



HAL
open science

Sym-Bobenko formula for minimal surfaces in Heisenberg space

Sébastien Cartier

► **To cite this version:**

Sébastien Cartier. Sym-Bobenko formula for minimal surfaces in Heisenberg space. *Comptes Rendus. Mathématique*, 2013, 351 (21-22), pp.825–827. 10.1016/j.crma.2013.10.014 . hal-00861418

HAL Id: hal-00861418

<https://hal.science/hal-00861418>

Submitted on 12 Sep 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sym-Bobenko formula for minimal surfaces in Heisenberg space*

Sébastien Cartier

September 12, 2013

Abstract

We give an immersion formula, the Sym-Bobenko formula, for minimal surfaces in the 3-dimensional Heisenberg space. Such a formula can be used to give a generalized Weierstrass type representation and construct explicit examples of minimal surfaces.

Mathematics Subject Classification: Primary 53A10, Secondary 53C42.

1 Introduction

A Sym-Bobenko formula is the expression of an immersion in terms of a one-parameter family of moving frames, called the *extended frame*. This idea was first used by A. Sym [9] in the case of surfaces with negative constant (Gauss) curvature in euclidean space. A. I. Bobenko applied the method to numerous cases [1] [2] [3], including constant mean curvature (*CMC* for short) surfaces in space forms — euclidean 3-space, 3-sphere and hyperbolic 3-space — and T. Taniguchi applied it to CMC spacelike surfaces in Minkowski 3-space [10]. In the mean time, the work of J. Dorfmeister, F. Pedit and H. Wu [8] and D. Brander, W. Rossman and N. Schmitt [4] show that Sym-Bobenko formulæ can be seen as generalized Weierstrass type representations for CMC surfaces, extended frames coming from holomorphic data.

In Heisenberg 3-space, the classical method does not apply, since the isometry group is of dimension only 4 — contrary to the ones of space forms that are 6-dimensional — and does not act transitively on orthonormal frames; there are “not enough” isometries to define a moving frame. We show that nevertheless, for minimal immersions a Sym-Bobenko formula can be established using an *ad-hoc* matrix-valued map.

In [7], J. F. Dorfmeister, J. Inoguchi and S. Kobayashi link this formula with pairs of meromorphic and anti-meromorphic 1-forms, which they call *pairs of normalized potentials*, in a way to get a generalized Weierstrass type representation for minimal surfaces.

*This work is part of the author’s Ph. D. thesis [5].

2 Surfaces in Heisenberg space

We see the 3-dimensional Heisenberg space Nil_3 as \mathbb{R}^3 , with generic coordinates (x_1, x_2, x_3) , endowed with the following riemannian metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + \left(\frac{1}{2}(x_2 dx_1 - x_1 dx_2) + dx_3 \right)^2.$$

We call *canonical frame* the orthonormal frame (E_1, E_2, E_3) defined by:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3},$$

and the Levi-Civita connection ∇ writes:

$$\begin{aligned} \nabla_{E_1} E_1 &= 0 & \nabla_{E_2} E_1 &= -\frac{1}{2} E_3 & \nabla_{E_3} E_1 &= -\frac{1}{2} E_2 \\ \nabla_{E_1} E_2 &= \frac{1}{2} E_3 & \nabla_{E_2} E_2 &= 0 & \nabla_{E_3} E_2 &= \frac{1}{2} E_1 \\ \nabla_{E_1} E_3 &= -\frac{1}{2} E_2 & \nabla_{E_2} E_3 &= \frac{1}{2} E_1 & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

Note that the vector field E_3 is a Killing field and that the projection $\pi : (x_1, x_2, x_3) \in \text{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$ on the first two coordinates is a Riemannian submersion. From now on, we identify \mathbb{R}^2 with \mathbb{C} .

We may also write Nil_3 as a subset of $\mathcal{M}_2(\mathbb{C})$. Consider the matrices:

$$\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The identification is the following:

$$\begin{aligned} (x_1, x_2, x_3) \in \text{Nil}_3 &\longleftrightarrow x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 \\ &= \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}). \end{aligned} \quad (1)$$

Note that this identification is purely formal and does not involve any manifold related structure.

Let Σ be a simply connected Riemann surface and z be a conformal parameter on Σ . A conformal immersion is denoted $f : \Sigma \rightarrow \text{Nil}_3$ with unit normal N and conformal factor $\rho : \Sigma \rightarrow (0, +\infty)$ meaning:

$$\begin{aligned} \langle f_z, f_z \rangle &= \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0, & \langle f_z, f_{\bar{z}} \rangle &= \frac{\rho}{2}, \\ \langle f_z, N \rangle &= \langle f_{\bar{z}}, N \rangle = 0 & \text{and} & \quad \langle N, N \rangle = 1. \end{aligned}$$

Consider also $\varphi = \langle N, E_3 \rangle : \Sigma \rightarrow (-1, 1)$ denote the angle function of N , $A = \langle f_z, E_3 \rangle : \Sigma \rightarrow \mathbb{C}$ the vertical part of f_z and $pdz^2 = \langle \nabla_{f_z} f_z, N \rangle dz^2$ the Hopf differential of f .

The Abresch-Rosenberg differential expresses $Qdz^2 = (ip + A^2)dz^2$, and a necessary and sufficient condition for f to be minimal is $\nabla_{f_z} f_{\bar{z}} = 0$.

We also decompose f into $f = (F, h)$ with $F = \pi \circ f : \Sigma \rightarrow \mathbb{C}$ the horizontal projection of f and $h : \Sigma \rightarrow \mathbb{R}$ its height function. We can express A in terms of F and h :

$$A = h_z - \frac{i}{4} (F\bar{F}_z - \bar{F}F_z). \quad (2)$$

In the matrix model (1) of Nil_3 , the map F is given by the non-diagonal coefficients — precisely the $(1, 2)$ -coefficient — and h by the diagonal ones.

The intuitive idea behind Sym-Bobenko formulæ in space forms is that, up to ambient isometries, the unit normal — or Gauss map — would locally determine the immersion up to ambient isometries. In Nil_3 such a map is defined as follows; see [6] for details. Since Nil_3 is a Lie group, the map $f^{-1}N$ takes values in the unit sphere \mathbb{S}^2 of the Lie algebra. Moreover, for a local study, we can suppose $\varphi > 0$ so that the values of $f^{-1}N$ are actually in the northern hemisphere of \mathbb{S}^2 . If s denotes the stereographic projection centered at the South Pole, we call *Gauss map* of an immersion f the map $g = s \circ (f^{-1}N)$ with values in the unit disk. Actually, endowing the unit disk with the Poincaré metric, we see the Gauss map g as a map with values into the hyperbolic disk \mathbb{H}^2 .

We use the following criterion to show that a conformal immersion in Heisenberg space is minimal:

Proposition 2.1 (Daniel [6]). *A conformal immersion $f = (F, h) : \Sigma \rightarrow \text{Nil}_3$ is minimal if and only if:*

$$F_{z\bar{z}} = \frac{i}{2} (\bar{A}F_z + AF_{\bar{z}}) \quad \text{and} \quad A_{\bar{z}} + \bar{A}_z = 0.$$

Furthermore, when f is minimal its Gauss map $g : \Sigma \rightarrow \mathbb{H}^2$ is harmonic.

3 The Sym-Bobenko formula

Consider the family $(\Psi_t)_{t \in \mathbb{R}}$ of matrix fields over Σ which are solutions of the system:

$$\left\{ \begin{array}{l} \Psi_t^{-1} d\Psi_t = \frac{1}{4} \left(\begin{array}{cc} (\log \rho_0)_z & i\sqrt{\rho_0} \\ -\frac{4iQ_0}{\sqrt{\rho_0}} e^{2it} & -(\log \rho_0)_z \end{array} \right) dz \\ \hspace{15em} + \frac{1}{4} \left(\begin{array}{cc} -(\log \rho_0)_{\bar{z}} & \frac{4i\bar{Q}_0}{\sqrt{\rho_0}} e^{-2it} \\ -i\sqrt{\rho_0} & (\log \rho_0)_{\bar{z}} \end{array} \right) d\bar{z} \\ \Psi_t(z = 0) = \sigma_3 \end{array} \right. ,$$

where $\rho_0 : \Sigma \rightarrow (0, +\infty)$ and $Q_0 : \Sigma \rightarrow \mathbb{C}$ are smooth. Such a family (Ψ_t) exists if and only if:

$$(\log \rho_0)_{z\bar{z}} = \frac{\rho_0}{8} - \frac{2|Q_0|^2}{\rho_0} \quad \text{and} \quad (Q_0)_{\bar{z}} = 0.$$

Theorem 3.1 (Sym-Bobenko formula). *Using the matrix model (1), define the map $f_t : \Sigma \rightarrow Nil_3$ for any $t \in \mathbb{R}$ as:*

$$f_t = -\frac{1}{2} \left(\sigma_0 \frac{\partial \hat{f}_t}{\partial t} \right)^d + (\hat{f}_t)^{nd} \quad \text{with} \quad \hat{f}_t = -2 \frac{\partial \Psi_t}{\partial t} \Psi_t^{-1} + 2 \Psi_t \sigma_0 \Psi_t^{-1}, \quad (3)$$

where the superscripts \cdot^d and \cdot^{nd} denote respectively the diagonal and non-diagonal terms. Then f_t is a conformal minimal immersion in Heisenberg space and the family (f_t) is the so-called associated family.

Proof. Fix $t \in \mathbb{R}$. From Equation (3), we get:

$$\begin{aligned} (f_t)^{nd} &= (\hat{f}_t)^{nd}, \quad ((f_t)_z)^d = \frac{1}{2} \left(\sigma_0 \left[(\hat{f}_t)_z, \hat{f}_t \right] \right)^d + 2 \left(\sigma_0 (\hat{f}_t)_z \right)^d \\ &\quad \text{and} \quad (\hat{f}_t)_{z\bar{z}} = \frac{i}{4} \left[(\hat{f}_t)_z, (\hat{f}_t)_{\bar{z}} \right], \end{aligned} \quad (4)$$

where $[\cdot, \cdot]$ denotes the commutator. From the first equation in (4), we have that matrices f_t and \hat{f}_t write:

$$f_t = \begin{pmatrix} h & F \\ \bar{F} & h \end{pmatrix} \quad \text{and} \quad \hat{f}_t = \begin{pmatrix} i\hat{h} & F \\ \bar{F} & -i\hat{h} \end{pmatrix},$$

with $F : \Sigma \rightarrow \mathbb{C}$ and $h, \hat{h} : \Sigma \rightarrow \mathbb{R}$ smooth. We show that F and h verify the conditions of Proposition 2.1. Using Equation (2) and the second identity in (4), we deduce $A = i\hat{h}_z$ and since \hat{h} is real-valued, we obtain $A_{\bar{z}} + \bar{A}_z = 0$. Finally, the (1,2)-coefficient of the third equation in (4) verify:

$$F_{z\bar{z}} = \frac{1}{2} \left(\hat{h}_{\bar{z}} F_z - \hat{h}_z F_{\bar{z}} \right) = \frac{i}{2} \left(\bar{A} F_z + A F_{\bar{z}} \right),$$

which concludes the proof. \square

References

- [1] A. I. Bobenko, *All constant mean curvature tori in \mathbb{R}^3 , \mathbb{S}^3 , \mathbb{H}^3 in terms of theta functions*, Math. Ann. **290** (1991), no. 2, 209–245.
- [2] A. I. Bobenko, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys **46** (1991), no. 4, 1–45.

- [3] A. I. Bobenko, *Surfaces in terms of 2 by 2 matrices. Old and new integrable cases*, Harmonic maps and integrable systems, Aspects Math., E23, Friedr. Vieweg, Braunschweig, 1994, pp. 83–127.
- [4] D. Brander, W. Rossman, and N. Schmitt, *Holomorphic representation of constant mean curvature surfaces in Minkowski space: consequences of non-compactness in loop group methods*, Adv. Math. **223** (2010), 949–986.
- [5] S. Cartier, *Surfaces des espaces homogènes de dimension 3*, Ph.D. thesis, Université Paris-Est, 2011, tel-00672332.
- [6] B. Daniel, *The Gauss map of minimal surfaces in the Heisenberg group*, Int. Math. Res. Not. IMRN (2011), no. 3, 674–695.
- [7] J. F. Dorfmeister, J. Inoguchi, and S. Kobayashi, *A loop group method for minimal surfaces in the three-dimensional Heisenberg group*, preprint arXiv:1210.7300, 2012.
- [8] J. F. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), no. 4, 633–668.
- [9] A. Sym, *Soliton surfaces and their applications (soliton geometry from spectral problems)*, Geometric aspects of the Einstein equations and integrable systems (Scheveningen, 1984), Lecture Notes in Phys., vol. 239, Springer, Berlin, 1985, pp. 154–231.
- [10] T. Taniguchi, *The Sym-Bobenko formula and constant mean curvature surfaces in Minkowski 3-space*, Tokyo J. Math. **20** (1997), no. 2, 463–473.

Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées
 (UMR 8050), UPEC, UPEMLV, CNRS, F-94010, Créteil, France
E-mail address: `sebastien.cartier@u-pec.fr`