tilde{Q}_n(z) = (\Sigma_n^* \Sigma_n - zI_n)^{-1}, \quad (2.1)$$

and by  $f_n(z)$  and  $\tilde{f}_n(z)$  their normalized traces which are the Stieltjes transforms of the empirical distribution of  $\Sigma_n \Sigma_n^*$ 's and  $\Sigma_n^* \Sigma_n$ 's eigenvalues:

$$f_n(z) = \frac{1}{N} \text{tr} Q_n(z), \quad \tilde{f}_n(z) = \frac{1}{n} \text{tr} \tilde{Q}_n(z). \quad (2.2)$$

The following canonical equation<sup>2</sup> admits a unique solution  $t_n$  in the class of Stieltjes transforms of probability measures (see for instance [5]):

$$t_n(z) = \frac{1}{N} \text{tr} (-zI_N + (1 - c_n)R_n - zc_n t_n(z)R_n)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}^+. \quad (2.3)$$

The function  $t_n$  being introduced, we can define the following  $N \times N$  matrix

$$T_n(z) = (-zI_N + (1 - c_n)R_n - zc_n t_n(z)R_n)^{-1}. \quad (2.4)$$

Matrix  $T_n(z)$  can be thought of as a *deterministic equivalent* of the resolvent  $Q_n(z)$  in the sense that it approximates the resolvent in various senses. For instance,

$$\frac{1}{N} \text{tr} T_n(z) - \frac{1}{N} \text{tr} Q_n(z) \xrightarrow{N, n \rightarrow \infty} 0, \quad z \in \mathbb{C} \setminus \mathbb{R}^+,$$

(in probability or almost surely). Otherwise stated,  $t_n(z) = N^{-1} \text{tr} T_n(z)$  is the deterministic equivalent of  $f_n(z)$ . As we shall see later in this paper, the following property holds true:

$$u_n^* Q_n(z) v_n - u_n^* T_n(z) v_n \xrightarrow{N, n \rightarrow \infty} 0 \quad (2.5)$$

where  $(u_n)$  and  $(v_n)$  are deterministic  $N \times 1$  vectors with uniformly bounded euclidian norms in  $N$ . As a consequence of (2.5), not only  $T_n$  conveys information on the limiting spectrum of the resolvent  $Q_n$  but also on the eigenvectors of  $Q_n$ .

If  $R_n = I_N$ , then  $t_n$  is simply the Stieltjes transform of Marčenko-Pastur's distribution [43] with parameter  $c_n$ .

---

<sup>2</sup>We borrow the name "canonical equation" from V.L. Girko who established in [23, 24] canonical equations associated to various models of large random matrices.

**2.3. Entries with non-null fourth cumulant and the limiting covariance for the trace of the resolvent.** As in [5], we first study the CLT for the trace of the resolvent. Let  $\mathcal{V}$  be the second moment of the random variable  $X_{ij}$  and  $\kappa$  its fourth cumulant:

$$\mathcal{V} = \mathbb{E}(X_{ij}^n)^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{ij}^n|^4 - |\mathcal{V}|^2 - 2.$$

If the entries are real or complex standard Gaussian, then  $\mathcal{V} = 1$  or  $0$  and  $\kappa = 0$ . Otherwise the fourth cumulant is a priori no longer equal to zero. This induces extra-terms in the computation of the limiting variance, mainly due to the following  $(\mathcal{V}, \kappa)$ -dependent identity:

$$\mathbb{E}(X_{\cdot 1}^* A X_{\cdot 1} - \text{tr } A)(X_{\cdot 1}^* B X_{\cdot 1} - \text{tr } B) = \text{tr } AB + |\mathcal{V}|^2 \text{tr } AB^T + \kappa \sum_{i=1}^N A_{ii} B_{ii}, \quad (2.6)$$

where  $X_{\cdot 1}$  stands for the first column (of dimension  $N \times 1$ ) of matrix  $X_n$  and where  $A, B$  are deterministic  $N \times N$  matrices. As a consequence, there will be three terms in the limiting covariance of the quantity (1.3); one will raise from the first term of the right hand side (r.h.s.) of (2.6), a second one will be proportional to  $|\mathcal{V}|^2$ , and a third one to  $\kappa$ . In order to describe these terms, let:

$$\tilde{t}_n(z) = -\frac{1 - c_n}{z} + c_n t_n(z). \quad (2.7)$$

The quantity  $\tilde{t}_n(z)$  is the deterministic equivalent associated to  $n^{-1} \text{tr}(\Sigma_n^* \Sigma_n - z I_n)^{-1}$ . Denote by  $R_n^T$  the transpose matrix of  $R_n$  (notice that since  $R_n$  is hermitian,  $R_n^T = \bar{R}_n$  and we shall use this latter notation) and by  $T_n^T$ , the transpose matrix<sup>3</sup> of  $T_n$ :

$$T_n^T(z) = (-z I_N + (1 - c_n) \bar{R}_n - z c_n t_n(z) \bar{R}_n)^{-1}; \quad (2.8)$$

notice that the definition of  $t_n(z)$  in (2.3) does not change if  $R_n$  is replaced by  $\bar{R}_n$  since the spectrum of both matrices  $R_n$  and  $\bar{R}_n$  is the same. We can now describe the limiting covariance of the trace of the resolvent:

$$\begin{aligned} \text{cov}(\text{tr } Q_n(z_1), \text{tr } Q_n(z_2)) &= \Theta_{0,n}(z_1, z_2) + |\mathcal{V}|^2 \Theta_{1,n}(z_1, z_2) + \kappa \Theta_{2,n}(z_1, z_2) + o(1), \\ &\triangleq \Theta_n(z_1, z_2) + o(1), \end{aligned} \quad (2.9)$$

where  $o(1)$  is a term that converges to zero as  $N, n \rightarrow \infty$  and

$$\Theta_{0,n}(z_1, z_2) \triangleq \left\{ \frac{\tilde{t}'_n(z_1) \tilde{t}'_n(z_2)}{(\tilde{t}_n(z_1) - \tilde{t}_n(z_2))^2} - \frac{1}{(z_1 - z_2)^2} \right\}, \quad (2.10)$$

$$\Theta_{1,n}(z_1, z_2) \triangleq \frac{\partial}{\partial z_2} \left\{ \frac{\partial \mathcal{A}_n(z_1, z_2)}{\partial z_1} \frac{1}{1 - |\mathcal{V}|^2 \mathcal{A}_n(z_1, z_2)} \right\}, \quad (2.11)$$

$$\Theta_{2,n}(z_1, z_2) \triangleq \frac{z_1^2 z_2^2 \tilde{t}'_n(z_1) \tilde{t}'_n(z_2)}{n} \sum_{i=1}^N \left( R_n^{1/2} T_n^2(z_1) R_n^{1/2} \right)_{ii} \left( R_n^{1/2} T_n^2(z_2) R_n^{1/2} \right)_{ii} \quad (2.12)$$

with

$$\mathcal{A}_n(z_1, z_2) = \frac{z_1 z_2}{n} \tilde{t}_n(z_1) \tilde{t}_n(z_2) \text{tr} \left\{ R_n^{1/2} T_n(z_1) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z_2) \bar{R}_n^{1/2} \right\}. \quad (2.13)$$

For alternative formulas for  $\Theta_{0,n}$  and  $\Theta_{2,n}$ , see Remarks 3.2 and 3.3.

At first sight, these formulas (established in Section 5) may seem complicated; however, much information can be inferred from them.

<sup>3</sup>Beware that  $T_n^T$  is not the entry-wise conjugate of  $T_n$ , due to the presence of  $z$ .



*The term  $\Theta_{0,n}$ .* This term is familiar as it already appears in Bai and Silverstein's CLT [5]. Notice that the quantities  $\tilde{t}_n$  and  $\tilde{t}'_n$  only depend on the spectrum of matrix  $R_n$ . Hence, under the additional assumption that:

$$c_n \xrightarrow{N,n \rightarrow \infty} c \in (0, \infty) \quad \text{and} \quad F^{R_n} \xrightarrow[N,n \rightarrow \infty]{\mathcal{D}} F^{\mathbf{R}}, \quad (2.14)$$

where  $F^{R_n}$  denotes the empirical distribution of  $R_n$ 's eigenvalues and  $F^{\mathbf{R}}$  is a probability measure, it can easily be proved that

$$\Theta_{0,n}(z_1, z_2) \xrightarrow{N,n \rightarrow \infty} \Theta_0(z_1, z_2) = \left\{ \frac{\tilde{t}'(z_1)\tilde{t}'(z_2)}{(\tilde{t}(z_1) - \tilde{t}(z_2))^2} - \frac{1}{(z_1 - z_2)^2} \right\}, \quad (2.15)$$

where  $\tilde{t}, \tilde{t}'$  are the limits of  $\tilde{t}_n, \tilde{t}'_n$  under (2.14).

*The term  $\Theta_{1,n}$ .* The interesting phenomenon lies in the fact that this term involves products of matrices  $R_n^{1/2}$  and its conjugate  $\bar{R}_n^{1/2}$ . These matrices have the same spectrum but conjugate eigenvectors. If  $R_n$  is not real, the convergence of  $\Theta_{1,n}$  is not granted, even under (2.14). If however  $R_n$  and  $X_n$ 's entries are real, i.e.  $\mathcal{V} = 1$ , then it can be easily proved that  $\Theta_{0,n} = \Theta_{1,n}$  hence the factor 2 in [5] between the complex and the real covariance.

*The term  $\Theta_{2,n}$ .* This term involves quantities of the type  $(R_n^{1/2} T_n R_n^{1/2})_{ii}$  which not only depend on the spectrum of matrix  $R_n$  but also on its eigenvectors. As a consequence, the convergence of such terms does not follow from an assumption such as (2.14), except in some particular cases (for instance if  $R_n$  is diagonal) and any assumption which enforces the convergence of such terms (as for instance in [46, Theorem 1.4]) implicitly implies an asymptotic joint behaviour between  $R_n$ 's eigenvectors and eigenvalues. We shall adopt a different point of view here and will not assume the convergence of these quantities.

**2.4. Representation of the linear statistics and limiting bias.** Recall that  $t_n(z)$  is the Stieltjes transform of a probability measure  $\mathcal{F}_n$ :

$$t_n(z) = \int_{\mathcal{S}_n} \frac{\mathcal{F}_n(d\lambda)}{\lambda - z} \quad (2.16)$$

with support  $\mathcal{S}_n$  included in a compact set. The purpose of this article is to describe the fluctuations of the linear statistics

$$L_n(f) = \sum_{i=1}^N f(\lambda_i) - N \int f(\lambda) \mathcal{F}_n(d\lambda) \quad (2.17)$$

as  $N, n \rightarrow \infty$ .

For a smooth enough function  $f$  of class  $C^{k+1}$  with bounded support, one can rely on Helffer-Sjöstrand's formula and write:

$$\begin{aligned} L_n(f) &= \text{tr} f(\Sigma_n \Sigma_n^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) \\ &= \frac{1}{\pi} \text{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(f)(z) \{ \text{tr} Q_n(z) - N t_n(z) \} \ell_2(dz). \end{aligned} \quad (2.18)$$

where  $\Phi_k(f)$  is defined in (1.6) and the last equality follows from the fact that

$$\int f(\lambda) \mathcal{F}_n(d\lambda) = \frac{1}{\pi} \text{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(f)(z) t_n(z) \ell_2(dz).$$

Based on (2.18), we shall first study the fluctuations of:

$$\mathrm{tr} Q_n(z) - Nt_n(z) = \{\mathrm{tr} Q_n(z) - \mathbb{E}\mathrm{tr} Q_n(z)\} + \{\mathbb{E}\mathrm{tr} Q_n(z) - Nt_n(z)\}$$

for  $z \in \mathbb{C}^+$ . The first difference in the r.h.s. will yield the fluctuations with a covariance  $\Theta_n(z_1, z_2)$  described in (2.9) while the second difference, deterministic, will yield the bias:

$$\begin{aligned} \mathbb{E}\mathrm{tr} Q_n(z) - Nt_n(z) &= |\mathcal{V}|^2 \mathcal{B}_{1,n}(z) + \kappa \mathcal{B}_{2,n}(z) + o(1) \\ &\triangleq \mathcal{B}_n(z) + o(1) \end{aligned} \quad (2.19)$$

where

$$\mathcal{B}_{1,n}(z) \triangleq -z^3 \bar{t}_n^3 \frac{\frac{1}{n} \mathrm{tr} R_n^{1/2} T_n^2(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}}{\left(1 - z^2 \bar{t}_n^2 \frac{1}{n} \mathrm{Tr} R_n^2 T_n^2\right) \left(1 - |\mathcal{V}|^2 z^2 \bar{t}_n^2 \frac{1}{n} \mathrm{Tr} R_n^{1/2} T_n(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}\right)}, \quad (2.20)$$

$$\mathcal{B}_{2,n}(z) \triangleq -z^3 \bar{t}_n^3 \frac{\frac{1}{n} \sum_{i=1}^N \left(R_n^{1/2} T_n R_n^{1/2}\right)_{ii} \left(R_n^{1/2} T_n^2 R_n^{1/2}\right)_{ii}}{1 - z^2 \bar{t}_n^2 \frac{1}{n} \mathrm{tr} R_n^2 T_n^2}. \quad (2.21)$$

The previous discussion on the terms  $\Theta_{1,n}$  and  $\Theta_{2,n}$  also applies to the terms  $\mathcal{B}_{1,n}$  and  $\mathcal{B}_{2,n}$  (whose expressions are established in Section 5) which are likely not to converge for similar reasons.

**2.5. Gaussian processes and the central limit theorem.** A priori, the mean  $\mathcal{B}_n$  and covariance  $\Theta_n$  of  $(\mathrm{tr} Q_n - Nt_n)$  do not converge. Hence, we shall express the Gaussian fluctuations of the linear statistics (2.17) in the following way: we first prove the existence of a family  $(G_n(z), z \in \mathcal{C})_{n \in \mathbb{N}}$  of tight Gaussian processes with mean and covariance:

$$\begin{aligned} \mathbb{E}G_n(z) &= \mathcal{B}_n(z), \\ \mathrm{cov}(G_n(z_1), G_n(z_2)) &= \Theta_n(z_1, z_2). \end{aligned}$$

We then express the fluctuations of the centralized trace as

$$d_{\mathcal{L}P}((\mathrm{tr} Q_n(z) - Nt_n(z)), G_n(z)) \xrightarrow{N, n \rightarrow \infty} 0.$$

with  $d_{\mathcal{L}P}$  the Lévy-Prohorov distance between  $P$  and  $Q$  probability measures over borel sets of  $\mathbb{R}, \mathbb{R}^d, \mathbb{C}$  or  $\mathbb{C}^d$ :

$$d_{\mathcal{L}P}(P, Q) = \inf \{\varepsilon > 0, P(A) \leq Q(A^\varepsilon) + \varepsilon \text{ for all Borel sets } A\}, \quad (2.22)$$

where  $A^\varepsilon$  is an  $\varepsilon$ -blow up of  $A$  (cf. [21, Section 11.3] for more details). If  $X$  is a random variable and  $\mathcal{L}(X)$  its distribution, denote (with a slight abuse of notation) by  $d_{\mathcal{L}P}(X, Y) \triangleq d_{\mathcal{L}P}(\mathcal{L}(X), \mathcal{L}(Y))$ .

Similarly, we will express the fluctuations of  $L_n(f)$  as:

$$d_{\mathcal{L}P}(L_n(f), \mathcal{N}_n(f)) \xrightarrow{N, n \rightarrow \infty} 0,$$

where  $\mathcal{N}_n(f)$  is a well-identified gaussian random variable.

**2.6. A meta-model argument.** As we need to cope with a sequence of Gaussian processes  $(G_n)$  instead of a single one, it will be necessary to establish various properties uniform in  $n, N$  such as:

- (1) the tightness of the sequence  $(G_n)$  (cf. Section 5.2);
- (2) a uniform bound over the variances of  $(\text{Tr } G_n(z))$  (cf. Section 6.2), needed to extend the CLT to non-analytic functionals;
- (3) a uniform bound over the biases of  $(\text{Tr } G_n(z))$  (cf. Section 7.1.1), needed to compute the bias for non-analytic functionals.

A direct approach based on the mere definition of process  $G_n$ 's parameters seems difficult, mainly due to the definitions of  $\Theta_n$  and  $\mathcal{B}_n$  which rely on quantities  $(t_n$  and  $\tilde{t}_n)$  defined as solutions of fixed-point equations. Since the previous properties will be established for the processes  $(\text{Tr } Q_n - Nt_n)$  anyway, the idea is to transfer them to  $G_n$  by means of the following matrix meta-model:

Let  $N, n$  and  $R_n$  be fixed and consider the  $NM \times NM$  matrix

$$R_n(M) = \begin{pmatrix} R_n & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & R_n \end{pmatrix}. \quad (2.23)$$

Matrix  $R_n(M)$  is a block matrix with  $N \times N$  diagonal blocks equal to  $R_n$ , and zero blocks elsewhere; for all  $M \geq 1$  the spectral norm of  $R_n(M)$  is equal to the spectral norm of  $R_n$  (which is fixed). In particular the sequence  $(R_n(M); M \geq 1)$  with  $N, n$  fixed satisfies assumption (A-2) with  $(R_n(M); M \geq 1)$  instead of  $(R_n)$ . Consider now the random matrix model:

$$\Sigma_n(M) = \frac{1}{\sqrt{Mn}} R_n(M)^{1/2} X_n(M) \quad (2.24)$$

where  $X_n(M)$  is a  $MN \times Mn$  matrix with i.i.d. random entries with the same distribution as the  $X_{ij}$ 's and satisfying (A-1). The interest of introducing matrix  $\Sigma_n(M)$  lies in the fact that matrices  $\Sigma_n(M)\Sigma_n(M)^*$  and  $\Sigma_n\Sigma_n^*$  have loosely speaking the same deterministic equivalents. Denote by  $t_n, T_n$  and  $\tilde{t}_n$  the deterministic equivalents of  $\Sigma_n\Sigma_n^*$  as defined in (2.3), (2.4) and (3.1), and by  $t_n(M), T_n(M)$  and  $\tilde{t}_n(M)$  their counterparts for the model  $\Sigma_n(M)\Sigma_n(M)^*$ . Taking advantage of the block structure of  $R_n(M)$ , a straightforward computation yields ( $N, n$  fixed):

$$\forall M \geq 1, \quad t_n(M) = t_n, \quad \tilde{t}_n(M) = \tilde{t}_n \quad \text{and} \quad T_n(M) = \begin{pmatrix} T_n & 0 & \cdots \\ & \ddots & \\ \cdots & 0 & T_n \end{pmatrix}.$$

Similarly, denote by  $\mathcal{B}_{n,M}$  and  $\Theta_{n,M}$  the quantities given by formulas (2.19) and (2.9) when replacing  $N, t_n, T_n$  and  $\tilde{t}_n$  by  $NM, t_n(M), T_n(M)$  and  $\tilde{t}_n(M)$ . Straightforward computation yields:

$$\forall M \geq 1, \quad \mathcal{B}_{n,M} = \mathcal{B}_n \quad \text{and} \quad \Theta_{n,M} = \Theta_n.$$

An interesting feature of this meta-model lies in the fact that all the quantities associated to  $\Sigma_n(M)\Sigma_n(M)^*$  converge as  $M \rightarrow \infty$  to the deterministic equivalents  $t_n, \tilde{t}_n$ , etc. As a consequence, one can easily transfer all the estimates obtained for

$$(\text{Tr} (\Sigma_n(M)\Sigma_n(M)^* - zI_{NM})^{-1} - NMt_n)$$

to the process  $(G_n)$ .

## 3. STATEMENT OF THE CLT FOR THE TRACE OF THE RESOLVENT

**3.1. Further notations.** If  $A$  is a  $N \times N$  matrix with real eigenvalues, denote by  $F^A$  the empirical distribution of the eigenvalues  $(\delta_i(A), i = 1 : N)$  of  $A$ , that is:

$$F^A(dx) = \frac{1}{n} \sum_{i=1}^N \delta_{\lambda_i(A)}(dx) .$$

Recall the definitions of  $Q_n, t_n, T_n$  and  $\tilde{t}_n$  (cf. (2.1), (2.3), (2.4) and (2.7)). The following relations hold true (see for instance [5]):

$$T_n(z) = -\frac{1}{z} (I_N + \tilde{t}_n(z)R_n)^{-1} \quad \text{and} \quad \tilde{t}_n(z) = -\frac{1}{z(1 + \frac{1}{n}\text{tr} R_n T_n(z))} . \quad (3.1)$$

Recall the definition of  $\mathcal{F}_n$  in (2.16) and let similarly  $\tilde{\mathcal{F}}_n$  be the probability distribution associated to  $\tilde{t}_n$ . The central object of study is the signed measure:

$$N \left( F^{\Sigma_n \Sigma_n^*} - \mathcal{F}_n \right) = n \left( F^{\Sigma_n^* \Sigma_n} - \tilde{\mathcal{F}}_n \right) ,$$

and its Stieltjes transform

$$M_n(z) = N(f_n(z) - t_n(z)) = n \left( \tilde{f}_n(z) - \tilde{t}_n(z) \right) . \quad (3.2)$$

Denote by  $o_P(1)$  any random variable which converges to zero in probability.

**3.2. Truncation.** In this section, we closely follow Bai and Silverstein [5]. We recall the framework developed there and introduce some additional notations.

Consider a sequence of positive numbers  $(\delta_n)$  which satisfies:

$$\delta_n \rightarrow 0, \quad \delta_n n^{1/4} \rightarrow \infty \quad \text{and} \quad \delta_n^{-4} \int_{\{|X_{11}| \geq \delta_n \sqrt{N}\}} |X_{11}|^4 \rightarrow 0$$

as  $N, n \rightarrow \infty$ . Let  $\hat{\Sigma}_n = n^{-1/2} R_n^{1/2} \hat{X}_n$  where  $\hat{X}_n$  is a  $N \times n$  matrix having  $(i, j)$ th entry  $X_{ij} 1_{\{|X_{ij}| < \delta_n \sqrt{N}\}}$ . This truncation step yields:

$$\mathbb{P} \left( \Sigma_n \Sigma_n^* \neq \hat{\Sigma}_n \hat{\Sigma}_n^* \right) \xrightarrow[N, n \rightarrow \infty]{} 0 \quad (3.3)$$

from which we deduce

$$\text{tr} (\Sigma_n \Sigma_n^* - zI_N)^{-1} - \text{tr} (\hat{\Sigma}_n \hat{\Sigma}_n^* - zI_N)^{-1} \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0 , \quad (3.4)$$

where  $\xrightarrow{\mathcal{P}}$  stands for the convergence in probability. Define  $\tilde{\Sigma}_n = n^{-1/2} R_n^{1/2} \tilde{X}_n$  where  $\tilde{X}_n$  is a  $N \times n$  matrix having  $(i, j)$ th entry  $(\hat{X}_{ij} - \mathbb{E}\hat{X}_{ij})/\sigma_n$ , where  $\sigma_n^2 = \mathbb{E}|\hat{X}_{ij} - \mathbb{E}\hat{X}_{ij}|^2$ . Using the fact that  $\lambda(\in \mathbb{R}) \mapsto \frac{1}{\lambda - z}$  is Lipschitz with Lipschitz constant  $|z|^{-2}$ , we obtain

$$\mathbb{E} \left| \text{tr} (\hat{\Sigma}_n \hat{\Sigma}_n^* - zI_N)^{-1} - \text{tr} (\tilde{\Sigma}_n \tilde{\Sigma}_n^* - zI_N)^{-1} \right| \leq \frac{1}{|z|^2} \sum_{i=1}^N \mathbb{E} \left| \tilde{\lambda}_i - \hat{\lambda}_i \right| \xrightarrow[N, n \rightarrow \infty]{(a)} 0 ,$$

where  $\tilde{\lambda}_i = \lambda_i(\tilde{\Sigma}_n \tilde{\Sigma}_n^*)$ ,  $\hat{\lambda}_i = \lambda_i(\hat{\Sigma}_n \hat{\Sigma}_n^*)$  and (a) follows from similar arguments as in [7, Section 9.7.1]. Hence

$$\text{tr} (\hat{\Sigma}_n \hat{\Sigma}_n^* - zI_N)^{-1} - \text{tr} (\tilde{\Sigma}_n \tilde{\Sigma}_n^* - zI_N)^{-1} \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0 , \quad (3.5)$$

Combining (3.4) and (3.5), we obtain

$$\mathrm{tr} (\Sigma_n \Sigma_n^* - z I_N)^{-1} - \mathrm{tr} (\tilde{\Sigma}_n \tilde{\Sigma}_n^* - z I_N)^{-1} \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0 .$$

Moreover, the moments are asymptotically not affected by these different steps:

$$\max \left( \left| \mathbb{E} \tilde{X}_{ij}^2 - \mathbb{E} X_{ij}^2 \right| ; \left( \mathbb{E} |\tilde{X}_{ij}|^2 - 1 \right) ; \left( \mathbb{E} |\tilde{X}_{ij}|^4 - \mathbb{E} |X_{ij}|^4 \right) \right) \xrightarrow[N, n \rightarrow \infty]{} 0 . \quad (3.6)$$

Note in particular that the fourth cumulant of  $\tilde{X}_{ij}$  converges to that of  $X_{ij}$ . Hence, it is sufficient to consider variables truncated at  $\delta_n \sqrt{n}$ , centralized and renormalized. This will be assumed in the sequel (we shall simply write  $X_{ij}$  and all related quantities with  $X_{ij}$ 's truncated, centralized, renormalized with no superscript any more).

**3.3. The Central Limit Theorem for the resolvent.** We extend below Bai and Silverstein's master lemma [5, Lemma 1.1]. Let  $A$  be such that

$$A > \lambda_R^+ \left( 1 + \sqrt{\ell^+} \right)^2 .$$

Denote by  $D$ ,  $D^+$  and  $D_\varepsilon$  the domains:

$$\begin{aligned} D &= [0, A] + \mathbf{i}[0, 1] , \\ D^+ &= [0, A] + \mathbf{i}(0, 1] , \\ D_\varepsilon &= [0, A] + \mathbf{i}[\varepsilon, 1] \quad (\varepsilon > 0) . \end{aligned} \quad (3.7)$$

**Theorem 1.** *Assume that (A-1) and (A-2) hold true , then*

- (1) *The process  $\{M_n(\cdot)\}$  as defined in (3.2) forms a tight sequence on  $D_\varepsilon$ , more precisely:*

$$\sup_{z_1, z_2 \in D_\varepsilon, n \geq 1} \frac{\mathbb{E} |M_n(z_1) - M_n(z_2)|^2}{|z_1 - z_2|^2} < \infty$$

- (2) *There exists a sequence  $(G_n(z), z \in D^+)$  of two-dimensional Gaussian processes with mean*

$$\mathbb{E} G_n(z) = |\mathcal{V}|^2 \mathcal{B}_{1,n}(z) + \kappa \mathcal{B}_{2,n}(z) \quad (3.8)$$

where  $\mathcal{B}_{1,n}(z)$  and  $\mathcal{B}_{2,n}(z)$  are defined in (2.20) and (2.21), and covariance:

$$\begin{aligned} \mathrm{cov} (G_n(z_1), G_n(z_2)) &= \mathbb{E} (G_n(z_1) - \mathbb{E} G_n(z_1)) (G_n(z_2) - \mathbb{E} G_n(z_2)) \\ &= \Theta_{0,n}(z_1, z_2) + |\mathcal{V}|^2 \Theta_{1,n}(z_1, z_2) + \kappa \Theta_{2,n}(z_1, z_2) , \end{aligned}$$

and

$$\mathrm{cov} \left( G_n(z_1), \overline{G_n(z_2)} \right) = \mathrm{cov} (G_n(z_1), G_n(\overline{z_2})) ,$$

with  $z_1, z_2 \in D^+ \cup \overline{D^+}$ , and where  $\Theta_{0,n}$ ,  $\Theta_{1,n}$  and  $\Theta_{2,n}$  are defined in (2.9), (2.10)-(2.12). Moreover  $(G_n(z), z \in D_\varepsilon)$  is tight.

- (3) *For any continuous functional  $F$  from  $C(D_\varepsilon; \mathbb{C})$  to  $\mathbb{C}$ ,*

$$\mathbb{E} F(M_n) - \mathbb{E} F(G_n) \xrightarrow[N, n \rightarrow \infty]{} 0$$

*Remark 3.1.* (1) The tightness of the process  $\{M_n\}$  immediately follows from Bai and Silverstein's lemma as this result has been proved in [5, Lemma 1.1] under Assumption (A-1) with no extra conditions on the moments of the entries.

- (2) Differences between Theorem 1 and [5, Lemma 1.1] appear in the bias and in the covariance where there are respectively two terms instead of one and three terms instead of one in [5, Lemma 1.1].
- (3) Since the extra terms do not converge, we need to consider a sequence of Gaussian processes instead of a single Gaussian process as in [5, Lemma 1.1].
- (4) In order to prove that the sequence of Gaussian processes is tight, we introduce a meta-matrix model to transfer the tightness of  $\{M_n\}$  to  $\{G_n\}$  (see for instance Section 5.2.1).
- (5) Following Bai and Silverstein [5], it is relatively straightforward with the help of Cauchy's formula to describe the fluctuations of  $L_n(f)$  for  $f$  analytic with Theorem 1 at hand. We skip this step since we will directly extend the CLT to non-analytic functions  $f$  in Section 4.

*Remark 3.2.* A closer look to Bai and Silverstein's proof [5, Sec.2 p.578] yields the following alternative expression for the term  $\Theta_{0,n}$ :

$$\Theta_{0,n}(z_1, z_2) = \frac{\partial}{\partial z_2} \left\{ \frac{\partial \mathcal{A}_{0,n}(z_1, z_2)}{\partial z_1} \frac{1}{1 - \mathcal{A}_{0,n}(z_1, z_2)} \right\}, \quad (3.9)$$

with

$$\mathcal{A}_{0,n}(z_1, z_2) = \frac{z_1 z_2}{n} \tilde{t}_n(z_1) \tilde{t}_n(z_2) \text{tr} \{R_n T_n(z_1) R_n T_n(z_2)\}. \quad (3.10)$$

Such an expression will be helpful in Section 6.2. As an interesting consequence: In the case where  $R_n$  and  $X_n$  have real entries (in particular  $\mathcal{V} = \mathbb{E}(X_{ij})^2 = 1$ ) then  $\mathcal{A}_{0,n} = \mathcal{A}_n$  and  $\Theta_{0,n} = \Theta_{1,n}$ .

*Remark 3.3.* A closer look to the proof below (see for instance (5.32)) yields the following formula for  $\Theta_{2,n}$  which will be of help in the sequel:

$$\Theta_{2,n}(z_1, z_2) = \frac{1}{n} \sum_{i=1}^N \frac{\partial}{\partial z_1} [z_1 T_n(z_1)]_{ii} \frac{\partial}{\partial z_2} [z_2 T_n(z_2)]_{ii}. \quad (3.11)$$

Proof of Theorem 1 is postponed to Section 5.

The end of the section is devoted to various specializations of Theorem 1 in the case where matrix  $R_n$  is diagonal. In this case, the results are simpler to express and comparisons can easily be made with related works.

**3.4. Covariance and bias in the special case of diagonal matrices ( $R_n$ ).** This case partially falls into the framework developed in Pan and Zhou [46] (note that the case  $\mathcal{V} \neq 0$  and 1 is not handled there). Matrix  $R_n$  being nonnegative definite hermitian, its entries are real positive if  $R_n$  is assumed to be diagonal. In this case, matrix  $T_n$  is diagonal as well (cf. (2.4)),  $T_n = T_n^T$  and simplifications occur for the following terms:

$$\begin{aligned} \mathcal{A}_n(z_1, z_2) &= \frac{z_1 z_2}{n} \tilde{t}_n(z_1) \tilde{t}_n(z_2) \text{tr} R_n T_n(z_1) R_n T_n(z_2), \\ \Theta_{2,n}(z_1, z_2) &= \frac{z_1^2 z_2^2 \tilde{t}'_n(z_1) \tilde{t}'_n(z_2)}{n} \text{tr} (R_n^2 T_n^2(z_1) T_n^2(z_2)), \\ \mathcal{B}_{1,n}(z) &= -z^3 \tilde{t}_n^3 \frac{\frac{1}{n} \text{tr} R_n^2 T_n^3}{(1 - z^2 \tilde{t}_n^2 \frac{1}{n} \text{Tr} R_n^2 T_n^2) (1 - |\mathcal{V}|^2 z^2 \tilde{t}_n^2 \frac{1}{n} \text{Tr} R_n^2 T_n^2)}, \\ \mathcal{B}_{2,n}(z) &= -z^3 \tilde{t}_n^3 \frac{\frac{1}{n} \text{tr} R_n^2 T_n^3}{1 - z^2 \tilde{t}_n^2 \frac{1}{n} \text{tr} R_n^2 T_n^2}. \end{aligned}$$

As one may notice, all the terms in the variance and the bias now only depend on the spectrum of  $R_n$ . Hence, the following convergence holds true under the extra assumption (2.14):

$$\begin{aligned} \mathcal{A}_n(z_1, z_2) &\xrightarrow{N, n \rightarrow \infty} \mathcal{A}(z_1, z_2) = c \tilde{t}(z_1) \tilde{t}(z_2) \int \frac{\lambda^2 F^{\mathbf{R}}(d\lambda)}{(1 + \lambda \tilde{t}(z_1))(1 + \lambda \tilde{t}(z_2))}, \\ \Theta_{1,n}(z_1, z_2) &\xrightarrow{N, n \rightarrow \infty} \Theta_1(z_1, z_2) = \frac{\partial}{\partial z_2} \left\{ \frac{\partial \mathcal{A}(z_1, z_2)}{\partial z_1} \frac{1}{1 - |\mathcal{V}|^2 \mathcal{A}(z_1, z_2)} \right\}, \\ \Theta_{2,n}(z_1, z_2) &\xrightarrow{N, n \rightarrow \infty} \Theta_2(z_1, z_2) = c \tilde{t}'(z_1) \tilde{t}'(z_2) \int \frac{\lambda^2 F^{\mathbf{R}}(d\lambda)}{(1 + \lambda \tilde{t}(z_1))^2 (1 + \lambda \tilde{t}(z_2))^2}, \\ \mathcal{B}_{1,n}(z) &\xrightarrow{N, n \rightarrow \infty} \mathcal{B}_1(z) = -\frac{cz^3 \tilde{t}^3(z)}{(1 - \mathcal{A}(z, z))(1 - |\mathcal{V}|^2 \mathcal{A}(z, z))} \int \frac{\lambda^2 F^{\mathbf{R}}(d\lambda)}{(1 + \lambda \tilde{t}(z))^3}, \\ \mathcal{B}_{2,n}(z) &\xrightarrow{N, n \rightarrow \infty} \mathcal{B}_2(z) = -\frac{cz^3 \tilde{t}^3(z)}{1 - \mathcal{A}(z, z)} \int \frac{\lambda^2 F^{\mathbf{R}}(d\lambda)}{(1 + \lambda \tilde{t}(z))^3}. \end{aligned}$$

where  $\tilde{t}, \tilde{t}'$  are the limits of  $\tilde{t}_n, \tilde{t}'_n$  under (2.14). This can be packaged into the following result:

**Corollary 3.1.** *Assume that (A-1) and (A-2) hold true. Assume moreover that  $R_n$  is diagonal and that the convergence assumption (2.14) holds true. Then  $M_n(\cdot)$  converges weakly on  $D_\varepsilon$  (defined in (3.7)) to a two-dimensional Gaussian process  $N(\cdot)$  satisfying:*

$$\mathbb{E}N(z) = \mathcal{B}(z) \quad \text{where } \mathcal{B} = |\mathcal{V}|^2 \mathcal{B}_1 + \kappa \mathcal{B}_2, \quad z \in D_\varepsilon$$

and  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are defined above and covariance

$$\text{cov}(N(z_1), N(z_2)) = \Theta(z_1, z_2) \quad \text{where } \Theta = \Theta_0 + |\mathcal{V}|^2 \Theta_1 + \kappa \Theta_2, \quad z_1, z_2 \in D_\varepsilon \cup \overline{D_\varepsilon}$$

and  $\Theta_0$  defined in (2.15) and  $\Theta_1, \Theta_2$  defined above.

**3.5. Additional computations in the case where  $R_n$  is the identity.** In this section, we assume that  $R_n = I_N$ .

The term proportional to  $|\mathcal{V}|^2$ . In this case, the quantity  $\mathcal{A}(z_1, z_2)$  takes the simplified form

$$\mathcal{A}(z_1, z_2) = \frac{c \tilde{t}_1 \tilde{t}_2}{(1 + \tilde{t}_1)(1 + \tilde{t}_2)}.$$

where we denote  $\tilde{t}_i = \tilde{t}(z_i)$ ,  $i = 1, 2$ . Straightforward computations yield:

$$\frac{\partial}{\partial z_i} \mathcal{A}(z_1, z_2) = \frac{\tilde{t}'_i}{(1 + \tilde{t}_i) \tilde{t}_i} \mathcal{A}(z_1, z_2), \quad i = 1, 2.$$

and

$$\Theta_1(z_1, z_2) = \frac{c \tilde{t}'_1 \tilde{t}'_2}{(1 + \tilde{t}_1)^2 (1 + \tilde{t}_2)^2 (1 - |\mathcal{V}|^2 \mathcal{A}(z_1, z_2))^2} = \frac{c \tilde{t}'_1 \tilde{t}'_2}{((1 + \tilde{t}_1)(1 + \tilde{t}_2) - |\mathcal{V}|^2 c \tilde{t}_1 \tilde{t}_2)^2}.$$

This formula is in accordance with [9, Formula (2.2)] (use [9, (3.4)] to equate both). If needed, one can then use the explicit expression of the Stieltjes transform of Marcenko-Pastur distribution (cf. also Proposition 4.2 below).

## 4. STATEMENT OF THE CLT FOR NON-ANALYTIC FUNCTIONALS

In order to lift the CLT from the trace of the resolvent to a smooth function  $f$ , the key ingredient is Helffer-Sjöstrand's formula (1.7). Let

$$\begin{aligned} L_n(f) &\stackrel{(a)}{=} \operatorname{Tr} f(\Sigma_n \Sigma_n^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) \\ &= \left( \operatorname{Tr} f(\Sigma_n \Sigma_n^*) - \mathbb{E} \operatorname{Tr} f(\Sigma_n \Sigma_n^*) \right) + \left( \mathbb{E} \operatorname{Tr} f(\Sigma_n \Sigma_n^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) \right) \\ &\triangleq L_n^1(f) + L_n^2(f), \end{aligned} \quad (4.1)$$

where  $\mathcal{F}_n$  in (a) is defined in (2.16). We describe the fluctuations of  $L_n^1(f)$  for non-analytic functions  $f$  in Section 4.1 and study the bias  $L_n^2(f)$  in Section 4.3.

**4.1. Fluctuations for the linear spectral statistics.** Denote by  $C_c^\infty(\mathbb{R}^d)$  (resp.  $C_c^m(\mathbb{R}^d)$ ) the set of infinitely differentiable (resp.  $C^m$ ) functions from  $\mathbb{R}^d$  to  $\mathbb{R}$  with compact support; by  $C_c^{mp}(\mathbb{R}^2)$  the set of functions from  $\mathbb{R}^2$  to  $\mathbb{R}$   $m$  times differentiable with respect to the first coordinate and  $p$  times with respect to the second one. As usual, if the subscript  $c$  is removed in the sets above, then the corresponding functions may no longer have a compact support.

**Theorem 2.** *Assume that (A-1) and (A-2) hold true. Let  $f_1, \dots, f_k$  be in  $C_c^3(\mathbb{R})$ . Consider the centered Gaussian random vector  $Z_n^1(\mathbf{f}) \triangleq (Z_n^1(f_1), \dots, Z_n^1(f_k))$  with covariance*

$$\begin{aligned} \operatorname{cov}(Z_n^1(f), Z_n^1(g)) &= \\ &= \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_2(f)(z_1) \overline{\partial \Phi_2(g)(z_2)} \Theta_n(z_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_2(f)(z_1) \bar{\partial} \Phi_2(g)(z_2) \Theta_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2), \end{aligned} \quad (4.2)$$

for  $f, g \in \{f_1, \dots, f_k\}$ , where  $\Phi_2(f)$  and  $\Phi_2(g)$  are defined as in (1.6), and where  $\Theta_n$  is defined in (2.9); let

$$L_n^1(\mathbf{f}) = (L_n^1(f_1), \dots, L_n^1(f_k)) \quad \text{with} \quad L_n^1(f) = \operatorname{tr} f(\Sigma_n \Sigma_n^*) - \mathbb{E} \operatorname{tr} f(\Sigma_n \Sigma_n^*).$$

Then, the sequence of  $\mathbb{R}^k$ -valued random vectors  $Z_n^1(\mathbf{f})$  is tight and the following convergence holds true:

$$d_{\mathcal{L}P}(L_n^1(\mathbf{f}), Z_n^1(\mathbf{f})) \xrightarrow{N, n \rightarrow \infty} 0, \quad (4.3)$$

or equivalently for every continuous bounded function  $F: \mathbb{R}^k \rightarrow \mathbb{C}$ ,

$$\mathbb{E} F(L_n^1(\mathbf{f})) - \mathbb{E} F(Z_n^1(\mathbf{f})) \xrightarrow{N, n \rightarrow \infty} 0. \quad (4.4)$$

Proof of Theorem 2 is postponed to Section 6.

We provide hereafter some information on the covariance operator.

Let  $N_1, N_2 \in \mathbb{N}$  and  $f \in C_c^{N_1+1, N_2+1}(\mathbb{R}^2)$ ; denote by  $z_1 = x + iu$ ,  $z_2 = y + iv$  and let  $\Phi_{N_1, N_2}(f)$  be defined as

$$\Phi_{N_1, N_2}(f)(z_1, z_2) = \sum_{\substack{n_1=0:N_1 \\ n_2=0:N_2}} \frac{\partial^{n_1+n_2}}{\partial x^{n_1} \partial y^{n_2}} f(x, y) \frac{(iu)^{n_1}}{n_1!} \frac{(iv)^{n_2}}{n_2!} \chi(u) \chi(v), \quad (4.5)$$



where  $\chi : \mathbb{R} \rightarrow \mathbb{R}^+$  is smooth, compactly supported with value 1 in a neighbourhood of the origin. Denote by  $\bar{\partial}_1 = \partial_x + i\partial_u$  and  $\bar{\partial}_2 = \partial_y + i\partial_v$ .

**Proposition 4.1.** *For every  $f \in C_c^{3,3}(\mathbb{R}^2)$ , denote by*

$$\begin{aligned} \Upsilon(f) = \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(z_1, z_2) \Theta_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2) \\ + \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(z_1, \bar{z}_2) \Theta_n(z_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) . \end{aligned}$$

Then  $\Upsilon(f)$  is a distribution (in the sense of L. Schwartz) on  $C_c^{3,3}(\mathbb{R}^2)$ . Moreover  $\Upsilon$  admits the following boundary value representation:

$$\begin{aligned} \Upsilon(f) = -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^2} f(x, y) \{ \Theta_n(x + i\varepsilon, y + i\varepsilon) + \Theta_n(x - i\varepsilon, y - i\varepsilon) \\ - \Theta_n(x - i\varepsilon, y + i\varepsilon) - \Theta_n(x + i\varepsilon, y - i\varepsilon) \} dx dy . \quad (4.6) \end{aligned}$$

Notice that for every  $f, g \in C_c^3(\mathbb{R})$  then  $f \otimes g \in C_c^{3,3}(\mathbb{R}^2)$  (where  $(f \otimes g)(x, y) = f(x)g(y)$ ) and

$$\Upsilon(f \otimes g) = \operatorname{cov}(Z_n^1(f), Z_n^1(g)) .$$

Proof of Proposition 4.1 is postponed to Section 6.3.

*Remark 4.1.* By relying on Tillmann's results [55], one may prove that the support of  $\Upsilon$  (as a distribution) is included in  $\mathcal{S}_n \times \mathcal{S}_n$ . We provide a more direct approach in a slightly simpler case in Section 4.2.

**4.2. More covariance formulas.** We provide here more explicit formulas for the variance than those given in Theorem 2 and Proposition 4.1; we also verify that these formulas are in agreement with other formulas available in the literature.

Recall that by [52, Theorem 1.1], the limit  $\lim_{\varepsilon \searrow 0} \tilde{t}_n(x + i\varepsilon)$  denoted by  $\tilde{t}_n(x)$  exists for all  $x \in \mathbb{R}$ ,  $x \neq 0$ ; the same holds true for  $t_n$ .

**Proposition 4.2.** *Assume that (A-1) and (A-2) hold true and let  $f, g \in C_c^3(\mathbb{R})$ ; assume moreover for simplicity that  $\mathcal{V} = \mathbb{E}X_{ij}^2$  is either equal to 0 or 1 and that  $R_n$  has real entries. Then the covariance of  $(Z_n(f), Z_n(g))$  in Theorem 2 writes*

$$\begin{aligned} \operatorname{cov}(Z_n^1(f), Z_n^1(g)) = \frac{1 + |\mathcal{V}|^2}{2\pi^2} \int_{\mathcal{S}_n^2} f'(x)g'(y) \ln \left| \frac{\tilde{t}_n(x) - \overline{\tilde{t}_n(y)}}{\tilde{t}_n(x) - \tilde{t}_n(y)} \right| dx dy \\ + \frac{\kappa}{\pi^2 n} \sum_{i=1}^N \left( \int_{\mathcal{S}_n} f'(x) \operatorname{Im}(x T_n(x))_{ii} dx \right) \left( \int_{\mathcal{S}_n} g'(y) \operatorname{Im}(y T_n(y))_{ii} dy \right) . \quad (4.7) \end{aligned}$$

Proof for Proposition 4.2 is postponed to Section 6.4.

*Remark 4.2.* Notice that the first term in the r.h.s. matches with the expression provided in [5, Eq. (1.17)] (see also [6, Eq. (9.8.8)]).

*Remark 4.3.* Concerning the cumulant term, we shall compare it with the explicit formula provided in [42] (see also [47]) in the case where  $R_n = I_N$ . Recall that in the context of

Marčenko-Pastur's theorem where  $R_n = I_N$ , we have  $\mathcal{S}_n = [\lambda^-, \lambda^+]$  where  $\lambda^- = (1 - \sqrt{c_n})^2$ ,  $\lambda^+ = (1 + \sqrt{c_n})^2$  and  $(T_n(x))_{ii} = t_n(x)$ . We will prove hereafter that:

$$\begin{aligned} & \frac{\kappa c_n}{\pi^2} \left( \int_{\lambda^-}^{\lambda^+} f'(x) \operatorname{Im}\{x t_n(x)\} dx \right) \left( \int_{\lambda^-}^{\lambda^+} g'(y) \operatorname{Im}\{y t_n(y)\} dy \right) \\ &= \frac{\kappa}{4c_n \pi^2} \left( \int_{\lambda^-}^{\lambda^+} f(x) \frac{x - (1 + c_n)}{\sqrt{(\lambda^+ - x)(x - \lambda^-)}} dx \right) \left( \int_{\lambda^-}^{\lambda^+} g(y) \frac{y - (1 + c_n)}{\sqrt{(\lambda^+ - y)(y - \lambda^-)}} dy \right) \end{aligned} \quad (4.8)$$

Notice that the l.h.s. of the equation above is the cumulant term as provided in (4.7) if  $R_n = I_N$  while the r.h.s. is the cumulant term as provided<sup>4</sup> in [42].

In the case where  $R_n = I_N$ , the Stieltjes transform of Marčenko-Pastur's distribution has an explicit form given by (see for instance [47, Chapter 7]):

$$t_n(z) = \frac{1}{2c_n z} \left\{ \sqrt{(z - (1 + c_n))^2 - 4c_n} - (z - (1 - c_n)) \right\}$$

where the branch of the square root is fixed by its asymptotics:  $z - (1 + c) + o(1)$  as  $z \rightarrow \infty$ . In particular, if  $x \in [\lambda^-, \lambda^+]$  then

$$\sqrt{(z - (1 + c))^2 - 4c} \Big|_{z=x+i0} = i \sqrt{(\lambda^+ - x)(x - \lambda^-)}.$$

Hence

$$\operatorname{Im}\{x t_n(x)\} = \frac{\sqrt{(\lambda^+ - x)(x - \lambda^-)}}{2c_n}.$$

It remains to perform an integration by parts to get

$$\begin{aligned} \int_{\lambda^-}^{\lambda^+} f'(x) \operatorname{Im}\{x t_n(x)\} dx &= - \int_{\lambda^-}^{\lambda^+} f'(x) \frac{\sqrt{(\lambda^+ - x)(x - \lambda^-)}}{2c_n} dx \\ &= \frac{1}{2c_n} \int_{\lambda^-}^{\lambda^+} f(x) \frac{(1 + c_n) - x}{\sqrt{(\lambda^+ - x)(x - \lambda^-)}} dx \end{aligned}$$

which yields (4.8).

As a corollary of Proposition 4.2, we obtain the following extension of Theorem 2.

Recall that  $\mathcal{S}_n$  is the support of the probability measure  $\mathcal{F}_n$ . Due to Assumption (A-2), it is clear that

$$\mathcal{S}_n \subset \mathcal{S}_\infty \triangleq \left[ 0, \lambda_R^+ \left( 1 + \sqrt{\ell^+} \right)^2 \right], \quad (4.9)$$

uniformly in  $n$ . Denote by  $\mathbf{h} \in C_c^\infty(\mathbb{R})$  a function whose value is 1 on a  $\eta$ -neighborhood  $\mathcal{S}_\infty^\eta$  of  $\mathcal{S}_\infty$ .

**Corollary 4.3.** *Assume that (A-1) and (A-2) hold true and let  $f_\ell \in C^3(\mathbb{R})$  with  $1 \leq \ell \leq k$ ; assume moreover that  $\mathcal{V} = \mathbb{E}X_{ij}^2$  is either equal to 0 or 1 and that  $R_n$  has real entries. Let  $\mathbf{h} \in C_c^\infty(\mathbb{R})$  be as above. Then (4.3)-(4.4) remain true with  $L_n^1(\mathbf{f})$  replaced by*

$$L_n^{1,\mathbf{h}}(\mathbf{f}) = \left( \operatorname{tr} f_\ell(\Sigma_n \Sigma_n^*) - \mathbb{E} \operatorname{tr} (f_\ell \mathbf{h})(\Sigma_n \Sigma_n^*); 1 \leq \ell \leq k \right)$$

and with the gaussian random vector  $Z_n^1(\mathbf{f}\mathbf{h})$  as in Theorem 2.

<sup>4</sup>Denote by the superscript  $^{\text{LP}}$  the quantities in [42] and use the correspondance  $c^{\text{LP}} \leftrightarrow 1/c$ ,  $a^{\text{LP}} \leftrightarrow c$  and  $\kappa_4^{\text{LP}} \leftrightarrow (a^{\text{LP}})^4 \kappa = c^2 \kappa$  to check that the r.h.s. of (4.8) equates the formula provided in [42].

Proof of Corollary 4.3 is postponed to Section 6.5.

#### 4.3. First-order expansions for the bias in the case of non-analytic functionals.

**Theorem 3.** *Assume (A-1) and (A-2) hold true and let  $f \in C_c^{18}(\mathbb{R})$ . Denote by*

$$Z_n^2(f) = \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_{17}(f)(z) \mathcal{B}_n(z) \ell_2(dz) , \quad (4.10)$$

where  $\mathcal{B}_n$  is defined in (2.19). Then

$$\mathbb{E} \operatorname{Tr}(f)(\Sigma_n \Sigma_n^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) - Z_n^2(f) \xrightarrow{N, n \rightarrow \infty} 0 .$$

Proof of Theorem 3 is postponed to Section 7.

*Remark 4.4* (Why eighteen?). A quick sketch of the proof of Theorem 3 provides some hints. Let  $f$  have a bounded support. By gaussian interpolation (whose cost is  $f \in C^8$ ), we only need to prove:

$$\mathbb{E} \operatorname{Tr} f(\Sigma_n^{\mathbb{C}}(\Sigma_n^{\mathbb{C}})^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) \rightarrow 0 ,$$

where  $\Sigma_n^{\mathbb{C}}$  is the counterpart of  $\Sigma_n$  with  $\mathcal{N}_{\mathbb{C}}(0, 1)$  i.i.d. entries. The proof of the latter is based on Helffer-Sjöstrand's formula:

$$\mathbb{E} \operatorname{Tr} f(\Sigma_n^{\mathbb{C}}(\Sigma_n^{\mathbb{C}})^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) = \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(f) \{ \operatorname{Tr} \mathbb{E} Q_n^{\mathbb{C}} - N t_n \} d\ell_2$$

where  $Q_n^{\mathbb{C}} = (\Sigma_n^{\mathbb{C}}(\Sigma_n^{\mathbb{C}})^* - z I_N)^{-1}$ , and on the following estimate, stated in Proposition 7.1:

$$| \mathbb{E} \operatorname{Tr} (\Sigma_n^{\mathbb{C}}(\Sigma_n^{\mathbb{C}})^* - z I_N)^{-1} - N t_n(z) | \leq \frac{1}{n} P_{12}(|z|) P_{17}(|\operatorname{Im}(z)|^{-1}) , \quad (4.11)$$

where  $P_k$  denotes a polynomial with degree  $k$  and positive coefficients. In view of Proposition 6.2,  $f$  needs to be of class  $C^{18}$ . If one can improve estimate (4.11) and decrease the powers of  $|\operatorname{Im}(z)|^{-1}$ , then one will automatically lower the regularity assumption over  $f$ . Notice that in the case of the Gaussian Unitary Ensemble, counterpart of (4.11) features  $|\operatorname{Im}(z)|^{-7}$  on its r.h.s. (cf. [27, Lemma 6.1]) hence the needed regularity is  $f \in C^8$  in this case.

**Proposition 4.4.** *Let  $Z_n^2(f)$  be defined as in (4.10), then  $Z_n^2$  is a distribution (in the sense of L. Schwartz) on  $C_c^{18}(\mathbb{R})$  and*

$$Z_n^2(f) = \frac{-i}{2\pi} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} f(x) \{ \mathcal{B}_n(x + i\varepsilon) - \mathcal{B}_n(x - i\varepsilon) \} dx . \quad (4.12)$$

Moreover, the singular points of  $\mathcal{B}_n(z)$  are included in  $\mathcal{S}_n$  and so is the support of  $Z_n^2$  (as a distribution). In particular, one can extend  $Z_n^2$  to  $C^{18}(\mathbb{R})$  by

$$\check{Z}_n^2(f) = Z_n^2(f\mathbf{h}) , \quad f \in C^{18}(\mathbb{R}) ,$$

where  $\check{Z}_n^2$  is the extension to  $C^{18}(\mathbb{R})$  and  $\mathbf{h} \in C_c^\infty(\mathbb{R})$  has value 1 on  $\mathcal{S}_n$ .

Proof of Proposition 4.4 is postponed to Section 7.2.

**Corollary 4.5.** *Assume (A-1) and (A-2) hold true. Let  $f \in C^{18}(\mathbb{R})$  and  $\mathbf{h} \in C_c^\infty(\mathbb{R})$  be a function whose value is 1 on a neighborhood of  $\mathcal{S}_\infty$ , then the following convergence holds true:*

$$\mathbb{E} \operatorname{Tr}(f\mathbf{h})(\Sigma_n \Sigma_n^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) - \check{Z}_n^2(f) \xrightarrow{N, n \rightarrow \infty} 0 .$$

The proof is straightforward and is therefore omitted.

## 5. PROOF OF THEOREM 1 (CLT FOR THE TRACE OF THE RESOLVENT)

Recall that  $M_n(z) = \text{tr} Q_n(z) - Nt_n(z)$ . It will be convenient to decompose  $M_n(z)$  as:

$$M_n(z) = M_n^1(z) + M_n^2(z) \quad \text{where} \quad \begin{cases} M_n^1(z) &= \text{tr} Q_n(z) - \text{tr} \mathbb{E} Q_n(z) \\ M_n^2(z) &= N(\mathbb{E} f_n(z) - t_n(z)) \end{cases} . \quad (5.1)$$

Denote by  $\xi_j$  the  $N \times 1$  vector

$$\xi_j = \Sigma_{\cdot j} = \frac{1}{\sqrt{n}} R^{1/2} X_{\cdot j}$$

and by  $\mathbb{E}_j$  the conditional expectation with respect to  $\mathcal{G}_j$ , the  $\sigma$ -field generated by  $\xi_1, \dots, \xi_j$ ; by convention,  $\mathbb{E}_0 = \mathbb{E}$ . We split Theorem 1 into intermediate results. Recall the definitions of  $D_\varepsilon, D^+$  and  $D$  in (3.7). Let

$$\Gamma = D^+ \cup \overline{D^+} \quad \text{where} \quad \overline{D^+} = \{\bar{z}, \quad z \in D^+\} .$$

**Proposition 5.1.** *Assume that (A-1) and (A-2) hold true; let  $z_1, z_2 \in \Gamma$ , then:*

$$M_n^1(z) = \sum_{j=1}^n Z_j^n(z) + o_P(1) ,$$

where the  $Z_j^n$ 's are martingale increments with respect to the  $\sigma$ -field  $\mathcal{G}_i$  and

$$\sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) Z_j^n(z_2) - \Theta_n(z_1, z_2) \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0 , \quad (5.2)$$

$$\sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) \overline{Z_j^n(z_2)} - \Theta_n(z_1, \bar{z}_2) \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0 , \quad (5.3)$$

where  $\Theta_n$  is defined in (2.9). Moreover,

$$M_n^2(z) - \mathcal{B}_n(z) \xrightarrow[N, n \rightarrow \infty]{} 0 ,$$

where  $\mathcal{B}_n$  is defined in (2.19).

**Proposition 5.2.** *There exists a sequence  $(G_n(z), z \in \Gamma)$  of two-dimensional Gaussian processes with mean  $\mathbb{E} G_n(z) = \mathcal{B}_n(z)$  and covariance*

$$\begin{aligned} \text{cov}(G_n(z_1), G_n(z_2)) &= \mathbb{E}(G_n(z_1) - \mathbb{E} G_n(z_1))(G_n(z_2) - \mathbb{E} G_n(z_2)) \\ &= \Theta_n(z_1, z_2) . \end{aligned}$$

Moreover,  $(G_n(z), z \in D_\varepsilon)$  is tight.

**5.1. Proof for Proposition 5.1.** The fact that  $(M_n)$  is a tight sequence has already been established in [5] (regardless of the assumption  $\kappa = 0$  and  $|\mathcal{V}| = 0/1$ ). In order to proceed, we shall heavily rely on the proof of [5, Lemma 1.1] which is the crux of Bai and Silverstein's paper. In Section 5.1.1 we recall the main steps of Bai and Silverstein's computations of the variance/covariance. In Sections 5.1.2 and 5.1.3, we compute the extra terms in the limiting variance. In Section 5.1.4, we compute the limiting bias (some details are postponed to Appendix A.1). In Section 5.3, we finally conclude the proof of Theorem 1 and address various subtleties which appear due to the existence of a sequence of Gaussian limiting processes.

In the sequel, we shall drop subscript  $n$  and write  $Q$  and  $R$  instead of  $Q_n$  and  $R_n$ . Denote by  $Q_j(z)$  the resolvent of matrix  $\Sigma\Sigma^* - \xi_j\xi_j^*$ , i.e.

$$Q_j(z) = (-zI + \Sigma\Sigma^* - \xi_j\xi_j^*)^{-1} .$$

The following quantities will be needed:

$$\begin{aligned} \beta_j(z) &= \frac{1}{1 + \xi_j^* Q_j(z) \xi_j} , \\ \bar{\beta}_j(z) &= \frac{1}{1 + \frac{1}{n} \text{tr} R_n Q_j(z)} , \\ b_n(z) &= \frac{1}{1 + \frac{1}{n} \mathbb{E} \text{tr} R_n Q_1(z)} , \\ \varepsilon_j(z) &= \xi_j^* Q_j(z) \xi_j - \frac{1}{n} \text{tr} R_n Q_j(z) , \\ \delta_j(z) &= \xi_j^* Q_j^2(z) \xi_j - \frac{1}{N} \text{tr} R_n Q_j^2(z) = \frac{d}{dz} \varepsilon_j(z) . \end{aligned}$$

5.1.1. *Preliminary variance computations.* We briefly review in this section the main steps related to the computation of the variance/covariance as presented in [5]. These standard steps will finally lead to Eq. (5.7) which will be the starting point of the computations associated to the  $|\mathcal{V}|^2$ - and  $\kappa$ -terms of the variance.

Let  $z \in \Gamma$ .

$$\begin{aligned} N(f_n(z) - \mathbb{E}f_n(z)) &= -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \beta_j(z) \xi_j^* Q_j^2(z) \xi_j \\ &= -\sum_{j=1}^n \mathbb{E}_j \left( \bar{\beta}_j(z) \delta_j(z) - \bar{\beta}_j^2(z) \varepsilon_j(z) \frac{1}{n} \text{tr} R Q_j^2(z) \right) + o_P(1) . \end{aligned}$$

Denote by

$$Z_j^n(z) = -\mathbb{E}_j \left( \bar{\beta}_j(z) \delta_j(z) - \bar{\beta}_j^2(z) \varepsilon_j(z) \frac{1}{n} \text{tr} R Q_j^2(z) \right) = -\mathbb{E}_j \frac{d}{dz} (\bar{\beta}_j(z) \varepsilon_j(z)) .$$

Hence,

$$M_n^1(z) = N(f_n(z) - \mathbb{E}f_n(z)) = \sum_{j=1}^n Z_j^n(z) + o_P(1)$$

The r.h.s. appears as a sum of martingale increments. Such a decomposition is important since it will enable us to rely on powerful CLTs for martingales (see [11, Theorem 35.12], and the variations below in Lemmas 5.6 and 5.7). These CLTs rely on the study of the terms:

$$\sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) Z_j^n(z_2) \quad \text{and} \quad \sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) \overline{Z_j^n(z_2)} .$$

Notice that since  $\overline{Z_j^n(z)} = Z_j^n(\bar{z})$ , we have  $\mathbb{E}_{j-1} Z_j^n(z_1) \overline{Z_j^n(z_2)} = \mathbb{E}_{j-1} Z_j^n(z_1) Z_j^n(\bar{z}_2)$ . Since the set  $\Gamma$  is stable by complex conjugation, it is sufficient to study the limiting behavior of:

$$\sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) Z_j^n(z_2) , \quad z_1, z_2 \in \Gamma$$

in order to prove (5.2) and (5.3). Now,

$$\sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) Z_j^n(z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \sum_{j=1}^n \mathbb{E}_{j-1} [\mathbb{E}_j (\bar{\beta}_j(z_1) \varepsilon_j(z_1)) \mathbb{E}_j (\bar{\beta}_j(z_2) \varepsilon_j(z_2))] \right\}. \quad (5.4)$$

Following the same arguments as in [5, pp. 571], one can prove that it is sufficient to study the convergence in probability of

$$\sum_{j=1}^n \mathbb{E}_{j-1} [\mathbb{E}_j (\bar{\beta}_j(z_1) \varepsilon_j(z_1)) \mathbb{E}_j (\bar{\beta}_j(z_2) \varepsilon_j(z_2))] .$$

Moreover,

$$\begin{aligned} & \sum_{j=1}^n \mathbb{E}_{j-1} [\mathbb{E}_j (\bar{\beta}_j(z_1) \varepsilon_j(z_1)) \mathbb{E}_j (\bar{\beta}_j(z_2) \varepsilon_j(z_2))] \\ &= \sum_{j=1}^n b_n(z_1) b_n(z_2) \mathbb{E}_{j-1} [\mathbb{E}_j \varepsilon_j(z_1) \mathbb{E}_j \varepsilon_j(z_2)] + o_P(1) , \\ &= \sum_{j=1}^n z_1 \tilde{t}_n(z_1) z_2 \tilde{t}_n(z_2) \mathbb{E}_{j-1} [\mathbb{E}_j \varepsilon_j(z_1) \mathbb{E}_j \varepsilon_j(z_2)] + o_P(1) . \end{aligned} \quad (5.5)$$

Hence, it is finally sufficient to study the limiting behaviour (in terms of convergence in probability) of the quantity:

$$\sum_{j=1}^n \mathbb{E}_{j-1} (\mathbb{E}_j \varepsilon_j(z_1) \mathbb{E}_j \varepsilon_j(z_2)) , \quad z_1, z_2 \in \Gamma . \quad (5.6)$$

Denote by  $A^T$  the transpose matrix of  $A$ . Applying (2.6) yields :

$$\begin{aligned} \sum_{j=1}^n \mathbb{E}_{j-1} (\mathbb{E}_j \varepsilon_j(z_1) \mathbb{E}_j \varepsilon_j(z_2)) &= \frac{1}{n^2} \sum_{j=1}^n \text{tr} \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R \mathbb{E}_j Q_j(z_2) R^{1/2} \right) \\ &\quad + \frac{|\mathcal{V}|^2}{n^2} \sum_{j=1}^n \text{tr} \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)^T \right) \\ &\quad + \frac{\kappa}{n^2} \sum_{j=1}^n \sum_{i=1}^N \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)_{ii} . \end{aligned} \quad (5.7)$$

The limiting behaviour of the first term of the r.h.s. has been completely described in [5] where it has been shown that:

$$\frac{\partial^2}{\partial z_1 \partial z_2} \left\{ z_1 z_2 \tilde{t}_n(z_1) \tilde{t}_n(z_2) \frac{1}{n^2} \sum_{j=1}^n \text{tr} \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R \mathbb{E}_j Q_j(z_2) R^{1/2} \right) \right\} = \Theta_{0,n}(z_1, z_2) + o_P(1) , \quad (5.8)$$

with  $\Theta_{0,n}(z_1, z_2)$  defined in (2.10).

We shall focus on the second and third terms.

5.1.2. *The term proportional to  $|\mathcal{V}|^2$  in the variance.* Notice first that the value of  $t_n$  and  $\tilde{t}_n$  is the same whether  $R$  is replaced by  $\bar{R}$  in (2.3) and (3.1) since  $t_n$  and  $\tilde{t}_n$  only depend on the spectrum of  $R$  (which is the same as the spectrum of  $\bar{R}$ ). Notice also that  $(R^{1/2})^T = \bar{R}^{1/2}$ , hence:

$$\left(R^{1/2}\mathbb{E}_j Q_j(z_2)R^{1/2}\right)^T = \bar{R}^{1/2}\mathbb{E}_j Q_j^T(z_2)\bar{R}^{1/2} .$$

Recall the definition of  $T_n^T(z)$  given by (2.8). Taking into account the fact that for a deterministic matrix  $A$ ,

$$\mathbb{E}\xi_j^T A \xi_j = \frac{\mathcal{V}}{n} \operatorname{tr}(\bar{R}^{1/2} A R^{1/2}) \quad \text{and} \quad \mathbb{E}\xi_j^* A \bar{\xi}_j = \frac{\bar{\mathcal{V}}}{n} \operatorname{tr}(R^{1/2} A \bar{R}^{1/2}) , \quad (5.9)$$

and following closely [5, Section 2], it is a matter of bookkeeping<sup>5</sup> to establish that:

$$\begin{aligned} & \frac{|\mathcal{V}|^2 z_1 z_2}{n^2} \tilde{t}_n(z_1) \tilde{t}_n(z_2) \sum_{j=1}^n \operatorname{tr} \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)^T \right) \\ &= |\mathcal{V}|^2 \mathcal{A}_n(z_1, z_2) \times \frac{1}{n} \sum_{j=1}^n \frac{1}{1 - \binom{j-1}{n} |\mathcal{V}|^2 \mathcal{A}_n(z_1, z_2)} + o_P(1) \\ &= \int_0^{|\mathcal{V}|^2 \mathcal{A}_n(z_1, z_2)} \frac{dz}{1-z} + o_P(1) \end{aligned} \quad (5.10)$$

where

$$\mathcal{A}_n(z_1, z_2) = \frac{z_1 z_2}{n} \tilde{t}_n(z_1) \tilde{t}_n(z_2) \operatorname{tr} \left\{ R^{1/2} T_n(z_1) R^{1/2} \bar{R}^{1/2} T_n^T(z_2) \bar{R}^{1/2} \right\} .$$

Finally,

$$\frac{\partial^2}{\partial z_1 \partial z_2} (5.10) = |\mathcal{V}|^2 \Theta_{1,n}(z_1, z_2) + o_P(1) = |\mathcal{V}|^2 \frac{\partial}{\partial z_2} \left\{ \frac{\partial \mathcal{A}_n(z_1, z_2) / \partial z_1}{1 - |\mathcal{V}|^2 \mathcal{A}_n(z_1, z_2)} \right\} + o_P(1) . \quad (5.11)$$

5.1.3. *The cumulant term in the variance.* We now handle the term proportional to  $\kappa$  in (5.7):

$$\frac{1}{n^2} \sum_{j=1}^n \sum_{i=1}^N \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)_{ii} . \quad (5.12)$$

The objective is to prove that  $\mathbb{E}_j Q_j(z)$  can be replaced by  $T_n(z)$  in the formula above, which boils down to prove a convergence of quadratic forms of the type (2.5). Such a convergence has already been established in [32] for large covariance matrices based on a non-centered matrix model with separable variance profile.

By interpolating between the quantity (5.12) and its counterpart when the entries are complex i.i.d. standard Gaussian, we will be able to rely on the results in [32] by using the unitary invariance of a Gaussian matrix (see Proposition 5.4 and Eq. (5.29) below).

Let  $\delta_z$  be the distance between the point  $z \in \mathbb{C}$  and the real nonnegative axis  $\mathbb{R}^+$ :

$$\delta_z = \operatorname{dist}(z, \mathbb{R}^+) . \quad (5.13)$$

---

<sup>5</sup>Similar computations for the term proportional to  $|\mathcal{V}|^2$  in the bias are outlined in Appendix A.1.

**Proposition 5.3.** *Assume that (A-1) and (A-2) hold true and let  $u_n$  be a deterministic  $N \times 1$  vector, then:*

$$\mathbb{E} |u_n^* Q(z) u_n - u_n^* \mathbb{E} Q(z) u_n|^2 \leq \frac{1}{n} \Phi(|z|) \Psi \left( \frac{1}{\delta_z} \right) \|u_n\|^2 ,$$

where  $\Phi$  and  $\Psi$  are fixed polynomials with coefficients independent from  $N, n, z$  and  $(u_n)$ .

Proof of Proposition 5.3 is an easy adaptation<sup>6</sup> of [32, Prop. 2.7] and is therefore omitted.

Denote by  $X_n^{\mathbb{C}}$  a  $N \times n$  matrix whose entries are independent standard complex circular Gaussian r.v. (i.e.  $X_{ij}^{\mathbb{C}} = U + iV$  where  $U, V$  are independent  $\mathcal{N}(0, 2^{-1})$  random variables); denote accordingly  $\Sigma_n^{\mathbb{C}} = n^{-1/2} R^{1/2} X_n^{\mathbb{C}}$ ,  $\xi_j^{\mathbb{C}} = (\Sigma_n^{\mathbb{C}})^* \cdot j$  and

$$Q_n^{\mathbb{C}}(z) = \left( -zI_N + \Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^* \right)^{-1} .$$

We now drop subscripts  $N$  and  $n$ .

**Proposition 5.4.** *Assume that (A-1) and (A-2) hold true and let , then:*

$$|u_n^* \mathbb{E} Q(z) u_n - u_n^* \mathbb{E} Q^{\mathbb{C}}(z) u_n| \leq \frac{1}{\sqrt{n}} \Phi(|z|) \Psi \left( \frac{1}{\delta_z} \right) \|u_n\|^2 , \quad (5.14)$$

where  $u_n$  is a deterministic  $N \times 1$  vector and  $\Phi$  and  $\Psi$  are fixed polynomials with coefficients independent from  $N, n, z$ . Moreover,

$$|\mathbb{E} \text{Tr} Q(z) - \mathbb{E} \text{Tr} Q^{\mathbb{C}}(z)| \leq K \frac{|z|^3}{\text{Im}(z)^7} , \quad (5.15)$$

where  $K$  is independent from  $N, n, z$ .

Notice that (5.14) is of direct use in this section while (5.15) will be used in Section 7.

*Proof.* We first prove (5.14). Consider the resolvent

$$Q^{(i)}(z) = \left( \sum_{\ell=1}^i \xi_{\ell}^{\mathbb{C}} \xi_{\ell}^{\mathbb{C}*} + \sum_{\ell=i+1}^n \xi_{\ell} \xi_{\ell}^* - zI_N \right)^{-1}$$

defined for  $1 \leq i \leq n-1$ . Denote by  $Q^{(0)} = Q$  and by  $Q^{(n)} = Q^{\mathbb{C}}$  and write

$$u^* \mathbb{E}(Q - Q^{\mathbb{C}}) u = \sum_{i=1}^n u^* \mathbb{E}(Q^{(i-1)} - Q^{(i)}) u . \quad (5.16)$$

We shall evaluate the difference  $u^* \mathbb{E}(Q^{(0)} - Q^{(1)}) u$ , the other ones being handled similarly. Denote by  $\check{Q}(z) = (\sum_{i=2}^n \xi_i \xi_i^* - zI_N)^{-1}$ , then:

$$Q^{(0)} = \check{Q} - \frac{\check{Q} \xi_1 \xi_1^* \check{Q}}{1 + \xi_1^* \check{Q} \xi_1} \quad \text{and} \quad Q^{(1)} = \check{Q} - \frac{\check{Q} \xi_1^{\mathbb{C}} \xi_1^{\mathbb{C}*} \check{Q}}{1 + \xi_1^{\mathbb{C}*} \check{Q} \xi_1^{\mathbb{C}}} . \quad (5.17)$$

<sup>6</sup>Notice in particular all the cancellations that appear when adapting the proof of [32, Prop. 2.7], due to the fact that  $\Sigma_n$  is centered here; notice also the fact that  $R$  not being diagonal has virtually no impact.



Dropping the subscript 1 to lighten the notations, we get:

$$\begin{aligned}
u^* \mathbb{E} \left( Q^{(0)} - Q^{(1)} \right) u &= u^* \mathbb{E} \left( \frac{\check{Q} \xi^{\mathbb{C}} \xi^{\mathbb{C}*} \check{Q}}{1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}}} - \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \xi^* \check{Q} \xi} \right) u \\
&= u^* \mathbb{E} \left( \frac{\check{Q} \xi^{\mathbb{C}} \xi^{\mathbb{C}*} \check{Q}}{1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}}} - \frac{\check{Q} \xi^{\mathbb{C}} \xi^{\mathbb{C}*} \check{Q}}{1 + \frac{1}{n} \text{tr} R \check{Q}} \right) u \\
&\quad + u^* \mathbb{E} \left( \frac{\check{Q} \xi^{\mathbb{C}} \xi^{\mathbb{C}*} \check{Q}}{1 + \frac{1}{n} \text{tr} R \check{Q}} - \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \frac{1}{n} \text{tr} R \check{Q}} \right) u \\
&\quad + u^* \mathbb{E} \left( \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \frac{1}{n} \text{tr} R \check{Q}} - \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \xi^* \check{Q} \xi} \right) u.
\end{aligned}$$

The second term in the r.h.s. above is zero (simply compute the conditional expectation with respect to  $\check{Q}$ ), the first and third term are of a similar nature; we therefore only estimate the third one denoted by  $\Delta_3$  below.

$$\begin{aligned}
|\Delta_3| &= \left| u^* \mathbb{E} \left( \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \frac{1}{n} \text{tr} R \check{Q}} - \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \xi^* \check{Q} \xi} \right) u \right| \\
&= \left| \mathbb{E} \frac{\xi^* \check{Q} u u^* \check{Q} \xi}{(1 + \xi^* \check{Q} \xi)(1 + \frac{1}{n} \text{tr} R \check{Q})} \left( \xi^* \check{Q} \xi - \frac{1}{n} \text{tr} R \check{Q} \right) \right| \\
&\leq \frac{|z|^2}{|\text{Im}(z)|^2} \left\{ \mathbb{E} \left| \xi^* \check{Q} \xi - \frac{1}{n} \text{tr} R \check{Q} \right|^2 \mathbb{E} |\xi^* \check{Q} u u^* \check{Q} \xi|^2 \right\}^{1/2}, \quad (5.18)
\end{aligned}$$

where the last inequality follows from Cauchy-Schwarz inequality plus the fact that both  $(-z(1 + \xi^* \check{Q} \xi))^{-1}$  and  $(-z(1 + n^{-1} \text{tr} R \check{Q}))^{-1}$  are Stieltjes transforms and hence upper-bounded in modulus by  $|\text{Im}(z)|^{-1}$ . A control for the first expectation in the above inequality directly follows from classical estimates (see for instance [7, Lemma B.26]):

$$\mathbb{E} \left| \xi^* \check{Q} \xi - \frac{1}{n} \text{tr} R \check{Q} \right|^2 \leq \frac{K}{n^2} \mathbb{E} |X_{11}|^4 \mathbb{E} (\text{tr} R \check{Q} R \check{Q}^*) \leq \frac{K}{n} \frac{\|R\|^2}{|\text{Im}(z)|^2} c_n \mathbb{E} |X_{11}|^4, \quad (5.19)$$

where  $K$  is a constant whose value may change from line to line but which remains independent from  $N, n$ . The second expectation can be handled in the following way:

$$\begin{aligned}
\mathbb{E} |\xi^* \check{Q} u u^* \check{Q} \xi|^2 &= \mathbb{E} \left| \xi^* \check{Q} u u^* \check{Q} \xi - \frac{1}{n} \text{tr} R \check{Q} u u^* \check{Q} + \frac{1}{n} \text{tr} R \check{Q} u u^* \check{Q} \right|^2 \\
&\leq 2 \mathbb{E} \left| \xi^* \check{Q} u u^* \check{Q} \xi - \frac{1}{n} \text{tr} R \check{Q} u u^* \check{Q} \right|^2 + 2 \mathbb{E} \left| \frac{1}{n} \text{tr} R \check{Q} u u^* \check{Q} \right|^2 \\
&\leq \frac{K}{n^2} \mathbb{E} |X_{11}|^4 \mathbb{E} \text{tr} (R^{1/2} \check{Q} u u^* \check{Q} R^{1/2}) (R^{1/2} \check{Q} u u^* \check{Q} R^{1/2})^* \\
&\quad + \frac{2}{n^2} \mathbb{E} |u^* \check{Q} R \check{Q} u|^2 \\
&\leq \frac{K}{n^2} \frac{\|R\|^2 \|u\|^4}{|\text{Im}(z)|^4}. \quad (5.20)
\end{aligned}$$

It now remains to gather (5.19) and (5.20) to get:

$$\left| u^* \mathbb{E} \left( Q^{(0)} - Q^{(1)} \right) u \right| \leq \frac{1}{n\sqrt{n}} \Phi(|z|) \Psi \left( \frac{1}{\delta_z} \right) \|u\|^2.$$

Finally, the result follows by upper-bounding each term of the sum in (5.16) and (5.14) is proved.

We now establish (5.15). Using a similar decomposition as in (5.16), we get:

$$\mathbb{E} \text{Tr} (Q - Q^{\mathbb{C}}) = \sum_{i=1}^n \mathbb{E} \text{Tr} (Q^{(i-1)} - Q^{(i)}).$$

We focus on the first term, use (5.17) and follow a similar notational convention by dropping subscript 1.

$$\begin{aligned} \mathbb{E} \text{Tr} \left( Q^{(0)} - Q^{(1)} \right) &= \mathbb{E} \text{Tr} \left( \frac{\check{Q} \xi^{\mathbb{C}} \xi^{\mathbb{C}*} \check{Q}}{1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}}} - \frac{\check{Q} \xi \xi^* \check{Q}}{1 + \xi^* \check{Q} \xi} \right) = \mathbb{E} \left( \frac{\xi^{\mathbb{C}*} \check{Q}^2 \xi^{\mathbb{C}}}{1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}}} - \frac{\xi^* \check{Q}^2 \xi}{1 + \xi^* \check{Q} \xi} \right) \\ &= \mathbb{E} \left( \frac{\xi^{\mathbb{C}*} \check{Q}^2 \xi^{\mathbb{C}} \xi^* \check{Q} \xi - \xi^* \check{Q}^2 \xi \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}}}{(1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}})(1 + \xi^* \check{Q} \xi)} \right) + \mathbb{E} \left( \frac{\xi^{\mathbb{C}*} \check{Q}^2 \xi^{\mathbb{C}} - \xi^* \check{Q}^2 \xi}{(1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}})(1 + \xi^* \check{Q} \xi)} \right). \end{aligned} \quad (5.21)$$

Denote by

$$\begin{aligned} A_1 &= (\xi^{\mathbb{C}*} \check{Q}^2 \xi^{\mathbb{C}})(\xi^* \check{Q} \xi) - (\xi^* \check{Q}^2 \xi)(\xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}}), \\ A_2 &= \xi^{\mathbb{C}*} \check{Q}^2 \xi^{\mathbb{C}} - \xi^* \check{Q}^2 \xi, \\ B_1 &= \frac{1}{(1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}})(1 + \xi^* \check{Q} \xi)} - \frac{1}{(1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}})(1 + \frac{1}{n} \text{tr} R \check{Q})}, \\ B_2 &= \frac{1}{(1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}})(1 + \frac{1}{n} \text{tr} R \check{Q})} - \frac{1}{(1 + \frac{1}{n} \text{tr} R \check{Q})^2}, \\ B_3 &= \frac{1}{(1 + \frac{1}{n} \text{tr} R \check{Q})^2}. \end{aligned}$$

With these notations at hand, we have:

$$B_1 + B_2 + B_3 = \frac{1}{(1 + \xi^{\mathbb{C}*} \check{Q} \xi^{\mathbb{C}})(1 + \xi^* \check{Q} \xi)}$$

and

$$\mathbb{E} \text{Tr} \left( Q^{(0)} - Q^{(1)} \right) = \mathbb{E} A_1 (B_1 + B_2 + B_3) + \mathbb{E} A_2 (B_1 + B_2 + B_3).$$

Notice that  $\mathbb{E} A_1 B_3 = \mathbb{E} A_2 B_3 = 0$ . By Cauchy-Schwarz inequality, Proof of Prop. 5.4 will be completed as long as we establish the following estimates:

$$\{\mathbb{E} |A_1|^2\}^{1/2} \leq \frac{K}{\sqrt{n}} \frac{1}{\text{Im}(z)^3}, \quad (5.22)$$

$$\{\mathbb{E} |A_2|^2\}^{1/2} \leq \frac{K}{\sqrt{n}} \frac{1}{\text{Im}(z)^2}, \quad (5.23)$$

$$\{\mathbb{E} |B_1|^2\}^{1/2} \leq \frac{K}{\sqrt{n}} \frac{|z|^3}{\text{Im}(z)^4}, \quad (5.24)$$

$$\{\mathbb{E} |B_2|^2\}^{1/2} \leq \frac{K}{\sqrt{n}} \frac{|z|^3}{\text{Im}(z)^4}. \quad (5.25)$$

Estimates (5.24)-(5.25) can be established as (5.18). In order to establish (5.22), we compute exactly the expectation  $\mathbb{E}|A_1|^2$  writing

$$\begin{aligned} & \mathbb{E} \left| \xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}} \xi^* \check{Q} \xi - \xi^* \check{Q}^2 \xi \xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}} \right|^2 \\ &= \mathbb{E} \left( \xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}} \xi^* \check{Q} \xi - \xi^* \check{Q}^2 \xi \xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}} \right) \overline{\left( \xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}} \xi^* \check{Q} \xi - \xi^* \check{Q}^2 \xi \xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}} \right)}, \end{aligned} \quad (5.26)$$

which splits into 4 terms:

$$\begin{aligned} (5.26) &= \mathbb{E} \left\{ \left| \xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}} \right|^2 \left| \xi^* \check{Q} \xi \right|^2 \right\} - \mathbb{E} \left\{ (\xi^* \check{Q}^2 \xi) \overline{(\xi^* \check{Q} \xi)} (\xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}}) \overline{(\xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}})} \right\} \\ &+ \mathbb{E} \left\{ \left| \xi^* \check{Q}^2 \xi \right|^2 \left| \xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}} \right|^2 \right\} - \mathbb{E} \left\{ (\xi^* \check{Q}^2 \xi) (\xi^* \check{Q} \xi) \overline{(\xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}})} (\xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}}) \right\}. \end{aligned}$$

Using the independence of  $\xi$ ,  $\xi^{\mathbb{C}}$  and  $\check{Q}$  together with formula (2.6), lengthy but straightforward computations yield the estimate

$$\mathbb{E} \left| \xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}} \xi^* \check{Q} \xi - \xi^* \check{Q}^2 \xi \xi^{\mathbb{C}^*} \check{Q} \xi^{\mathbb{C}} \right|^2 \leq \frac{K}{n \operatorname{Im}(z)^6}.$$

Similar computations yield

$$\mathbb{E} \left| \xi^{\mathbb{C}^*} \check{Q}^2 \xi^{\mathbb{C}} - \xi^* \check{Q}^2 \xi \right|^2 \leq \frac{K}{n \operatorname{Im}(z)^4}, \quad (5.27)$$

and (5.22)-(5.23) are established. Estimate (5.15) is established and proof of Prop. 5.4 is completed.  $\square$

**Corollary 5.5.** *Assume that (A-1) and (A-2) hold true, then the following convergence holds true:*

$$\begin{aligned} & \frac{1}{n^2} \sum_{j=1}^n \sum_{i=1}^N \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)_{ii} \\ & - \frac{1}{n} \sum_{i=1}^N \left( R^{1/2} T(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} T(z_2) R^{1/2} \right)_{ii} \xrightarrow[n, N \rightarrow \infty]{\mathcal{P}} 0. \end{aligned}$$

*Proof.* We first transform the sum to be calculated:

$$\frac{1}{n^2} \sum_{j=1}^n \sum_{i=1}^N \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)_{ii}. \quad (5.28)$$

Using Proposition 5.3 enables us to replace the conditional expectation  $\mathbb{E}_i$  by the true expectation in every term  $(R^{1/2} \mathbb{E}_j Q_j(z) R^{1/2})_{ii}$ . Now using the fact that

$$Q = Q_j - \frac{Q_j \xi_j \xi_j^* Q_j}{1 + \xi_j^* Q_j \xi_j}$$

and computations similar to those made in Proposition 5.4, one can replace  $\mathbb{E} Q_j$  by  $\mathbb{E} Q$ . Finally, by Proposition 5.4,  $\mathbb{E} Q$  can be replaced by  $\mathbb{E} Q^{\mathbb{C}}$ . We are led to study the sum:

$$\frac{1}{n} \sum_{i=1}^N \left( R^{1/2} \mathbb{E} Q^{\mathbb{C}}(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} \mathbb{E} Q^{\mathbb{C}}(z_2) R^{1/2} \right)_{ii}.$$

Denote by  $R_n = U_n \Delta U_n^*$  the spectral decomposition of covariance matrix  $R_n$ . Since matrix  $U_n$  is unitary, then  $Y_n = U_n^* X_n^{\mathbb{C}}$  has i.i.d. standard complex Gaussian entries and the resolvent writes:

$$\begin{aligned} Q^{\mathbb{C}}(z) &= \left( R_n^{1/2} X_n^{\mathbb{C}} (X_n^{\mathbb{C}})^* R_n^{1/2} - z I_N \right)^{-1} = U_n \left( \Delta^{1/2} Y_n Y_n^* \Delta^{1/2} - z I_N \right)^{-1} U_n^* \\ &\triangleq U_n Q_{\Delta}(z) U_n^* . \end{aligned} \quad (5.29)$$

Denote by  $T_{\Delta}(z)$  the matrix

$$T_{\Delta}(z) = (-z I_N + (1 - c_n) \Delta - z c_n t_n(z) \Delta)^{-1} ,$$

where  $t_n(z)$  is defined in (2.3); notice that the definition of  $t_n(z)$  only depends on the spectrum of  $R_n$  (or equivalently  $\Delta$ ); notice also that

$$T_n(z) = U_n T_{\Delta}(z) U_n^* . \quad (5.30)$$

It has been proved in [32, Theorem 1.1] that for every deterministic  $N \times 1$  vector  $v_n$ :

$$\mathbb{E} |v_n^* (Q_{\Delta}(z) - T_{\Delta}(z)) v_n|^2 \leq \frac{1}{n} \Phi_2(|z|) \Psi_2 \left( \frac{1}{\delta_z} \right) \|v_n\|^4 .$$

Hence,

$$\begin{aligned} |v_n^* \mathbb{E} Q_{\Delta}(z) v_n - v_n^* T_{\Delta}(z) v_n| &\leq \left( \mathbb{E} |v_n^* (Q_{\Delta}(z) - T_{\Delta}(z)) v_n|^2 \right)^{1/2} \\ &\leq \frac{\|v_n\|^2}{\sqrt{n}} \sqrt{\Phi_2(|z|) \Psi_2(\delta_z^{-1})} \leq \frac{\|v_n\|^2}{\sqrt{n}} \left( \frac{1 + \Phi_2(|z|)}{2} \right) \left( \frac{1 + \Psi_2(\delta_z^{-1})}{2} \right) . \end{aligned}$$

In particular, let  $e_i$  be the  $i^{\text{th}}$  coordinate vector, then

$$\begin{aligned} &\left| \left( R^{1/2} \mathbb{E} Q(z) R^{1/2} \right)_{ii} - \left( R^{1/2} T(z) R^{1/2} \right)_{ii} \right| \\ &= \left| \left( R^{1/2} U \mathbb{E} Q_{\Delta}(z) U^* R^{1/2} \right)_{ii} - \left( R^{1/2} U T_{\Delta}(z) U^* R^{1/2} \right)_{ii} \right| \\ &\leq \frac{\|R^{1/2} U^* e_i\|^2}{\sqrt{n}} \left( \frac{1 + \Phi_2(|z|)}{2} \right) \left( \frac{1 + \Psi_2(\delta_z^{-1})}{2} \right) , \end{aligned}$$

which completes the proof.  $\square$

Combining the result in Corollary 5.5 together with (5.5) and (5.7), we have proved so far that:

$$\begin{aligned} &\frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \frac{z_1 z_2 \tilde{t}_n(z_1) \tilde{t}_n(z_2)}{n^2} \sum_{j=1}^n \sum_{i=1}^N \left( R^{1/2} \mathbb{E}_j Q_j(z_1) R^{1/2} \right)_{ii} \left( R^{1/2} \mathbb{E}_j Q_j(z_2) R^{1/2} \right)_{ii} \right\} \\ &= \frac{1}{n} \sum_{i=1}^N \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ z_1 z_2 \tilde{t}_n(z_1) \tilde{t}_n(z_2) \left( R_n^{1/2} T_n(z_1) R_n^{1/2} \right)_{ii} \left( R_n^{1/2} T_n(z_2) R_n^{1/2} \right)_{ii} \right\} + o_P(1) . \end{aligned} \quad (5.31)$$

Taking into account (3.1) and the matrix identity  $U(I + VU)^{-1}V = 1 - (I + UV)^{-1}$ , we obtain:

$$\begin{aligned}
(5.31) &= \frac{1}{n} \sum_{i=1}^N \frac{\partial^2}{\partial z_1 \partial z_2} (I_N - (I_N + \tilde{t}_n(z_1)R_n)^{-1})_{ii} (I_N - (I_N + \tilde{t}_n(z_2)R_n)^{-1})_{ii} + o_P(1), \\
&= \frac{1}{n} \sum_{i=1}^N \frac{\partial}{\partial z_1} [z_1 T_n(z_1)]_{ii} \frac{\partial}{\partial z_2} [z_2 T_n(z_2)]_{ii} + o_P(1), \\
&= \frac{z_1^2 z_2^2 \tilde{t}'_n(z_1) \tilde{t}'_n(z_2)}{n} \sum_{i=1}^N \left( R_n^{1/2} T_n^2(z_1) R_n^{1/2} \right)_{ii} \left( R_n^{1/2} T_n^2(z_2) R_n^{1/2} \right)_{ii} + o_P(1), \\
&= \Theta_{2,n}(z_1, z_2) + o_P(1), \tag{5.32}
\end{aligned}$$

where  $\Theta_{2,n}$  is given by formula (2.12).

Now gathering (5.8), (5.11) and (5.32), we have established so far:

$$\sum_{j=1}^n \mathbb{E}_{j-1} Z_j^n(z_1) Z_j^n(z_2) = \Theta_n(z_1, z_2) + o_P(1)$$

which is the first part of Proposition 5.1.

5.1.4. *Computations for the bias.* In this section, we are interested in the computation of  $N(\mathbb{E}f_n(z) - t_n(z))$ . As

$$\tilde{f}_n(z) = -\frac{(1-c_n)}{z} + c_n f_n(z) \quad \text{and} \quad \tilde{t}_n(z) = -\frac{(1-c_n)}{z} + c_n t_n(z),$$

we immediately obtain  $N(\mathbb{E}f_n(z) - t_n(z)) = n(\mathbb{E}\tilde{f}_n(z) - \tilde{t}_n(z))$ . Combining (2.7) and (3.1) yields:

$$-z - \frac{1}{\tilde{t}_n(z)} + \frac{1}{n} \text{tr} R_n (I_N + \tilde{t}_n(z)R_n)^{-1} = 0. \tag{5.33}$$

Following Bai and Silverstein [5, Section 4], we introduce the quantity  $A_n(z)$  defined as:

$$\begin{aligned}
A_n(z) &= z \mathbb{E}\tilde{f}_n(z) + 1 + \frac{1}{n} \text{tr} \left( I_N + \mathbb{E}\tilde{f}_n(z)R_n \right)^{-1} - c_n \\
&= z \mathbb{E}\tilde{f}_n(z) + 1 + \frac{1}{n} \text{tr} \left( I_N + \mathbb{E}\tilde{f}_n(z)R_n \right)^{-1} - \frac{1}{n} \text{tr} I_N^{-1} \\
&= -\mathbb{E}\tilde{f}_n(z) \left( -z - \frac{1}{\mathbb{E}\tilde{f}_n(z)} + \frac{1}{n} \text{tr} R_n (I_N + \mathbb{E}\tilde{f}_n(z)R_n)^{-1} \right),
\end{aligned}$$

hence

$$-\frac{A_n(z)}{\mathbb{E}\tilde{f}_n(z)} = -z - \frac{1}{\mathbb{E}\tilde{f}_n(z)} + \frac{1}{n} \text{tr} R_n (I_N + \mathbb{E}\tilde{f}_n(z)R_n)^{-1}. \tag{5.34}$$

Subtracting (5.33) to (5.34) finally yields:

$$\mathbb{E}\tilde{f}_n(z) - \tilde{t}_n(z) = -A_n(z) \tilde{t}_n(z) \left[ 1 - \frac{\tilde{t}_n(z) \mathbb{E}\tilde{f}_n(z)}{n} \text{tr} R_n^2 \left( I_N + \mathbb{E}\tilde{f}_n(z)R_n \right)^{-1} \left( I_N + \tilde{t}_n(z)R_n \right)^{-1} \right]^{-1},$$

which is the counterpart of [5, Eq. (4.12)]. The same arguments as in [5] now yields:

$$n \left( \mathbb{E}\tilde{f}_n(z) - \tilde{t}_n(z) \right) = -n A_n(z) \tilde{t}_n(z) \left[ 1 - \frac{\tilde{t}_n^2(z)}{n} \text{tr} R_n^2 \left( I_N + \tilde{t}_n(z)R_n \right)^{-2} \right]^{-1} + o(1). \tag{5.35}$$

It remains to study the behaviour of  $nA_n(z)$ . Following [5, Eq. (4.10)], we obtain:

$$\begin{aligned} nA_n(z) &= \\ &= \frac{b_n^2}{n} \mathbb{E} \text{tr} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n Q_1 R_n - b_n^2 n \mathbb{E} \left[ \left( \xi_1^* Q_1 \xi_1 - \frac{1}{n} \text{tr} Q_1 R_n \right) \right. \\ &\quad \left. \times \left( \xi_1^* Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} \xi_1 - \frac{1}{n} \text{tr} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n \right) \right] + o(1). \end{aligned}$$

Applying (2.6) to the right term to the r.h.s. of the previous equation (recall that  $R^T = \bar{R}$ ), we obtain:

$$\begin{aligned} nA_n(z) &= -|\mathcal{V}|^2 \frac{b_n^2}{n} \mathbb{E} \text{tr} R_n^{1/2} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n^{1/2} \bar{R}_n^{1/2} Q_1^T \bar{R}_n^{1/2} \\ &\quad - \kappa \frac{b_n^2}{n} \sum_{i=1}^N \left( R_n^{1/2} Q_1 R_n^{1/2} \right)_{ii} \left( R_n^{1/2} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n^{1/2} \right)_{ii} + o(1). \end{aligned} \quad (5.36)$$

The first term of the r.h.s. has been fully analyzed in [5] in the case where  $R_n$  and  $X_n$  are real matrices. We adapt these computations to the general case and outline in Appendix A.1 the proof of the identity:

$$\begin{aligned} &- |\mathcal{V}|^2 \frac{b_n^2}{n} \mathbb{E} \text{tr} R_n^{1/2} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n^{1/2} \bar{R}_n^{1/2} Q_1^T \bar{R}_n^{1/2} \\ &= |\mathcal{V}|^2 \frac{\frac{z^3 \bar{t}_n^2}{n} \text{tr} R_n^{1/2} T_n^2(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}}{1 - \frac{|\mathcal{V}|^2 z^2 \bar{t}_n^2}{n} \text{tr} R_n^{1/2} T_n(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}} + o(1), \end{aligned} \quad (5.37)$$

where  $T_n^T(z)$  is defined in (2.8). The term proportional to the cumulant in (5.36) can be analyzed as in Section 5.1.3, and one can prove that:

$$\begin{aligned} &- \kappa \frac{b_n^2}{n} \sum_{i=1}^N \left( R_n^{1/2} Q_1 R_n^{1/2} \right)_{ii} \left( R_n^{1/2} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n^{1/2} \right)_{ii} \\ &= -\kappa \frac{z^2 \bar{t}_n^2}{n} \sum_{i=1}^N \left( R_n^{1/2} T_n R_n^{1/2} \right)_{ii} \left( R_n^{1/2} T_n \left( \tilde{t}_n R_n + I_N \right)^{-1} R_n^{1/2} \right)_{ii} + o(1). \end{aligned} \quad (5.38)$$

We now plug (5.37) and (5.38) into (5.35) to conclude.

$$\begin{aligned} n \left( \mathbb{E} \tilde{f}_n(z) - \tilde{t}_n(z) \right) &= -|\mathcal{V}|^2 \frac{z^3 \bar{t}_n^3}{n} \frac{\text{tr} R_n^{1/2} T_n^2(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}}{\left( 1 - \frac{|\mathcal{V}|^2 z^2 \bar{t}_n^2}{n} \text{tr} R_n^{1/2} T_n(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2} \right) \left( 1 - \frac{\bar{t}_n^2}{n} \text{tr} R_n^2 T_n^2 \right)} \\ &\quad - \kappa \frac{z^3 \bar{t}_n^3}{n} \sum_{i=1}^N \frac{\left( R_n^{1/2} T_n R_n^{1/2} \right)_{ii} \left( R_n^{1/2} T_n^2 R_n^{1/2} \right)_{ii}}{1 - \frac{z^2 \bar{t}_n^2}{n} \text{tr} R_n^2 T_n^2} + o(1). \end{aligned}$$

Proof of Proposition 5.1 is completed.

**5.2. Proof of Proposition 5.2.** Recall the meta-model introduced in Section 2.6.

5.2.1. *The Gaussian process  $G_n$ .* Let

$$M_{n,M}(z) = \text{tr}(\Sigma_n(M)\Sigma_n(M)^* - zI_{NM})^{-1} - MNt_n(z) .$$

Applying Proposition 5.1 to the matrix model  $\Sigma_n(M)\Sigma_n(M)^*$  yields:

$$\forall z \in \Gamma , \quad M_{n,M}^1(z) = \sum_{j=1}^{nM} Z_j^M(z) + o_P(1) ,$$

where the  $Z_j^M$ 's are martingale increments and

$$\begin{aligned} \sum_{j=1}^{nM} \mathbb{E}_{j-1} Z_j^M(z_1) Z_j^M(z_2) &\xrightarrow[N, n \text{ fixed}, M \rightarrow \infty]{\mathcal{P}} \Theta_n(z_1, z_2) , \\ M_{n,M}^2(z) &\xrightarrow[N, n \text{ fixed}, M \rightarrow \infty]{} \mathcal{B}_n(z) . \end{aligned}$$

Notice that there is a genuine limit in the previous convergence. Applying the central limit theorem for martingales [11, Theorem 35.12] plus the tightness argument for  $(M_{n,M}(z), z \in \Gamma)$  provided by Proposition 5.1 immediately yields the fact that  $M_{n,M}$  converges in distribution to a Gaussian process  $(G_n(z), z \in \Gamma)$  with mean  $\mathcal{B}_n(z)$  and covariance function  $\Theta_n(z_1, z_2)$ .

5.2.2. *Tightness of the sequence of Gaussian processes  $(G_n)$ .* In order to prove that the sequence of Gaussian processes  $(G_n)$  is tight, we shall prove, according to Prohorov's theorem, that it is relatively compact in distribution. Consider the set of matrices:

$$\{(R_n(M), M \geq 1); R_n \text{ is a } N \times n \text{ matrix, } N = N(n); n \geq 1\}$$

where  $R_n(M)$  is defined in (2.23). Since  $\|R_n(M)\| = \|R_n\|$  for all  $M \geq 1$ , we have

$$\sup_{M \geq 1, N, n \rightarrow \infty} \|R_n(M)\| = \sup_{N, n \rightarrow \infty} \|R_n\| < \infty$$

by Assumption (A-2). Hence, by Proposition 5.1, the family  $\{M_{n,M}; M \geq 1\}_{N, n \rightarrow \infty}$  is tight, hence relatively compact in distribution. As the distribution  $\mathcal{L}(G_n)$  of the Gaussian process  $G_n$  is the limit (in  $M$ ) of the distribution  $\mathcal{L}(M_{n,M})$  of  $M_{n,M}$ ,  $\mathcal{L}(G_n)$  belongs to the closure of  $\{\mathcal{L}(M_{n,M})\}$ , which is compact. Finally,  $\{\mathcal{L}(G_n)\}$  is included in a compact set, hence is relatively compact. In particular, the family of Gaussian processes  $(G_n)$  is tight.

5.3. **Proof of Theorem 1.** The two propositions below are minor variations of known results. They will be helpful to conclude the proof of Theorem 1.

**Lemma 5.6** (CLT for martingales I). *Suppose that for each  $n$   $Y_{n1}, Y_{n2}, \dots, Y_{nr_n}$  is a real martingale difference sequence with respect to the increasing  $\sigma$ -field  $\{\mathcal{G}_{n,j}\}$  having second moments. Assume moreover that  $(\Theta_n^2)$  is a sequence of nonnegative real numbers, uniformly bounded. If*

$$\sum_{j=1}^{r_n} \mathbb{E}(Y_{nj}^2 | \mathcal{G}_{n,j-1}) - \Theta_n^2 \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0 ,$$

and for each  $\varepsilon > 0$ ,

$$\sum_{j=1}^{r_n} \mathbb{E}(Y_{nj}^2 1_{|Y_{nj}| > \varepsilon}) \xrightarrow[n \rightarrow \infty]{} 0 ,$$

then, for every bounded continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}f\left(\sum_{j=1}^{r_n} Y_{nj}\right) - \mathbb{E}f(Z_n) \xrightarrow[n \rightarrow \infty]{} 0, \quad (5.39)$$

where  $Z_n$  is a centered Gaussian random variable with variance  $\Theta_n^2$ .

Hereafter is the multidimensional and complex extension of Lemma 5.6 we shall rely on in the sequel:

**Lemma 5.7** (CLT for martingales II). *Suppose that for each  $n$   $(Y_{nj}; 1 \leq j \leq r_n)$  is a  $\mathbb{C}^d$ -valued martingale difference sequence with respect to the increasing  $\sigma$ -field  $\{\mathcal{G}_{n,j}; 1 \leq j \leq r_n\}$  having second moments. Write:*

$$Y_{nj}^T = (Y_{nj}^1, \dots, Y_{nj}^d).$$

Assume moreover that  $(\Theta_n(k, \ell))_n$  and  $(\tilde{\Theta}_n(k, \ell))_n$  are uniformly bounded sequences of complex numbers, for  $1 \leq k, \ell \leq d$ . If

$$\sum_{j=1}^{r_n} \mathbb{E}(Y_{nj}^k \bar{Y}_{nj}^d | \mathcal{G}_{n,j-1}) - \Theta_n(k, \ell) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0, \quad (5.40)$$

$$\sum_{j=1}^{r_n} \mathbb{E}(Y_{nj}^k Y_{nj}^\ell | \mathcal{G}_{n,j-1}) - \tilde{\Theta}_n(k, \ell) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0, \quad (5.41)$$

and for each  $\varepsilon > 0$ ,

$$\sum_{j=1}^{r_n} \mathbb{E}(|Y_{nj}|^2 1_{|Y_{nj}| > \varepsilon}) \xrightarrow[n \rightarrow \infty]{} 0, \quad (5.42)$$

then, for every bounded continuous function  $f : \mathbb{C}^d \rightarrow \mathbb{R}$

$$\mathbb{E}f\left(\sum_{j=1}^{r_n} Y_{nj}\right) - \mathbb{E}f(Z_n) \xrightarrow[n \rightarrow \infty]{} 0, \quad (5.43)$$

where  $Z_n$  is a  $\mathbb{C}^d$ -valued centered Gaussian random vector with parameters

$$\mathbb{E}Z_n Z_n^* = (\Theta_n(k, \ell))_{k, \ell} \quad \text{and} \quad \mathbb{E}Z_n Z_n^T = (\tilde{\Theta}_n(k, \ell))_{k, \ell}.$$

Lemmas 5.6 and 5.7 are variations around the Central Limit Theorem for martingales (see Billingsley [11, Theorem 35.12]) which enables us to prove (in the real case):

$$\forall t \in \mathbb{R}, \quad \mathbb{E}e^{it \sum_{j=1}^{r_n} Y_{nj}} - e^{-\frac{t^2 \Theta_n^2}{2}} \rightarrow 0$$

and Lévy theorem for the weak convergence criterion via characteristic functions (see Kallenberg [38, Theorem 5.3 and Theorem 5.5]) which yields (5.43) from the above convergence. Details of the proof are omitted.

**Lemma 5.8** (Tightness and weak convergence). *Let  $K$  be a compact set in  $\mathbb{C}$ ; let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be random elements in  $C(K, \mathbb{C})$ . Assume that for all  $d \geq 1$ , for all  $z_1, \dots, z_d \in K$ , for all  $f \in C(\mathbb{C}^d, \mathbb{C})$  we have:*

$$\mathbb{E}f(X_n(z_1), \dots, X_n(z_d)) - \mathbb{E}f(Y_n(z_1), \dots, Y_n(z_d)) \xrightarrow[n \rightarrow \infty]{} 0.$$



Assume moreover that  $(X_n)$  and  $(Y_n)$  are tight, then for every continuous and bounded functional  $F : C(K, \mathbb{C}) \rightarrow \mathbb{C}$ , we have:

$$\mathbb{E}F(X_n) - \mathbb{E}F(Y_n) \xrightarrow{n \rightarrow \infty} 0 .$$

Lemma 5.8 can be proved as [38, Lemma 16.2]; the proof is therefore omitted.

We are now in position to conclude.

In order to apply Lemma 5.7, it remains to check that  $\Theta_n$  as defined in (2.9) is uniformly bounded for  $z_1, z_2 \in \Gamma$  fixed but this is an easy byproduct of Proposition 5.2.

Proposition 5.1 together with Lemma 5.7 (notice that condition (5.42) can be proved as in [5]) yield the fact that for every  $z_1, \dots, z_d \in \Gamma$  and for every bounded continuous function  $f : \Gamma^d \rightarrow \mathbb{C}$ :

$$\mathbb{E}f(M_n(z_1), \dots, M_n(z_d)) - \mathbb{E}f(G_n(z_1), \dots, G_n(z_d)) \xrightarrow{N, n \rightarrow \infty} 0 ,$$

where  $G_n$  is well-defined by Proposition 5.2. Now the tightness of  $M_n$  and  $G_n$  together with Lemma 5.8 yield the last statement of Theorem 1.

## 6. PROOF OF THEOREM 2 (FLUCTUATIONS FOR NON-ANALYTIC FUNCTIONALS)

In this section, we will assume that the random variables  $(X_{ij}^n)$  are truncated, centered and normalized, following Section 3.2.

**6.1. Useful properties.** Recall that  $\mathcal{S}_n \subset \mathcal{S}_\infty \triangleq \left[0, \lambda_R^+ \left(1 + \sqrt{\ell^+}\right)^2\right]$  uniformly in  $n$ . Denote by  $\mathbf{h} \in C_c^\infty(\mathbb{R})$  a function whose value is 1 on a  $\eta$ -neighborhood  $\mathcal{S}_\infty^\eta$  of  $\mathcal{S}_\infty$ .

**Proposition 6.1.** (1) Assume that (A-1) and (A-2) hold true; let the random variables  $(X_{ij}^n)$  be truncated as in Section 3.2, function  $\mathbf{h}$  be defined as above and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Then

$$\mathrm{tr} f(\Sigma_n \Sigma_n^*) - \mathrm{tr} (f\mathbf{h})(\Sigma_n \Sigma_n^*) \xrightarrow[N, n \rightarrow \infty]{a.s.} 0 .$$

(2) Let  $\mathbf{h}_n$  be a smooth function on  $\mathbb{R}$  with compact support and whose value is 1 on a  $\eta$ -neighborhood  $\mathcal{S}_n^\eta$  of  $\mathcal{S}_n$ ; then:

$$\int_{\mathbb{R}} f(\lambda) \mathcal{F}_n(d\lambda) = \int_{\mathbb{R}} (f\mathbf{h}_n)(\lambda) \mathcal{F}_n(d\lambda) , .$$

Proof of Proposition 6.1 is straightforward and is based on the fact that almost surely,

$$\limsup_{N, n \rightarrow \infty} \|\Sigma_n \Sigma_n^*\| < \lambda_R^+ \left(1 + \sqrt{\ell^+}\right)^2 + \eta ,$$

a fact that can be found in [7] for instance. Details are left to the reader.

The following proposition underlines how a sufficient regularity of function  $f$  compensates the singularity in  $\mathrm{Im}(z)^{-1}$  near the real axis.

**Proposition 6.2.** Let  $\mu, \nu$  be two probability measures on  $\mathbb{R}$  and  $g_\mu$  and  $g_\nu$  their associated Stieltjes transforms. Assume that

$$|g_\mu(z) - g_\nu(z)| \leq \frac{|h(z)|}{\mathrm{Im}(z)^k} , \quad z \in \mathbb{C}^+ ,$$

where  $h$  is a continuous function over  $\text{cl}(\mathbb{C}^+)$ , the closure of  $\mathbb{C}^+$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function of order  $C^{k+1}$  with bounded support; recall the definition of  $\Phi_k(f)$  in (1.6) and denote by

$$\|f\|_{k+1} = \sup_{0 \leq \ell \leq k+1} \|f^{(\ell)}\|_{\infty} \quad \text{where} \quad \|g\|_{\infty} = \sup_{x \in \mathbb{R}} |g(x)| .$$

Then

$$\begin{aligned} \left| \int f d\mu - \int f d\nu \right| &\leq \frac{1}{\pi} \left| \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(f)(z) \{g_{\mu}(z) - g_{\nu}(z)\} \ell_2(dz) \right| , \\ &\leq K \|f\|_{k+1} \int_{\text{supp}(f) \times \text{supp}(\chi)} |h(z)| \ell_2(dz) , \\ &\leq K' \|f\|_{k+1} . \end{aligned} \tag{6.1}$$

*Proof.* Write

$$\begin{aligned} \bar{\partial} \Phi_k(f)(x + iy) &= \partial_x \Phi_k(f)(x + iy) + i \partial_y \Phi_k(f)(x + iy) \\ &= \frac{(iy)^k f^{(k+1)}(x)}{k!} \chi(y) + i \sum_{\ell=0}^k \frac{(iy)^{\ell} f^{(\ell)}(x)}{\ell!} \chi'(y) . \end{aligned}$$

From this and the fact that  $\chi$  is equal to 1 for  $y$  small enough, we deduce that

$$\bar{\partial} \Phi_k(f)(x + iy) = \frac{(iy)^k f^{(k+1)}(x)}{k!}$$

near the real axis. Hence  $|\bar{\partial} \Phi_k(f)(x + iy)| \leq 1_{\text{supp}(f) \times \text{supp}(\chi)}(x, y) K \|f\|_{k+1} y^k$  near the real axis, which yields (6.1).  $\square$

**6.2. Proof of Theorem 2.** Recall the definition of the sets  $D$ ,  $D^+$  and  $D_{\varepsilon}$  given in (3.7) and the fact that constant  $A > \lambda_R^+ (1 + \sqrt{\ell^+})^2$ .

**Lemma 6.3.** *Let  $(\varphi_n(z), z \in D^+ \cup \overline{D^+})_{n \in \mathbb{N}}$  and  $(\psi_n(z), z \in D^+ \cup \overline{D^+})_{n \in \mathbb{N}}$  be centered complex-valued continuous random processes an such that  $\varphi(\bar{z}) = \overline{\varphi(z)}$  and  $\psi(\bar{z}) = \overline{\psi(z)}$ . Assume that:*

(i) *The following convergence in distribution holds true: for all  $d \geq 1$  and  $(z_1, \dots, z_d) \in D^+$ ,*

$$d_{\mathcal{L}P} \left( (\varphi_n(z_1), \dots, \varphi_n(z_d)), (\psi_n(z_1), \dots, \psi_n(z_d)) \right) \xrightarrow{n \rightarrow \infty} 0$$

(ii) *For all  $\varepsilon > 0$ ,  $\varphi_n(z)$  and  $\psi_n(z)$  are tight on  $D_{\varepsilon}$ .*

(iii) *The process  $(\psi_n(z))$  is gaussian with covariance matrix  $\kappa_n(z_1, z_2)$ ,  $(z_1, z_2 \in D^+ \cup \overline{D^+})$ .*

(iv) *The following estimates hold true*

$$\forall n \in \mathbb{N} , \forall z \in D^+ , \quad \text{var } \varphi_n(z) \leq \frac{1}{\text{Im}(z)^{2k}} \quad \text{and} \quad \text{var } \psi_n(z) \leq \frac{1}{\text{Im}(z)^{2k}} .$$

(v) *Let functions  $g_{\ell} : \mathbb{R} \rightarrow \mathbb{R}$  ( $1 \leq \ell \leq L$ ) be  $C^{k+1}$  and have compact support.*

Then,

$$d_{\mathcal{L}P} \left( \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(\mathbf{g})(z) \varphi_n(z) \ell_2(dz) , \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(\mathbf{g})(z) \psi_n(z) \ell_2(dz) \right) \xrightarrow{n \rightarrow \infty} 0 ,$$

where

$$\bar{\partial} \Phi_k(g_j)(z) = (\partial_x + i\partial_y) \sum_{\ell=0}^k \frac{(iy)^\ell}{\ell!} g_j^{(\ell)}(x) \chi(y) \quad \text{and} \quad \bar{\partial} \Phi_k(\mathbf{g}) = (\bar{\partial} \Phi_k(g_j)) ; 1 \leq j \leq L$$

with  $\chi$  being smooth, compactly supported with value 1 in a neighbourhood of 0. Moreover,

$$\frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(\mathbf{g})(z) \psi_n(z) \ell_2(dz)$$

is centered gaussian with covariance matrix:

$$\begin{aligned} \operatorname{cov} \left( \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(g_k)(z) \psi_n(z) \ell_2(dz) , \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(g_\ell)(z) \psi_n(z) \ell_2(dz) \right) \\ = \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_k(g_k)(z_1) \overline{\bar{\partial} \Phi_k(g_\ell)(z_2)} \kappa_n(z_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) \\ + \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_k(g_k)(z_2) \bar{\partial} \Phi_k(g_\ell)(z_2) \kappa_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2) , \end{aligned} \quad (6.2)$$

for  $1 \leq k, \ell \leq L$ .

Proof of Lemma 6.3 is provided in Appendix A.2.

The strategy to prove Theorem 2 closely follows this lemma. Denote by

$$\varphi_n(z) = \operatorname{tr} Q_n(z) - \mathbb{E} \operatorname{tr} Q_n(z) \quad \text{and} \quad \psi_n(z) = G_n(z) - \mathbb{E} G_n(z) ,$$

the process  $G_n$  being defined in Theorem 1, then conditions (i), (ii) and (iii) are immediate consequences of Theorem 1. In order to check condition (iv), we establish the following proposition:

**Proposition 6.4.** *Assume that (A-1) and (A-2) hold true, then:*

(i) (Bordenave [12], Hachem et al. [31, Lemma 6.3], Shcherbina [50]) For all  $z \in \mathbb{C}^+$ ,

$$\operatorname{var} \operatorname{tr} Q_n(z) \leq \frac{C}{\operatorname{Im}(z)^4} ,$$

(ii) For all  $z \in \mathbb{C}^+$ ,

$$\operatorname{var} G_n(z) \leq \frac{C}{\operatorname{Im}(z)^4} ,$$

where  $C$  is a constant that may depend polynomially on  $|z|$ .

Proof of Proposition 6.4 is postponed to Appendix A.3.

Taking into account the estimates established in Proposition 6.4 immediatly yields the first part of Theorem 2 in the case where functions  $(g_\ell)$  have a bounded support and satisfy (v) with  $k = 2$ , i.e. are  $C^3$ . It remains to prove the equivalence between (4.3) and (4.4). but this immediatly follows from:

**Proposition 6.5.** *Let  $(X_n)$  and  $(Y_n)$  be  $\mathbb{C}^d$ -valued random variables and assume that both sequences are tight, then the following are equivalent:*

- (i) *the following convergence holds true:*  $d_{\mathcal{L}P}(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{} 0$  .  
(ii) *for every continuous bounded function  $f : \mathbb{C}^d \rightarrow \mathbb{C}$ ,*  $\mathbb{E} f(X_n) - \mathbb{E} f(Y_n) \xrightarrow[n \rightarrow \infty]{} 0$  ,

Proposition 6.5 can be proved easily by contradiction using the fact that  $d_{\mathcal{L}P}$  metrizes the convergence of laws; its proof is hence omitted.

**6.3. Proof of Proposition 4.1.** Let  $f \in C_c^\infty(\mathbb{R}^2)$ . A simple but lengthy computation yields the fact that

$$\bar{\partial}_2 \bar{\partial}_1 \Phi_{N_1, N_2}(f)(x + iu, y + iv) = \frac{\partial^{N_1 + N_2 + 2}}{\partial x^{N_1 + 1} \partial y^{N_2 + 1}} f(x, y) \times \frac{(iu)^{N_1}}{N_1!} \frac{(iv)^{N_2}}{N_2!} \quad (6.3)$$

for  $u, v$  small enough. Let now  $N_1 = N_2 = 2$ . Since  $|\Theta_n(z_1, z_2)| \leq K|z_1 z_2|^{-2}$  for any  $z_1, z_2 \in \mathbb{C}^+$  and  $z_1, z_2$  in a compact set (use Cauchy-Schwarz and apply Proposition 6.4),  $\Upsilon(f)$  is well-defined. Let  $K$  be a compact set in  $\mathbb{R}^2$  and let  $f \in C_c^\infty(\mathbb{R}^2)$  with support included in  $K$ , then one can easily prove that

$$|\Upsilon(f)| \leq C_K \|f\|_{3,3} \quad \text{with} \quad \|f\|_{3,3} = \sup_{\substack{\ell, p \leq 3 \\ (x, y) \in K}} \|\partial_x^\ell \partial_y^p f(x, y)\|_\infty .$$

This in particular implies that  $\Upsilon$  is a distribution on  $C_c^\infty(\mathbb{R}^2)$ , of finite order (3, 3) and hence uniquely extends as a distribution on  $C_c^{3,3}(\mathbb{R}^2)$ .

Moreover,  $\Upsilon(f)$  can be written as:

$$\begin{aligned} \Upsilon(f) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(z_1, z_2) \Theta_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi^2} \operatorname{Re} \int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(z_1, \bar{z}_2) \Theta_n(z_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) , \end{aligned}$$

where  $\mathbb{C}_\varepsilon^+ = \{z \in \mathbb{C}, \operatorname{Im}(z) \geq \varepsilon\}$ . Taking into account the facts that:

$$\overline{\bar{\partial}_2 \bar{\partial}_1 \Phi_{n_1, n_2}(f)(z_1, z_2)} = \bar{\partial}_2 \bar{\partial}_1 \Phi_{n_1, n_2}(f)(\bar{z}_1, \bar{z}_2) \quad \text{and} \quad \overline{\Theta_n(z_1, z_2)} = \Theta_n(\bar{z}_1, \bar{z}_2) ,$$

we can expand  $\Upsilon(f)$  as:

$$\begin{aligned} \Upsilon(f) &= \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(z_1, z_2) \Theta_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(\bar{z}_1, \bar{z}_2) \Theta_n(\bar{z}_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(z_1, \bar{z}_2) \Theta_n(z_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{2,2}(f)(\bar{z}_1, z_2) \Theta_n(\bar{z}_1, z_2) \ell_2(dz_1) \ell_2(dz_2) . \end{aligned}$$

We now apply twice Green's formula to each integral and obtain

$$\begin{aligned}
\Upsilon(f) &= -\lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \Phi_{2,2}(f)(x_1 + i\varepsilon, x_2 + i\varepsilon) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 \\
&\quad - \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \Phi_{2,2}(f)(x_1 - i\varepsilon, x_2 - i\varepsilon) \Theta_n(x_1 - i\varepsilon, x_2 - i\varepsilon) dx_1 dx_2 \\
&\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \Phi_{2,2}(f)(x_1 + i\varepsilon, x_2 - i\varepsilon) \Theta_n(x_1 + i\varepsilon, x_2 - i\varepsilon) dx_1 dx_2 \\
&\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \Phi_{2,2}(f)(x_1 - i\varepsilon, x_2 + i\varepsilon) \Theta_n(x_1 - i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 .
\end{aligned}$$

Notice that the sign changes in the two last integrals follow from the contour orientations in Green's formula. We now prove

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^2} \Phi_{2,2}(f)(x_1 + i\varepsilon, x_2 + i\varepsilon) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 \\
= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^2} f(x_1, x_2) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 . \quad (6.4)
\end{aligned}$$

The three other integrals can be handled similarly and this will achieve the boundary value representation (4.6) for  $\Upsilon(f)$ .

Using the mere definition of  $\Phi_{N_1, N_2}(f)$  (cf. (4.5)) and Green's formula we get:

$$\begin{aligned}
&\int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{1,0}(f)(z_1, z_2) \Theta_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2) \\
&= - \int_{\mathbb{R}^2} \Phi_{1,0}(f)(x_1 + i\varepsilon, x_2 + i\varepsilon) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 \\
&= - \int_{\mathbb{R}^2} f(x_1, x_2) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 - i\varepsilon \int_{\mathbb{R}^2} \partial_x f(x_1, x_2) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 .
\end{aligned}$$

We extract the first term of the r.h.s. from the equation above. Taking into account (6.3) and the fact that  $|\Theta_n(z_1, z_2)| \leq |z_1 z_2|^{-2}$  for  $z_1, z_2$  in a compact set of  $\mathbb{C} \setminus \mathbb{R}$ , we obtain:

$$\limsup_{\varepsilon \downarrow 0} \left| \varepsilon^3 \int_{\mathbb{R}^2} f(x_1, x_2) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 \right| < \infty .$$

By applying the same argument to the quantity

$$\int_{(\mathbb{C}_\varepsilon^+)^2} \bar{\partial}_2 \bar{\partial}_1 \Phi_{4-\ell, 0}(f)(z_1, z_2) \Theta_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2)$$

for  $\ell = 2$  then  $\ell = 1$  and  $\ell = 0$ , we can similarly prove that

$$\limsup_{\varepsilon \downarrow 0} \left| \varepsilon^\ell \int_{\mathbb{R}^2} f(x_1, x_2) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 \right| < \infty \quad \text{for } \ell = 2, 1, 0.$$

We finally obtain

$$\limsup_{\varepsilon \downarrow 0} \left| \int_{\mathbb{R}^2} f(x_1, x_2) \Theta_n(x_1 + i\varepsilon, x_2 + i\varepsilon) dx_1 dx_2 \right| < \infty .$$

Expanding  $\Phi_{2,2}(f)$  into (6.4) and using the above estimate immediatly yields (6.4).

Proof of Proposition 4.1 is completed.

6.4. **Proof of Proposition 4.2.** The covariance writes (in short)

$$\text{cov}(Z_n^1(f), Z_n^1(g)) = -\frac{1}{4\pi^2} \lim_{\varepsilon \downarrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int f(x)g(y) \Theta_n(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy ,$$

where  $\pm_1, \pm_2 \in \{+, -\}$  and  $\pm_1 \pm_2$  is the sign resulting from the product  $\pm_1 1$  by  $\pm_2 1$ . Unfolding  $\Theta_n = \Theta_{0,n} + |\mathcal{V}|^2 \Theta_{1,n} + \kappa \Theta_{2,n}$ , we have three terms to compute. According to the assumptions of Proposition 4.2, either  $|\mathcal{V}|^2$  equals 1 or 0. In the latter case, the term corresponding to  $\Theta_{1,n}$  vanishes; if  $|\mathcal{V}|^2 = 1$ , then the quantities  $\mathcal{A}_n$  and  $\mathcal{A}_{0,n}$  (respectively defined in (2.13) and (3.10)) are equal, and so are  $\Theta_{0,n}$  and  $\Theta_{1,n}$ . We first establish

$$\begin{aligned} -\frac{1}{4\pi^2} \lim_{\varepsilon \downarrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int f(x)g(y) \Theta_{0,n}(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy \\ = \frac{1}{2\pi^2} \int_{S_n^2} f'(x)g'(y) \ln \left| \frac{\tilde{t}_n(x) - \overline{\tilde{t}_n(y)}}{\tilde{t}_n(x) - \tilde{t}_n(y)} \right| dx dy . \end{aligned} \quad (6.5)$$

The proof relies on formula (3.9) and the following expression of  $\mathcal{A}_{0,n}$

$$1 - \mathcal{A}_{0,n}(z_1, z_2) = \frac{(z_1 - z_2)\tilde{t}_n(z_1)\tilde{t}_n(z_2)}{\tilde{t}_n(z_1) - \tilde{t}_n(z_2)} \quad (6.6)$$

which can be obtained using (3.1). Using (3.9) and performing a double integration by parts yields

$$\begin{aligned} \int f(x)g(y) \Theta_{0,n}(x + \mathbf{i}\varepsilon, y + \mathbf{i}\varepsilon) dx dy &= \int f'(x)g'(y) \ln |1 - \mathcal{A}_{0,n}(x + \mathbf{i}\varepsilon, y + \mathbf{i}\varepsilon)| dx dy \\ &\quad + \mathbf{i} \int f'(x)g'(y) \text{Arg}(1 - \mathcal{A}_{0,n}(x + \mathbf{i}\varepsilon, y + \mathbf{i}\varepsilon)) dx dy \end{aligned}$$

Following [5, Section 5], we need only to consider the logarithm term and show its convergence since the Arg term will eventually disappear (functions  $f$  and  $g$  being real, the covariance will be real as well). Using (6.6), we obtain

$$\int f'(x)g'(y) \ln |1 - \mathcal{A}_{0,n}(x + \mathbf{i}\varepsilon, y + \mathbf{i}\varepsilon)| dx dy = \int f'(x)g'(y) \ln \left| \frac{(x-y)\tilde{t}_n(x + \mathbf{i}\varepsilon)\tilde{t}_n(y + \mathbf{i}\varepsilon)}{\tilde{t}_n(x + \mathbf{i}\varepsilon) - \tilde{t}_n(y + \mathbf{i}\varepsilon)} \right| dx dy$$

and the sum writes

$$\begin{aligned} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int f(x)g(y) \Theta_n(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy \\ = 2 \int f'(x)g'(y) \ln \left\{ \left| \frac{(x-y)\tilde{t}_n(x + \mathbf{i}\varepsilon)\tilde{t}_n(y + \mathbf{i}\varepsilon)}{\tilde{t}_n(x + \mathbf{i}\varepsilon) - \tilde{t}_n(y + \mathbf{i}\varepsilon)} \right| \times \left| \frac{\tilde{t}_n(x + \mathbf{i}\varepsilon) - \tilde{t}_n(y - \mathbf{i}\varepsilon)}{(x-y + 2\mathbf{i}\varepsilon)\tilde{t}_n(x + \mathbf{i}\varepsilon)\tilde{t}_n(y - \mathbf{i}\varepsilon)} \right| \right\} dx dy , \\ \stackrel{(a)}{=} 2 \int f'(x)g'(y) \left\{ \ln \left| \frac{x-y}{x-y + 2\mathbf{i}\varepsilon} \right| + \ln \left| \frac{\tilde{t}_n(x + \mathbf{i}\varepsilon) - \tilde{t}_n(y - \mathbf{i}\varepsilon)}{\tilde{t}_n(x + \mathbf{i}\varepsilon) - \tilde{t}_n(y + \mathbf{i}\varepsilon)} \right| \right\} dx dy , \end{aligned}$$

where (a) follows from the fact that  $\tilde{t}_n(\bar{z}) = \overline{\tilde{t}_n(z)}$  and  $|z| = |\bar{z}|$ . It is straightforward to prove that the first integral of the r.h.s. vanishes as  $\varepsilon \rightarrow 0$ . Using similar arguments as in [5, Section 5], one can prove that

$$\sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int f(x)g(y) \Theta_n(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy = 2 \int f'(x)g'(y) \ln \left| \frac{\tilde{t}_n(x) - \overline{\tilde{t}_n(y)}}{\tilde{t}_n(x) - \tilde{t}_n(y)} \right| dx dy ,$$

which is the desired result. We now establish

$$\begin{aligned} & -\frac{\kappa}{4\pi^2} \lim_{\varepsilon \downarrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int f(x)g(y) \Theta_{2,n}(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy \\ & = \frac{\kappa}{\pi^2 n} \sum_{i=1}^N \left( \int_{\mathcal{S}_n} f'(x) \operatorname{Im}(x T_n(x))_{ii} dx \right) \left( \int_{\mathcal{S}_n} g'(y) \operatorname{Im}(y T_n(y))_{ii} dy \right) \end{aligned} \quad (6.7)$$

Due to formula (3.11), we only need to prove

$$\frac{\mathbf{i}}{2\pi} \lim_{\varepsilon \downarrow 0} \sum_{\pm \in \{+, -\}} \pm \int f(x) \frac{\partial}{\partial x} [(x \pm \mathbf{i}\varepsilon) T_n(x \pm \mathbf{i}\varepsilon)]_{ii} dx = \frac{1}{\pi} \int_{\mathcal{S}_n} f'(x) \operatorname{Im}(x T_n(x))_{ii} dx . \quad (6.8)$$

Performing an integration by parts and taking into account the fact that  $T_n(\bar{z}) = \overline{T_n(z)}$  yields

$$\begin{aligned} & \frac{\mathbf{i}}{2\pi} \lim_{\varepsilon \downarrow 0} \sum_{\pm \in \{+, -\}} \pm \int f(x) \frac{\partial}{\partial x} [(x \pm \mathbf{i}\varepsilon) T_n(x \pm \mathbf{i}\varepsilon)]_{ii} dx \\ & = -\frac{\mathbf{i}}{2\pi} \lim_{\varepsilon \downarrow 0} \int f'(x) 2\mathbf{i} \operatorname{Im} [(x + \mathbf{i}\varepsilon) T_n(x + \mathbf{i}\varepsilon)]_{ii} dx \stackrel{(a)}{=} \frac{1}{\pi} \int_{\mathcal{S}_n} f'(x) \operatorname{Im}(x T_n(x))_{ii} dx , \end{aligned}$$

where step (a) follows from the fact that

$$\inf_{\substack{1 \leq i \leq N, \\ z \in (0, A] \times (0, B]}} |(1 + \tilde{t}_n(z) \lambda_i)| > 0 \quad (6.9)$$

where the  $\lambda_i$ 's stand for  $R_n$ 's eigenvalues. In fact, assume that (6.9) holds true, then using the spectral decomposition of  $R_n$ , the pointwise convergence of  $\tilde{t}_n(z)$  to  $\tilde{t}_n(x)$  as  $\mathbb{C}^+ \ni z \rightarrow x \in \mathbb{R}$  (see for instance [52]) and formula (3.1), then one obtains the pointwise convergence

$$\operatorname{Im} [(x + \mathbf{i}\varepsilon) T_n(x + \mathbf{i}\varepsilon)]_{ii} \xrightarrow{\varepsilon \rightarrow 0} \operatorname{Im} [x T_n(x)]_{ii}$$

for  $x > 0$ . Since  $\operatorname{Im}(\tilde{t}(x)) = 0$  outside  $\mathcal{S}_n$ , so is  $\operatorname{Im}[x T_n(x)]_{ii}$ . Finally, (6.9) provides a uniform bound for  $\operatorname{Im} [(x + \mathbf{i}\varepsilon) T_n(x + \mathbf{i}\varepsilon)]_{ii}$  and (a) follows from the dominated convergence theorem. It remains to prove (6.9). Assume that the infimum is zero, then there exists  $\lambda^* \in \{\lambda_1, \dots, \lambda_N\}$  with  $\lambda^* \neq 0$  and a sequence  $(z_\ell)$  such that  $\tilde{t}_n(z_\ell) \rightarrow -\frac{1}{\lambda^*}$  and  $z_\ell \rightarrow x^* \in \mathbb{R}$ . Formula (3.1) yields

$$\forall z \in \mathbb{C}^+, \tilde{t}_n(z) = \frac{1}{-z + \frac{1}{n} \sum_{i=1}^N \frac{\lambda_i}{1 + \tilde{t}_n(z) \lambda_i}} \Leftrightarrow \frac{1}{n} \sum_{i=1}^N \frac{\lambda_i}{1 + \tilde{t}_n(z) \lambda_i} = \frac{1}{\tilde{t}_n(z)} + z .$$

Taking  $z = z_\ell$  yields a contradiction since since the l.h.s. goes to infinity while the r.h.s. remains bounded. Necessarily, (6.9) holds true and (6.7) is proved.

Proof of Proposition 4.2 is completed by gathering (6.5), (6.7) and using the fact that  $\Theta_{0,n} = \Theta_{1,n}$ .

**6.5. Proof of Corollary 4.3.** In order to establish the fluctuations in the case where functions  $(f_\ell)$  are  $C^3$  in a neighborhood of  $\mathcal{S}_\infty$  but may not have a bounded support, we proceed as following: Write

$$\begin{aligned} \text{tr } f_\ell(\Sigma_n \Sigma_n^*) - \mathbb{E} \text{tr } (f_\ell \mathbf{h})(\Sigma_n \Sigma_n^*) \\ = \underbrace{\text{tr } f_\ell(\Sigma_n \Sigma_n^*) - \text{tr } (f_\ell \mathbf{h})(\Sigma_n \Sigma_n^*)}_{\Gamma_\ell^1} + \underbrace{\text{tr } (f_\ell \mathbf{h})(\Sigma_n \Sigma_n^*) - \mathbb{E} \text{tr } (f_\ell \mathbf{h})(\Sigma_n \Sigma_n^*)}_{\Gamma_\ell^2} . \end{aligned}$$

By Proposition 6.1, the vector  $(\Gamma_\ell^1)$  almost surely converges to zero while the fluctuations for vector  $(\Gamma_\ell^2)$  are described by Theorem 2 with covariance given by Proposition 4.2, where functions  $f_k$  and  $f_\ell$  must be replaced by  $(f_k \mathbf{h})$  and  $(f_\ell \mathbf{h})$ . The variance formula provided in this proposition shows that  $\text{cov}(Z_1^n(f_\ell \mathbf{h}), Z_1^n(f_{\ell'} \mathbf{h}))$  does not depend on function  $\mathbf{h}$  as long as  $\mathbf{h}$  has value 1 on  $\mathcal{S}_n$ .

## 7. PROOF OF THEOREM 3 (BIAS FOR NON-ANALYTIC FUNCTIONALS)

**7.1. Proof of Theorem 3.** Recall the notations  $\Sigma_n^{\mathbb{C}}, Q_n^{\mathbb{C}}$ , etc. introduced in Section 5. We split the bias into two terms

$$\begin{aligned} \mathbb{E} \text{Tr } f(\Sigma_n \Sigma_n^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) &= \mathbb{E} \text{Tr } f(\Sigma_n \Sigma_n^*) - \mathbb{E} \text{Tr } f(\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^*) \\ &\quad + \mathbb{E} \text{Tr } f(\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) , \\ &\triangleq T_1 + T_2 . \end{aligned}$$

We will prove the following. Provided that function  $f$  is of class  $C^8$  with bounded support, then:

$$\mathbb{E} \text{Tr } f(\Sigma_n \Sigma_n^*) - \mathbb{E} \text{Tr } f(\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^*) - \frac{1}{\pi} \text{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_7(f)(z) \mathcal{B}_n(z) \ell_2(dz) \xrightarrow{N, n \rightarrow \infty} 0 . \quad (7.1)$$

Provided that function  $f$  is of class  $C^{18}$  with bounded support, then:

$$\mathbb{E} \text{Tr } f(\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^*) - N \int f(\lambda) \mathcal{F}_n(d\lambda) \xrightarrow{N, n \rightarrow \infty} 0 . \quad (7.2)$$

As one can check, it is much more demanding in terms of assumptions to prove (7.2) than (7.1). Convergence in (7.2) should be compared to the results in Haagerup and Thørbjørnsen [27] (counterpart in the GUE case), Schultz [49] (GOE), Capitaine and Donati-Martin [17], Loubaton et al. [56] ('signal plus noise' model), etc.

**7.1.1. Proof of (7.1).** The heart of the proof lies in Helffer-Sjöstrand's formula, in Theorem 1 (bias part) and in a dominated convergence argument. By Theorem 1,

$$\mathbb{E} \text{Tr } (\Sigma_n \Sigma_n^* - z I_N)^{-1} - N t_n(z) - \mathcal{B}_n(z) \xrightarrow{N, n \rightarrow \infty} 0 .$$

The same argument yields:

$$\mathbb{E} \text{Tr } (\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^* - z I_N)^{-1} - N t_n(z) \xrightarrow{N, n \rightarrow \infty} 0 ,$$

because in the later case  $\mathcal{V} = \kappa = 0$ , hence the bias is zero for the matrix model  $\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^*$ . Subtracting yields:

$$\mathbb{E} \text{Tr } Q_n(z) - \mathbb{E} \text{Tr } Q_n^{\mathbb{C}}(z) - \mathcal{B}_n(z) \xrightarrow{N, n \rightarrow \infty} 0 .$$



Recall that by Proposition 5.4,

$$\left| \mathbb{E} \text{Tr} Q_n(z) - \mathbb{E} \text{Tr} Q_n^{\mathbb{C}}(z) \right| \leq K \frac{|z|^3}{\text{Im}(z)^7} . \quad (7.3)$$

In order to transfer this bound to  $\mathcal{B}_n(z)$ , we invoke a meta-model argument (cf. Section 2.6): Consider matrix  $\Sigma_n(M)$  and its counterpart  $\Sigma_n^{\mathbb{C}}(M)$  as defined in (2.24) and recall that in this case, we have a genuine limit:

$$\mathbb{E} \text{Tr} (\Sigma_n(M) \Sigma_n^*(M) - z I_{NM})^{-1} - \mathbb{E} \text{Tr} (\Sigma_n^{\mathbb{C}}(M) (\Sigma_n^{\mathbb{C}}(M))^* - z I_{NM})^{-1} \xrightarrow[\substack{M \rightarrow \infty \\ N, n \text{ fixed}}]{} \mathcal{B}_n(z) .$$

Since the estimate (7.3) remains true for all  $M \geq 1$ , we obtain:

$$\begin{aligned} |\mathcal{B}_n(z)| &= \lim_{M \rightarrow \infty} \left| \mathbb{E} \text{Tr} (\Sigma_n(M) \Sigma_n(M)^* - z I_{MN})^{-1} - \mathbb{E} \text{Tr} (\Sigma^{\mathbb{C}}(M)_n (\Sigma_n^{\mathbb{C}}(M))^* - z I_{NM})^{-1} \right| \\ &\leq K \frac{|z|^3}{\text{Im}(z)^7} . \end{aligned} \quad (7.4)$$

Write

$$\begin{aligned} \mathbb{E} \text{Tr} f(\Sigma_n \Sigma_n^*) - \mathbb{E} \text{Tr} f(\Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^*) &- \frac{1}{\pi} \text{Re} \int_{\mathbb{C}_+} \bar{\partial} \Phi(f)(z) \mathcal{B}_n(z) \ell_2(dz) \\ &= \frac{1}{\pi} \text{Re} \int_{\mathbb{C}_+} \bar{\partial} \Phi(f)(z) \{ \mathbb{E} \text{Tr} Q_n(z) - \mathbb{E} \text{Tr} Q_n^{\mathbb{C}}(z) - \mathcal{B}_n(z) \} \ell_2(dz) . \end{aligned} \quad (7.5)$$

In view of (7.5), we need a dominated convergence argument in order to prove (7.1); such an argument follows from Proposition 6.2, (7.3) and (7.4) as long as  $f$  is of class  $C^8$  with large but bounded support. This concludes the proof of (7.1).

7.1.2. *Proof of (7.2).* The gist of the proof lies in the following proposition whose proof is postponed to Appendix A.4:

**Proposition 7.1.** *Denote by  $P_\ell(X)$  a polynomial in  $X$  with degree  $\ell$  and positive coefficients, then:*

$$\left| \mathbb{E} \text{Tr} \left( \Sigma_n^{\mathbb{C}} (\Sigma_n^{\mathbb{C}})^* - z I_N \right)^{-1} - N t_n(z) \right| \leq \frac{1}{n} P_{12}(|z|) P_{17}(|\text{Im}(z)|^{-1}) .$$

Using Helffer-Sjöstrand's formula, Proposition 7.1 together with Proposition 6.2 immediately yield (7.2) for any  $f$  of class  $C^{18}$  with large but bounded support.

7.2. **Proof of Proposition 4.4.** One can easily prove that  $Z_n^2$  is a distribution on  $C_c^{18}(\mathbb{R})$  following the lines of proof of Proposition 4.1. Similarly, one can establish the boundary value representation (4.12). It remains to prove that the singular points of  $\mathcal{B}_n(z)$  are included in  $\mathcal{S}_n$ . Following the definitions of  $\mathcal{B}_{1,n}$  and  $\mathcal{B}_{2,n}$  cf. (2.20) and (2.21), we simply need to prove that the quantities

$$\left( 1 - z^2 \tilde{t}_n^2 \frac{1}{n} \text{Tr} R_n^2 T_n^2 \right) \quad \text{and} \quad \left( 1 - |\mathcal{V}|^2 z^2 \tilde{t}_n^2 \frac{1}{n} \text{Tr} R_n^{1/2} T_n(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2} \right)$$

are invertible for  $z \notin \mathcal{S}_n$ . We focus on the first one. Assume first that  $z \in \mathbb{C} \setminus \mathbb{R}$ . Using the inequality  $|\text{tr}(AB)| \leq (\text{tr}(AA^*) \text{tr}(BB^*))^{1/2}$  yields:

$$\left| z^2 \tilde{t}_n^2(z) \frac{1}{n} \text{Tr} R_n^2 T_n^2(z) \right| \leq \frac{|z|^2 |\tilde{t}_n(z)|^2}{n} \text{tr} R_n T_n(z) R_n T_n^*(z) .$$

Since  $T_n^*(z) = T_n(\bar{z})$ , we can assume without loss of generality that  $z_1, z_2 \in \mathbb{C}^+$ .

$$\left| 1 - z^2 \tilde{t}_n^2(z) \frac{1}{n} \text{Tr} R_n^2 T_n^2(z) \right| \geq 1 - \frac{|z|^2 |\tilde{t}_n(z)|^2}{n} \text{tr} R_n T_n(z) R_n T_n^*(z) = |\tilde{t}_n(z)|^2 \frac{\text{Im}(z)}{\text{Im}(\tilde{t}_n(z))} \quad (7.6)$$

where the last identity follows from (A.15). In order to extend the previous estimate to  $z \in \mathbb{R} \setminus \mathcal{S}_n$ , let  $z = x + iy$  with  $x \in \mathbb{R} \setminus \mathcal{S}_n$ ; then a direct computation yields:

$$\frac{\text{Im}(\tilde{t}_n(z))}{\text{Im}(z)} = \int \frac{\tilde{F}_n(d\lambda)}{|\lambda - z|^2} \xrightarrow{y \searrow 0} \int \frac{\tilde{F}_n(d\lambda)}{|\lambda - x|^2} \neq 0.$$

Therefore, by continuity  $(z) \mapsto 1 - z^2 \tilde{t}_n^2(z) \frac{1}{n} \text{Tr} R_n^2 T_n^2(z)$  does not vanish on  $\mathbb{C} \setminus \mathcal{S}_n$  and  $\mathcal{B}_{1,n}$  is analytic on this set. We can similarly prove that  $\mathcal{B}_{2,n}$  is also analytic on the same set. Consider now a function  $f \in C_c^{18}(\mathbb{R})$  whose support is disjoint from  $\mathcal{S}_n$ , then it is straightforward to check that  $Z_n^2(f) = 0$  and the proof of the proposition is completed.

## APPENDIX A. REMAINING PROOFS

**A.1. Proof of Proposition 5.1: remaining computations for the bias.** In this section, we outline the proof of identity (5.37) which we recall below:

$$\begin{aligned} & -|\mathcal{V}|^2 \frac{b_n^2}{n} \mathbb{E} \text{tr} R_n^{1/2} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n^{1/2} \bar{R}_n^{1/2} Q_1^T \bar{R}_n^{1/2} \\ &= |\mathcal{V}|^2 \frac{\frac{z^3 \tilde{t}_n^2}{n} \text{tr} R_n^{1/2} T_n^2(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}}{1 - \frac{|\mathcal{V}|^2 z^2 \tilde{t}_n^2}{n} \text{tr} R_n^{1/2} T_n(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2}} + o(1). \quad (\text{A.1}) \end{aligned}$$

The proof closely follows computations in [5, Section 4] and is essentially a matter of book-keeping; in particular, all the estimates established there remain valid in the context where  $R_n$  and  $X_n$  are not real. We shall focus here on the algebraic identities.

We first replace  $Q_1$  by  $Q$  and approximate  $Q$  by (cf. [5, Eq. 4.13]):

$$Q(z) = -(zI_N - b_n(z)R_n)^{-1} + b_n(z)A(z) + B(z) + C(z) \quad (\text{A.2})$$

where

$$A(z) = \sum_{j=1}^n (zI_N - b_n(z)R_n)^{-1} (\xi_j \xi_j^* - n^{-1}R_n) Q_j(z).$$

The terms  $B(z)$  and  $C(z)$  will not contribute in the sequel. Denote by

$$\begin{aligned} M &= (\mathbb{E} \tilde{f}_n R_n + I_N)^{-1} R_n^{1/2} \bar{R}_n^{1/2}, \\ \mathcal{T} &= \frac{1}{n} \mathbb{E} \text{tr} R_n^{1/2} Q_1 \left( \mathbb{E} \tilde{f}_n R_n + I_N \right)^{-1} R_n^{1/2} \bar{R}_n^{1/2} Q_1^T \bar{R}_n^{1/2}. \end{aligned}$$

We have:

$$\begin{aligned} \mathcal{T} &= \frac{1}{n} \mathbb{E} \text{tr} R_n^{1/2} Q_1 M Q_1^T \bar{R}_n^{1/2} \\ &= \frac{1}{n} \mathbb{E} \text{tr} R_n^{1/2} Q M Q^T \bar{R}_n^{1/2} + o(1) \\ &= -\frac{1}{n} \mathbb{E} \text{tr} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} M Q^T R_n^{1/2} + \frac{b_n(z)}{n} \mathbb{E} \text{tr} R_n^{1/2} A(z) M Q^T \bar{R}_n^{1/2} + o(1) \\ &\triangleq \mathcal{T}_1 + \mathcal{T}_2 + o(1) \quad (\text{A.3}) \end{aligned}$$

In order to compute  $\mathcal{T}_1$ , we approximate  $Q^T$  in the same way as in (A.2); we take into account the fact that for some deterministic matrix  $\Gamma$ ,  $\mathbb{E} \operatorname{tr}(\Gamma A) = 0$ ; we also use the approximation  $b_n(z) = -z\bar{t}_n(z) + o(1)$  and equation (3.1). The computation of  $\mathcal{T}_1$  then easily follows:

$$\begin{aligned} \mathcal{T}_1 &= -\frac{1}{n} \mathbb{E} \operatorname{tr} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} M Q^T \bar{R}_n^{1/2} \\ &= \frac{1}{n} \operatorname{tr} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} M (zI_N - b_n(z)\bar{R}_n)^{-1} \bar{R}_n^{1/2} + o(1) \\ &= -\frac{z}{n} \operatorname{tr} R_n^{1/2} T_n^2(z) R_n^{1/2} \bar{R}_n^{1/2} T_n^T(z) \bar{R}_n^{1/2} + o(1). \end{aligned}$$

We now focus on the term

$$\begin{aligned} \mathcal{T}_2 &= \frac{b_n(z)}{n} \mathbb{E} \operatorname{tr} R_n^{1/2} A(z) M Q^T \bar{R}_n^{1/2} \\ &= \frac{b_n(z)}{n} \mathbb{E} \operatorname{tr} R_n^{1/2} \sum_{j=1}^n (zI_N - b_n(z)R_n)^{-1} (\xi_j \xi_j^* - n^{-1}R_n) Q_j(z) M Q^T \bar{R}_n^{1/2} \\ &= \frac{b_n(z)}{n} \mathbb{E} \operatorname{tr} R_n^{1/2} \sum_{j=1}^n (zI_N - b_n(z)R_n)^{-1} \{ \xi_j \xi_j^* Q_j(z) M (Q^T - Q_j^T) + D(z) + E(z) \} \bar{R}_n^{1/2} \end{aligned}$$

where

$$\begin{aligned} D(z) &= \xi_j \xi_j^* Q_j M Q_j^T - n^{-1} R_n M Q_j M Q_j^T \\ E(z) &= n^{-1} R_n M (Q_j^T - Q^T) \end{aligned}$$

will not contribute. Using the rank-one perturbation identity for  $Q^T - Q_j^T$ , we obtain:

$$\begin{aligned} \mathcal{T}_2 &= \frac{b_n(z)}{n} \mathbb{E} \operatorname{tr} R_n^{1/2} \sum_{j=1}^n (zI_N - b_n(z)R_n)^{-1} \xi_j \xi_j^* Q_j(z) M (Q^T - Q_j^T) \bar{R}_n^{1/2} + o(1) \\ &= -\frac{b_n(z)}{n} \mathbb{E} \operatorname{tr} R_n^{1/2} \sum_{i=1}^n (zI_N - b_n(z)R_n)^{-1} \xi_j \xi_j^* Q_j(z) M \frac{Q_j^T \bar{\xi}_j \bar{\xi}_j^* Q_j^T}{1 + \bar{\xi}_j^* Q_j^T \bar{\xi}_j} \bar{R}_n^{1/2} + o(1) \\ &= -\frac{b_n(z)}{n} \sum_{j=1}^n \mathbb{E} \frac{1}{1 + \bar{\xi}_j^* Q_j^T \bar{\xi}_j} \left( \bar{\xi}_j^* Q_j^T \bar{R}_n^{1/2} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} \xi_j \right) \left( \xi_j^* Q_j(z) M Q_j^T \bar{\xi}_j \right) + o(1). \end{aligned}$$

In order to pursue the computation of  $\mathcal{T}_2$ , we shall perform the following approximations: The quantity  $(1 + \bar{\xi}_j^* Q_j^T \bar{\xi}_j)^{-1}$  can be replaced by  $b_n$  and the two remaining quadratic forms in the expectation can be decorrelated. Now, using formulas (5.9), we obtain:

$$\begin{aligned} \mathcal{T}_2 &= -\frac{b_n^2(z)}{n} \sum_{j=1}^n \mathbb{E} \left( \bar{\xi}_j^* Q_j^T \bar{R}_n^{1/2} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} \xi_j \right) \mathbb{E} \left( \xi_j^* Q_j(z) M Q_j^T \bar{\xi}_j \right) + o(1), \\ &= -\frac{|\mathcal{V}|^2 b_n^2(z)}{n} \sum_{j=1}^n \mathbb{E} \operatorname{tr} \left( \bar{R}_n^{1/2} Q_j^T \bar{R}_n^{1/2} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} R_n^{1/2} \right) \mathbb{E} \operatorname{tr} \left( R_n^{1/2} Q_j(z) M Q_j^T \bar{R}_n^{1/2} \right) + o(1). \end{aligned}$$

We can now replace  $Q_j$  by  $Q$  with no loss and use equation (A.2) to obtain:

$$\begin{aligned} \mathcal{T}_2 &= -\frac{|\mathcal{V}|^2 b_n^2(z)}{n^2} \mathbb{E} \operatorname{tr} \left( \bar{R}_n^{1/2} Q^T(z) \bar{R}_n^{1/2} R_n^{1/2} (zI_N - b_n(z)R_n)^{-1} R_n^{1/2} \right) \\ &\quad \times \mathbb{E} \operatorname{tr} \left( R_n^{1/2} (-zI_N + b_n(z)R_n)^{-1} + b_n(z)A(z) \right) M Q^T \bar{R}_n^{1/2} + o(1), \\ &= |\mathcal{V}|^2 b_n^2(z) \frac{1}{n} \operatorname{tr} R^{1/2} T(z) R^{1/2} \bar{R}_n^{1/2} T^T(z) \bar{R}_n^{1/2} (\mathcal{T}_1 + \mathcal{T}_2) + o(1). \end{aligned} \tag{A.4}$$

Denote by

$$\mathcal{T}_3 = \frac{1}{n} \text{tr} R^{1/2} T(z) R^{1/2} \bar{R}^{1/2} T^T(z) \bar{R}^{1/2} .$$

We now extract  $\mathcal{T}_2$  from (A.4) and plug it into (A.3). We finally obtain:

$$\mathcal{T} = \mathcal{T}_1 + |\mathcal{V}|^2 b_n^2(z) \frac{\mathcal{T}_1 \mathcal{T}_3}{1 - |\mathcal{V}|^2 b_n^2(z) \mathcal{T}_3} + o(1) = \frac{\mathcal{T}_1}{1 - |\mathcal{V}|^2 b_n^2(z) \mathcal{T}_3} + o(1) .$$

Multiplying  $\mathcal{T}$  by  $-|\mathcal{V}|^2 b_n^2(z) = -|\mathcal{V}|^2 z^2 \bar{t}_n^2(z)$  finally yields (A.1).

**A.2. Proof of Lemma 6.3.** By Proposition 6.2,

$$\begin{aligned} \mathbb{E} \left| \int_D \bar{\partial} \Phi(g)(z) \varphi_n(z) \ell_2(dz) \right| &\leq \int_D |\bar{\partial} \Phi(g)(z)| \mathbb{E} |\varphi_n(z)| \ell_2(dz) \\ &\leq \|g\|_{k+1, \infty} \int_D \text{Im}(z)^k \{\text{var } \varphi_n(z)\}^{1/2} \ell_2(dz) < \infty , \end{aligned}$$

by (iii) and (iv). Hence  $\frac{1}{\pi} \text{Re} \int_D \bar{\partial} \Phi(g)(z) \varphi_n(z) \ell_2(dz)$  is a well-defined a.s. finite random variable. This estimate, uniform in  $n$ , readily implies the tightness of

$$\left( \frac{1}{\pi} \text{Re} \int_D \bar{\partial} \Phi(g)(z) \varphi_n(z) \ell_2(dz) ; n \in \mathbb{N} \right) .$$

Notice that the integrals with  $\psi_n$  instead of  $\varphi_n$  are similarly well-defined and tight.

By conditions (i) and (ii), we obtain:

$$d_{\mathcal{L}P} \left( \frac{1}{\pi} \text{Re} \int_{D_\varepsilon} \bar{\partial} \Phi(\mathbf{g})(z) \varphi_n(z) \ell_2(dz) , \frac{1}{\pi} \text{Re} \int_{D_\varepsilon} \bar{\partial} \Phi(\mathbf{g})(z) \psi_n(z) \ell_2(dz) \right) \xrightarrow{N, n \rightarrow \infty} 0 \quad (\text{A.5})$$

(apply Lemma 5.8).

Let  $\mathbf{g} = (g_\ell ; 1 \leq \ell \leq L)$  and  $f : \mathbb{C}^L \rightarrow \mathbb{C}$  be bounded and continuous. Consider the following notations:

$$\begin{aligned} \xi_n &= \frac{1}{\pi} \text{Re} \int_D \bar{\partial} \Phi(\mathbf{g})(z) \varphi_n(z) \ell_2(dz) , & \xi_n^\varepsilon &= \frac{1}{\pi} \text{Re} \int_{D_\varepsilon} \bar{\partial} \Phi(\mathbf{g})(z) \varphi_n(z) \ell_2(dz) , \\ \eta_n &= \frac{1}{\pi} \text{Re} \int_D \bar{\partial} \Phi(\mathbf{g})(z) \psi_n(z) \ell_2(dz) , & \eta_n^\varepsilon &= \frac{1}{\pi} \text{Re} \int_{D_\varepsilon} \bar{\partial} \Phi(\mathbf{g})(z) \psi_n(z) \ell_2(dz) . \end{aligned}$$

We have

$$\begin{aligned} |\mathbb{E} f(\xi_n) - \mathbb{E} f(\eta_n)| \\ \leq |\mathbb{E} f(\xi_n) - \mathbb{E} f(\xi_n^\varepsilon)| + |\mathbb{E} f(\xi_n^\varepsilon) - \mathbb{E} f(\eta_n^\varepsilon)| + |\mathbb{E} f(\eta_n^\varepsilon) - \mathbb{E} f(\eta_n)| . \end{aligned} \quad (\text{A.6})$$

Given  $\rho > 0$ , we first prove that for all  $n \geq 1$ ,

$$|\mathbb{E} f(\xi_n) - \mathbb{E} f(\xi_n^\varepsilon)| \leq (4\|f\|_\infty + 1)\rho \quad (\text{A.7})$$

for  $\varepsilon$  small enough.

We have

$$\mathbb{P}\{|\xi_n - \xi_n^\varepsilon| > \delta\} \leq \frac{1}{\delta} \left( \int_{[0, A] + i[0, \varepsilon]} |\bar{\partial} \Phi(\mathbf{g})(z)| \mathbb{E} |\varphi_n(z)| \ell_2(dz) \right) \quad (\text{A.8})$$

which can be made arbitrarily small if  $\varepsilon$  is small enough, independently from  $n$ . Now,

$$\begin{aligned} |\mathbb{E}f(\xi_n) - \mathbb{E}f(\xi_n^\varepsilon)| &\leq |\mathbb{E}f(\xi_n) - \mathbb{E}f(\xi_n^\varepsilon)| 1_{\{|\xi_n - \xi_n^\varepsilon| > \eta\}} \\ &\quad + |\mathbb{E}f(\xi_n) - \mathbb{E}f(\xi_n^\varepsilon)| 1_{\{|\xi_n - \xi_n^\varepsilon| \leq \eta, |\xi_n| \vee |\xi_n^\varepsilon| > K\}} \\ &\quad + |\mathbb{E}f(\xi_n) - \mathbb{E}f(\xi_n^\varepsilon)| 1_{\{|\xi_n - \xi_n^\varepsilon| \leq \eta, |\xi_n| \vee |\xi_n^\varepsilon| \leq K\}}. \end{aligned}$$

First invoke the tightness of  $|\xi_n| \vee |\xi_n^\varepsilon|$  and choose  $K$  large enough so that the second term of the r.h.s. is lower than  $2\|f\|_\infty \rho$ ; then choose  $\eta > 0$  small enough so that  $f$  being absolutely continuous over  $\{z \in \mathbb{C}^+, |z| \leq K\}$ , the third term of the r.h.s. is lower than  $\rho$ ; finally for such  $K$  and  $\eta$ , take advantage of (A.8) and choose  $\varepsilon$  small enough so that the first term of the r.h.s. is lower than  $2\|f\|_\infty \rho$ . Eq. (A.7) is proved.

One can similarly prove that  $|\mathbb{E}f(\eta_n) - \mathbb{E}f(\eta_n^\varepsilon)| \leq (4\|f\|_\infty + 1)\rho$  for  $\varepsilon > 0$  small enough. Such  $\varepsilon$  being fixed, it remains to control the second term of the r.h.s. of (A.6), but this immediately follows from (A.5).

In order to prove that  $\eta_n$  is multivariate gaussian with prescribed covariance (6.2), we first consider  $\eta_n^\varepsilon$ . Approximating the integral in  $\eta_n^\varepsilon$  by Riemann sums and using the fact that weak limits of gaussian vectors are gaussian immediately yields that  $\eta_n^\varepsilon$  is a gaussian vector with covariance matrix:

$$[\text{cov}(\eta_n^\varepsilon)]_{k\ell} = \frac{1}{\pi^2} \mathbb{E} \left\{ \text{Re} \int_{D^\varepsilon} \bar{\partial}\Phi(g_k)(z) \psi_n(z) \ell_2(dz) \text{Re} \int_{D^\varepsilon} \bar{\partial}\Phi(g_\ell)(z) \psi_n(z) \ell_2(dz) \right\}$$

for  $1 \leq k, \ell \leq L$ . Using the elementary identity:

$$\text{Re}(z)\text{Re}(z') = \frac{\text{Re}(z\bar{z}') + \text{Re}(z'z)}{2},$$

we obtain:

$$\begin{aligned} [\text{cov}(\eta_n^\varepsilon)]_{k\ell} &= \frac{1}{2\pi^2} \text{Re} \int_{(D^\varepsilon)^2} \bar{\partial}\Phi(g_k)(z_1) \overline{\bar{\partial}\Phi(g_\ell)(z_2)} \mathbb{E} \psi_n(z_1) \overline{\psi_n(z_2)} \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \frac{1}{2\pi^2} \text{Re} \int_{(D^\varepsilon)^2} \bar{\partial}\Phi(g_k)(z_1) \bar{\partial}\Phi(g_\ell)(z_2) \mathbb{E} \psi_n(z_1) \psi_n(z_2) \ell_2(dz_1) \ell_2(dz_2). \end{aligned}$$

Using the fact that  $\overline{\psi_n(z_2)} = \psi_n(\bar{z}_2)$  yields:

$$\begin{aligned} [\text{cov}(\eta_n^\varepsilon)]_{k\ell} &= \frac{1}{2\pi^2} \text{Re} \int_{(D^\varepsilon)^2} \bar{\partial}\Phi(g_k)(z_1) \overline{\bar{\partial}\Phi(g_\ell)(z_2)} \kappa_n(z_1, \bar{z}_2) \ell_2(dz_1) \ell_2(dz_2) \\ &\quad + \frac{1}{2\pi^2} \text{Re} \int_{(D^\varepsilon)^2} \bar{\partial}\Phi(g_k)(z_1) \bar{\partial}\Phi(g_\ell)(z_2) \kappa_n(z_1, z_2) \ell_2(dz_1) \ell_2(dz_2). \end{aligned}$$

In order to lift the gaussianity from  $\eta_n^\varepsilon$  to  $\eta_n$  and to extend the covariance formula from the one above to formula (6.2), we rely on the approximation theorem [38, Theorem 4.28] and on assumptions (iv) and (v) on the variance estimates and on the regularity of functions  $g_k, g_\ell$  in Lemma 6.3.

Proof of Lemma 6.3 is completed.

**A.3. Proof of Proposition 6.4.** Recall the notations  $\xi_i, Q_i$  introduced in Section 5. Denote by  $\Sigma^{\{i\}} = (\xi_1, \dots, \xi_{i-1}, \check{\xi}_i, \xi_{i+1}, \dots, x_n)$ , where  $\check{\xi}_i$  is an independent copy of  $\xi_i$ . Let  $Q^{\{i\}}$  be the associated resolvent:

$$Q^{\{i\}} = \left( \Sigma^{\{i\}} (\Sigma^{\{i\}})^{-1} - zI_N \right)^{-1}.$$

Rank-one perturbation formulas yield:

$$Q = Q_i - \frac{Q_i \xi_i \xi_i^* Q_i}{1 + \xi_i^* Q_i \xi_i} \quad \text{and} \quad Q^{\{i\}} = Q_i - \frac{Q_i \check{\xi}_i \check{\xi}_i^* Q_i}{1 + \check{\xi}_i^* Q_i \check{\xi}_i}.$$

We are now in position to apply Efron-Stein's inequality (cf. [14, Theorem 3.1]):

$$\begin{aligned} \text{var Tr } Q(z) &\leq \frac{1}{2} \sum_{i=1}^n \mathbb{E} \left| \text{Tr } Q(z) - \text{Tr } Q^{\{i\}}(z) \right|^2 \\ &= \frac{1}{2} \sum_{i=1}^n \mathbb{E} \left| \frac{\xi_i^* Q_i^2 \xi_i}{1 + \xi_i^* Q_i \xi_i} - \frac{\check{\xi}_i^* Q_i^2 \check{\xi}_i}{1 + \check{\xi}_i^* Q_i \check{\xi}_i} \right|^2 \\ &= \sum_{i=1}^n \mathbb{E} \left[ \text{var}_i \left( \frac{\xi_i^* Q_i^2 \xi_i}{1 + \xi_i^* Q_i \xi_i} \right) \right] \stackrel{(a)}{\leq} \sum_{i=1}^n \mathbb{E} \left[ \mathbb{E}_{\{i\}} \left| \frac{\xi_i^* Q_i^2 \xi_i}{1 + \xi_i^* Q_i \xi_i} - \frac{\mathbb{E}_{\{i\}} \xi_i^* Q_i^2 \xi_i}{1 + \mathbb{E}_{\{i\}} \xi_i^* Q_i \xi_i} \right|^2 \right] \end{aligned}$$

where  $\text{var}_i$  is the variance under the expectation  $\mathbb{E}_{\{i\}}$  with respect to  $\xi_i$ , and (a) follows from the fact that  $\text{var}(X) = \inf_a \mathbb{E}|X - a|^2$ . Denote by

$$\overset{\circ}{X} = X - \mathbb{E}_{\{i\}} X, \quad B = \xi_i^* Q_i^2 \xi_i, \quad A = 1 + \xi_i^* Q_i \xi_i.$$

We have

$$\frac{B}{A} - \frac{\mathbb{E}_{\{i\}} B}{\mathbb{E}_{\{i\}} A} = \frac{\overset{\circ}{B}}{\mathbb{E}_{\{i\}} A} - \frac{B}{A} \frac{\overset{\circ}{A}}{\mathbb{E}_{\{i\}} A}.$$

Notice that  $\text{Im}(A) = \text{Im}(z) (\xi_i^* Q_i Q_i^* \xi_i)$ , hence

$$\left| \frac{B}{A} \right| \leq \frac{|\xi_i^* Q_i^2 \xi_i|}{\text{Im}(A)} \leq \frac{\xi_i^* Q_i Q_i^* \xi_i}{\text{Im}(A)} \leq \frac{1}{\text{Im}(z)}.$$

Now

$$\begin{aligned} \mathbb{E}_{\{i\}} \left( \frac{\xi_i^* Q_i^2 \xi_i}{1 + \xi_i^* Q_i \xi_i} - \frac{\mathbb{E}_{\{i\}} \xi_i^* Q_i^2 \xi_i}{1 + \mathbb{E}_{\{i\}} \xi_i^* Q_i \xi_i} \right)^2 &\leq K \mathbb{E}_{\{i\}} \left| \frac{\overset{\circ}{B}}{\mathbb{E}_{\{i\}} A} \right|^2 + K \mathbb{E}_{\{i\}} \left| \frac{B}{A} \frac{\overset{\circ}{A}}{\mathbb{E}_{\{i\}} A} \right|^2, \\ &\leq K \mathbb{E}_{\{i\}} \left| \frac{\overset{\circ}{B}}{\mathbb{E}_{\{i\}} A} \right|^2 + \frac{K}{\text{Im}(z)^2} \mathbb{E}_{\{i\}} \left| \frac{\overset{\circ}{A}}{\mathbb{E}_{\{i\}} A} \right|^2. \end{aligned}$$

We first focus on the estimation of  $\mathbb{E}_{\{i\}} |\overset{\circ}{A}|^2$ ; we have:

$$\mathbb{E}_{\{i\}} |\overset{\circ}{A}|^2 \leq \frac{K}{n^2} \text{Tr}(Q_i R Q_i^* R) \leq \frac{K}{n^2} \|R\| \text{Tr}(Q_i R Q_i^*).$$

Before dividing by  $|\mathbb{E}_{\{i\}} A|^2$ , notice that  $\text{Im}(\mathbb{E}_{\{i\}}(A)) = \text{Im}(z) \frac{1}{n} \text{Tr}(Q_i R Q_i^*)$  and that  $(-z \mathbb{E}_{\{i\}} A)^{-1}$  is the Stieltjes transform of a probability measure and hence that

$$\frac{1}{|\mathbb{E}_{\{i\}} A|} \leq \frac{|z|}{\text{Im}(z)}.$$

We now divide and get:

$$\frac{\mathbb{E}_{\{i\}} |\overset{\circ}{A}|^2}{|\mathbb{E}_{\{i\}} A|^2} \leq \frac{K}{n} \left[ \frac{\frac{1}{n} \text{Tr}(Q_i R Q_i^*)}{\text{Im}(\mathbb{E}_{\{i\}} A)} \right] \frac{1}{|\mathbb{E}_{\{i\}}(A)|} \leq \frac{K}{n} \frac{1}{\text{Im}(z)} \frac{|z|}{\text{Im}(z)} = \frac{K}{n} \frac{1}{\text{Im}(z)^2}.$$

Similarly, one can prove that

$$\frac{\mathbb{E}_{\{i\}}|\overset{\circ}{B}|^2}{|\mathbb{E}_{\{i\}}A|^2} \leq \frac{K}{n} \frac{1}{\text{Im}(z)^4}.$$

From this we get the desired estimate:

$$\text{var tr } Q(z) \leq \frac{K}{\text{Im}(z)^4}.$$

In order to prove the second part of condition (iii), we rely on a meta-model argument (cf. Section 2.6). Denote by

$$M_{n,M}^1(z) = \text{Tr} (\Sigma_n(M)\Sigma_n^*(M) - zI_N)^{-1} - \mathbb{E}\text{Tr} (\Sigma_n(M)\Sigma_n^*(M) - zI_N)^{-1},$$

then

$$\text{var} \left\{ \text{tr} (\Sigma_n(M)\Sigma_n^*(M) - zI_N)^{-1} \right\} \leq \frac{C}{\text{Im}(z)^4},$$

moreover  $M_{n,M}^1(z)$  converges in distribution to  $\psi_n(z)$  as  $M \rightarrow \infty$ ,  $N$  and  $n$  being fixed (see for instance the details in Section 5.2). Consider the continuous bounded function  $h_K(x) = |x|^2 \wedge K$ , then

$$\mathbb{E}h_K(\psi_n(z)) = \lim_{M \rightarrow \infty} \mathbb{E}h_K(M_{n,M}^1(z)) \leq \limsup_{M \rightarrow \infty} \mathbb{E}|M_{n,M}^1(z)|^2 \leq \frac{C}{\text{Im}(z)^4}.$$

Now letting  $K \rightarrow \infty$  yields the desired bound by monotone convergence theorem:

$$\text{var}(\psi_n(z)) \leq \frac{C}{\text{Im}(z)^4}.$$

**A.4. Proof of Proposition 7.1.** In the whole section, we consider the matrix model  $\Sigma_n^{\mathbb{C}}(\Sigma_n^{\mathbb{C}})^*$ ; we however simply write  $f_n$ ,  $\tilde{f}_n$ ,  $Q_n$  while the underlying random variables are  $\mathcal{N}_{\mathbb{C}}(0, 1)$ .

Recall that  $\tilde{f}_n(z) = -\frac{1-c_n}{z} + c_n f_n(z)$ , where  $f_n(z) = \frac{1}{n} \text{Tr } Q_n(z)$ . Denote by

$$S_n(z) = \left( -z \left( I_N + \mathbb{E}\tilde{f}_n(z) R_n \right) \right)^{-1} \quad \text{and} \quad \tilde{\tau}_n(z) = -\frac{1}{z \left( 1 + \frac{1}{n} \mathbb{E}\text{Tr } R_n Q_n(z) \right)}.$$

Let

$$\varepsilon_n(z) = n(\mathbb{E}\tilde{f}_n(z) - \tilde{\tau}_n(z)), \tag{A.9}$$

$$\tilde{\varepsilon}_n(z) = \mathbb{E}\text{Tr } R_n Q_n(z) - \text{Tr } R_n S_n(z). \tag{A.10}$$

Taking into account the definition of  $\tilde{\tau}_n$  and  $\tilde{t}_n$ , we get:

$$\begin{aligned} n \left( \mathbb{E}\tilde{f}_n - \tilde{t}_n \right) &= n \left( \tilde{\tau}_n - \tilde{t}_n \right) + \varepsilon_n \\ &= \frac{1}{z} \frac{\text{Tr } R_n T_n - \text{Tr } R_n \mathbb{E}Q_n}{\left( 1 + \frac{1}{n} \mathbb{E}\text{Tr } R_n Q_n \right) \left( 1 + \frac{1}{n} \text{Tr } R_n T_n \right)} + \varepsilon_n. \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\mathrm{Tr} R_n Q_n - \mathrm{Tr} R_n T_n &= \mathrm{Tr} R_n S_n - \mathrm{Tr} R_n T_n + \tilde{\varepsilon}_n \\
&= -\frac{1}{z} \mathrm{Tr} R_n \left[ (I_N + \mathbb{E} \tilde{f}_n R_n)^{-1} - (I_N + \tilde{t}_n R_n)^{-1} \right] + \tilde{\varepsilon}_n \\
&= \frac{1}{z} \mathrm{Tr} R_n (I_N + \mathbb{E} \tilde{f}_n R_n)^{-1} R_n (I_N + \tilde{t}_n R_n)^{-1} \left( \mathbb{E} \tilde{f}_n - \tilde{t}_n \right) + \tilde{\varepsilon}_n .
\end{aligned}$$

Gathering the two previous equations, we finally end up with the  $2 \times 2$  linear system:

$$\begin{pmatrix} n(\mathbb{E} \tilde{f}_n - \tilde{t}_n) \\ \mathbb{E}\mathrm{Tr} R_n Q_n - \mathrm{Tr} R_n T_n \end{pmatrix} = D_0(z) \begin{pmatrix} n(\mathbb{E} \tilde{f}_n - \tilde{t}_n) \\ \mathbb{E}\mathrm{Tr} R_n Q_n - \mathrm{Tr} R_n T_n \end{pmatrix} + \begin{pmatrix} \varepsilon_n \\ \tilde{\varepsilon}_n \end{pmatrix}$$

where

$$D_0(z) = \begin{pmatrix} 0 & z \tilde{\tau}_n \tilde{t}_n \\ \frac{z}{n} \mathrm{Tr} R_n S_n R_n T_n & 0 \end{pmatrix} ,$$

from which we extract

$$n \left( \mathbb{E} \tilde{f}_n - \tilde{t}_n \right) = \frac{1}{\det(I_2 - D_0(z))} \left( \varepsilon_n(z) + z \tilde{\tau}_n \tilde{t}_n \tilde{\varepsilon}_n(z) \right) . \quad (\text{A.11})$$

It remains to bound  $\varepsilon_n$  and  $\tilde{\varepsilon}_n$  and to lowerbound  $|\det(I_2 - D_0)|$ . The first task relies on standard gaussian calculus for random matrices (Poincaré-Nash inequality, integration by part formula, etc. - see for instance [47]) and yields:

$$|\varepsilon_n(z)| \leq \frac{1}{n} P_2(|z|) P_5(|\mathrm{Im}(z)|^{-1}) , \quad (\text{A.12})$$

$$|\tilde{\varepsilon}_n(z)| \leq \frac{1}{n} P_3(|z|) P_7(|\mathrm{Im}(z)|^{-1}) . \quad (\text{A.13})$$

Details are omitted and can be found in [58, Chapter 3].

The second task is more involved and goes along the lines developed in [27] and [56]; it relies on the following proposition:

**Proposition A.1.** *There exist  $\eta > 0$ , polynomials  $P_{12}^\dagger$  and  $P_{16}^\dagger$  and an integer  $N_0$  such that for every  $z$  in the set*

$$\mathcal{E}_n = \left\{ z \in \mathbb{C}^+ , 1 - \frac{1}{n^2} P_{12}^\dagger(|z|) P_{16}^\dagger(|\mathrm{Im}(z)|^{-1}) > 0 \right\}$$

and for  $N \geq N_0$

$$|\det(I_2 - D_0(z))| > \frac{K |\mathrm{Im}(z)|^4}{(\eta^2 + |z|^2)^2} ,$$

where  $K$  is some constant independent from  $N, n$ .

Assume for a while that Proposition A.1 holds true, and let  $z \in \mathcal{E}_n$ ; then taking into account (A.12)-(A.13) and the fact that  $|z \tilde{t}_n \tilde{\tau}_n| \leq |z| |\mathrm{Im}(z)|^{-2}$ , (A.11) yields:

$$|n(\mathbb{E} \tilde{f}_n(z) - \tilde{t}_n(z))| \leq P_8(|z|) P_{13}(|\mathrm{Im}(z)|^{-1}) .$$



If  $z \notin \mathcal{E}_n$ , then  $1 \leq \frac{1}{n^2} P_{12}^\dagger(|z|) P_{16}^\dagger(|\text{Im}(z)|^{-1})$  and

$$\begin{aligned} |n(\mathbb{E}\tilde{f}_n(z) - \tilde{t}_n(z))| &\leq n \left( |\mathbb{E}\tilde{f}_n(z)| + |\tilde{t}_n(z)| \right) \leq \frac{2n}{|\text{Im}(z)|^{-1}} \times 1 \\ &\leq \frac{2n}{|\text{Im}(z)|^{-1}} \times \frac{1}{n^2} P_{12}^\dagger(|z|) P_{16}^\dagger(|\text{Im}(z)|^{-1}) \\ &\leq \frac{1}{n} P_{12}^\dagger(|z|) P_{17}(|\text{Im}(z)|^{-1}) . \end{aligned}$$

Proof of Proposition 7.1 is completed as long as Proposition A.1 holds true.

*Proof of Proposition A.1.* In order to lowerbound  $\det(I_2 - D_0(z))$ , we introduce two auxiliary systems.

Consider matrix

$$D(z) = \begin{pmatrix} 0 & |\tilde{t}_n(z)|^2 \\ \frac{|z|^2}{n} \text{Tr} R_n T_n R_n T_n^* & 0 \end{pmatrix} ,$$

In order to evaluate the determinant of matrix  $I_2 - D$ , we need to find an equation where such a matrix appears. One can easily prove that:

$$\begin{pmatrix} \text{Im}(\tilde{t}_n) \\ \text{Im}\left(\frac{z}{n} \text{Tr} R_n T_n\right) \end{pmatrix} = D(z) \begin{pmatrix} \text{Im}(\tilde{t}_n) \\ \text{Im}\left(\frac{z}{n} \text{Tr} R_n T_n\right) \end{pmatrix} + \begin{pmatrix} |\tilde{t}_n|^2 \\ 0 \end{pmatrix} \text{Im}(z) , \quad (\text{A.14})$$

from which we extract the determinant:

$$\det(I_2 - D(z)) = 1 - |z|^2 |\tilde{t}_n|^2 \frac{1}{n} \text{Tr} R_n T_n R_n T_n^* = |\tilde{t}|^2 \frac{\text{Im}(z)}{\text{Im}(\tilde{t}_n)} . \quad (\text{A.15})$$

Recall that  $|\tilde{t}_n(z)| \leq (\text{Im}(z))^{-1}$ ; in order to lowerbound  $\text{Im}(\tilde{t}_n(z))$ , recall that the associated probability measures  $\tilde{\mathcal{F}}_n$  form a tight family and in particular there exists  $\eta > 0$  such that  $\tilde{\mathcal{F}}_n([0, \eta]) > 1/2$  for every  $n \geq 1$ . Write:

$$\text{Im}(\tilde{t}_n(z)) = \text{Im}(z) \int_{\mathbb{R}^+} \frac{\tilde{\mathcal{F}}_n(d\lambda)}{|\lambda - z|^2} \geq \int_0^\eta \frac{\tilde{\mathcal{F}}_n(d\lambda)}{|\lambda - z|^2} \geq \frac{1}{4(\eta^2 + |z|^2)} . \quad (\text{A.16})$$

Plugging these estimates into (A.15) yields the bound:

$$\det(I_2 - D(z)) = 1 - |z|^2 |\tilde{t}_n|^2 \frac{1}{n} \text{Tr} R_n T_n R_n T_n^* \geq \frac{\text{Im}(z)^4}{16(\eta^2 + |z|^2)^2} . \quad (\text{A.17})$$

Consider now matrix

$$D'(z) = \begin{pmatrix} 0 & |\tilde{\tau}_n(z)|^2 \\ \frac{|z|^2}{n} \text{Tr} R_n S_n R_n S_n^* & 0 \end{pmatrix} .$$

In order to evaluate the determinant of matrix  $I_2 - D'$ , we need to find an equation where such a matrix appears. Recall the definition of  $\varepsilon_n$  and  $\tilde{\varepsilon}_n$  in (A.9)-(A.10). Taking their imaginary parts, we obtain:

$$\begin{pmatrix} \text{Im}(\mathbb{E}\tilde{f}_n) \\ \text{Im}\left(\frac{z}{n} \text{Tr} \mathbb{E} R_n Q_n\right) \end{pmatrix} = \begin{pmatrix} \text{Im}(\tilde{\tau}_n) \\ \text{Im}\left(\frac{z}{n} \text{Tr} R_n S_n\right) \end{pmatrix} + \begin{pmatrix} \frac{1}{n} \text{Im}(\varepsilon_n) \\ \frac{1}{n} \text{Im}(\tilde{\varepsilon}_n) \end{pmatrix} .$$

Computing the imaginary part of  $\tilde{\tau}_n$  and  $n^{-1} \text{Tr} R_n S_n$ , we get:

$$\begin{aligned} \text{Im}(\tilde{\tau}_n) &= |\tilde{\tau}_n|^2 \text{Im}(z) + |\tilde{\tau}_n|^2 \text{Im}\left(\frac{z}{n} \text{Tr} \mathbb{E} R_n Q_n\right) , \\ \text{Im}\left(\frac{z}{n} \text{Tr} R_n S_n\right) &= \frac{|z|^2}{n} \text{Tr}(R_n S_n R_n S_n^*) \text{Im}(\mathbb{E}\tilde{f}_n) . \end{aligned}$$

Hence the system:

$$\begin{pmatrix} \operatorname{Im}(\mathbb{E}\tilde{f}_n) \\ \operatorname{Im}(\frac{z}{n}\operatorname{Tr}\mathbb{E}R_nQ_n) \end{pmatrix} = D'(z) \begin{pmatrix} \operatorname{Im}(\mathbb{E}\tilde{f}_n) \\ \operatorname{Im}(\frac{z}{n}\operatorname{Tr}\mathbb{E}R_nQ_n) \end{pmatrix} + \begin{pmatrix} |\tilde{\tau}_n|^2 \\ 0 \end{pmatrix} \operatorname{Im}(z) + \begin{pmatrix} \frac{1}{n}\operatorname{Im}(\varepsilon_n) \\ \frac{1}{n}\operatorname{Im}(\tilde{\varepsilon}_n) \end{pmatrix}. \quad (\text{A.18})$$

This system is similar to (A.14) with two extra error terms  $\frac{1}{n}\operatorname{Im}(\varepsilon_n)$  and  $\frac{1}{n}\operatorname{Im}(\tilde{\varepsilon}_n)$ ; but these terms are controlled by (A.12)-(A.13).

It is well-known that for all  $z \in \mathbb{C}^+$ ,  $\mathbb{E}\tilde{f}_n(z) - \tilde{t}_n(z) \rightarrow 0$  as  $N, n \rightarrow \infty$ . The mere definition of  $\varepsilon_n$  together with estimate (A.12) yields  $\tilde{\tau}_n - \mathbb{E}\tilde{f}_n \rightarrow 0$ ; therefore  $\tilde{\tau}_n(z) - \tilde{t}_n(z) \rightarrow 0$  for  $z \in \mathbb{C}^+$  as  $N, n \rightarrow 0$ . This, together with (A.16) yields:

$$\operatorname{Im}(\tilde{\tau}_n(z)) \geq \frac{\operatorname{Im}(z)}{8(\eta^2 + |z|^2)}$$

for  $n$  large enough. Taking into account the fact that  $\operatorname{Im}(\frac{z}{n}\operatorname{Tr}(\mathbb{E}R_nQ_n)) \geq 0$ , the first equation in (A.18) yields:

$$\begin{aligned} \operatorname{Im}(\mathbb{E}\tilde{f}_n(z)) &\geq |\tilde{\tau}_n|^2 \operatorname{Im}(z) + \frac{1}{n}\operatorname{Im}(\varepsilon_n(z)), \\ &\stackrel{(a)}{\geq} \frac{|\operatorname{Im}(z)|^3}{64(\eta^2 + |z|^2)^2} - \frac{1}{n^2}P_2(|z|)P_5(|\operatorname{Im}(z)|^{-1}), \\ &\geq \frac{|\operatorname{Im}(z)|^3}{64(\eta^2 + |z|^2)^2} \left(1 - \frac{1}{n^2}P_6(|z|)P_8(|\operatorname{Im}(z)|^{-1})\right) \end{aligned}$$

for  $n$  large enough, where (a) follows from the previous estimate over  $\tilde{\tau}_n$  and (A.12). Polynomials  $P_6$  and  $P_8$  being fixed as in the previous estimate, consider the set:

$$\mathcal{E}_n^1 = \left\{ z \in \mathbb{C}^+; 1 - \frac{1}{n^2}P_6(|z|)P_8(|\operatorname{Im}(z)|^{-1}) \geq \frac{1}{2} \right\}.$$

Then for  $n$  large enough and  $z \in \mathcal{E}_n^1$ ,

$$\operatorname{Im}(\mathbb{E}\tilde{f}_n(z)) \geq \frac{|\operatorname{Im}(z)|^3}{128(\eta^2 + |z|^2)^2}.$$

From (A.18), we have:

$$\begin{aligned} \det(I_2 - D'(z)) &= |\tilde{\tau}_n|^2 \frac{\operatorname{Im}(z)}{\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} + |\tilde{\tau}_n|^2 \frac{\operatorname{Im}(\frac{z}{n}\tilde{\varepsilon}_n(z))}{\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} + \frac{\operatorname{Im}(\frac{1}{n}\varepsilon_n(z))}{\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} \\ &\geq |\tilde{\tau}_n|^2 |\operatorname{Im}(z)|^2 - |\tilde{\tau}_n|^2 \frac{|z\tilde{\varepsilon}_n(z)|}{n\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} - \frac{|\varepsilon_n(z)|}{n\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} \end{aligned}$$

Taking into account the various estimates previously established (recall that  $|\tilde{\tau}_n| \leq |\operatorname{Im}(z)|^{-1}$ ), we obtain:

$$\begin{aligned} \det(I_2 - D'(z)) &\geq \frac{|\operatorname{Im}(z)|^4}{64(\eta^2 + |z|^2)^2} - \frac{1}{|\operatorname{Im}(z)|^2} \frac{|z\tilde{\varepsilon}_n(z)|}{n\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} - \frac{|\varepsilon_n(z)|}{n\operatorname{Im}(\mathbb{E}\tilde{f}_n(z))} \\ &\geq \frac{|\operatorname{Im}(z)|^4}{64(\eta^2 + |z|^2)^2} - \frac{128|z|(\eta^2 + |z|^2)^2 P_3(|z|)P_7(|\operatorname{Im}(z)|^{-1})}{n^2 |\operatorname{Im}(z)|^5} \\ &\quad - \frac{128(\eta^2 + |z|^2)^2 P_2(|z|)P_5(|\operatorname{Im}(z)|^{-1})}{n^2 |\operatorname{Im}(z)|^3} \\ &\geq \frac{|\operatorname{Im}(z)|^4}{64(\eta^2 + |z|^2)^2} \left(1 - \frac{1}{n^2}P_{12}(|z|)P_{16}(|\operatorname{Im}(z)|^{-1})\right), \end{aligned}$$

valid for  $n$  large enough and  $z \in \mathcal{E}_n^1$ . Polynomials  $P_{12}$  and  $P_{16}$  being fixed as in the previous estimates, consider the set:

$$\mathcal{E}_n = \mathcal{E}_n^1 \cap \mathcal{E}_n^2 \quad \text{where} \quad \mathcal{E}_n^2 = \left\{ z \in \mathbb{C}^+; 1 - \frac{1}{n^2} P_{12}(|z|) P_{16}(|\text{Im}(z)|^{-1}) \geq \frac{1}{2} \right\}.$$

Then for  $n$  large enough and  $z \in \mathcal{E}_n$ , we have:

$$\det(I_2 - D'(z)) \geq \frac{|\text{Im}(z)|^4}{128(\eta^2 + |z|^2)^2}. \quad (\text{A.19})$$

We are now in position to lowerbound  $\det(I_2 - D_0(z))$ . Notice that

$$\begin{aligned} |\det(I_2 - D_0(z))| &= \left| 1 - \frac{z^2}{n} \tilde{\tau}_n \tilde{t}_n \text{Tr} R_n S_n R_n T_n \right|, \\ &\geq 1 - \frac{|z|^2}{n} |\tilde{\tau}_n \tilde{t}_n| (\text{Tr} R_n S_n R_n S_n^*)^{1/2} (\text{Tr} R_n T_n R_n T_n^*)^{1/2}, \\ &\geq (\det(I_2 - D(z)))^{1/2} (\det(I_2 - D'(z)))^{1/2} \end{aligned}$$

by [32, Proposition 6.1]. Proof of Proposition A.1 is completed. □

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