



Topological sensitivity analysis of piezoelectric actuator

A.A. Novotny, Jan Sokolowski

► To cite this version:

A.A. Novotny, Jan Sokolowski. Topological sensitivity analysis of piezoelectric actuator. [Research Report] LNCC/MCT; IECL. 2013. hal-00861461

HAL Id: hal-00861461

<https://hal.science/hal-00861461>

Submitted on 12 Sep 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TOPOLOGICAL SENSITIVITY ANALYSIS OF PIEZOELECTRIC ACTUATOR

A.A. NOVOTNY AND J. SOKOŁOWSKI

ABSTRACT. The coupled model of a piezoelectric actuator is considered in the paper. For the specific shape optimization problem the shape-topological sensitivity analysis is performed. The obtained result can be used within the optimum design procedure of a micromechanism which transforms the electrical energy supplemented via its piezoceramic part into elastic energy of an actuator. The domain decomposition technique is applied in order to obtain the topological derivatives of the tracking-type functional defined on a part of the boundary of the elastic body under consideration. The proposed method is general and can be used for the purposes of the shape-topological sensitivity analysis for a broad class of multiphysics models.

1. INTRODUCTION

Applications of the asymptotic analysis in singularly perturbed geometrical domains combined with the domain decomposition technique for the shape-topological sensitivity analysis of coupled partial differential equations is proposed in the paper. The geometrical domain Ω for the model under consideration is decomposed into subdomains with the different physical properties. The influence of the subdomains each on the another is effected by the transmission conditions on the interfaces. In other words, the fields of mechanical and electric natures are coupled by the transmission conditions as well as by the mathematical models in the interior of each subdomain. The resulting model is complex, and its topological sensitivity analysis should be performed in such a way that the obtained formula can be directly used in numerical methods of shape optimization. To this end the fictitious domain decomposition is employed for the purposes of the shape-topological analysis. This means that in the elastic material of the body the fictitious ring domain $C(R, \varepsilon) := \{\varepsilon < |x| < R\}$ is introduced for the purposes of the asymptotic analysis with respect to $\varepsilon \rightarrow 0$, and the result of the analysis is expressed on the boundary of the ball $B_R = \{|x| < R\}$. From the asymptotic expansions of elastic energy in the interior of B_R or of $C(R, \varepsilon)$ the expansion of the Dirichlet-to-Neumann map associated with the ball or with the ring is obtained.

Remark 1. *In the paper the nonlocal boundary operator is considered on the fictitious boundary ∂B_R . The operator is defined in the ball B_R as the Dirichlet-to-Neumann map for the linear elasticity boundary value problem. Its asymptotic analysis is performed in B_R with respect to small parameter $\varepsilon \rightarrow 0$ which governs the material properties of the elastic body. Having the asymptotic expansion in hand, the operator is employed as the Steklov-Poincaré operator in the truncated domain $\Omega_R := \Omega \setminus \overline{B}_R$. In this way, the influence of singularity associated with the limit passage $\varepsilon \rightarrow 0$ is modeled in the truncated domain via nonlocal boundary conditions, and the subdomain B_R is eliminated from the shape-topological analysis of the shape functional under considerations. This approach simplifies the asymptotic analysis performed in the paper. Therefore, there is a double notation for the same boundary pseudodifferential operator on ∂B_R depending on the fact if ∂B_R is considered as the exterior boundary of the fictitious subdomain B_R or the interior boundary of truncated domain Ω_R . First, the Dirichlet-to-Neumann map is considered for B_R or for $C(R, \varepsilon)$, see the details below, and its expansion is obtained with respect to the small parameter which governs the size of a inclusion or a hole in B_R . Then, the same expansion is used in the nonlocal boundary conditions for the truncated domain. More precisely, the nonlocal boundary operator stands for the Steklov-Poincaré operator when acting on the interior boundary ∂B_R of the truncated domain $\Omega_R := \Omega \setminus \overline{B}_R$. Thus, the elasticity boundary value problem in B_R is called the interior problem of fictitious domain decomposition, and the coupled model*

Key words and phrases. Asymptotic analysis, Steklov-Poincaré operator, Dirichlet-to-Neumann map, domain decomposition, topological derivative, piezoelectric actuator, topology design.

in Ω_R is called the exterior problem of fictitious domain decomposition in the notation employed in the paper. The boundary ∂B_R is used as an interface of the fictitious domain decomposition introduced exclusively for the purposes of asymptotic analysis in the singularly perturbed domain Ω .

In the ball B_R the contrast parameter $0 < \gamma < \infty$ is considered for inclusions and the hole is obtained by the limit passage $\gamma \rightarrow 0$. The domain decomposition method combined with the asymptotic expansions in the ring associated with the small parameter $\varepsilon \rightarrow 0$ allows us to find the topological derivative of the shape functional defined in the truncated domain $\Omega \setminus \overline{B_R}$ for all $R > \varepsilon \rightarrow 0$. Otherwise, the asymptotic analysis of the coupled model in singularly perturbed geometrical domain Ω should be performed which would not be the best idea because of the complexity of such an approach.

In particular, we are interested in design of piezoelectric actuators, which consist of multi-flexible structures actuated by piezoceramic devices that generate an output displacement in a specified direction on the boundary of the actuated part. The multi-flexible structure transforms the piezoceramic output displacement by amplifying and changing its direction. This kind of mechanism can be manufactured in a very small scale. Therefore, the spectrum of applications of such microtools becomes broader in recent years including microsurgery, nanotechnology processing, cell manipulation, among others. Yet, the development of microtools requires the design of actuated multi-flexible structures which are able to produce complex movements originated from simple expansion/contraction of the piezoceramic actuator. The performance of microtools can be strongly enhanced by optimizing the actuated multi-flexible structures with respect to their shape and their topology. The shape sensitivity analysis of such coupled models has been fully developed in [10] and [9] for quasi-electrostatic layered piezoelectric devices and for non-stationary elastic, piezoelectric and acoustic coupled system, respectively. However, a more general approach to deal with shape and topology optimization design is based on the topological derivative. In fact, this relatively new concept represents the first term of the asymptotic expansion of a given shape functional with respect to the small parameter which measures the size of singular domain perturbations, such as holes, inclusions, source-terms and cracks. The topological asymptotic analysis was introduced in the fundamental paper [13] and has been successfully applied in the treatment of problems such as topology optimization [3], inverse analysis [7], image processing [6], multi-scale constitutive modeling [2], fracture mechanics sensitivity analysis [4] and damage evolution modeling [1]. For an account of new developments in this branch of shape optimization we refer to [12].

In this paper the topological derivative is applied in the context of topology optimization of piezoelectric actuated multi-flexible structures. The basic idea consists in maximizing the performance of the microtool by introducing of small inclusions in the multi-flexible elastic part. Since this problem is modeled by a coupled electro-mechanical system, the domain decomposition technique combined with the Steklov-Poincaré pseudo-differential boundary operator is used to derive the first order term of the asymptotic expansion of the shape functional with respect to the small parameter measuring the size of the inclusions. In particular, the closed form of the topological derivative is obtained, which can be used e.g., as a steepest descent direction in the microtools design by topology optimization method.

The paper is organized as follows. In Section 2 the Steklov-Poincaré pseudo-differential boundary operator is introduced. The electro-mechanical coupled system modeling the piezoelectric actuators as well as the adopted shape functional are presented in Section 3. Finally, in Section 4 the associated topological asymptotic expansion is rigorously derived. The closed form of topological derivatives for inclusion and holes are obtained for two and three spatial dimensions [12]. Some concluding remarks and perspectives are given in Section 5.

2. THE STEKLOV-POINCARÉ AND DIRICHLET-TO-NEUMANN BOUNDARY OPERATORS

In order to perform the shape-topological sensitivity analysis of a coupled model defined in Ω , in the presence of singular domain perturbations resulting from the insertion of cavities, holes or inclusions in the elastic subdomain, the fictitious domain decomposition $\Omega := \Omega_R \cup \overline{B_R}$ is introduced into the model. The small region B_R which includes the singular domain perturbation B_ε is selected for the asymptotic analysis performed e.g., by the method of compound asymptotic expansions for the singularity depending on the small parameter $\varepsilon \rightarrow 0$. The remaining part Ω_R of Ω is called

the truncated domain. The result of asymptotic analysis in B_R is specified on the fictitious boundary $\Gamma_R := \partial B_R$ for the Dirichlet-to-Neumann operator associated with B_R . The same operator is called the Steklov-Poincaré operator for the boundary value problem defined in Ω_R . In this way the asymptotic analysis is performed in a simple geometry with radial symmetry of the ball B_R , and it is separated from the shape-topological sensitivity analysis of the functional which is performed in Ω_R with the nonlocal boundary conditions defined by the Steklov-Poincaré operator on Γ_R , and in the absence of the singularity B_ε inside of the domain.

The asymptotic expansion of Dirichlet-to-Neumann boundary operators with respect to $\varepsilon \rightarrow 0$ is performed for the nonhomogeneous Dirichlet boundary value problem of linear elasticity defined in the ring $C(R, \varepsilon)$. The case of an inclusion B_ε in B_R can be considered as a regular perturbation of the bilinear form associated with the boundary value problem in Ω_R , depending on the contrast parameter $0 < \gamma < \infty$. The limit case obtained for $\gamma \rightarrow 0$ with the inclusion which is replaced by a hole or a cavity of the radius $\varepsilon \rightarrow 0$ is considered in $C(R, \varepsilon)$ as the singular perturbation of B_R .

Thus we present the detailed asymptotic analysis following [15] of the representative singular domain perturbation by the insertion of a hole in two spatial dimensions. The case of an inclusion can be analysed in the same manner by using the Kolosov complex potentials in two spatial dimensions. The asymptotic analysis in three spatial dimensions can be performed by an application of the method of compound asymptotic expansions.

Thus, we introduce the Steklov-Poincaré pseudo-differential boundary operator obtained explicitly from the closed solutions of the two dimensional linear elasticity boundary value problems.

2.1. Dirichlet-to-Neumann map on Γ_R . Let us consider an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and let $\Omega_R = \Omega \setminus \overline{B_R}$ represent an elastic body, where $B_R = \{|x| < R\}$ is a ball of radius R and center at the origin \mathcal{O} . We define the Dirichlet-to Neumann map in B_R .

The nonhomogeneous Dirichlet boundary value problem of linear elasticity is considered in B_R , namely

$$u = \varphi \text{ on } \Gamma_R := \partial B_R, \quad u \in H^1(B_R; \mathbb{R}^d) : a(u, v) = 0 \quad \forall v \in H_0^1(B_R; \mathbb{R}^d), \quad (2.1)$$

where $\sigma(u) = C \nabla^s u$, with C the Hooke's tensor of elastic constants, and

$$a(u, v) = \int_{B_R} \sigma(u) \cdot \nabla^s v. \quad (2.2)$$

With the solution of the nonhomogeneous Dirichlet boundary value problem is associated its Neumann trace $T(u)$ such that the Green's formula is valid

$$a(u, v) = (Lu, v) - (T(u), v)_{\Gamma_R}. \quad (2.3)$$

Since there is no source in B_R , and taking into account the Dirichlet condition, the Green's formula becomes

$$a(u, v) = -(T(\varphi), v)_{\Gamma_R}. \quad (2.4)$$

The Dirichlet-to-Neumann map $\mathcal{A} : H^{1/2}(\Gamma_R; \mathbb{R}^d) \rightarrow H^{-1/2}(\Gamma_R; \mathbb{R}^d)$ is defined and it takes the form

$$(\mathcal{A}(\varphi), v)_{\Gamma_R} := -(T(\varphi), v)_{\Gamma_R} \equiv a(u, \varphi). \quad (2.5)$$

Thus

$$(\mathcal{A}(\varphi), \varphi)_{\Gamma_R} \equiv a(u, u), \quad (2.6)$$

where $(\varphi, \varphi) \rightarrow (\mathcal{A}(\varphi), \varphi)_{\Gamma_R}$ is a symmetric and coercive bilinear form on the space of traces $H^{1/2}(\Gamma_R; \mathbb{R}^d)$.

2.2. Dirichlet-to-Neumann map on Γ_R for B_R with an inclusion ω_ε . Let $\varepsilon \rightarrow 0$ be a small parameter and assume that ω_ε , of the characteristic function $x \rightarrow \chi_\varepsilon(x) \in \{0, 1\}$, is a small inclusion of radius ε and centre at the origin. In particular, $\omega_\varepsilon := B_\varepsilon$ is considered here.

Define in the perturbed domain, denoted by the same symbol B_R , with the variable Hooke's tensor of elastic constants

$$x \rightarrow C_\varepsilon(x) = (1 - \chi_\varepsilon(x))C + \gamma\chi_\varepsilon(x)C \quad (2.7)$$

with the contrast parameter γ . Hence the constitutive relation of linear elasticity $\sigma_\varepsilon(u) = C_\varepsilon \nabla^s u$ is now dependent on the small parameter ε . The boundary value problem of linear elasticity is considered in B_R

$$u_\varepsilon = \varphi \text{ on } \Gamma_R := \partial B_R, \quad u_\varepsilon \in H^1(B_R; \mathbb{R}^d) : a_\varepsilon(u_\varepsilon, v) = 0 \quad \forall v \in H_0^1(B_R; \mathbb{R}^d), \quad (2.8)$$

where

$$a_\varepsilon(u_\varepsilon, v) = \int_{B_R} \sigma_\varepsilon(u_\varepsilon) \cdot \nabla^s v. \quad (2.9)$$

With the solution u_ε of the nonhomogeneous Dirichlet boundary value problem is associated its Neumann trace $T_\varepsilon(u_\varepsilon)$ such that the Green's formula is valid

$$a_\varepsilon(u_\varepsilon, v) = (L_\varepsilon u_\varepsilon, v) - (T_\varepsilon(u_\varepsilon), v)_{\Gamma_R}. \quad (2.10)$$

Thus, the Dirichlet-to-Neumann map takes the form

$$(\mathcal{A}_\varepsilon(\varphi), \varphi)_{\Gamma_R} \equiv a_\varepsilon(u_\varepsilon, u_\varepsilon), \quad (2.11)$$

since there is no source in B_R .

Now we are going to recall some results of [15] on the asymptotic expansion of the Steklov-Poincaré operator for the singular perturbations of a ring, i.e. in the limit case of $\gamma \rightarrow 0$ and for the interior radius of the ring which tends to zero. This means that in such a case of singular perturbations the inclusion ω_ε becomes a hole, the case of an inclusion for $0 < \gamma < \infty$ is considered as a regular perturbation in coefficients of the elliptic operators and it can be analysed from the point of view of asymptotic analysis in the similar way.

2.3. Singular perturbations of solutions in the ring. Let us consider a ring $C(R, \varepsilon) := B_R \setminus \overline{B_\varepsilon} = \{\varepsilon < |x| < R\}$. In order to establish the exact formula for the Steklov-Poincaré operator in plane elasticity we derive the analytic form of the solution for the elasticity system in the ring, with general displacement condition on the outer boundary and traction free inner boundary, parameterized by the (small) inner radius ε .

Let us assume for simplicity that the center of the ring lies at origin of the coordinate system, and take polar coordinates (r, θ) with \mathbf{e}_r pointing outwards and \mathbf{e}_θ perpendicularly in the counter-clockwise direction. Then the displacement on the outer boundary $r = R$ is given in the form of the Fourier series

$$2\mu(u_r + iu_\theta) = \sum_{k=-\infty}^{k=+\infty} U_k e^{ik\theta} \quad (2.12)$$

The required regularity condition for the boundary data translates into some inequalities for coefficients U_k , as it will be made precise later.

The solution in the ring must be compared with the solution in the full circle, so we will have to construct it as well. Probably the best tool for obtaining both exact solutions is the complex variable method, described in [11].

Proposition 2. *For plane domains with one hole solutions of elasticity boundary value problems take the form*

$$\begin{aligned} \sigma_{rr} - i\sigma_{r\theta} &= 2\Re\phi' - e^{2i\theta}(\bar{z}\phi'' + \psi') \\ \sigma_{rr} + i\sigma_{\theta\theta} &= 4\Re\phi' \\ 2\mu(u_r + iu_\theta) &= e^{-i\theta}(\kappa\phi - z\bar{\phi}' - \bar{\psi}) \end{aligned} \quad (2.13)$$

where ϕ, ψ are given by complex series

$$\begin{aligned} \phi &= A \log(z) + \sum_{k=-\infty}^{k=+\infty} a_k z^k \\ \psi &= -\kappa \bar{A} \log(z) + \sum_{k=-\infty}^{k=+\infty} b_k z^k \end{aligned} \quad (2.14)$$

Here μ is the Lamé constant, ν the Poisson ratio, $\kappa = 3 - 4\nu$ in the plane strain case, and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress. In addition, $\Re\varphi$ is used to denote the real part of a complex function φ , while $\Im\varphi$ is going to be used to denote its imaginary counterpart.

Now we can substitute displacement condition for $r = R$ into

$$2\mu(u_r + iu_\theta) = 2\kappa Ar \log(r) \frac{1}{z} - \bar{A} \frac{1}{r} z + \sum_{p=-\infty}^{p=+\infty} [\kappa r a_{p+1} - (1-p)\bar{a}_{1-p} r^{-2p+1} - \bar{b}_{-(p+1)} r^{-2p-1}] z^p$$

and obtain the infinite system of linear equations.

$$\begin{aligned} p = -1: \quad & 2\kappa Ar \log(r) + (\kappa a_0 - \bar{b}_0) - 2\bar{a}_2 r^2 = U_{-1} \\ p = 1: \quad & -\bar{A} + \kappa r^2 a_2 - \bar{b}_{-2} \frac{1}{r^2} = U_1 \\ p \notin \{-1, 1\}: \quad & \kappa r^{p+1} a_{p+1} - (1-p)\bar{a}_{1-p} r^{-p+1} - \bar{b}_{-(p+1)} r^{-(p+1)} = U_p \end{aligned} \quad (2.15)$$

The traction-free condition

$$\sigma \mathbf{e}_r = [\sigma_{rr}, \sigma_{r\theta}]^\top$$

on some circle means $\sigma_{rr} = \sigma_{r\theta} = 0$. Hence, assuming $r := \varepsilon$, we have another infinite system.

$$\begin{aligned} p = -1: \quad & 2A + 2\bar{a}_2 r^2 + 2\frac{1}{r^2} b_{-2} = 0 \\ p = 1: \quad & (\kappa + 1) \frac{1}{r^2} \bar{A} = 0 \\ p \notin \{-1, 1\}: \quad & (1+p)a_{p+1} + \bar{a}_{1-p} r^{-2p} + \frac{1}{r^2} b_{p-1} = 0 \end{aligned} \quad (2.16)$$

Denote $d_0 = \kappa a_0 - \bar{b}_0$ since a_0, b_0 appear only in this combination. Using (2.15) we may recover the solution for the full circle. Because in this case the singularities must vanish, we have $b_{-k} = a_{-k} = A = 0$ for $k = 1, 2, \dots$ and comparing the same powers of r :

$$\begin{aligned} d_0^0 &= U_{-1} + \frac{2}{\kappa} \bar{U}_1, \quad \Re a_1^0 = \frac{1}{(\kappa - 1)R} \Re U_0, \quad \Im a_1^0 = \frac{1}{(\kappa + 1)R} \Im U_0 \\ a_k^0 &= \frac{1}{\kappa R^k} U_{k-1}, \quad b_k^0 = -\frac{1}{R^k} [(k+2) \frac{1}{\kappa} U_{k+1} + \bar{U}_{-(k+1)}], \quad k > 1 \end{aligned} \quad (2.17)$$

Now let us repeat the same procedure for the ring. Here the singularities may be present, because 0 does not belong to the domain. Hence, from (2.15) for $r = R$ and (2.16) for $r = \varepsilon$ we obtain $A = 0$ and the formulas

$$\begin{aligned} d_0 &= A_{-1} + \frac{2R^4}{\kappa R^4 + \varepsilon^4} \bar{U}_1, \quad a_2 = \frac{R^2}{\kappa R^4 + \varepsilon^4} U_1 \\ \Re a_1 &= \frac{R}{(\kappa - 1)R^2 + 2\varepsilon^2} \Re U_0, \quad \Im a_1 = \frac{1}{\kappa + 1} \Im A_0 \\ b_{-1} &= -\frac{2\varepsilon^2 R}{(\kappa - 1)R^2 + 2\varepsilon^2} \Re U_0, \quad b_{-2} = -\frac{\varepsilon^4 R^2}{\kappa R^4 + \varepsilon^4} \bar{U}_1 \end{aligned}$$

The rest of the coefficients will be computed later. However, we may at this stage compare the results with known solutions for the uniformly stretched circle or ring obtained in another way. In such a case $U_0 = 2\mu u_r(R)$ does not vanish and, for the full circle, $\psi = 0$, $\phi = a_1^0 z$ with

$$a_1^0 = \frac{2\mu}{(\kappa - 1)R} u_r(R).$$

For the ring we have $\phi = a_1 z$, $\psi = b_{-1} \frac{1}{z}$ where

$$a_1 = \frac{1}{(\kappa - 1) + 2\varepsilon^2} 2\mu u_R(1), \quad b_{-1} = -\frac{2\varepsilon^2}{(\kappa - 1) + 2\varepsilon^2} 2\mu u_R(1).$$

After substitutions we obtain, in both cases, the same results as given in [8]. Similarly the comparison with the solution for the ring with displacement conditions on both boundaries, obtained in [5] also using complex method, confirms the correctness of the formulas.

There remains to compute the rest of the coefficients a_k, b_k for the case of the ring. Taking $p = -k$, $k = 2, 3, \dots$ in conditions on both boundaries gives the system

$$\begin{aligned} \kappa a_{-(k-1)} R^{-(k-1)} - (k+1) \bar{a}_{k+1} R^{k+1} - \bar{b}_{k-1} R^{k-1} &= U_{-k} \\ -(k-1) a_{-(k-1)} \varepsilon^2 + \bar{a}_{k+1} \varepsilon^{2(k+1)} + b_{-(k+1)} &= 0, \end{aligned} \quad (2.18)$$

while $p = +k$, $k = 2, 3, \dots$ results in

$$\begin{aligned} \kappa a_{k+1} R^{k+1} + (k-1) \bar{a}_{-(k-1)} R^{-(k-1)} - \bar{b}_{-(k+1)} R^{-(k+1)} &= U_k \\ (k+1) a_{k+1} \varepsilon^{2(k+1)} + \bar{a}_{-(k-1)} \varepsilon^2 + b_{k-1} \varepsilon^{2k} &= 0 \end{aligned} \quad (2.19)$$

These systems may be represented in a recursive form, convenient for numerical computations and further analysis. Namely,

$$\begin{bmatrix} S_k(\varepsilon)_{11} & S_k(\varepsilon)_{12} \\ S_k(\varepsilon)_{21} & S_k(\varepsilon)_{22} \end{bmatrix} \begin{bmatrix} a_{k+1} \\ b_{k-1} \end{bmatrix} = \begin{bmatrix} U_k \\ \bar{U}_{-k} \end{bmatrix} \quad (2.20)$$

where the entries $S_k(\varepsilon)_{ij}$ are given by

$$\begin{aligned} S_k(\varepsilon)_{11} &= \kappa R^{k+1} - (k^2 - 1) R^{1-k} \varepsilon^{2k} + k^2 R^{-(k+1)} \varepsilon^{2(k+1)} \\ S_k(\varepsilon)_{12} &= -(k-1) (R^{1-k} \varepsilon^{2(k-1)} - R^{-(k+1)} \varepsilon^{2k}) \\ S_k(\varepsilon)_{21} &= -(k+1) (R^{k+1} + \kappa R^{1-k} \varepsilon^{2k}) \\ S_k(\varepsilon)_{22} &= -R^{k-1} - \kappa R^{1-k} \varepsilon^{2(k-1)} \end{aligned}$$

as well as

$$\begin{bmatrix} a_{-(k-1)} \\ b_{-(k+1)} \end{bmatrix} = \begin{bmatrix} -(k+1) \varepsilon^{2k} & -\varepsilon^{2(k-1)} \\ -k^2 \varepsilon^{2(k+1)} & -(k-1) \varepsilon^{2k} \end{bmatrix} \begin{bmatrix} \bar{a}_{k+1} \\ \bar{b}_{k-1} \end{bmatrix}, \quad (2.21)$$

In fact the formulas (2.21), (2.20) are correct also for $k = 0, 1$ and in the limit $\varepsilon \rightarrow 0+$, but the derivation must separate these cases. Thus for, given $k > 1$ and using some initial a_k, b_k obtained earlier, we may first compute a_{k+1}, b_{k-1} using (2.20) and then $a_{-(k-1)}, b_{-(k+1)}$ from (2.21).

We may now use the above results for the asymptotic analysis of the solution. To simplify the formulas, we assume $R = 1$, which means only re-scaling and does not diminish generality (in general case ε would be replaced by ε/R). Then by direct computation we get the following bounds for the differences between the coefficients on the full circle and the ring. For the initial values of k they read

$$\begin{aligned} d_0 - d_0^0 &= -\varepsilon^4 \frac{2}{\kappa(\kappa R^4 + \varepsilon^4)} \bar{U}_1 \\ a_1 - a_1^0 &= -\varepsilon^2 \frac{2}{(\kappa - 1)R((\kappa - 1)R^2 + 2\varepsilon^2)} \Re U_0 \\ a_2 - a_2^0 &= -\varepsilon^4 \frac{1}{\kappa R^2(\kappa R^4 + \varepsilon^4)} U_1 \end{aligned} \quad (2.22)$$

and for higher values

$$|a_3 - a_3^0| \leq \Lambda (|U_2| \varepsilon^4 + |U_{-2}| \varepsilon^2) \quad (2.23)$$

and for $k = 4, 5, \dots$

$$|a_k - a_k^0| \leq \Lambda \left(|U_{k-1}| \varepsilon^{3(k-1)/2} + |U_{1-k}| \varepsilon^{3(k-2)/2} \right) \quad (2.24)$$

where the exponent $k/2$ has been used to counteract the growth of k^2 in terms like $k^2 \varepsilon^{k/2}$. Similarly

$$|b_1 - b_1^0| \leq \Lambda (|U_2| \varepsilon^4 + |U_{-2}| \varepsilon^2) \quad (2.25)$$

and for $k = 2, 3, \dots$

$$|b_k - b_k^0| \leq \Lambda \left(|U_{k+1}| \varepsilon^{3(k+1)/2} + |U_{-(k+1)}| \varepsilon^{3k/2} \right) \quad (2.26)$$

From relation (2.21) we get another estimates

$$\begin{aligned} |a_{-k}| &\leq \Lambda \varepsilon^{2k} (|U_{k+1}| + |U_{-(k+1)}|), \quad k = 1, 2, \dots \\ |b_{-k}| &\leq \Lambda \varepsilon^{2(k-1)} (|U_{k-1}| + |U_{1-k}|), \quad k = 3, 4, \dots \end{aligned} \quad (2.27)$$

Here Λ is a constant independent of ε and U_i . Observe that the corrections proportional to ε^2 are present only in $a_1, b_1, a_3, b_{-1}, a_{-1}$. The rest is of the order at least $O(\varepsilon^3)$ (in fact $O(\varepsilon^4)$).

The obtained estimates are now translated into the following theorem concerning the solution of the elasticity system in the ring.

Theorem 3. *If the nonhomogeneous Dirichlet boundary condition φ satisfies*

$$\|\varphi\|_{H^{1/2}(\partial B_R; \mathbb{R}^2)} \leq \Lambda_0,$$

which in terms of the Fourier coefficients U_i means

$$\sum_{k=-\infty}^{k=+\infty} \sqrt{1+k^2} |U_k|^2 \leq \Lambda_0,$$

then the elastic energy concentrated in the ring $C(R, \varepsilon) = B_R \setminus \overline{B_\varepsilon}$ splits into the energy of the disk, the first correction term of order ε^2 and the remainder, which is uniformly of the order $\Lambda_0 \varepsilon^3$.

Corollary 4. *In the case of the regular perturbations of the disk by an elastic inclusion ω_ε of the size $\varepsilon \rightarrow 0$, and with the contrast parameter $0 < \gamma < \infty$, the elastic energy splits into the energy of the unperturbed disk, the first correction term of order ε^2 and the remainder, which is uniformly of the order $\Lambda_0 \varepsilon^3$. This means that the associated Steklov-Poincaré operator admits the asymptotic expansion*

$$\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon^2 \mathcal{B} + o(\varepsilon^2) \quad (2.28)$$

in the operator norm $\mathcal{L}(H^{1/2}(\partial B_R; \mathbb{R}^2); H^{-1/2}(\partial B_R; \mathbb{R}^2))$, where $\partial B_R \equiv \Gamma_R$.

3. PROBLEM FORMULATION

Now, we assume that $\Omega = \Omega^M \cup \Gamma \cup \Omega^P$, where the mutually disjoint open domains Ω^P and Ω^M have the common interface Γ , as shown in fig. 1. In our notation, Ω^M and Ω^P represent the regions where mechanical and piezoelectric devices are located, respectively.

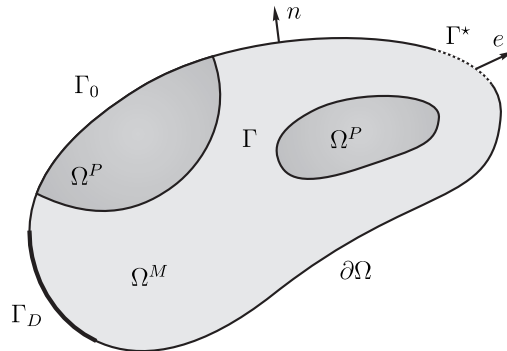


FIGURE 1. Piezo-elastic coupled problem.

3.1. The mechanical model . We are interested in the following system

$$\begin{cases} \operatorname{div} \sigma(u) &= 0 & \text{in } \Omega^M \\ \operatorname{div} S(w, q) &= 0 \\ \operatorname{div} \phi(w, q) &= 0 \end{cases} \quad \text{in } \Omega^P \quad (3.1)$$

where the first equation describes the linear elasticity system and the second one is the coupled system representing the electromechanical interaction phenomenon. The equations are coupled at interface Γ . In particular, $\sigma(u)$ is the mechanical stress tensor, $S(w, q)$ is the electromechanical stress tensor and $\phi(w, q)$ is the electric displacement. The constitutive laws describing the elastic behavior and piezoelectric effects, both in the linearized case of small mechanical deformations and electric fields, are

$$\begin{cases} \sigma(u) &= C \nabla^s u , \\ S(w, q) &= A \nabla^s w + P \nabla q , \\ \phi(w, q) &= P^T \nabla^s w - K \nabla q , \end{cases} \quad (3.2)$$

where u and w are the mechanical and electromechanical displacements, respectively, and q is the electric potential. In addition, C and A are the elasticity fourth-order tensors respectively associated to the elastic and electromechanical parts, P the piezoelectric coupling third-order tensor and K the dielectric second-order tensor. As usual C , A and K satisfy the symmetry conditions $C_{ijkl} = C_{jikl} = C_{klij}$, $A_{ijkl} = A_{jikl} = A_{klij}$, and $K_{ij} = K_{ji}$, whereas P satisfies $P_{ijk} = P_{jik}$. It is assumed for simplicity that all constitutive tensors are piecewise constant, i.e., constant in each sub-domain Ω^M and Ω^P . In the case of isotropic elasticity, the tensor C has the form

$$C = 2\mu \mathbb{I} + \lambda(I \otimes I) , \quad (3.3)$$

where μ and λ are the Lamé's coefficients, I and \mathbb{I} are the second and fourth orders identity tensors, respectively. We complement the system (3.1) with the following boundary conditions

$$u = 0 \text{ on } \Gamma_D , \quad \sigma(u)n = ku \text{ on } \Gamma^* \quad \text{and} \quad \begin{cases} q &= 0 \text{ on } \Gamma_0 \\ q &= \bar{q} \text{ on } \Gamma \end{cases} , \quad (3.4)$$

where Γ^* , Γ_D and Γ_0 are parts of the boundary $\partial\Omega$ and n is the outward unit normal vector pointing toward the exterior of Ω . If it is not specified, we consider homogeneous natural (Neumann) boundary conditions of the form $\sigma(u)n = 0$, $S(w, q)n = 0$ and $\phi(w, q) \cdot n = 0$ on a part of $\partial\Omega$. Finally, we consider the following transmission conditions

$$\begin{cases} u &= w \\ \sigma(u)n &= S(w, q)n \end{cases} \quad \text{on } \Gamma , \quad (3.5)$$

where n is the unit normal vector pointing toward the exterior of Ω^M . The variational formulation of the above coupled system reads:

Problem 5. Find $u \in \mathcal{V}$ and $q \in \mathcal{Q}$ such that

$$\begin{cases} \int_{\Omega^M} \sigma(u) \cdot \nabla^s \eta - \int_{\Gamma^*} ku \cdot \eta + \int_{\Omega^P} S(w, q) \cdot \nabla^s \eta &= 0 \quad \forall \eta \in \mathcal{V} , \\ \int_{\Omega^P} \phi(w, q) \cdot \nabla \xi &= 0 \quad \forall \xi \in \mathcal{Q}_0 . \end{cases} \quad (3.6)$$

where $u = u$ in Ω^M and $u = w$ in Ω^P . The space \mathcal{V} of displacements fields is defined as

$$\mathcal{V} = \left\{ v \in H^1(\Omega; \mathbb{R}^d) : v|_{\Gamma_D} = 0 \right\} , \quad (3.7)$$

while the electric potentials sets \mathcal{Q} and \mathcal{Q}_0 are respectively defined as

$$\mathcal{Q} = \{ q \in H^1(\Omega^P) : q|_{\Gamma} = \bar{q}, \quad q|_{\Gamma_0} = 0 \} , \quad \mathcal{Q}_0 = \{ q \in H^1(\Omega^P) : q|_{\partial\Omega^P} = 0 \} . \quad (3.8)$$

3.2. The shape functional . We are interested in minimization of the tracking-type shape functional

$$J(u) = - \int_{\Gamma^*} u \cdot e . \quad (3.9)$$

where e is used to denote a given direction on the boundary Γ^* (see fig. 1).

3.3. The adjoint state. We are going to evaluate the shape gradient of functional (3.9). For further simplifications, we introduce an adjoint system of the form

$$\left\{ \begin{array}{lcl} \operatorname{div} \sigma(u^a) & = & 0 \quad \text{in } \Omega^M \\ \operatorname{div} S^a(w^a, q^a) & = & 0 \\ \operatorname{div} \phi^a(w^a, q^a) & = & 0 \end{array} \right\} \quad \text{in } \Omega^P, \quad (3.10)$$

where $\sigma(u^a)$, $S^a(w^a, q^a)$ and $\phi^a(w^a, q^a)$ respectively are the adjoints mechanical stress tensor, electromechanical stress tensor and electrical displacement, given by

$$\left\{ \begin{array}{lcl} \sigma(u^a) & = & C \nabla^s u^a, \\ S^a(w^a, q^a) & = & A \nabla^s w^a - P \nabla q^a, \\ \phi^a(w^a, q^a) & = & -P^T \nabla^s w^a - K \nabla q^a, \end{array} \right. \quad (3.11)$$

where u^a , w^a and q^a are the adjoint mechanical displacement, electromechanical displacement and electric potential, respectively. The system (3.10) has the following boundary conditions

$$u^a = 0 \text{ on } \Gamma_D \quad \text{and} \quad q^a = 0 \text{ on } \partial\Omega^P \quad (3.12)$$

and the transmission conditions

$$\left\{ \begin{array}{lcl} u^a & = & w^a \\ \sigma(u^a)n & = & S^a(w^a, q^a)n \end{array} \right. \quad \text{on } \Gamma. \quad (3.13)$$

Finally,

$$\sigma(u^a)n = ku^a + e \text{ on } \Gamma^*. \quad (3.14)$$

The variational formulation of the coupled system for adjoint state equations reads:

Problem 6. Find $u^a \in \mathcal{V}$ and $q^a \in \mathcal{Q}_0$ such that

$$\left\{ \begin{array}{lcl} \int_{\Omega^M} \sigma(u^a) \cdot \nabla^s \eta - \int_{\Gamma^*} ku^a \cdot \eta + \int_{\Omega^P} S^a(w^a, q^a) \cdot \nabla^s \eta & = & \int_{\Gamma^*} e \cdot \eta \quad \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi^a(w^a, q^a) \cdot \nabla \xi & = & 0 \quad \forall \xi \in \mathcal{Q}_0, \end{array} \right. \quad (3.15)$$

where $u^a = u^a$ in Ω^M and $u^a = w^a$ in Ω^P .

4. TOPOLOGICAL DERIVATIVE

The topological derivative of functional (3.9) is evaluated for the insertion of a small inclusion in Ω^M with the material properties depending on the contrast. To describe the topological perturbation of Ω^M we introduce a piecewise constant function γ_ε of the form

$$\gamma_\varepsilon = \gamma_\varepsilon(x) := \begin{cases} 1 & \text{if } x \in \Omega^M \setminus \overline{B_\varepsilon} \\ \gamma & \text{if } x \in B_\varepsilon \end{cases}, \quad (4.1)$$

where $0 < \gamma < \infty$ is the contrast parameter of the material properties in matrix and inclusion, and $B_\varepsilon(\hat{x}) := \{|x - \hat{x}| < \varepsilon\}$ for $\hat{x} \in \Omega^M$.

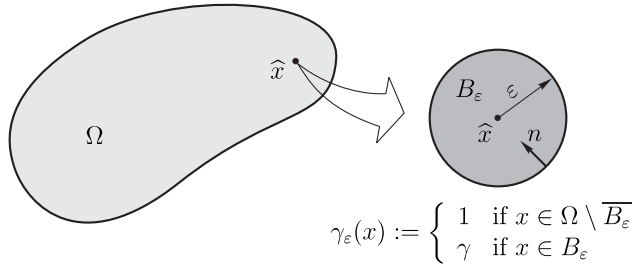


FIGURE 2. topologically perturbed domain by the nucleation of a small circular inclusion.

Note that in this case the topologies of the original and perturbed domains are preserved. However, we are introducing a non-smooth perturbation in the coefficients of the differential operator through

the contrast γ_ε , by changing the material property of the background in a small region $B_\varepsilon \subset \Omega^M$. Therefore, the sensitivity of the shape functional with respect to the nucleation of an inclusion can also be studied through the topological asymptotic analysis concept.

The variational formulation associated to the perturbed coupled system reads:

Problem 7. Find $\mathbf{u}_\varepsilon \in \mathcal{V}$ and $q_\varepsilon \in \mathcal{Q}$ such that

$$\begin{cases} \int_{\Omega^M} \sigma_\varepsilon(u_\varepsilon) \cdot \nabla^s \eta - \int_{\Gamma^*} k u_\varepsilon \cdot \eta + \int_{\Omega^P} S(w_\varepsilon, q_\varepsilon) \cdot \nabla^s \eta = 0 & \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi(w_\varepsilon, q_\varepsilon) \cdot \nabla \xi = 0 & \forall \xi \in \mathcal{Q}_0, \end{cases} \quad (4.2)$$

where $\sigma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon C \nabla^s u_\varepsilon$. In addition, $\mathbf{u}_\varepsilon = u_\varepsilon$ in Ω^M and $\mathbf{u}_\varepsilon = w_\varepsilon$ in Ω^P .

4.1. Preliminaries. In the paper a coupled model is considered in the domain $\Omega := \Omega^M \cup \Gamma \cup \Omega^P$, where Ω^M , Ω^P are the elastic and piezoelectric subdomains, and Γ stands for an interface, as shown in fig. 1. The coupled system is well-posed, and can be written in the strong form as an abstract equation for the unknown functions $U := (u, w, q)$,

$$LU = F \quad (4.3)$$

in the appropriate function spaces over the domain Ω . The weak form reads

$$\mathcal{L}(U, \Phi) = (F, \Phi), \quad (4.4)$$

with the test functions Φ . The bilinear form associated with the elastic component of the coupled model in the subdomain Ω^M is simply given by standard expression of linear elasticity

$$(u, \eta) \rightarrow a(\Omega^M; u, \eta) := \int_{\Omega^M} \sigma(u) \cdot \nabla^s \eta \quad (4.5)$$

in the unperturbed subdomain Ω^M , as well as by

$$(u, \eta) \rightarrow a_\varepsilon(\Omega^M; u, \eta) := \int_{\Omega^M} \sigma_\varepsilon(u) \cdot \nabla^s \eta \quad (4.6)$$

in the perturbed subdomain by an inclusion or a cavity. Here $\varepsilon \rightarrow 0$ is the parameter which governs the size of the topological perturbation. In the latter case, the weak formulation of the coupled model also depends on the parameter, and can be rewritten as follows

$$\mathcal{L}_\varepsilon(U_\varepsilon, \Phi) = (F, \Phi), \quad (4.7)$$

or written in the strong form $L_\varepsilon U_\varepsilon = F$. The perturbed system is also well-posed for $\varepsilon \in [0, \varepsilon_0)$, with $\varepsilon_0 > 0$, i.e., the inverse operator is uniformly bounded: $\|U_\varepsilon\| \leq C\|F\|$ in appropriate norms. However, in the case of a cavity the associated function spaces are obviously dependent on the perturbation, the case of an inclusion is therefore the regular perturbation of the problem in the fixed function spaces setting.

We proceed further with the domain decomposition technique in the subdomain $\Omega^M := B_R \cup \Gamma_R \cup \Omega_R^M$, where B_R is the ball of radius R which contains the topological perturbation denoted by B_ε , $\Gamma_R \equiv \partial B_R$ stands for the boundary of B_R , and the remaining subdomain $\Omega_R^M := \Omega^M \setminus \overline{B_R}$ is far from the singular topological domain perturbation.

Let u_ε denote the solution of coupled equations in the perturbed domain Ω i.e., including the inclusion B_ε . We are going to show, that the restriction of u_ε to the truncated domain solves a boundary value problem with the nonlocal boundary conditions on Γ_R defined by the Steklov-Poincaré operator.

Since for all $\varepsilon \in [0, \varepsilon_0)$,

$$a_\varepsilon(\Omega^M; u_\varepsilon, \eta) = a(\Omega_R^M; u_\varepsilon, \eta) + a_\varepsilon(B_R; u_\varepsilon, \eta) \quad (4.8)$$

and the last term is equivalent to the Steklov-Poincaré component by construction

$$a_\varepsilon(B_R; u_\varepsilon, \eta) \equiv (\mathcal{A}_\varepsilon(u_\varepsilon), \eta)_{\Gamma_R}. \quad (4.9)$$

Henceforth the bilinear form in the topologically perturbed domain Ω^M can be replaced by the bilinear form in the unperturbed domain Ω_R^M , however with a nonlocal pseudo-differential operator

$$a_\varepsilon(\Omega^M; u_\varepsilon, \eta) = a(\Omega_R^M; u_\varepsilon, \eta) + (\mathcal{A}_\varepsilon(u_\varepsilon), \eta)_{\Gamma_R}. \quad (4.10)$$

This replacement is in fact crucial for the proofs of topological differentiability for the shape functionals defined for the coupled system. Actually, the asymptotic expansion of the solutions to the coupled model in the truncated domain can be easily deduced from the well-posedness of the model. It means that for the linear model, and the regular perturbations of the differential operator, the asymptotic expansion of the Steklov-Poincaré operator

$$\mathcal{A}_\varepsilon = \mathcal{A} + f(\varepsilon)\mathcal{A}' + \mathcal{R}_\varepsilon(f(\varepsilon)) \quad (4.11)$$

implies the asymptotic expansion of the solutions to the coupled model in the truncated domain $\Omega_R^M \cup \Gamma_R \cup \Omega^P$ of the same form

$$U_\varepsilon = U + f(\varepsilon)U' + \tilde{U}_\varepsilon, \quad (4.12)$$

where \tilde{U}_ε is the remainder, namely $\|\tilde{U}_\varepsilon\| = o(f(\varepsilon))$ in appropriated norms.

4.2. Topological Asymptotic Expansion of the Steklov-Poincaré operator. The proposed method of asymptotic analysis is employed now to the coupled system defined in Ω_R . The dependence of the model on the small parameter $\varepsilon \rightarrow 0$ occurs in the nonlocal boundary conditions imposed on Γ_R . The variational form of (4.2) restricted to Ω_R is obtained,

$$\left\{ \begin{array}{l} \int_{\Omega^M \setminus \overline{B_R}} \sigma_\varepsilon(u_\varepsilon) \cdot \nabla^s \eta + \int_{\partial B_R} \mathcal{A}_\varepsilon(u_\varepsilon) \cdot \eta - \\ \int_{\Gamma^\star} k u_\varepsilon \cdot \eta + \int_{\Omega^P} S(w_\varepsilon, q_\varepsilon) \cdot \nabla^s \eta = 0 \quad \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi(w_\varepsilon, q_\varepsilon) \cdot \nabla \xi = 0 \quad \forall \xi \in \mathcal{Q}_0, \end{array} \right. \quad (4.13)$$

where $B_R = B_R(\hat{x})$ is a ball of radius R and center at $\hat{x} \in \Omega^M$, as shown in fig 3.

The Steklov-Poincaré operator on the interior boundary ∂B_R of the truncated domain Ω_R

$$\mathcal{A}_\varepsilon : \varphi \in H^{1/2}(\partial B_R; \mathbb{R}^d) \rightarrow \sigma_\varepsilon(u_\varepsilon)n \in H^{-1/2}(\partial B_R; \mathbb{R}^d), \quad (4.14)$$

by construction coincides with the Dirichlet-to-Neumann map of the linear elasticity on the ball B_R ,

$$\left\{ \begin{array}{l} \operatorname{div} \sigma_\varepsilon(v_\varepsilon) = 0 \quad \text{in } B_R, \\ \sigma_\varepsilon(v_\varepsilon) = \gamma_\varepsilon C \nabla^s v_\varepsilon, \\ v_\varepsilon = \varphi \quad \text{on } \partial B_R, \\ \llbracket v_\varepsilon \rrbracket = 0 \\ \llbracket \sigma_\varepsilon(v_\varepsilon) \rrbracket n = 0 \end{array} \right\} \quad \text{on } \partial B_\varepsilon, \quad (4.15)$$

with $\mathcal{A}_\varepsilon(\varphi) = \sigma_\varepsilon(v_\varepsilon)n$, which assures the identity $v_\varepsilon = u_\varepsilon|_{B_R}$ in B_R , where u_ε is the solution of the perturbed problem in Ω .

Proposition 8. *If the Steklov-Poincaré operator $\varphi \rightarrow \mathcal{A}_\varepsilon(\varphi)$ of problem (4.13) is the Dirichlet-to-Neumann map defined by (4.15), then the solution to (4.13) coincides with the restriction to Ω_R of the solution to perturbed problem in Ω .*

The identity for the energy functional of (4.15) holds

$$\begin{aligned} 0 = - \int_{B_R} \operatorname{div} \sigma_\varepsilon(v_\varepsilon) \cdot v_\varepsilon &= \int_{B_R} \sigma_\varepsilon(v_\varepsilon) \cdot \nabla^s v_\varepsilon - \int_{\partial B_R} \sigma_\varepsilon(v_\varepsilon)n \cdot v_\varepsilon \\ &= \int_{B_R} \sigma_\varepsilon(v_\varepsilon) \cdot \nabla^s v_\varepsilon - \int_{\partial B_R} \mathcal{A}_\varepsilon(\varphi) \cdot \varphi, \end{aligned} \quad (4.16)$$

hence the elastic energy in B_R equals to the energy of the Steklov-Poincaré operator on the boundary. Thus we conclude that the asymptotic expansion of the Steklov-Poincaré operator on the common boundary ∂B_R equals to the asymptotic expansion of the elastic energy in the domain B_R . Namely

$$\int_{B_R} \sigma_\varepsilon(v_\varepsilon) \cdot \nabla^s v_\varepsilon = \int_{\partial B_R} \mathcal{A}_\varepsilon(\varphi) \cdot \varphi, \quad (4.17)$$

for the mapping defined by (4.15)

$$\varphi \in H^{-1/2}(\partial B_R; \mathbb{R}^d) \rightarrow \sigma_\varepsilon(v_\varepsilon)n \in H^{-1/2}(\partial B_R; \mathbb{R}^d). \quad (4.18)$$

Since the operator \mathcal{A}_ε is symmetric, we can also write

$$\int_{B_R} \sigma_\varepsilon(v_\varepsilon) \cdot \nabla^s v_\varepsilon = \langle \mathcal{A}_\varepsilon(\varphi), \varphi \rangle_{(H^{-1/2} \times H^{1/2})(\partial B_R; \mathbb{R}^d)}. \quad (4.19)$$

It is well-known that the topological asymptotic expansion for the energy functional takes the following form [12]:

$$\int_{B_R} \sigma_\varepsilon(v_\varepsilon) \cdot \nabla^s v_\varepsilon = \int_{B_R} \sigma(v) \cdot \nabla^s v + f(\varepsilon) \mathbb{P}_\gamma \sigma(v(\hat{x})) \cdot \nabla^s v(\hat{x}) + \mathcal{R}(f(\varepsilon)), \quad (4.20)$$

where $v = u|_{B_R}$ is the solution to the original (unperturbed) problem (3.6) and \mathbb{P}_γ is the Pólya-Szegő polarization tensor. According to [14] we have the following expansion of the Steklov-Poincaré operator

$$\mathcal{A}_\varepsilon = \mathcal{A} + f(\varepsilon)\mathcal{B} + \mathcal{R}_\varepsilon \quad (4.21)$$

in the operator norm $\mathcal{L}(H^{1/2}(\partial B_R; \mathbb{R}^d); H^{-1/2}(\partial B_R; \mathbb{R}^d))$. By symmetry of the operator, the expansion of the energy functional can also be written as

$$\langle \mathcal{A}_\varepsilon(\varphi), \vartheta \rangle = \langle \mathcal{A}(\varphi), \vartheta \rangle + f(\varepsilon) \langle \mathcal{B}(\varphi), \vartheta \rangle + \langle \mathcal{R}_\varepsilon(\varphi), \vartheta \rangle, \quad (4.22)$$

where $\langle \mathcal{R}_\varepsilon(\varphi), \vartheta \rangle = \mathcal{R}(f(\varepsilon))$. Then, from the asymptotic expansion of the energy functional, we get

$$\langle \mathcal{B}(\varphi), \vartheta \rangle = \mathbb{P}_\gamma \sigma(\varphi(\hat{x})) \cdot \nabla^s \vartheta(\hat{x}) \quad \forall \hat{x} \in \Omega^M. \quad (4.23)$$

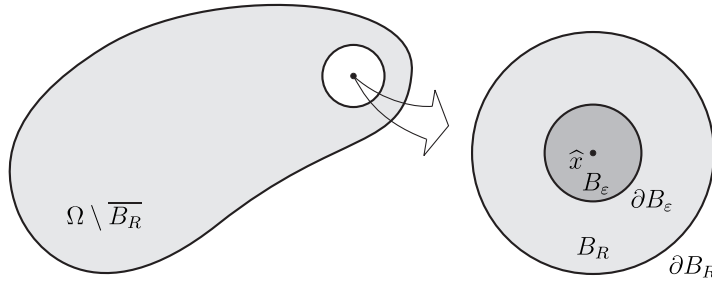


FIGURE 3. Truncated domain.

4.3. Topological Asymptotic Expansion of the Solution. We consider the following ansätze for the solutions $u_\varepsilon, w_\varepsilon, q_\varepsilon$ to the topologically perturbed coupled system (4.2)

$$u_\varepsilon = u + f(\varepsilon)g + \tilde{u}_\varepsilon, \quad (4.24)$$

$$w_\varepsilon = w + f(\varepsilon)h + \tilde{w}_\varepsilon, \quad (4.25)$$

$$q_\varepsilon = q + f(\varepsilon)p + \tilde{q}_\varepsilon, \quad (4.26)$$

where u, w, q are solutions to the original (unperturbed) coupled system (3.6), g, h, p are the first order asymptotic correction terms and $\tilde{u}_\varepsilon, \tilde{w}_\varepsilon, \tilde{q}_\varepsilon$ are the remainders. Now, we plug these ansätze in (4.2)

and collect the terms with the same powers of ε to obtain three boundary value problems. The first problem for u , w and q

$$\left\{ \begin{array}{l} \int_{\Omega^M \setminus \overline{B_R}} \sigma(u) \cdot \nabla^s \eta + \int_{\partial B_R} \mathcal{A}(u) \cdot \eta - \\ \int_{\Gamma^*} k u \cdot \eta + \int_{\Omega^P} S(w, q) \cdot \nabla^s \eta = 0 \quad \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi(w, q) \cdot \nabla \xi = 0 \quad \forall \xi \in \mathcal{Q}_0. \end{array} \right. \quad (4.27)$$

The second problem for g , h and p

$$\left\{ \begin{array}{l} \int_{\Omega^M \setminus \overline{B_R}} \sigma(g) \cdot \nabla^s \eta + \int_{\partial B_R} (\mathcal{A}(g) + \mathcal{B}(u)) \cdot \eta - \\ \int_{\Gamma^*} k g \cdot \eta + \int_{\Omega^P} S(h, p) \cdot \nabla^s \eta = 0 \quad \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi(h, p) \cdot \nabla \xi = 0 \quad \forall \xi \in \mathcal{Q}_0, \end{array} \right. \quad (4.28)$$

and the third problem for the remainders \tilde{u}_ε , \tilde{w}_ε and \tilde{q}_ε

$$\left\{ \begin{array}{l} \int_{\Omega^M \setminus \overline{B_R}} \sigma_\varepsilon(\tilde{u}_\varepsilon) \cdot \nabla^s \eta + \int_{\partial B_R} \mathcal{A}_\varepsilon(\tilde{u}_\varepsilon) \cdot \eta - \\ \int_{\Gamma^*} k \tilde{u}_\varepsilon \cdot \eta + \int_{\Omega^P} S(\tilde{w}_\varepsilon, \tilde{q}_\varepsilon) \cdot \nabla^s \eta = \int_{\partial B_R} \mathfrak{F}_\varepsilon \cdot \eta \quad \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi(\tilde{w}_\varepsilon, \tilde{q}_\varepsilon) \cdot \nabla \xi = 0 \quad \forall \xi \in \mathcal{Q}_0, \end{array} \right. \quad (4.29)$$

where the source \mathfrak{F}_ε is given by

$$\mathfrak{F}_\varepsilon = -(\mathcal{R}_\varepsilon(u) + f(\varepsilon)\mathcal{R}_\varepsilon(g) + f(\varepsilon)^2\mathcal{B}(g)). \quad (4.30)$$

The estimations $\|\tilde{u}_\varepsilon\|_{H^1(\Omega^M \setminus \overline{B_R}; \mathbb{R}^d)} = o(f(\varepsilon))$, $\|\tilde{w}_\varepsilon\|_{H^1(\Omega^P; \mathbb{R}^d)} = o(f(\varepsilon))$ and $\|\tilde{q}_\varepsilon\|_{H^1(\Omega^P)} = o(f(\varepsilon))$ hold true for the remainders.

4.4. Topological Asymptotic Expansion of the Shape Functional. Now we are in position to establish the asymptotic expansion of the functional and obtain its topological derivative.

After introducing the first ansatz in the shape functional associated to the perturbed problem, we have

$$\begin{aligned} J(u_\varepsilon) &= - \int_{\Gamma^*} (u + f(\varepsilon)g + \tilde{u}_\varepsilon) \cdot e \\ &= - \int_{\Gamma^*} u \cdot e - f(\varepsilon) \int_{\Gamma^*} g \cdot e - \int_{\Gamma^*} \tilde{u}_\varepsilon \cdot e \\ &= J(u) - f(\varepsilon) \int_{\Gamma^*} g \cdot e + \mathcal{R}(f(\varepsilon)). \end{aligned} \quad (4.31)$$

Now, let us rewrite the adjoint system (3.15) as

$$\left\{ \begin{array}{l} \int_{\Omega^M \setminus \overline{B_R}} \sigma(u^a) \cdot \nabla^s \eta + \int_{B_R} \mathcal{A}(u^a) \cdot \eta - \\ \int_{\Gamma^*} k u^a \cdot \eta + \int_{\Omega^P} S^a(w^a, q^a) \cdot \nabla^s \eta = \int_{\Gamma^*} e \cdot \eta \quad \forall \eta \in \mathcal{V}, \\ \int_{\Omega^P} \phi^a(w^a, q^a) \cdot \nabla \xi = 0 \quad \forall \xi \in \mathcal{Q}_0. \end{array} \right. \quad (4.32)$$

By taking g , h and p as the test functions in (4.32) we have the following equalities

$$\begin{aligned} \int_{\Omega^M \setminus \overline{B_R}} \sigma(u^a) \cdot \nabla^s g + \int_{B_R} \mathcal{A}(u^a) \cdot g - \int_{\Gamma^\star} k u^a \cdot g &+ \\ \int_{\Omega^P} A \nabla^s w^a \cdot \nabla^s h - \int_{\Omega^P} P \nabla q^a \cdot \nabla^s h &= \int_{\Gamma^\star} e \cdot g, \end{aligned} \quad (4.33)$$

$$- \int_{\Omega^P} P^\top \nabla^s w^a \cdot \nabla p - \int_{\Omega^P} K \nabla q^a \cdot \nabla p = 0. \quad (4.34)$$

On the other hand, by taking u^a , w^a and q^a as the test functions in (4.28) we obtain

$$\begin{aligned} \int_{\Omega^M \setminus \overline{B_R}} \sigma(g) \cdot \nabla^s u^a + \int_{B_R} (\mathcal{A}(g) + \mathcal{B}(u)) \cdot u^a - \int_{\Gamma^\star} k g \cdot u^a &+ \\ \int_{\Omega^P} A \nabla^s h \cdot \nabla^s w^a + \int_{\Omega^P} P \nabla p \cdot \nabla^s w^a &= 0, \end{aligned} \quad (4.35)$$

$$\int_{\Omega^P} P^\top \nabla^s h \cdot \nabla q^a - \int_{\Omega^P} K \nabla p \cdot \nabla q^a = 0. \quad (4.36)$$

Combining the above equalities yields the following important result

$$\begin{aligned} \int_{\Gamma^\star} e \cdot g &= - \int_{B_R} \mathcal{B}(u) \cdot u^a \\ &= - \langle \mathcal{B}(u), u^a \rangle_{(H^{-1/2} \times H^{1/2})(\partial B_R; \mathbb{R}^d)} \\ &= - \mathbb{P}_\gamma \sigma(u(\hat{x})) \cdot \nabla^s u^a(\hat{x}), \end{aligned} \quad (4.37)$$

where we have considered the symmetry of the bilinear forms. Finally, the topological asymptotic expansion of the shape functional leads to

$$J(u_\varepsilon) = J(u) + f(\varepsilon) \mathbb{P}_\gamma \sigma(u(\hat{x})) \cdot \nabla^s u^a(\hat{x}) + \mathcal{R}(f(\varepsilon)). \quad (4.38)$$

By assuming that the inclusion is far from the piezoelectric part, the topological derivative is given by the following closed formula

$$\mathcal{T}(\hat{x}) = \mathbb{P}_\gamma \sigma(u(\hat{x})) \cdot \nabla^s u^a(\hat{x}) \quad \forall \hat{x} \in \Omega^M. \quad (4.39)$$

Corollary 9. *In two spatial dimensions, the function $f(\varepsilon) = \varepsilon^2$ and the polarization tensor for inclusions ($0 < \gamma < \infty$) reads [12]*

$$\mathbb{P}_\gamma = \pi \frac{1-\gamma}{1+\gamma\beta} \left((1+\beta) \mathbb{I} + \frac{1}{2}(\alpha-\beta) \frac{1-\gamma}{1+\gamma\alpha} I \otimes I \right). \quad (4.40)$$

with the constants α and β given by

$$\alpha = \frac{\lambda + \mu}{\mu} \quad \text{and} \quad \beta = \frac{\lambda + 3\mu}{\lambda + \mu}. \quad (4.41)$$

In three spatial dimensions, the function $f(\varepsilon) = \varepsilon^3$ and the polarization tensor for holes ($\gamma = 0$) yields [12]

$$\mathbb{P}_0 = 2\pi \frac{1-\nu}{7-5\nu} \left(10\mathbb{I} - \frac{1-5\nu}{1-2\nu} I \otimes I \right), \quad (4.42)$$

where ν is the Poisson ratio.

5. CONCLUSIONS

In the paper the topological derivatives of the tracking-type shape functional for the coupled models of elasto-piezoelectric type are derived in two and three spatial dimensions. The associated shape optimization problems are already analyzed from the point of view of shape optimization in the former papers. In this paper the preceding results are completed by the topological sensitivity analysis. The remarkable simplicity of the closed form sensitivity given by (4.39) is to be noted. In fact, once the solutions u, w, q to the original (unperturbed) coupled system (3.6) have been obtained, the

topological derivative $\mathcal{T}(\hat{x})$ can be evaluated for all $\hat{x} \in \Omega^M$. The information provided by $\mathcal{T}(\hat{x})$ can be potentially used in a number of practical applications such as, for example, the shape-topological design of microtools. In the forthcoming paper the theoretical results already derived are used in numerical results and some results of computations are reported. The applications include the design of elasto-piezoelectric devices with the prescribed mechanical behavior on a part of the boundary of elastic body.

REFERENCES

- [1] G. Allaire, F. Jouve, and N. Van Goethem. Damage and fracture evolution in brittle materials by shape optimization methods. *Journal of Computational Physics*, 230(12):5010–5044, 2011.
- [2] S. Amstutz, S.M. Giusti, A.A. Novotny, and E.A. de Souza Neto. Topological derivative for multi-scale linear elasticity models applied to the synthesis of microstructures. *International Journal for Numerical Methods in Engineering*, 84:733–756, 2010.
- [3] S. Amstutz, A.A. Novotny, and E.A. de Souza Neto. Topological derivative-based topology optimization of structures subject to Drucker-Prager stress constraints. *Computer Methods in Applied Mechanics and Engineering*, 233–236:123–136, 2012.
- [4] N. Van Goethem and A.A. Novotny. Crack nucleation sensitivity analysis. *Mathematical Methods in the Applied Sciences*, 33(16):197–1994, 2010.
- [5] W.A. Gross. The second fundamental problem of elasticity applied to a plane circular ring. *Zeitschrift für Angewandte Mathematik und Physik*, 8:71–73, 1957.
- [6] M. Hintermüller and A. Laurain. Multiphase image segmentation and modulation recovery based on shape and topological sensitivity. *Journal of Mathematical Imaging and Vision*, 35:1–22, 2009.
- [7] M. Hintermüller, A. Laurain, and A.A. Novotny. Second-order topological expansion for electrical impedance tomography. *Advances in Computational Mathematics*, 36(2):235–265, 2012.
- [8] M. Kachanov, B. Shafiro, and I. Tsukrov. *Handbook of Elasticity Solutions*. Kluwer Academic Publishers, Dordrecht, 2003.
- [9] G. Leugering, A. A. Novotny, G. Perla-Menzala, and J. Sokołowski. On shape optimization for an evolution coupled system. *Applied Mathematical & Optimization*, 64:441–466, 2011.
- [10] G. Leugering, A.A. Novotny, G. Perla-Menzala, and J. Sokołowski. Shape sensitivity analysis of a quasi-electrostatic piezoelectric system in multilayered media. *Mathematical Methods in the Applied Sciences*, 33(17):2118–2131, 2010.
- [11] N.I. Muskhelishvili. *Some Basic Problems on the Mathematical Theory of Elasticity*. Noordhoff, Groningen, 1952.
- [12] A. A. Novotny and J. Sokołowski. *Topological derivatives in shape optimization*. Interaction of Mechanics and Mathematics. Springer, 2013.
- [13] J. Sokołowski and A. Żochowski. On the topological derivative in shape optimization. *SIAM Journal on Control and Optimization*, 37(4):1251–1272, 1999.
- [14] J. Sokołowski and A. Żochowski. Modelling of topological derivatives for contact problems. *Numerische Mathematik*, 102(1):145–179, 2005.
- [15] J. Sokołowski and A. Żochowski. Topological derivatives in plane elasticity. In *23rd IFIP TC 7 Conference on System Modeling and Optimization, IFIP Advances in Information and Communication Technology Volume 312, 2009*, pp 459–475, Berlin, Germany, 2009. Springer.

LABORATÓRIO NACIONAL DE COMPUTAÇÃO CIENTÍFICA LNCC/MCT, AV. GETÚLIO VARGAS 333, 25651-075 PETRÓPOLIS - RJ, BRASIL.

E-mail address: novotny@lncc.br

UNIVERSITÉ DE LORRAINE, CNRS, INRIA, INSTITUTE ÉLIE CARTAN NANCY, UMR7502, BP 239 - 54506 VANDOEUVRE LÈS NANCY CEDEX, FRANCE

E-mail address: Jan.Sokolowski@univ-lorraine.fr