

J-CLASS OF FINITELY GENERATED ABELIAN SEMIGROUPS OF AFFINE MPAS ON \mathbb{C}^n AND HYPERCYCLICITY

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ABSTRACT. We give a characterization of hypercyclic using (locally hypercyclic) of semigroup \mathcal{G} of affine maps of \mathbb{C}^n . We prove the existence of a \mathcal{G} -invariant open subset of \mathbb{C}^n in which any locally hypercyclic orbit is dense in \mathbb{C}^n .

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in \mathbb{C} and $GL(n, \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call A the *linear part* of f . The map f is invertible if $A \in GL(n, \mathbb{C})$. Denote by $MA(n, \mathbb{C})$ the vector space of all affine maps on \mathbb{C}^n and $GA(n, \mathbb{C})$ the group of all invertible affine maps of $MA(n, \mathbb{C})$.

Let \mathcal{G} be an abelian affine sub-semigroup of $MA(n, \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of \mathcal{G} through v : $\mathcal{G}(v) = \{f(v) : f \in \mathcal{G}\} \subset \mathbb{C}^n$. Denote by \overline{E} the closure of a subset $E \subset \mathbb{C}^n$. The semigroup \mathcal{G} is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $\overline{\mathcal{G}(v)} = \mathbb{C}^n$. For an account of results and bibliography on hypercyclicity, we refer to the book [4] by Bayart and Matheron. We refer the reader to the recent book [4] and [6] for a thorough account on hypercyclicity. Costakis and Manoussos in [5] “localize“ the concept of hypercyclicity using J-sets. By analogy, we generalize this notion to affine case as follow: Suppose that \mathcal{G} is generated by p affines maps f_1, \dots, f_p ($p \geq 1$) then for $x \in \mathbb{C}^n$, we define the extended limit set $J_{\mathcal{G}}(x)$ to be the set of $y \in \mathbb{C}^n$ for which there exists a sequence of vectors $(x_m)_m$ with $x_m \rightarrow x$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ with

$$(1.1) \quad k_m^{(1)} + k_m^{(2)} + \dots + k_m^{(p)} \rightarrow +\infty$$

such that

$$f_1^{k_m^{(1)}} f_2^{k_m^{(2)}} \dots f_p^{k_m^{(p)}} x_m \rightarrow y.$$

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Note that condition (1.1) is equivalent to having at least one of the sequences $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ containing a strictly increasing subsequence tending to $+\infty$. We say that \mathcal{G} is *locally hypercyclic* if there exists a vector $v \in \mathbb{C}^n \setminus \{0\}$ such that $J_{\mathcal{G}}(v) = \mathbb{C}^n$. So, the question to investigate is the following: When an abelian sub-semigroup of $MA(n, \mathbb{C})$ can be hypercyclic?

The main purpose of this paper is twofold: firstly, we give a general characterization of the above question for any abelian *sub-semigroup* of $MA(n, \mathbb{C})$ using J-sets. Secondly, we generalize the results proved in [3], by A.Ayadi and H.Marzougi, for linear semigroups, which answers negatively, the question raised in the paper of Costakis and Manoussos [5]: is it true that a locally hypercyclic abelian semigroup H generated by matrices A_1, \dots, A_n is hypercyclic whenever $J_H(x) = \mathbb{C}^n$ for a finite set of $x \in \mathbb{C}^n$ whose vector space is equal \mathbb{C}^n ? Similarly for \mathbb{R}^n .

Denote by $\mathcal{B}_0 = (e_1, \dots, e_n)$ the canonical basis of \mathbb{C}^n . Let $P \in GL(n, \mathbb{C})$.

Let introduce the following notations and definitions. Denote by:

Let $n \in \mathbb{N}_0$ be fixed, denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.
- $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$ the canonical basis of \mathbb{C}^{n+1} and I_{n+1} the identity matrix of $GL(n+1, \mathbb{C})$.

For each $m = 1, 2, \dots, n+1$, denote by:

- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over \mathbb{C} of the form

$$(1.1) \quad \begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix}$$

- $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1.1) with $\mu \neq 0$.

Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $n_1 + \dots + n_r = n+1$. In particular, $r \leq n+1$. Write

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r = 1$, then $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$ and $\eta = (n+1)$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n+1, \mathbb{C})$.
- $u_0 = (e_{1,1}, \dots, e_{r,1}) \in \mathbb{C}^{n+1}$ where $e_{k,1} = (1, 0, \dots, 0) \in \mathbb{C}^{n_k}$, for $k = 1, \dots, r$. So $u_0 \in \{1\} \times \mathbb{C}^n$.
- $p_2 : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ the second projection defined by $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$.
- Define the map $\Phi : GA(n, \mathbb{C}) \rightarrow GL(n+1, \mathbb{C})$

$$f = (A, a) \mapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then Φ is an injective homomorphism of groups. Write

- $G = \Phi(\mathcal{G})$, it is an abelian subgroup of $GL(n + 1, \mathbb{C})$.

Let consider the normal form of G : By Proposition 2.1, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ and a partition η of $(n + 1)$ such that $G' = P^{-1}GP \subset \mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$. For such a choice of matrix P , we let

- $v_0 = Pu_0$. So $v_0 \in \{1\} \times \mathbb{C}^n$, since $P \in \Phi(GA(n, \mathbb{C}))$.
- $w_0 = p_2(v_0) \in \mathbb{C}^n$. We have $v_0 = (1, w_0)$.
- $\varphi = \Phi^{-1}(P) \in GA(n, \mathbb{C})$.

Denote by $V' := \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$, $V = P(V')$ and $U' = p_2(V')$, then

$$(1) \quad U' = \begin{cases} \mathbb{C}^{n_1-1} \times \prod_{k=2}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1} & \text{if } r \geq 2 \\ \mathbb{C}^{n_1-1} & \text{if } r = 1. \end{cases}$$

For such choice of the matrix $P \in \Phi(GA(n + 1, \mathbb{C}))$, we can write

$$P = \begin{bmatrix} 1 & 0 \\ d & Q \end{bmatrix}, \quad \text{with } Q \in GL(n, \mathbb{C}).$$

We have $\varphi = (Q, d) \in GA(n, \mathbb{C})$, $U = \varphi(U')$ and $G^* = G \cap GL(n + 1, \mathbb{C})$. We have G^* is an abelian semigroup of $GL(n + 1, \mathbb{C})$.

Our principal results are the following:

Theorem 1.1. *Let \mathcal{G} be a finitely generated abelian semigroup of affine maps on \mathbb{C}^n . If $J_{\mathcal{G}}(v) = \mathbb{C}^n$ for some $v \in U$ then $\overline{\mathcal{G}(v)} = \mathbb{C}^n$.*

Corollary 1.2. *Under the hypothesis of Theorem 1.1, the following are equivalent:*

- \mathcal{G} is hypercyclic.
- $\overline{J_{\mathcal{G}}(w_0)} = \mathbb{C}^n$.
- $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$.

Corollary 1.3. *Under the hypothesis of Theorem 1.1, set $\mathcal{A} = \{x \in \mathbb{C}^n : J_{\mathcal{G}}(x) = \mathbb{C}^n\}$. If \mathcal{G} is not hypercyclic then $\mathcal{A} \subset \bigcup_{k=1}^r H_k$, ($r \leq n$) where H_k are \mathcal{G} -invariant affine subspaces of \mathbb{C}^n with dimension $n - 1$.*

2. Preliminaries and basic notions

Proposition 2.1. *Let \mathcal{G} be an abelian sub-semigroup of $MA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a sub-semigroup of $\mathcal{K}_{\eta, r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.*

To prove Proposition 2.1, we need the following proposition.

Proposition 2.2. ([2], Proposition 2.1) *Let \mathcal{G} be an abelian subgroup of $GA(n, \mathbb{C})$ and $G = \Phi(\mathcal{G})$. Then there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.*

Proof of Proposition 2.1. Suppose first, $G \subset GL(n + 1, \mathbb{C})$. Let \widehat{G} be the group generated by G . Then \widehat{G} is abelian and by Proposition 2.2, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\widehat{G}P$ is an abelian subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$, for some $r \in \{1, \dots, n + 1\}$ and $\eta \in (\mathbb{N}_0)^r$. In particular, $P^{-1}GP \subset \mathcal{K}_{\eta, r}^*(\mathbb{C})$.

Suppose now, $G \subset M_{n+1}(\mathbb{C})$. For every $A \in G$, there exists $\lambda_A \in \mathbb{C}$ such that $(A - \lambda_A I_{n+1}) \in GL(n + 1, \mathbb{C})$ (one can take λ_A non eigenvalue of A). Write \widehat{L} be the group generated by $L := \{A - \lambda_A I_{n+1} : A \in G\}$. Then \widehat{L} is an abelian subsemigroup of $GL(n + 1, \mathbb{C})$. Hence by above, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\widehat{L}P \subset \mathcal{K}_{\eta, r}^*(\mathbb{C})$, for some $\eta \in (\mathbb{N}_0)^r$. As

$$P^{-1}LP = \{P^{-1}AP - \lambda_A I_{n+1} : A \in G\}$$

then $P^{-1}GP \subset \mathcal{K}_{\eta, r}(\mathbb{C})$. This proves the proposition. \square

Let \widetilde{G} be the semigroup generated by G and $\mathbb{C}I_{n+1} = \{\lambda I_{n+1} : \lambda \in \mathbb{C}\}$. Then \widetilde{G} is an abelian sub-semigroup of $M(n + 1, \mathbb{C})$. By Proposition 2.1, there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a sub-semigroup of $\mathcal{K}_{\eta, r}(\mathbb{C})$ for some $r \leq n + 1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ and this also implies that $P^{-1}\widetilde{G}P$ is a sub-semigroup of $\mathcal{K}_{\eta, r}(\mathbb{C})$.

Lemma 2.3. *Let $x \in \mathbb{C}^n$ and $G = \Phi(\mathcal{G})$. The following are equivalent:*

- (i) $\overline{\mathcal{G}(x)} = \mathbb{C}^n$.
- (ii) $\overline{G(1, x)} = \{1\} \times \mathbb{C}^n$.
- (iii) $\widetilde{G}(1, x) = \mathbb{C}^{n+1}$.

Proof. (i) \iff (ii) : is obvious since $\{1\} \times \mathcal{G}(x) = G(1, x)$ by construction.
 (iii) \implies (ii) : Let $y \in \mathbb{C}^n$ and $(B_m)_m$ be a sequence in \tilde{G} such that $\lim_{m \rightarrow +\infty} B_m(1, x) = (1, y)$. One can write $B_m = \lambda_m \Phi(f_m)$, with $f_m \in \mathcal{G}$ and $\lambda_m \in \mathbb{C}^*$, thus $B_m(1, x) = (\lambda_m, \lambda_m f_m(x))$, so $\lim_{m \rightarrow +\infty} \lambda_m = 1$. Therefore,
 $\lim_{m \rightarrow +\infty} \Phi(f_m)(1, x) = \lim_{m \rightarrow +\infty} \frac{1}{\lambda_m} B_m(1, x) = (1, y)$. Hence, $(1, y) \in \overline{G(1, x)}$.
 (ii) \implies (iii) : Since $\mathbb{C}^{n+1} \setminus (\{0\} \times \mathbb{C}^n) = \bigcup_{\lambda \in \mathbb{C}^*} \lambda(\{1\} \times \mathbb{C}^n)$ and for every $\lambda \in \mathbb{C}^*$, $\lambda G(1, x) \subset \tilde{G}(1, x)$, we get

$$\begin{aligned} \mathbb{C}^{n+1} &= \overline{\mathbb{C}^{n+1} \setminus (\{0\} \times \mathbb{C}^n)} \\ &= \overline{\bigcup_{\lambda \in \mathbb{C}^*} \lambda(\{1\} \times \mathbb{C}^n)} \\ &= \overline{\bigcup_{\lambda \in \mathbb{C}^*} \lambda \overline{G(1, x)}} \subset \overline{\tilde{G}(1, x)} \end{aligned}$$

Hence $\mathbb{C}^{n+1} = \overline{\tilde{G}(1, x)}$. □

We will use the following theorem, to prove Lemma 2.5:

Theorem 2.4. ([1], Theorem 1.1) *Let \tilde{G} be a finitely generated abelian semi-group of matrices of $M_{n+1}(\mathbb{C})$. If $J_{\tilde{G}}(v) = \mathbb{C}^{n+1}$ for some $v \in V$ then $\overline{\tilde{G}(v)} = \mathbb{C}^{n+1}$.*

Lemma 2.5. *Let \mathcal{G} be an abelian semigroup of affine maps generated by f_1, \dots, f_p and $x \in \mathbb{C}^n$. Then the following assertions are equivalent:*

- (i) $y \in J_{\mathcal{G}}(x)$.
- (ii) $(1, y) \in J_{\tilde{G}}(1, x)$.

Proof. (i) \implies (ii) : Since $y \in J_{\mathcal{G}}(x)$, then there exists a sequence of vectors $(x_m)_m$ with $x_m \rightarrow x$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ with

$$(1.1) \quad k_m^{(1)} + k_m^{(2)} + \dots + k_m^{(p)} \rightarrow +\infty$$

such that

$$f_1^{k_m^{(1)}} f_2^{k_m^{(2)}} \dots f_p^{k_m^{(p)}} x_m \rightarrow y.$$

Therefore, $(1, x_m) \rightarrow (1, x)$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ such that

$$\Phi(f_1)^{k_m^{(1)}} \Phi(f_2)^{k_m^{(2)}} \dots \Phi(f_p)^{k_m^{(p)}} (1, x_m) \rightarrow (1, y).$$

Hence $(1, y) \in J_{\tilde{G}}(1, x)$.

(ii) \implies (i) : Since $(1, y) \in J_{\tilde{G}}(1, x)$, then there exists a sequence of vectors

$(\lambda_m, x_m)_m \subset \mathbb{C} \times \mathbb{C}^n$ with $(\lambda_m, x_m) \longrightarrow (1, x)$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ with

$$(1.2) \quad k_m^{(1)} + k_m^{(2)} + \dots + k_m^{(p)} \longrightarrow +\infty$$

such that

$$\alpha_1^{k_m^{(1)}} \dots \alpha_p^{k_m^{(p)}} \Phi(f_1)^{k_m^{(1)}} \Phi(f_2)^{k_m^{(2)}} \dots \Phi(f_p)^{k_m^{(p)}} (\lambda_m, x_m) \longrightarrow (1, y).$$

Denote by $c_m = \alpha_1^{k_m^{(1)}} \dots \alpha_p^{k_m^{(p)}}$ and

$$g_m = \alpha_1^{k_m^{(1)}} \dots \alpha_p^{k_m^{(p)}} \Phi(f_1)^{k_m^{(1)}} \Phi(f_2)^{k_m^{(2)}} \dots \Phi(f_p)^{k_m^{(p)}}.$$

Then $\lim_{m \rightarrow +\infty} c_m \lambda_m = 1$. As $\lim_{m \rightarrow +\infty} \lambda_m = 1$, so $\lim_{m \rightarrow +\infty} c_m = 1$. It follows that

$$\lim_{m \rightarrow +\infty} \frac{1}{c_m} g_m(1, x_m) = (1, y). \text{ Hence } \lim_{m \rightarrow +\infty} f_1^{k_m^{(1)}} f_2^{k_m^{(2)}} \dots f_p^{k_m^{(p)}} (x_m) = y. \quad \square$$

Proof of Theorem 1.1. Suppose that $J_{\mathcal{G}}(v) = \mathbb{C}^n$ with $v \in U$. By Proposition 2.1, we can assume that $G \subset \mathcal{K}_{\eta, r}(\mathbb{C})$ and $P = I_{n+1}$. By Lemma 2.5, we have $J_{\widetilde{G}}(1, v) = \mathbb{C}^{n+1}$. Then by Theorem 2.4, $\overline{G}(1, v) = \mathbb{C}^{n+1}$. It follows by Lemma 2.3, that $\overline{\mathcal{G}}(v) = \mathbb{C}^n$. \square

Now to prove Corollary 1.2, we need to use the following proposition:

Proposition 2.6. [5] *Let G be an abelian sub-semigroup of $M_{n+1}(\mathbb{C})$ generated by A_1, \dots, A_p , $p \geq 1$. Then G is hypercyclic if and only if $J_G(x) = \mathbb{C}^{n+1}$ for every $x \in \mathbb{C}^{n+1}$.*

Proof of Corollary 1.2. (i) \implies (ii) follows from Proposition 2.6 and (ii) \implies (iii) results from Theorem 1.1. (iii) \implies (i) : is trivial.

Proof of Corollary 1.3. If \mathcal{G} is not hypercyclic then by Theorem 1.1, $J_{\mathcal{G}}(v) \neq \mathbb{C}^n$ for any $v \in U'$, thus $U \cap \mathcal{A} = \emptyset$ and therefore $\mathcal{A} \subset \mathbb{C}^n \setminus U$. On the other hand, $U = \varphi(U')$, so

$$\begin{aligned} \mathbb{C}^n \setminus U &= \mathbb{C}^n \setminus \varphi(U') \\ &= \varphi(\mathbb{C}^n \setminus U') \\ &= \varphi \left(\bigcup_{k=1}^r L_k \right) \quad (\text{by (1)}) \\ &= \bigcup_{k=1}^r \varphi(L_k) \end{aligned}$$

with $L_k = \{x = [x_1, \dots, x_r]^T, x_k \in \{0\} \times \mathbb{C}^{n_k-1}, x_i \in \mathbb{C}^{n_i}, \text{ if } i \neq k\}$. It follows that $\mathcal{A} \subset \bigcup_{k=1}^r H_k$ with $H_k = \varphi(L_k)$ is an affine space with dimension $n - 1$. \square

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