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# Sym-Bobenko formula for minimal surfaces in Heisenberg space\*

Sébastien Cartier

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## Abstract

*We give an immersion formula, the Sym-Bobenko formula, for minimal surfaces in the 3-dimensional Heisenberg space. Such a formula can be used to give a generalized Weierstrass type representation and construct explicit examples of minimal surfaces.*

*Mathematics Subject Classification: Primary 53A10, Secondary 53C42.*

## 1 Introduction

A Sym-Bobenko formula is the expression of an immersion in terms of a one-parameter family of moving frames, called the *extended frame*. This idea was first used by A. Sym [9] in the case of surfaces with negative constant (Gauss) curvature in euclidean space. A. I. Bobenko applied the method to numerous cases [1] [2] [3], including constant mean curvature (*CMC* for short) surfaces in space forms — euclidean 3-space, 3-sphere and hyperbolic 3-space — and T. Taniguchi applied it to CMC spacelike surfaces in Minkowski 3-space [10]. In the mean time, the work of J. Dorfmeister, F. Pedit and H. Wu [8] and D. Brander, W. Rossman and N. Schmitt [4] show that Sym-Bobenko formulæ can be seen as generalized Weierstrass type representations for CMC surfaces, extended frames coming from holomorphic data.

In Heisenberg 3-space, the classical method does not apply, since the isometry group is of dimension only 4 — contrary to the ones of space forms that are 6-dimensional — and does not act transitively on orthonormal frames; there are “not enough” isometries to define a moving frame. We show that nevertheless, for minimal immersions a Sym-Bobenko formula can be established using an *ad-hoc* matrix-valued map.

In [7], J. F. Dorfmeister, J. Inoguchi and S. Kobayashi link this formula with pairs of meromorphic and anti-meromorphic 1-forms, which they call *pairs of normalized potentials*, in a way to get a generalized Weierstrass type representation for minimal surfaces.

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\*This work is part of the author’s Ph. D. thesis [5].

## 2 Surfaces in Heisenberg space

We see the 3-dimensional Heisenberg space  $\text{Nil}_3$  as  $\mathbb{R}^3$ , with generic coordinates  $(x_1, x_2, x_3)$ , endowed with the following riemannian metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + \left( \frac{1}{2}(x_2 dx_1 - x_1 dx_2) + dx_3 \right)^2.$$

We call *canonical frame* the orthonormal frame  $(E_1, E_2, E_3)$  defined by:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3},$$

and the Levi-Civita connection  $\nabla$  writes:

$$\begin{aligned} \nabla_{E_1} E_1 &= 0 & \nabla_{E_2} E_1 &= -\frac{1}{2} E_3 & \nabla_{E_3} E_1 &= -\frac{1}{2} E_2 \\ \nabla_{E_1} E_2 &= \frac{1}{2} E_3 & \nabla_{E_2} E_2 &= 0 & \nabla_{E_3} E_2 &= \frac{1}{2} E_1 \\ \nabla_{E_1} E_3 &= -\frac{1}{2} E_2 & \nabla_{E_2} E_3 &= \frac{1}{2} E_1 & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

Note that the vector field  $E_3$  is a Killing field and that the projection  $\pi : (x_1, x_2, x_3) \in \text{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$  on the first two coordinates is a Riemannian submersion. From now on, we identify  $\mathbb{R}^2$  with  $\mathbb{C}$ .

We may also write  $\text{Nil}_3$  as a subset of  $\mathcal{M}_2(\mathbb{C})$ . Consider the matrices:

$$\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The identification is the following:

$$\begin{aligned} (x_1, x_2, x_3) \in \text{Nil}_3 &\longleftrightarrow x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 \\ &= \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}). \end{aligned} \quad (1)$$

Note that this identification is purely formal and does not involve any manifold related structure.

Let  $\Sigma$  be a simply connected Riemann surface and  $z$  be a conformal parameter on  $\Sigma$ . A conformal immersion is denoted  $f : \Sigma \rightarrow \text{Nil}_3$  with unit normal  $N$  and conformal factor  $\rho : \Sigma \rightarrow (0, +\infty)$  meaning:

$$\begin{aligned} \langle f_z, f_z \rangle &= \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0, & \langle f_z, f_{\bar{z}} \rangle &= \frac{\rho}{2}, \\ \langle f_z, N \rangle &= \langle f_{\bar{z}}, N \rangle = 0 & \text{and} & \quad \langle N, N \rangle = 1. \end{aligned}$$

Consider also  $\varphi = \langle N, E_3 \rangle : \Sigma \rightarrow (-1, 1)$  denote the angle function of  $N$ ,  $A = \langle f_z, E_3 \rangle : \Sigma \rightarrow \mathbb{C}$  the vertical part of  $f_z$  and  $pdz^2 = \langle \nabla_{f_z} f_z, N \rangle dz^2$  the Hopf differential of  $f$ .

The Abresch-Rosenberg differential expresses  $Qdz^2 = (ip + A^2)dz^2$ , and a necessary and sufficient condition for  $f$  to be minimal is  $\nabla_{f_z} f_{\bar{z}} = 0$ .

We also decompose  $f$  into  $f = (F, h)$  with  $F = \pi \circ f : \Sigma \rightarrow \mathbb{C}$  the horizontal projection of  $f$  and  $h : \Sigma \rightarrow \mathbb{R}$  its height function. We can express  $A$  in terms of  $F$  and  $h$ :

$$A = h_z - \frac{i}{4} (F\bar{F}_z - \bar{F}F_z). \quad (2)$$

In the matrix model (1) of  $\text{Nil}_3$ , the map  $F$  is given by the non-diagonal coefficients — precisely the (1, 2)-coefficient — and  $h$  by the diagonal ones.

The intuitive idea behind Sym-Bobenko formulæ in space forms is that, up to ambient isometries, the unit normal — or Gauss map — would locally determine the immersion up to ambient isometries. In  $\text{Nil}_3$  such a map is defined as follows; see [6] for details. Since  $\text{Nil}_3$  is a Lie group, the map  $f^{-1}N$  takes values in the unit sphere  $\mathbb{S}^2$  of the Lie algebra. Moreover, for a local study, we can suppose  $\varphi > 0$  so that the values of  $f^{-1}N$  are actually in the northern hemisphere of  $\mathbb{S}^2$ . If  $s$  denotes the stereographic projection centered at the South Pole, we call *Gauss map* of an immersion  $f$  the map  $g = s \circ (f^{-1}N)$  with values in the unit disk. Actually, endowing the unit disk with the Poincaré metric, we see the Gauss map  $g$  as a map with values into the hyperbolic disk  $\mathbb{H}^2$ .

We use the following criterion to show that a conformal immersion in Heisenberg space is minimal:

**Proposition 2.1** (Daniel [6]). *A conformal immersion  $f = (F, h) : \Sigma \rightarrow \text{Nil}_3$  is minimal if and only if:*

$$F_{z\bar{z}} = \frac{i}{2} (\bar{A}F_z + AF_{\bar{z}}) \quad \text{and} \quad A_{\bar{z}} + \bar{A}_z = 0.$$

Furthermore, when  $f$  is minimal its Gauss map  $g : \Sigma \rightarrow \mathbb{H}^2$  is harmonic.

### 3 The Sym-Bobenko formula

Consider the family  $(\Psi_t)_{t \in \mathbb{R}}$  of matrix fields over  $\Sigma$  which are solutions of the system:

$$\left\{ \begin{array}{l} \Psi_t^{-1} d\Psi_t = \frac{1}{4} \begin{pmatrix} (\log \rho_0)_z & i\sqrt{\rho_0} \\ -\frac{4iQ_0}{\sqrt{\rho_0}} e^{2it} & -(\log \rho_0)_z \end{pmatrix} dz \\ \quad \quad \quad + \frac{1}{4} \begin{pmatrix} -(\log \rho_0)_{\bar{z}} & \frac{4i\bar{Q}_0}{\sqrt{\rho_0}} e^{-2it} \\ -i\sqrt{\rho_0} & (\log \rho_0)_{\bar{z}} \end{pmatrix} d\bar{z} \\ \Psi_t(z=0) = \sigma_3 \end{array} \right. ,$$

where  $\rho_0 : \Sigma \rightarrow (0, +\infty)$  and  $Q_0 : \Sigma \rightarrow \mathbb{C}$  are smooth. Such a family  $(\Psi_t)$  exists if and only if:

$$(\log \rho_0)_{z\bar{z}} = \frac{\rho_0}{8} - \frac{2|Q_0|^2}{\rho_0} \quad \text{and} \quad (Q_0)_{\bar{z}} = 0.$$

**Theorem 3.1** (Sym-Bobenko formula). *Using the matrix model (1), define the map  $f_t : \Sigma \rightarrow Nil_3$  for any  $t \in \mathbb{R}$  as:*

$$f_t = -\frac{1}{2} \left( \sigma_0 \frac{\partial \hat{f}_t}{\partial t} \right)^d + (\hat{f}_t)^{nd} \quad \text{with} \quad \hat{f}_t = -2 \frac{\partial \Psi_t}{\partial t} \Psi_t^{-1} + 2 \Psi_t \sigma_0 \Psi_t^{-1}, \quad (3)$$

where the superscripts  $\cdot^d$  and  $\cdot^{nd}$  denote respectively the diagonal and non-diagonal terms. Then  $f_t$  is a conformal minimal immersion in Heisenberg space and the family  $(f_t)$  is the so-called associated family.

*Proof.* Fix  $t \in \mathbb{R}$ . From Equation (3), we get:

$$\begin{aligned} (f_t)^{nd} &= (\hat{f}_t)^{nd}, \quad ((f_t)_z)^d = \frac{1}{2} \left( \sigma_0 \left[ (\hat{f}_t)_z, \hat{f}_t \right] \right)^d + 2 \left( \sigma_0 (\hat{f}_t)_z \right)^d \\ &\quad \text{and} \quad (\hat{f}_t)_{z\bar{z}} = \frac{i}{4} \left[ (\hat{f}_t)_z, (\hat{f}_t)_{\bar{z}} \right], \end{aligned} \quad (4)$$

where  $[\cdot, \cdot]$  denotes the commutator. From the first equation in (4), we have that matrices  $f_t$  and  $\hat{f}_t$  write:

$$f_t = \begin{pmatrix} h & F \\ \bar{F} & h \end{pmatrix} \quad \text{and} \quad \hat{f}_t = \begin{pmatrix} i\hat{h} & F \\ \bar{F} & -i\hat{h} \end{pmatrix},$$

with  $F : \Sigma \rightarrow \mathbb{C}$  and  $h, \hat{h} : \Sigma \rightarrow \mathbb{R}$  smooth. We show that  $F$  and  $h$  verify the conditions of Proposition 2.1. Using Equation (2) and the second identity in (4), we deduce  $A = i\hat{h}_z$  and since  $\hat{h}$  is real-valued, we obtain  $A_{\bar{z}} + \bar{A}_z = 0$ . Finally, the (1,2)-coefficient of the third equation in (4) verify:

$$F_{z\bar{z}} = \frac{1}{2} \left( \hat{h}_{\bar{z}} F_z - \hat{h}_z F_{\bar{z}} \right) = \frac{i}{2} \left( \bar{A} F_z + A F_{\bar{z}} \right),$$

which concludes the proof.  $\square$

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