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Gersende Fort. Central Limit Theorems for Stochastic Approximation with controlled Markov chain dynamics. 2013. hal-00861097

HAL Id: hal-00861097

<https://hal.science/hal-00861097>

Preprint submitted on 11 Sep 2013

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Central Limit Theorems for Stochastic Approximation with controlled Markov chain dynamics

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Keywords: Stochastic Approximation, Limit Theorems, Controlled Markov chain

Abstract

This paper provides a Central Limit Theorem (CLT) for a process $\{\theta_n, n \geq 0\}$ satisfying a stochastic approximation (SA) equation of the form $\theta_{n+1} = \theta_n + \gamma_{n+1}H(\theta_n, X_{n+1})$; a CLT for the associated average sequence is also established. The originality of this paper is to address the case of controlled Markov chain dynamics $\{X_n, n \geq 0\}$ and the case of multiple targets. The framework also accomodates (randomly) truncated SA algorithms.

Sufficient conditions for CLT's to hold are provided as well as comments on how these conditions extend previous works (such as independent and identically distributed dynamics, the Robbins-Monro dynamic or the single target case). The paper gives a special emphasis on how these conditions hold for SA with controlled Markov chain dynamics and multiple targets; it is proved that this paper improves on existing works.

Acknowledgements. I gratefully acknowledge Prof. P. Priouret for fruitful discussions.

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1 Introduction

Stochastic Approximation (SA) algorithms were introduced for finding roots of an unknown function h (for recent surveys on SA, see e.g. [8, 26, 20, 6, 19]). SA defines iteratively a sequence $\{\theta_n, n \geq 0\}$ by the update rule

$$\theta_{n+1} = \theta_n + \gamma_{n+1} \Xi_{n+1}, \quad (1)$$

where $\{\gamma_n, n \geq 1\}$ is a sequence of deterministic step-size and Ξ_{n+1} is a random variable (r.v.) standing for a noisy measurement of the unknown quantity $h(\theta_n)$.

Our aim is to establish the rate of convergence of the sequence $\{\theta_n, n \geq 0\}$ to a limiting point θ_* in the following framework.

Let $\Theta \subseteq \mathbb{R}^d$; the sequence $\{\theta_n, n \geq 0\}$ is a Θ -valued random sequence defined on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \{\mathcal{F}_n, n \geq 0\})$ and given by

$$\theta_{n+1} = \theta_n + \gamma_{n+1} (h(\theta_n) + e_{n+1} + r_{n+1}), \quad \theta_0 \in \Theta;$$

where $h : \Theta \rightarrow \mathbb{R}^d$ is a measurable function, $\{e_n, n \geq 1\}$ is a \mathcal{F}_n -adapted \mathbb{P} -martingale increment sequence and $\{r_n, n \geq 1\}$ is a vanishing \mathcal{F}_n -adapted random sequence. Such a general description covers many SA algorithms: as discussed below (see Section 2.1), it covers the case when Ξ_{n+1} is of the form $H(\theta_n, X_{n+1})$ where $\{X_n, n \geq 1\}$ are independent and identically distributed (i.i.d.) r.v. such that (s.t.) $\mathbb{E}[H(\theta, X)] = h(\theta)$; and the more general case when $\{X_n, n \geq 1\}$ is an adapted (non stationary) Markov chain with transition kernel driven by the current value of the SA sequence $\{\theta_n, n \geq 0\}$. It also covers the case of fixed truncated and randomly truncated SA algorithms i.e. situations when given a (possibly random) sequence of subsets $\{\mathcal{K}_n, n \geq 0\}$ of Θ , the update rule is given by

$$\theta_{n+1} = \begin{cases} \theta_n + \gamma_{n+1} \Xi_{n+1}, & \text{if } \theta_n + \gamma_{n+1} \Xi_{n+1} \in \mathcal{K}_{n+1} \\ \theta_0 & \text{otherwise .} \end{cases} \quad (2)$$

Such a truncated algorithm is used for example to solve optimization problem on a constraint set Θ (in this case, $\mathcal{K}_n = \Theta$ for any n), or to ensure stability of the random sequence $\{\theta_n, n \geq 0\}$ in situations where the location of the sought-for root is unknown (in this case, \mathcal{K}_n is an increasing sequence of sets, see [9] and [8, Chapter 2]).

Our second aim is to extend the previous results to the case of multiple targets: we provide asymptotic convergence rates of $\{\theta_n, n \geq 0\}$ to a point θ_* given the event $\{\lim_q \theta_q = \theta_*\}$ for some θ_* in the interior of Θ . Note that this paper is devoted to convergence rates so that sufficient conditions for the convergence is out of the scope of the paper; for convergence, the interested reader can refer to [4, 11, 3, 8, 2, 6].

The originality of this paper consists in deriving rates of convergence in a new framework characterized by (i) general assumptions on the noisy measurement Ξ_{n+1} of $h(\theta_n)$ which weaken the conditions in the literature and (ii) the multiple targets problem. In Section 2.2, our framework will be carefully compared to the literature.

We derive sufficient conditions on the step-size sequence $\{\gamma_n, n \geq 1\}$, on the random sequences $\{e_n, r_n, n \geq 1\}$ and on the limiting point θ_* so that $\gamma_n^{-1/2}(\theta_n - \theta_*)$ converges in distribution under the conditional probability $\mathbb{P}(\cdot | \lim_q \theta_q = \theta_*)$. The limiting distribution is a (mixture of) centered Gaussian distribution(s) and this distribution is explicitly characterized. We also address the rate of convergence of the associated averaged process $\{\bar{\theta}_n, n \geq 0\}$ defined by

$$\bar{\theta}_n \stackrel{\text{def}}{=} \frac{1}{n+1} \sum_{k=0}^n \theta_k. \quad (3)$$

We prove that this averaged sequence reaches the optimal rate and the optimal variance (in a sense discussed below); such a result was already established in the literature in a more restrictive framework.

The paper is organized as follows. Section 2 (resp. Section 3) is devoted to the SA sequence $\{\theta_n, n \geq 0\}$ (resp. the averaged SA sequence $\{\bar{\theta}_n, n \geq 0\}$). We successively introduce the assumptions, comment these conditions, compare our framework to the literature and state a Central Limit Theorem (CLT). In Section 4, our results are applied to a randomly truncated SA algorithm with controlled Markov chain dynamics; since our conditions are quite weak, we are able to obtain better convergence rates than the rates obtained in Delyon [10]. All the proofs are postponed in Section 5.

2 A Central Limit Theorem for Stochastic Approximation

2.1 Assumptions

Let $\Theta \subseteq \mathbb{R}^d$. We consider the \mathbb{R}^d -valued sequence satisfying for $n \geq 0$,

$$\theta_{n+1} = \theta_n + \gamma_{n+1}h(\theta_n) + \gamma_{n+1}e_{n+1} + \gamma_{n+1}r_{n+1}, \quad \theta_0 \in \Theta; \quad (4)$$

and we establish a Central Limit Theorem along sequences $\{\theta_n, n \geq 0\}$ converging to some point $\theta_* \in \Theta$ which is a root of the function h . We assume the following conditions on the attractive target θ_* .

- C1** (a) θ_* is in the interior of Θ and $h(\theta_*) = 0$.

- (b) The mean field $h : \Theta \rightarrow \mathbb{R}^d$ is measurable and twice continuously differentiable in a neighborhood of θ_* .
- (c) The gradient $\nabla h(\theta_*)$ is a Hurwitz matrix. Denote by $-L$, $L > 0$, the largest real part of its eigenvalues.

Let $\{e_n, n \geq 1\}$ be a \mathbb{R}^d -valued random variables defined on the filtered space $(\Omega, \mathcal{A}, \mathbb{P}, \{\mathcal{F}_n, n \geq 0\})$. We will denote by $|\cdot|$ the Euclidean norm on \mathbb{R}^d ; and by x^T the transpose of a matrix x . By convention, vectors are column-vectors. For a set A , 1_A is the indicator function. It is assumed

- C2** (a) $\{e_n, n \geq 1\}$ is a \mathcal{F}_n -adapted \mathbb{P} -martingale-increment sequence i.e. $\mathbb{E}[e_n | \mathcal{F}_{n-1}] = 0$ \mathbb{P} -almost surely.
- (b) For any $m \geq 1$, there exists a sequence of measurable sets $\{\mathcal{A}_{m,k}, k \geq 0\}$ such that $\mathcal{A}_{m,k} \in \mathcal{F}_k$ and there exists $\tau > 0$ such that

$$\sup_{k \geq 0} \mathbb{E}[|e_{k+1}|^{2+\tau} 1_{\mathcal{A}_{m,k}}] < \infty.$$

In addition, for any $m \geq 1$, $\lim_k 1_{\mathcal{A}_{m,k}} 1_{\lim_q \theta_q = \theta_*} = 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_*}$ and the limiting set satisfies $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_*) = 1$.

- (c) $\mathbb{E}[e_{k+1} e_{k+1}^T | \mathcal{F}_k] = U_* + D_k^{(1)} + D_k^{(2)}$ where U_* is a symmetric positive definite (random) matrix and

$$\begin{cases} D_k^{(1)} \xrightarrow{\text{a.s.}} 0, & \text{on the set } \{\lim_q \theta_q = \theta_*\} \\ \lim_n \gamma_n \mathbb{E}\left[\sum_{k=1}^n D_k^{(2)} \middle| 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m}\right] = 0; \end{cases} \quad (5)$$

the sequence $\{\mathcal{A}_m, m \geq 1\}$ is defined in C2b.

We will show (see remark 5.3 in Section 5) that the condition on the r.v. $\{D_k^{(2)}, k \geq 1\}$ can be replaced with: $D_k^{(2)} = D_k^{(2,a)} + D_k^{(2,b)}$

$$\lim_n \gamma_n \mathbb{E}\left[\left|\sum_{k=1}^n D_k^{(2,a)} 1_{\mathcal{A}_{m,k}} 1_{A_k}\right| + \left|\sum_{k=1}^n D_k^{(2,b)}\right| 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_*}\right] = 0, \quad \forall m \geq 1, \quad (6)$$

where $\{A_k, k \geq 1\}$ is any \mathcal{F}_k -adapted sequence of sets satisfying $\lim_k 1_{A_k} = 1_{\lim_q \theta_q = \theta_*}$; and $\mathcal{A}_{m,k}$ is given by C2b.

For a sequence of \mathbb{R}^d -valued r.v. $\{Z_n, n \geq 0\}$, we write $Z_n = O_{w.p.1}(1)$ if $\sup_n |Z_n| < \infty$ w.p.1; and $Z_n = o_{L^p}(1)$ if $\lim_n \mathbb{E}[|Z_n|^p] = 0$. Let $\{r_n, n \geq 1\}$ be a \mathbb{R}^d -valued random variables defined on the filtered space $(\Omega, \mathcal{A}, \mathbb{P}, \{\mathcal{F}_n, n \geq 0\})$.

C3 r_n is \mathcal{F}_n -adapted. $r_n = r_n^{(1)} + r_n^{(2)}$ with, for any $m \geq 1$,

$$\begin{cases} \gamma_n^{-1/2} r_n^{(1)} \mathbb{1}_{\lim_q \theta_q = \theta_*} \mathbb{1}_{\mathcal{A}_m} = O_{w.p.1}(1) o_{L^1}(1), \\ \sqrt{\gamma_n} \sum_{k=1}^n r_k^{(2)} \mathbb{1}_{\lim_q \theta_q = \theta_*} \mathbb{1}_{\mathcal{A}_m} = O_{w.p.1}(1) o_{L^1}(1). \end{cases}$$

The sequence $\{\mathcal{A}_m, m \geq 1\}$ is defined in C2b.

The last assumption is on the step-size sequence.

C4 One of the following conditions is satisfied:

- (a) $\sum_k \gamma_k = +\infty, \sum_k \gamma_k^2 < \infty$ and $\log(\gamma_{k-1}/\gamma_k) = o(\gamma_k)$.
- (b) $\sum_k \gamma_k = +\infty, \sum_k \gamma_k^2 < \infty$ and there exists $\gamma_* > 1/(2L)$ such that $\log(\gamma_{k-1}/\gamma_k) \sim \gamma_k/\gamma_*$.

2.2 Comments on the assumptions

The framework described by (4) and the conditions C1 to C4 is general enough to cover many scenarios studied in the literature and to address new ones.

For SA algorithms (1) with $\Xi_{n+1} = H(\theta_n, X_{n+1})$, $\{X_n, n \geq 1\}$ i.i.d. r.v. (and independent of θ_0) such that $h(\theta) = \mathbb{E}[H(\theta, X)]$, Eq. (4) is satisfied with

$$e_{n+1} = H(\theta_n, X_{n+1}) - h(\theta_n), \quad r_{n+1} = 0;$$

and $\mathbb{E}[e_{n+1}|\mathcal{F}_n] = 0$. Our framework also addresses the case when $\{X_n, n \geq 1\}$ is a \mathcal{F}_n -adapted controlled Markov chain i.e. when there exists a family of transition kernels $\{Q_\theta, \theta \in \Theta\}$ such that

$$\mathbb{P}(X_{n+1} \in \cdot | \mathcal{F}_n) = Q_{\theta_n}(X_n, \cdot),$$

each kernel possessing an invariant probability distribution π_θ and $h(\theta) = \int H(\theta, x) \pi_\theta(dx)$ - hereafter, these algorithms will be called ‘‘SA with controlled Markov chain dynamics’’. Introduce the solution \hat{H}_θ of the Poisson equation $H(\theta, \cdot) - h(\theta) = \hat{H}_\theta - Q_\theta \hat{H}_\theta$ (see e.g. [16, Chapter 8] or [22, Chapter 17]), and set

$$e_{n+1} = \hat{H}_{\theta_n}(X_{n+1}) - Q_{\theta_n} \hat{H}_{\theta_n}(X_n), \quad r_{n+1} = Q_{\theta_n} \hat{H}_{\theta_n}(X_n) - Q_{\theta_n} \hat{H}_{\theta_n}(X_{n+1});$$

then $\mathbb{E}[e_{n+1}|\mathcal{F}_n] = 0$ \mathbb{P} -almost surely. We will provide in Section 4 sufficient conditions on the transition kernels Q_θ so that these sequences $\{e_n, r_n, n \geq 1\}$ exist and satisfy the conditions C2 and C3. Note that the i.i.d. case is a special case of the controlled Markov chain framework (set $Q_\theta = \pi_\theta = \pi$ for any θ); and the so-called Robbins-Monro case corresponds to $Q_\theta = \pi_\theta$ for any θ .

Truncated SA algorithms (2) can be written as

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\Xi_{n+1} + (\theta_0 - \theta_n - \gamma_{n+1}\Xi_{n+1}) 1_{\theta_n + \gamma_{n+1}\Xi_{n+1} \notin \mathcal{K}_{n+1}} ;$$

in most (if not any) proof of convergence of this sequence to limiting points in the interior of Θ , the first step consists in proving that \mathbb{P} -almost-surely, the number of truncations is finite (see e.g. Andrieu et al.[2, Theorem 1]). Therefore, the term $(\theta_0 - \theta_n - \gamma_{n+1}\Xi_{n+1}) 1_{\theta_n + \gamma_{n+1}\Xi_{n+1} \notin \mathcal{K}_{n+1}}$ is null for any large n on the set $\{\lim_q \theta_q = \theta_\star\}$ thus showing that it is part of $\gamma_{n+1}r_{n+1}^{(1)}$ in the expansion (4).

The condition C1 considers a limiting target θ_\star which is assumed to be stable and such that the linear term in the Taylor's expansion of h at θ_\star does not vanish (see condition C1c). Results for the case of vanishing linear term can be found in Chen [8, Section 3.2]. When h is a gradient function so that the SA algorithm is a stochastic gradient procedure, the condition C1a assumes that θ_\star is a root of the gradient. Therefore, our assumptions do not cover the case of constrained optimization problem with solutions on the boundaries of the constraint set Θ . For rates of convergence for these constrained SA algorithms, see e.g. Buche and Kushner [7].

The conditions C2 and C3 are designed to address the case of multiple targets, a framework which improves on many published results. It is usually assumed in the literature that there is an unique limiting target (see e.g. Fabian [12], Buche and Kushner [7], Chen [8, Chapter 3] and Lelong [21]). While we are interested in proving a Central Limit Theorem given the tail event $\{\lim_q \theta_q = \theta_\star\}$, it is assumed in C2a that the r.v. e_{n+1} in the expansion (4) is a martingale increment with respect to (w.r.t.) the probability \mathbb{P} . As discussed above, such an expansion is easily verified. Note that since the event $\{\lim_q \theta_q = \theta_\star\}$ is in the tail σ -field $\sigma(\bigvee_n \mathcal{F}_n)$, it is not true that $\{e_n, n \geq 1\}$ are martingale-increments w.r.t. the probability $\mathbb{P}(\cdot | \lim_q \theta_q = \theta_\star)$. Therefore, our framework is not a special case of the single target framework.

The main use of C2 is to prove that the $\{e_n, n \geq 1\}$ satisfies a CLT under the conditional distribution $\mathbb{P}(\cdot | \lim_q \theta_q = \theta_\star)$. We could weaken some of the assumptions, for example by relaxing the $2 + \tau$ -moment condition C2b which is a way to easily check the Lindeberg condition for martingale difference array. Nevertheless, our goal is not only to state a theorem with weaker assumptions but also to present easy-to-check conditions.

When there exists $\tau > 0$ such that $\sup_{k \geq 1} \mathbb{E} [|e_k|^{2+\tau}] < \infty$, C2b is satisfied with $\mathcal{A}_m = \mathcal{A}_{m,k} = \Omega$. When there exist $\tau, \delta > 0$ such that

$$\sup_{k \geq 0} \mathbb{E} [|e_{k+1}|^{2+\tau} 1_{|\theta_k - \theta_\star| \leq \delta}] < \infty , \quad (7)$$

then C2b is satisfied with $\mathcal{A}_{m,k} = \bigcap_{m \leq j \leq k} \{|\theta_j - \theta_\star| \leq \delta\}$ and $\mathcal{A}_m = \bigcap_{j \geq m} \{|\theta_j -$

$|\theta_\star| \leq \delta$. In most contributions, rates of convergence are derived under the condition (7) (see e.g. the recent works by Pelletier [23] and Lelong [21]). This framework is too restrictive to address the case of SA with controlled Markov chain dynamics when the ergodic properties of the transition kernels $\{Q_\theta, \theta \in \Theta\}$ are not uniform in θ . Our assumption C2b is designed to address this framework as it will be shown in Section 4.

C2c is an assumption on the conditional variance of the martingale-increment term e_n , which is more general than what is usually assumed. In Zhu [27], Pelletier [23], Chen [8] and Leling [21] (resp. in Delyon [10]), a CLT is proved under the assumption that $\mathbb{E}[e_{k+1}e_{k+1}^T | \mathcal{F}_k] = U_\star + D_k^{(1)}$ (resp. $\mathbb{E}[e_{k+1}e_{k+1}^T | \mathcal{F}_k] = U_\star + D_k^{(2)}$) where $D_k^{(1)}, D_k^{(2)}$ satisfy (5) and U_\star is a deterministic symmetric positive definite matrix. A first improvement is to remove the assumption that U_\star is deterministic. A second improvement is in the combination $D_k^{(1)} + D_k^{(2)}$. The introduction of the term $D_k^{(2)}$ is a strong improvement since it covers the case of SA with controlled Markov chain dynamic: observe indeed that in this case $\mathbb{E}[e_{k+1}e_{k+1}^T | \mathcal{F}_k]$ is a function of (X_k, θ_k) and it is really unlikely that this term converges almost-surely to a (random) variable along the set $\{\lim_q \theta_q = \theta_\star\}$. Allowing an additional term $D_k^{(2)}$ such that the sum $\sum_{k=1}^n D_k^{(2)}$ converges in some sense to zero introduces more flexibility (see Section 4 for more details). We will also show in Section 4 how our framework improves on Delyon [10]. Examples of SA algorithm where C2c holds with resp. Robbins-Monro and controlled Markov chain dynamics can be found resp. in Bianchi et al. [5] and Fort et al. [13].

Examples of sequences satisfying the condition C4 are the polynomial ones. The step size $\gamma_n \sim \gamma_\star n^{-a}$ for $a \in (1/2, 1)$ satisfies C4a. The step size $\gamma_n \sim \gamma_\star/n$ satisfies C4b; note that the condition on (γ_\star, L) is well known in the literature (see e.g. Chen [8, Assumption A3.1.4]).

2.3 Main result

Theorem 2.1. *Choose $\theta_0 \in \Theta$ and consider the sequence $\{\theta_n, n \geq 0\}$ given by (4). Assume C1, C2, C3 and C4. Let V be the positive definite matrix satisfying w.p.1 on the set $\{\lim_q \theta_q = \theta_\star\}$,*

$$\begin{cases} V\nabla h(\theta_\star)^T + \nabla h(\theta_\star)V = -U_\star, & \text{in case C4a,} \\ V(\text{Id} + 2\gamma_\star \nabla h(\theta_\star)^T) + (\text{Id} + 2\gamma_\star \nabla h(\theta_\star))V = -2\gamma_\star U_\star, & \text{in case C4b.} \end{cases}$$

Under the conditional probability $\mathbb{P}(\cdot | \lim_q \theta_q = \theta_\star)$, $\{\gamma_n^{-1/2}(\theta_n - \theta_\star), n \geq 1\}$ converges in distribution to a r.v. with characteristic function given for any $t \in \mathbb{R}^d$

by

$$\frac{1}{\mathbb{P}(\lim_q \theta_q = \theta_\star)} \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp\left(-\frac{1}{2} t^T V t\right) \right] .$$

When the matrix U_\star in Assumption C2c is deterministic, the limiting distribution is a centered multidimensional Gaussian distribution with (deterministic) covariance matrix V .

Given matrices A, E , existence of a solution to the equation $VA + A^T V = -E$ is solved by the Lyapunov theorem (see e.g. Horn and Johnson [18, Theorem 2.2.1.]). When A is a (negative) stable real matrix and E is positive definite, then there exists a unique positive definite matrix V satisfying the Lyapunov equation $VA + A^T V = -E$ (see e.g. Horn and Johnson [18, Theorem 2.2.3.]).

Sketch of the proof of Theorem 2.1 The proof of Theorem 2.1 is detailed in Section 5. The key ingredient is the Central Limit Theorem for martingale arrays.

As commented in Section 2.2, e_n is not a martingale-increment w.r.t. the conditional probability $\mathbb{P}(\cdot | \lim_q \theta_q = \theta_\star)$. To overcome this technical difficulty, we use that

$$e_{n+1} = e_{n+1} 1_{A_n} + e_{n+1} (1 - 1_{A_n}) \quad (8)$$

where $\{A_n, n \geq 1\}$ is a \mathcal{F}_n -adapted sequence of sets converging to $\{\lim_q \theta_q = \theta_\star\}$ (such a sequence always exists, see Lemma 5.7). Along the event $\{\lim_q \theta_q = \theta_\star\}$, the second term in the right hand side (rhs) of (8) is null for any n larger than some almost-surely finite random time.

We write $\theta_n - \theta_\star = \mu_n + \rho_n$, where μ_n satisfies the equation

$$\mu_{n+1} = (\text{Id} + \gamma_{n+1} \nabla h(\theta_\star)) \mu_n + \gamma_{n+1} e_{n+1} ; \quad \mu_0 = 0 .$$

Id denotes the $d \times d$ identity matrix. Roughly speaking, the sequence $\{\mu_n, n \geq 0\}$ captures the linear approximation of $h(\theta_n)$ and the martingale-increment noise sequence $\{e_n, n \geq 1\}$.

We prove that $\gamma_n^{-1/2} \rho_n 1_{\lim_q \theta_q = \theta_\star}$ converges to zero in probability so that $\{\mu_n, n \geq 0\}$ is the leading term. We then establish that for any $t \in \mathbb{R}^d$,

$$\lim_n \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp\left(i \gamma_n^{-1/2} t^T \mu_n\right) \right] = \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp\left(-\frac{1}{2} t^T V t\right) \right] .$$

3 A Central Limit Theorem for Iterate Averaging

Theorem 2.1 shows that the rate of convergence of the sequence $\{\theta_n, n \geq 0\}$ to θ_\star is $O(n^{a/2})$ when $\gamma_n \sim \gamma_\star/n^a$ for some $a \in (1/2, 1]$. The maximal rate is reached by choosing $\gamma_n \sim \gamma_\star/n$, for some γ_\star satisfying the conditions C4b. The main

drawback with such a choice of the step-size sequence $\{\gamma_n, n \geq 1\}$ is that in practice, $-L$ i.e. the largest real part of the eigenvalues of $\nabla h(\theta_*)$ is unknown so that the condition C4b is difficult to check.

The second comment is on the limiting covariance matrix when the rate is maximal (i.e. in the case $\gamma_n \sim \gamma_*/n$). For any non-singular matrix Γ , we could define the algorithm

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n + \gamma_{n+1}\Gamma h(\tilde{\theta}_n) + \gamma_{n+1}\Gamma e_{n+1} + \gamma_{n+1}\Gamma r_{n+1}, \quad \tilde{\theta}_0 \in \Theta.$$

This equation is of the form (4) with a mean field $\tilde{h} = \Gamma h$ and noises $\{e_n, r_n, n \geq 1\}$ replaced with $\{\Gamma e_n, \Gamma r_n, n \geq 1\}$. Then, Theorem 2.1 gives sufficient conditions so that a CLT for the sequence $\{\tilde{\theta}_n, n \geq 0\}$ holds: the matrix V is replaced with $\tilde{V} = \tilde{V}(\Gamma)$ satisfying

$$\tilde{V}(\text{Id} + 2\gamma_*\nabla h(\theta_*)^T\Gamma^T) + (\text{Id} + 2\gamma_*\nabla h(\theta_*)\Gamma)\tilde{V} = -2\gamma_*\Gamma U_*\Gamma^T.$$

A natural question is the “optimal” choice of the gain matrix Γ , defined as the matrix Γ_* such that for any $\lambda \in \mathbb{R}^d$, $\lambda^T\tilde{V}(\Gamma)\lambda \geq \lambda^T\tilde{V}(\Gamma_*)\lambda$. Following the same lines as in Benveniste et al. [4, Proposition 4, Chapter 3, Part I], it can be proved that $\Gamma_* = -\gamma_*^{-1}\nabla h(\theta_*)^{-1}$ and in this case,

$$\tilde{V}(\Gamma_*) = \gamma_*^{-1}\nabla h(\theta_*)^{-1}U_*\nabla h(\theta_*)^{-T}.$$

Theorem 3.2 below shows that by considering the averaged sequence $\{\bar{\theta}_n, n \geq 0\}$, the optimal rate of convergence (i.e. the rate \sqrt{n}) and the optimal asymptotic covariance matrix (optimal in the sense discussed above) can be reached whatever the sequence $\{\gamma_n, n \geq 1\}$ satisfying C4a used in the basic SA sequence (4). Therefore, such an optimality can be obtained even when $\nabla h(\theta_*)$ is unknown. Note also that on a practical point of view, slow decreasing step-size γ_n are better (see e.g. Spall [26, Section 4.4.]) and this simple averaging procedure improves the rate of convergence of the estimate of θ_* .

These properties of the averaged sequence were simultaneously established by Ruppert [25] and Polyak and Juditsky [24] under more restrictive conditions than those stated below.

3.1 Assumptions

- AVER1** (a) $\{e_n, n \geq 1\}$ is a \mathcal{F}_n -adapted \mathbb{P} -martingale-increment sequence.
 (b) There exists a sequence $\{\mathcal{A}_m, m \geq 1\}$ such that $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_*) = 1$, and for any $m \geq 1$,

$$\sup_k \mathbb{E} [|e_k|^2 1_{\mathcal{A}_m, k-1}] < \infty,$$

where $\mathcal{A}_{m,k-1} \in \mathcal{F}_{k-1}$ and $\lim_k 1_{\mathcal{A}_{m,k}} = 1_{\mathcal{A}_m}$ almost-surely on the set $\{\lim_q \theta_q = \theta_\star\}$.

(c) Let

$$\mathcal{E}_{n+1} = \frac{1}{\sqrt{n+1}} \sum_{k=0}^n e_{k+1}.$$

There exists a random matrix U_\star , positive definite w.p.1. on the set $\{\lim_q \theta_q = \theta_\star\}$, such that for any $t \in \mathbb{R}^d$,

$$\lim_n \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp(it^T \mathcal{E}_{n+1}) \right] = \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp\left(-\frac{1}{2} t^T U_\star t\right) \right].$$

We prove in Lemma 5.5 that when $\lim_n n\gamma_n > 0$, assumption C2 implies AVER1. Note also that since $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_\star) = 1$, AVER1c is equivalent to the condition: for any $m \geq 1$,

$$\lim_n \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp(it^T \mathcal{E}_{n+1}) 1_{\mathcal{A}_m} \right] = \mathbb{E} \left[1_{\lim_q \theta_q = \theta_\star} \exp\left(-\frac{1}{2} t^T U_\star t\right) 1_{\mathcal{A}_m} \right].$$

For a sequence of \mathbb{R}^d -valued r.v. $\{Z_n, n \geq 0\}$, we write $Z_n = O_{L^p}(1)$ if $\sup_n \mathbb{E}[|Z_n|^p] < \infty$.

AVER2 r_n is \mathcal{F}_n -adapted. $r_n = r_n^{(1)} + r_n^{(2)}$ with for any $m \geq 1$,

- (a) $\gamma_n^{-1/2} r_n^{(1)} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} = O_{w.p.1}(1) O_{L^2}(1)$.
- (b) $\sqrt{\gamma_n} \sum_{k=1}^n r_k^{(2)} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} = O_{w.p.1}(1) O_{L^2}(1)$.
- (c) $n^{-1/2} \sum_{k=0}^n r_{k+1} 1_{\lim_q \theta_q = \theta_\star} \xrightarrow{\mathbb{P}} 0$.

The sequence $\{\mathcal{A}_m, m \geq 1\}$ is defined in AVER1b.

Note that AVER2c is equivalent to $n^{-1/2} \sum_{k=0}^n r_{k+1} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \xrightarrow{\mathbb{P}} 0$ for any $m \geq 1$.

AVER3 $\lim_n n\gamma_n = +\infty$ and

$$\lim_n \frac{1}{\sqrt{n}} \sum_{k=1}^n \gamma_k^{-1/2} \left| 1 - \frac{\gamma_k}{\gamma_{k+1}} \right| = 0, \quad \lim_n \frac{1}{\sqrt{n}} \sum_{k=1}^n \gamma_k = 0.$$

The step size $\gamma_n \sim \gamma_\star n^{-a}$ for $a \in (1/2, 1)$ satisfies AVER3 but the step size $\gamma_n \sim \gamma_\star/n$ does not. Observe that if the sequence $\{\gamma_n, n \geq 0\}$ is non-increasing (or ultimately non-increasing) then (see Lemma 5.13)

$$\lim_n n\gamma_n = +\infty \implies \lim_n \frac{1}{\sqrt{n}} \sum_{k=1}^n \gamma_k^{-1/2} \left| 1 - \frac{\gamma_k}{\gamma_{k+1}} \right| = 0.$$

3.2 Main results

We show that the above conditions allow a control of the L^2 -moment of the errors $\{\theta_n - \theta_*, n \geq 0\}$. This result is a cornerstone for the proof of Theorem 3.2. The proof is given in Section 5.

Proposition 3.1. *Assume C1, C4, AVER1a-b and AVER2a-b. Then, for any $m \geq 1$*

$$\gamma_n^{-1} \|\theta_n - \theta_*\|^2 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} = O_{w.p.1}(1) O_{L^1}(1) .$$

Theorem 3.2. *Choose $\theta_0 \in \Theta$ and consider the averaged sequence given by (3). Assume C1, C4a, AVER1, AVER2 and AVER3. Then for any $t \in \mathbb{R}^d$,*

$$\begin{aligned} \lim_n \mathbb{E} [1_{\lim_q \theta_q = \theta_*} \exp(i\sqrt{n} t^T (\bar{\theta}_n - \theta_*))] \\ = \mathbb{E} \left[1_{\lim_q \theta_q = \theta_*} \exp \left(-\frac{1}{2} t^T \nabla h(\theta_*)^{-1} U_* (\nabla h(\theta_*)^{-1})^T t \right) \right] . \end{aligned}$$

Sketch of the proof of Theorem 3.2 The proof is detailed in Section 5. Since $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_*) = 1$, we only have to prove that for any $m \geq 1$ and $t \in \mathbb{R}^d$,

$$\begin{aligned} \lim_n \mathbb{E} [1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \exp(i\sqrt{n} t^T (\bar{\theta}_n - \theta_*))] \\ = \mathbb{E} \left[1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \exp \left(-\frac{1}{2} t^T \nabla h(\theta_*)^{-1} U_* (\nabla h(\theta_*)^{-1})^T t \right) \right] . \end{aligned}$$

We write

$$\bar{\theta}_n - \theta_* = -\frac{\nabla h(\theta_*)^{-1}}{n+1} \sum_{k=0}^n e_{k+1} + Z_n .$$

We show that $\sqrt{n} Z_n 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m}$ converges to zero in probability for any $m \geq 1$; for this step, the main tool is Proposition 3.1. The proof is then concluded by AVER1c.

4 Application to SA with controlled Markov chain dynamics

Let $\{\mathcal{K}_n, n \geq 0\}$ be a sequence of compact subsets of Θ such that

$$\mathcal{K}_n \subseteq \mathcal{K}_{n+1} , \quad \bigcup_{n \geq 0} \mathcal{K}_n = \Theta .$$

Let $\{Q_\theta, \theta \in \Theta\}$ be a family of Markov transition kernels onto $(\mathbb{X}, \mathcal{X})$. We consider the following SA algorithm with truncation at randomly varying bounds: $\theta_0 \in \mathcal{K}_0, \sigma_0 = 0$ and for $n \geq 0$,

$$\text{set } \theta_{n+1/2} = \theta_n + \gamma_{n+1} H(\theta_n X_{n+1}).$$

update

$$(\theta_{n+1}, \sigma_{n+1}) = \begin{cases} (\theta_{n+1/2}, \sigma_n), & \text{if } \theta_{n+1/2} \in \mathcal{K}_{\sigma_n}, \\ (\theta_0, \sigma_n + 1) & \text{otherwise,} \end{cases}$$

where $\{X_n, n \geq 0\}$ is a controlled Markov chain on $(\Omega, \mathcal{A}, \mathbb{P})$ with conditional distribution given by

$$\mathbb{P}(X_{n+1} \in A | \mathcal{F}_n) = Q_{\theta_n}(X_n, A), \quad \mathcal{F}_n = \sigma(\theta_0, X_0, \dots, X_n). \quad (9)$$

The random sequence $\{\sigma_n, n \geq 0\}$ is a non-negative integer-valued sequence counting the number of truncations. Such a truncated SA was introduced by Chen et al. [9] (see also Chen [8, Chapter 2]) to address the boundedness problem of the SA sequence $\{\theta_n, n \geq 0\}$. A more general truncated SA algorithm with controlled Markov chain dynamics is introduced in Andrieu et al. [2]: when truncation occurs, both the parameter $\theta_{n+1/2}$ and the draw X_n used to obtain the next point X_{n+1} are modified.

The key point of the proof of convergence of this algorithm is to show that the number of truncations is finite with probability one, so that after some random time, the sequence $\{\theta_n, n \geq 0\}$ is almost-surely bounded and obeys the update rule $\theta_{n+1} = \theta_n + \gamma_{n+1} H(\theta_n, X_{n+1})$. Conditions implying almost-sure boundedness and almost-sure convergence of the sequence $\{\theta_n, n \geq 0\}$ when $\{X_n, n \geq 0\}$ is a controlled Markov chain can be found in Andrieu et al. [2, Section 3]. Since in this paper we are interested in CLT's, we will assume that

- A1** (a) For any $\theta \in \Theta$, there exists a probability distribution π_θ on $(\mathbb{X}, \mathcal{X})$ such that $\pi_\theta Q_\theta = \pi_\theta$. Set

$$h(\theta) = \int H(\theta, x) \pi_\theta(dx). \quad (10)$$

- (b) the number of truncations is finite with probability one: $\mathbb{P}(\limsup_n \sigma_n < \infty) = 1$ and there exists $\theta_\star \in \Theta$ satisfying C1 such that $\mathbb{P}(\lim_n \theta_n = \theta_\star) > 0$.

For simplicity, we consider the case when H is bounded and the step-size is polynomially decreasing. Extensions to the case H is unbounded can be done along the same lines as in Andrieu et al. [2].

- A2** (a) for any compact set $\mathcal{K} \subseteq \Theta$, $\sup_{\theta \in \mathcal{K}} \sup_{x \in \mathbb{X}} |H(\theta, x)| < \infty$.
 (b) There exists $a \in (1/2, 1]$ such that $\gamma_n = \gamma_*/n^a$. When $a = 1$, γ_* satisfies the condition C4b.

We assume that the transition kernels $\{Q_\theta, \theta \in \Theta\}$ satisfy

- A3** (a) For any $\theta \in \Theta$, there exists a measurable function $\widehat{H}_\theta : (\mathbb{X}, \mathcal{X}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$H(\theta, x) - h(\theta) = \widehat{H}_\theta(x) - Q_\theta \widehat{H}_\theta(x). \quad (11)$$

There exists a function $V_1 : \mathcal{X} \rightarrow [1, \infty)$ such that for any compact subset $\mathcal{K} \subseteq \Theta$,

$$\sup_{\theta \in \mathcal{K}} \sup_{x \in \mathbb{X}} \frac{|\widehat{H}_\theta(x)| + |Q_\theta \widehat{H}_\theta(x)|}{V_1(x)} < \infty. \quad (12)$$

- (b) For any $\theta \in \Theta$, there exists a measurable function $U_\theta : (\mathbb{X}, \mathcal{X}) \rightarrow (\mathbb{R}^{d^2}, \mathcal{B}(\mathbb{R}^{d^2}))$ such that

$$F_\theta(x) - \int F_\theta(x) \pi_\theta(dx) = U_\theta(x) - Q_\theta U_\theta(x), \quad (13)$$

where $F_\theta(x) = \int Q_\theta(x, dy) \widehat{H}_\theta(y) \widehat{H}_\theta(y)^T - Q_\theta \widehat{H}_\theta(x) \left(Q_\theta \widehat{H}_\theta(x) \right)^T$. There exists a function $V_2 : \mathcal{X} \rightarrow [1, \infty)$ such that for any compact subset $\mathcal{K} \subseteq \Theta$,

$$\sup_{\theta \in \mathcal{K}} \sup_{x \in \mathbb{X}} \frac{|U_\theta(x)| + |Q_\theta U_\theta(x)|}{V_2(x)} < \infty. \quad (14)$$

- (c) There exist $\delta, \tau, \bar{\tau} > 0$ such that for any $m \geq 1$,

$$\begin{aligned} \sup_{k \geq m} \mathbb{E} \left[(V_1^{2+\tau}(X_{k+1}) + V_2^{1+\bar{\tau}}(X_{k+1})) 1_{\cap_{m \leq j \leq k} \{|\theta_j - \theta_*| \leq \delta\}} \right] &< \infty, \\ \mathbb{E} [V_1^{2+\tau}(X_m) + V_2^{1+\bar{\tau}}(X_m)] &< \infty. \end{aligned}$$

- (d) For any compact subset $\mathcal{K} \subseteq \Theta$, there exist $b, C > 0$ such that for any $\theta, \theta' \in \mathcal{K}$,

$$\begin{aligned} \left| Q_\theta \widehat{H}_\theta(x) - Q_{\theta'} \widehat{H}_{\theta'}(x) \right| &\leq C |\theta - \theta'|^{1/2+b} V_1(x), \\ |U_\theta(x) - U_{\theta'}(x)| &\leq C |\theta - \theta'|^b V_2(x). \end{aligned}$$

Furthermore, almost-surely

$$\lim_n \left(\int F_{\theta_n}(x) \pi_{\theta_n}(dx) - \int F_{\theta_*}(x) \pi_{\theta_*}(dx) \right) 1_{\lim_q \theta_q = \theta_*} = 0.$$

Conditions implying the existence of π_θ and solutions to the Poisson equations (11) and (13) can be found e.g. in Hernandez-Lerma and Lasserre [16, Chapter 8] or in Meyn and Tweedie [22, Chapter 17]. When the transition kernel Q_θ is uniformly ergodic, then $V_1 = V_2$ and is equal to the constant function 1. When the kernel is V -geometrically ergodic, we can choose $V_1 = V^{1/p}, V_2 = V^{2/p}$ for any $p \geq 2$. Sufficient conditions for (12) and (14) based on Lyapunov drift inequalities when the chain is geometrically ergodic (resp. subgeometrically ergodic) are given by Fort et al. [14, Lemma 2.3] (resp. Andrieu et al. [1]. Andrieu et al. [2, Proposition 6.1.] gives sufficient conditions to check A3c (compare this assumption with the condition A3(ii) of Andrieu et al.) when the kernels are V -geometrically ergodic: in this case, for any $p \geq 2$ we can choose $V_1 = V^{1/p}, V_2 = V^{2/p}$ and $2(1 + \bar{\tau})/p = 1$. The first set of conditions in A3d is an assumption on the regularity-in- θ of the solution to the Poisson equation. Andrieu et al. [2, Proposition 6.1.] give sufficient conditions in terms of the regularity-in- θ of the transition kernels Q_θ . When $\pi_\theta = \pi$ for any θ , the second set of conditions can be established by combining smoothness-in- θ properties of the function F_θ and the dominated convergence theorem. When π_θ depends on θ , Fort et al. [14, Theorem 2.11 and Proposition 4.3] give sufficient conditions for this condition to hold.

We now show how these assumptions imply the conditions C1 to C4. Under A1b, the condition C1 holds; note also that the conditional probability $\mathbb{P}(\cdot | \lim_q \theta_q = \theta_\star)$ is well defined. By using (10) and (11), we write the truncated SA algorithm on the form (4) by setting

$$\begin{aligned} e_{n+1} &= \widehat{H}_{\theta_n}(X_{n+1}) - Q_{\theta_n} \widehat{H}_{\theta_n}(X_n), \\ r_{n+1} &= Q_{\theta_n} \widehat{H}_{\theta_n}(X_n) - Q_{\theta_n} \widehat{H}_{\theta_n}(X_{n+1}) + (\theta_0 - \theta_{n+1/2}) 1_{\theta_{n+1/2} \notin \mathcal{K}_{\sigma_n}}. \end{aligned}$$

Let us prove that the condition C2 holds. Since $\theta_n \in \mathcal{F}_n$, Eq. (9) implies C2a. Fix δ such that $B(\theta_\star, \delta) = \{\theta \in \mathbb{R}^d, |\theta - \theta_\star| \leq \delta\} \subseteq \Theta$. Set

$$\mathcal{A}_{m,k} = \begin{cases} \emptyset & \text{if } k < m, \\ \bigcap_{m \leq j \leq k} \{|\theta_j - \theta_\star| \leq \delta, \theta_j = \theta_{j-1/2}\} & \text{otherwise.} \end{cases}$$

Then for any k, m , $\mathcal{A}_{m,k} \in \mathcal{F}_k$; $\lim_k \mathcal{A}_{m,k} = \mathcal{A}_m$ where $\mathcal{A}_m = \bigcap_{j \geq m} \{|\theta_j - \theta_\star| \leq \delta, \theta_j = \theta_{j-1/2}\}$; and $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_\star) = 1$ by A1b. Fix $m \geq 1$; by (12) applied with $\mathcal{K} = B(\theta_\star, \delta)$, there exists a constant C such that

$$\mathbb{E} [|e_{k+1}|^{2+\tau} 1_{\mathcal{A}_{m,k}}] \leq C \mathbb{E} [(V_1^{2+\tau}(X_k) + V_1^{2+\tau}(X_{k+1})) 1_{\mathcal{A}_{m,k}}],$$

A3c concludes the proof of C2b. Observe that $\mathbb{E}[e_{k+1}e_{k+1}^T|\mathcal{F}_k] = F_{\theta_k}(X_k)$. By using (13), we write $\mathbb{E}[e_{k+1}e_{k+1}^T|\mathcal{F}_k] = U_\star + D_k^{(1)} + D_k^{(2,a)} + D_k^{(2,b)}$ with

$$\begin{aligned} U_\star &= \int F_{\theta_\star}(x) \pi_{\theta_\star}(dx), \\ D_k^{(1)} &= \int F_{\theta_k}(x) \pi_{\theta_k}(dx) - \int F_{\theta_\star}(x) \pi_{\theta_\star}(dx), \\ D_k^{(2,a)} &= U_{\theta_k}(X_{k+1}) - Q_{\theta_k}U_{\theta_k}(X_k), \\ D_k^{(2,b)} &= U_{\theta_k}(X_k) - U_{\theta_k}(X_{k+1}). \end{aligned}$$

By A3d, $D_k^{(1)} \xrightarrow{\text{a.s.}} 0$ on the set $\{\lim_q \theta_q = \theta_\star\}$. By (9), $\mathbb{E}[D_k^{(2,a)}|\mathcal{F}_{k-1}] = 0$; by application of the Burkholder inequality (see e.g. Hall and Heyde [15, Theorem 2.10]), it holds for any $A_k \in \mathcal{F}_k$ such that $\lim_k A_k = \{\lim_q \theta_q = \theta_\star\}$

$$\mathbb{E} \left[\left| \sum_{k=1}^n D_k^{(2,a)} 1_{A_k} 1_{\mathcal{A}_{m,k}} \right|^{1+\bar{\tau}} \right] \leq C n^{1 \vee (1+\bar{\tau})/2}.$$

The constant C is finite since under A3c, $\sup_k \mathbb{E} \left[|D_k^{(2,a)}|^{1+\bar{\tau}} 1_{\mathcal{A}_{m,k}} \right] < \infty$. Furthermore,

$$\sum_{k=m}^n D_k^{(2,b)} = U_{\theta_m}(X_m) - U_{\theta_n}(X_{n+1}) + \sum_{k=m+1}^n (U_{\theta_k}(X_k) - U_{\theta_{k-1}}(X_k))$$

so that by A3c-d, there exists a constant C such that

$$\mathbb{E} \left[\left| \sum_{k=1}^n D_k^{(2,b)} \right|^{1+\bar{\tau}} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \right] \leq C \left(1 + \sum_{k=m}^{n-1} \gamma_k^{b(1+\bar{\tau})} \right).$$

The above discussion shows that C2c is verified if $a > 1/2 \vee 1/(1+\bar{\tau})$.

Finally, let us study r_n . We write $r_{n+1} = r_{n+1}^{(1)} + r_{n+1}^{(2)}$ with

$$r_{n+1}^{(1)} = (\theta_0 - \theta_{n+1/2}) 1_{\theta_{n+1/2} \notin \mathcal{K}_{\sigma_n}} + Q_{\theta_{n+1}} \hat{H}_{\theta_{n+1}}(X_{n+1}) - Q_{\theta_n} \hat{H}_{\theta_n}(X_{n+1}).$$

By A1b and A3d, $\gamma_n^{-1/2} r_n^{(1)} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} = o_{w.p.1}(1) + o_{L^1}(1)$ for any fixed $m \geq 1$. In addition, by (12), there exists a constant C such that

$$\mathbb{E} \left[\left| \sum_{k=1}^n r_k^{(2)} \right| 1_{\mathcal{A}_m} \right] \leq \mathbb{E}[V_1(X_1)] + \mathbb{E}[V(X_{n+1}) 1_{\mathcal{A}_m}];$$

and by A3c, this term is uniformly bounded in n .

The above discussion is summarized in the following proposition

Proposition 4.1. *Assume A1, A2 and A3. If $a \in (1/2 \vee 1/(1 + \bar{\tau}), 1]$, the conditions C1 to C4 are satisfied and*

$$U_\star = \int \pi_{\theta_\star}(dx) \left(\widehat{H}_{\theta_\star}(x) \widehat{H}_{\theta_\star}(x)^T - Q_{\theta_\star} \widehat{H}_{\theta_\star}(x) \left(Q_{\theta_\star} \widehat{H}_{\theta_\star}(x) \right)^T \right) .$$

By application of Theorem 2.1, we obtain a CLT for randomly truncated SA with controlled Markov chain dynamics. Our result improves on Delyon [10, Theorem 25]. Under stronger conditions (for example, it is assumed that V_1 and V_2 are bounded functions; there is a single target θ_\star), Delyon [10] establishes a CLT in the case $\gamma_n = \gamma_\star/n^a$ with the condition $a \in (2/3, 1]$. Note that if V_1, V_2 are bounded then A3c holds with any $\bar{\tau} > 0$ so that our approach only requires $a \in (1/2, 1]$ which is the usual range of values for SA algorithms.

Using similar tools, the conditions of Theorem 3.2 can be verified; details are left to the interested reader.

5 Proof

5.1 Definitions and Notations

Let $\{A_n, n \geq 0\}$ be a sequence of sets such that

$$A_n \in \mathcal{F}_n, \quad \lim_n 1_{A_n} = 1_{\lim_q \theta_q = \theta_\star} \quad \text{w.p.1} . \quad (15)$$

Such a sequence exists by Lemma 5.7. Define recursively two sequences

$$\mu_{n+1} = (\text{Id} + \gamma_{n+1} \nabla h(\theta_\star)) \mu_n + \gamma_{n+1} e_{n+1}, \quad \mu_0 = 0; \quad (16)$$

$$\rho_{n+1} = \theta_{n+1} - \theta_\star - \mu_{n+1}, \quad \rho_0 = \theta_0 - \theta_\star; \quad (17)$$

and the matrices $\psi_\star(n, k)$ for $1 \leq k \leq n$,

$$\psi_\star(n, k) = \prod_{j=k}^n (\text{Id} + \gamma_j \nabla h(\theta_\star)) . \quad (18)$$

By convention, $\psi_\star(n, n+1) = \text{Id}$. Under C1a-b, there exist a set of random $d \times d$ symmetric matrices $\{R_i^{(n)}, i \leq d\}$ such that the entry i of the column vector $\{h(\theta_n) - \nabla h(\theta_\star)(\theta_n - \theta_\star)\}$ is equal to $(\theta_n - \theta_\star)^T R_i^{(n)} (\theta_n - \theta_\star)$. More precisely,

$$R_i^{(n)}(k, l) = \int_0^1 \frac{1}{2} (1-t)^2 \frac{\partial^2 h_i}{\partial \theta_k \partial \theta_l} (\theta_n + t(\theta_n - \theta_\star)) dt . \quad (19)$$

Let $R_{\bullet}^{(n)}$ be the tensor such that

$$h(\theta_n) = \nabla h(\theta_{\star})(\theta_n - \theta_{\star}) + (\theta_n - \theta_{\star})^T R_{\bullet}^{(n)}(\theta_n - \theta_{\star}). \quad (20)$$

Finally, for $1 \leq k \leq n$, define the $d \times d$ matrices

$$\psi(n, k) = \prod_{j=k}^n (\text{Id} + \gamma_j \{ \nabla h(\theta_{\star}) + 2\mu_{j-1}^T R_{\bullet}^{(j-1)} + \rho_{j-1}^T R_{\bullet}^{(j-1)} \}), \quad (21)$$

with the convention that $\psi(n, n+1) = \text{Id}$.

5.2 Preliminary results on the sequence $\{\mu_n, n \geq 0\}$

By iterating (16), we have by definition of ψ_{\star} (see (18))

$$\mu_{n+1} = \sum_{k=1}^{n+1} \gamma_k \psi_{\star}(n+1, k+1) e_k. \quad (22)$$

Proposition 5.1. *Assume C1b-c, C2a-b and C4. Then*

- (i) $\mu_n 1_{\lim_q \theta_q = \theta_{\star}} \xrightarrow{\text{a.s.}} 0$ when $n \rightarrow \infty$.
- (ii) for any $m \geq 1$, $\gamma_k^{-1} |\mu_k|^2 1_{\lim_q \theta_q = \theta_{\star}} 1_{\mathcal{A}_m} = O_{L^1}(1) + o_{w.p.1}(1)$.

Proof. Let $m \geq 1$ be fixed. Set $\mu_{n+1} = \mu_{n+1}^{(1)} + \mu_{n+1}^{(2)}$, with

$$\mu_{n+1}^{(1)} = \sum_{k=1}^{n+1} \gamma_k \psi_{\star}(n+1, k+1) e_k 1_{\mathcal{A}_{m, k-1}}.$$

(i) Since

$$\mathbb{P} \left(\bigcup_m \mathcal{A}_m \mid \lim_q \theta_q = \theta_{\star} \right) \geq \lim_M \mathbb{P} \left(\mathcal{A}_M \mid \lim_q \theta_q = \theta_{\star} \right) = 1,$$

we only have to prove that for any $m \geq 1$, $\lim_n \mu_n 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_{\star}} \xrightarrow{\text{a.s.}} 0$. Let $m \geq 1$. Let us first consider $\mu_n^{(1)}$ and define $S_n \stackrel{\text{def}}{=} \sum_{k \geq n} \gamma_k e_k 1_{\mathcal{A}_{m, k-1}}$. By (22) and

the Abel transform, we have

$$\begin{aligned}
\mu_{n+1}^{(1)} &= \sum_{k=1}^{n+1} \psi_{\star}(n+1, k+1)(S_k - S_{k+1}) \\
&= (S_{n+1} - S_{n+2}) + \sum_{k=1}^n \psi_{\star}(n+1, k+1)S_k - \sum_{k=2}^{n+1} \psi_{\star}(n+1, k)S_k \\
&= -S_{n+2} + \psi_{\star}(n+1, 2)S_1 + \sum_{k=2}^n (\psi_{\star}(n+1, k+1) - \psi_{\star}(n+1, k))S_k \\
&= -S_{n+2} + \psi_{\star}(n+1, 2)S_1 - \sum_{k=2}^n \gamma_k \psi_{\star}(n+1, k+1) \nabla h(\theta_{\star}) S_k \quad (23)
\end{aligned}$$

where we used (18) in the last equality. Under C1b-c and C4, Lemmas 5.8 and 5.9 yield for any fixed $\ell \geq 1$

$$\limsup_n \sum_{k=2}^n \gamma_k |\psi_{\star}(n+1, k+1)| < \infty, \quad \lim_n |\psi_{\star}(n+1, \ell)| = 0. \quad (24)$$

Under C2a, for any $\ell \geq 1$, $\mathbb{E}[|S_{\ell}|^2] \leq \sum_{k \geq \ell} \gamma_k^2 \mathbb{E}[|e_k|^2 1_{\mathcal{A}_{m, k-1}}]$. By C4 and C2b, the rhs is finite for any $\ell \geq 1$, thus implying that (a) S_{ℓ} is finite w.p.1. and (b) $\lim_n S_n = 0$ w.p.1. (23), (24) and these properties of S_n imply that $\mu_n^{(1)} 1_{\mathcal{A}_m} \xrightarrow{\text{a.s.}} 0$ when $n \rightarrow \infty$.

Let us now consider $\mu_n^{(2)}$. By C2b, there exists a random index K such that for any $k \geq K$, $(1 - 1_{\mathcal{A}_{m, k}}) 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_{\star}} = 0$. Hence, for any $n \geq K$,

$$\mu_{n+1}^{(2)} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_{\star}} = \sum_{k=1}^K \gamma_k \psi_{\star}(n+1, k+1) e_k (1 - 1_{\mathcal{A}_{m, k-1}}) 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_{\star}}. \quad (25)$$

Then, by (24), $\mu_n^{(2)} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_{\star}} = o_{w.p.1.}(1)$. This concludes the proof of item (i).

(ii) Under C2a, (16) implies

$$\mathbb{E} \left[|\mu_{n+1}^{(1)}|^2 \right] \leq \sum_{k=1}^{n+1} \gamma_k^2 \mathbb{E} \left[|\psi_{\star}(n+1, k+1) e_k 1_{\mathcal{A}_{m, k-1}}|^2 \right].$$

By C1c, C2b, C4 and Lemma 5.8, there exist positive constants C, L' such that

$$\begin{aligned} \mathbb{E} \left[|\mu_{n+1}^{(1)}|^2 \right] &\leq \sum_{k=1}^{n+1} \gamma_k^2 |\psi_\star(n+1, k+1)|^2 \mathbb{E} [|e_k|^2 1_{\mathcal{A}_{m, k-1}}] \\ &\leq C \sup_k \mathbb{E} [|e_k|^2 1_{\mathcal{A}_{m, k-1}}] \sum_{k=1}^{n+1} \gamma_k^2 \exp(-2L' \sum_{j=k+1}^{n+1} \gamma_j). \end{aligned}$$

Therefore, by Lemma 5.9 and C4, $\limsup_k \gamma_k^{-1} \mathbb{E} [|\mu_k|^2] < +\infty$. Consider now $\mu_{n+1}^{(2)}$. By C4 and Lemma 5.8, $\lim_n \gamma_n^{-1} |\psi_\star(n, \ell)|^2 \rightarrow 0$ for any fixed ℓ . Therefore, by (25), $\gamma_n^{-1} |\mu_{n+1}^{(2)}|^2 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} = o_{w.p.1}(1)$. This concludes the proof of the second item. \square

5.3 Preliminary results on the sequence $\{\rho_n, n \geq 0\}$

By (17) and (20),

$$\begin{aligned} \rho_{n+1} &= (\text{Id} + \gamma_{n+1} \nabla h(\theta_\star)) \rho_n + \gamma_{n+1} r_{n+1} + \gamma_{n+1} (\theta_n - \theta_\star)^T R_\bullet^{(n)} (\theta_n - \theta_\star) \\ &= (\text{Id} + \gamma_{n+1} \nabla h(\theta_\star)) \rho_n + \gamma_{n+1} r_{n+1} + \gamma_{n+1} (\mu_n + \rho_n)^T R_\bullet^{(n)} (\mu_n + \rho_n) \\ &= \left(\text{Id} + \gamma_{n+1} \nabla h(\theta_\star) + 2\gamma_{n+1} \mu_n^T R_\bullet^{(n)} + \gamma_{n+1} \rho_n^T R_\bullet^{(n)} \right) \rho_n \\ &\quad + \gamma_{n+1} r_{n+1} + \gamma_{n+1} \mu_n^T R_\bullet^{(n)} \mu_n. \end{aligned}$$

By induction, this yields

$$\rho_n = \psi(n, 1) \rho_0 + \sum_{k=1}^n \gamma_k \psi(n, k+1) \left(r_k + \mu_{k-1}^T R_\bullet^{(k-1)} \mu_{k-1} \right), \quad (26)$$

where $\psi(n, k)$ is given by (21).

Proposition 5.2. *Assume C1, C2a-b and C4. Let $\theta_0 \in \Theta$. Then, for any $m \geq 1$,*

$$\left\{ \rho_n - \sum_{k=1}^n \gamma_k \psi(n, k+1) r_k \right\} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} = \gamma_n^{1 \wedge (1/2 + \kappa)} O_{w.p.1}(1) O_{L^1}(1),$$

with $\kappa = 1/2$ under C4a and $\kappa \in (0, L\gamma_\star - 1/2)$ under C4b.

Assume in addition C3. Then, for any $m \geq 1$,

$$\sum_{k=1}^n \gamma_k \psi(n, k+1) r_k 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} = \gamma_n^{1/2} O_{w.p.1}(1) O_{L^1}(1).$$

Proof. The proof is given under C4b. The case C4a - which is simpler - is on the same lines and is omitted. Let $m \geq 1$ be fixed.

(i) Let $\eta > 0$ and $\kappa \in (0, L\gamma_* - 1/2)$ such that

$$(L - \eta)\gamma_* > 1/2 + \kappa. \quad (27)$$

Note that such (η, κ) exist under C4b. This implies that

$$\limsup_n \gamma_n^{-(1/2+\kappa)} \exp(-(L - \eta) \sum_{j=1}^n \gamma_j) < +\infty. \quad (28)$$

We now prove by application of Lemma 5.8 that there exists an almost-surely finite positive r.v. U_η such that for any $1 \leq k \leq n$,

$$|\psi(n, k)| 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \leq U_\eta \exp(-(L - \eta) \sum_{j=k}^n \gamma_j). \quad (29)$$

To that goal, let us prove w.p.1. $\lim_n (\rho_n^T R_\bullet^{(n)} + 2\mu_n^T R_\bullet^{(n)}) 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} = 0$. By Proposition 5.1, $\lim_n \mu_n 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_*} = 0$ w.p.1. and this implies that w.p.1.,

$$\lim_n \rho_n 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} = \lim_n (\theta_n - \theta_* - \mu_n) 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} = 0.$$

In addition, under C1b, $R_\bullet^{(n)} 1_{\lim_q \theta_q = \theta_*} = O_{w.p.1.}(1)$. This concludes the proof of (29). Set $\kappa' \stackrel{\text{def}}{=} 1 \wedge (1/2 + \kappa)$. By (26), we have

$$\rho_n - \sum_{k=1}^n \gamma_k \psi(n, k+1) r_k = \psi(n, 1) \rho_0 + \sum_{k=1}^n \gamma_k \psi(n, k+1) \left(\mu_{k-1}^T R_\bullet^{(k-1)} \mu_{k-1} \right). \quad (30)$$

Consider the first term. By (29),

$$\gamma_n^{-(1/2+\kappa)} |\psi(n, 1)| |\rho_0| 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \leq \gamma_n^{-(1/2+\kappa)} U_\eta \exp(-(L - \eta) \sum_{j=1}^n \gamma_j) |\rho_0|,$$

and by (28), this term is $O_{w.p.1.}(1)$. For the second term, it holds by (29)

$$\begin{aligned} & \gamma_n^{-\kappa'} \left| \sum_{k=1}^n \gamma_k \psi(n, k+1) \mu_{k-1}^T R_\bullet^{(k-1)} \mu_{k-1} \right| 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \\ & \leq \gamma_n^{-\kappa'} \sum_{k=1}^n \gamma_k^2 |\psi(n, k+1)| (\gamma_k^{-1/2} |\mu_{k-1}|)^2 |R_\bullet^{(k-1)}| 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \\ & \leq O_{w.p.1.}(1) \gamma_n^{-\kappa'} \sum_{k=1}^n \gamma_k^{1+\kappa'} \exp(-(L - \eta) \sum_{j=k+1}^n \gamma_j) (\gamma_k^{-1/2} |\mu_{k-1}|)^2 |R_\bullet^{(k-1)}| 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m}. \end{aligned}$$

By Proposition 5.1, $\gamma_n^{-1}|\mu_n|^2 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} = O_{L^1}(1) + o_{w.p.1}(1)$ and under C1b, $|R_\bullet^{(k)}| 1_{\lim_q \theta_q = \theta_\star} = O_{w.p.1}(1)$. By Lemma 5.9 and (27), this term is $O_{w.p.1}(1)O_{L^1}(1)$.

(ii) Set $r_k = r_k^{(1)} + r_k^{(2)}$ as in C3. It holds,

$$\begin{aligned} \gamma_n^{-1/2} \left| \sum_{k=1}^n \gamma_k \psi(n, k+1) r_k^{(1)} \right| 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \\ \leq \gamma_n^{-1/2} \sum_{k=1}^n \gamma_k^{3/2} |\psi(n, k+1)| |\gamma_k^{-1/2} r_k^{(1)}| 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \end{aligned}$$

and by (27), (29), C3, C4 and Lemma 5.9, this term is $O_{w.p.1}(1)o_{L^1}(1)$. For the second term, we use the Abel lemma: set $\Xi_n \stackrel{\text{def}}{=} \sum_{k=1}^n r_k^{(2)} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star}$. Then

$$\begin{aligned} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \gamma_n^{-1/2} \sum_{k=1}^n \gamma_k \psi(n, k+1) r_k^{(2)} \\ = \sqrt{\gamma_n} \Xi_n + \gamma_n^{-1/2} \sum_{k=1}^{n-1} \gamma_k \gamma_{k+1} \psi(n, k+2) \left\{ \left(\frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} \right) I + H_k \right\} \Xi_k \end{aligned}$$

where $H_k = \nabla h(\theta_\star) + 2\mu_k^T R_\bullet^{(k)} + \rho_k^T R_\bullet^{(k)}$. Following the same lines as above, along the event $\{\lim_q \theta_q = \theta_\star\} \cap \mathcal{A}_m$, $\sup_k |H_k|$ is finite w.p.1. Hence, o

$$\begin{aligned} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \gamma_n^{-1/2} \left| \sum_{k=1}^n \gamma_k \psi(n, k+1) r_k^{(2)} \right| \leq O_{w.p.1}(1) \left\{ \sqrt{\gamma_n} |\Xi_n| \right. \\ \left. + \gamma_n^{-1/2} \sum_{k=1}^{n-1} \sqrt{\gamma_k} \gamma_{k+1} |\psi(n, k+2)| \sqrt{\gamma_k} |\Xi_k| \right\} \end{aligned}$$

and the rhs is $O_{w.p.1}(1)o_{L^1}(1)$ by Lemma 5.9 and C3. \square

5.4 Proof of Theorem 2.1

By (17), $\gamma_n^{-1/2}(\theta_n - \theta_\star) = \gamma_n^{-1/2}\mu_n + \gamma_n^{-1/2}\rho_n$. We first prove that on $\{\lim_q \theta_q = \theta_\star\}$, the second term tends to zero in probability. By C2b, for any $\epsilon > 0$ there exists $m \geq 1$ such that $\mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_\star) \geq 1 - \epsilon$. Therefore, it is sufficient to prove that for any $m \geq 1$, $\gamma_n^{-1/2}\rho_n 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \xrightarrow{\mathbb{P}} 0$ when $n \rightarrow \infty$. This property holds by Proposition 5.2.

We now prove a CLT for the sequence $\{\gamma_n^{-1/2}\mu_n, n \geq 0\}$. It is readily seen that

$$\lim_n \mathbb{E} \left[\exp(i\gamma_n^{-1/2} t^T \mu_n) 1_{\lim_q \theta_q = \theta_\star} \right] = \mathbb{E} \left[\exp(-0.5t^T V t) 1_{\lim_q \theta_q = \theta_\star} \right]$$

if and only if

$$\lim_n \mathbb{E} \left[\exp(i\gamma_n^{-1/2} t^T \mu_n 1_{\lim_q \theta_q = \theta_*}) \right] = \mathbb{E} \left[\exp(-0.5 t^T V t 1_{\lim_q \theta_q = \theta_*}) \right]$$

Furthermore, by C4 and Lemma 5.8, for any fixed $\ell \geq 1$, $\lim_n \gamma_n^{-1/2} |\psi_*(n, \ell)| = 0$ (where ψ_* is given by (18)); this property, together with (22) and (15) imply that

$$\lim_n \mathbb{E} \left[\exp(i\gamma_n^{-1/2} t^T \mu_n 1_{\lim_q \theta_q = \theta_*}) \right] = \lim_n \mathbb{E} \left[\exp \left(i t^T \sum_{k=1}^n X_{n+1,k} 1_{A_{k-1}} \right) \right]$$

where $X_{n+1,k} = \gamma_{n+1}^{-1/2} \gamma_k \psi_*(n+1, k+1) e_k$. By C2a and (15), $\mathbb{E} [X_{n+1,k} 1_{A_{k-1}} | \mathcal{F}_{k-1}] = 0$ and the limit in distribution is obtained by standard results on CLT for martingale-arrays (see e.g. Hall and Heyde [15, Corollary 3.1.]).

Lindeberg condition we have to prove that for any $\epsilon > 0$,

$$\sum_{k=1}^n \mathbb{E} \left[|X_{n+1,k}|^2 1_{|X_{n+1,k}| \geq \epsilon} | \mathcal{F}_{k-1} \right] 1_{A_{k-1}} \xrightarrow{\mathbb{P}} 0 .$$

Following the same lines as above, it can be proved that equivalently, we have to prove for any $m \geq 1$,

$$1_{A_m} 1_{\lim_q \theta_q = \theta_*} \sum_{k=1}^n \mathbb{E} \left[|X_{n+1,k}|^2 1_{|X_{n+1,k}| \geq \epsilon} | \mathcal{F}_{k-1} \right] \xrightarrow{\mathbb{P}} 0 .$$

Let $m \geq 1$ be fixed and set $X_{n+1,k} = X_{n+1,k}^{(1)} + X_{n+1,k}^{(2)}$ with

$$X_{n+1,k}^{(1)} = X_{n+1,k} 1_{A_{m,k-1}} , \quad X_{n+1,k}^{(2)} = X_{n+1,k} (1 - 1_{A_{m,k-1}}) .$$

We can assume without loss of generality that τ given by C2b is small enough so that $(2 + \tau)L\gamma_* > 1 + \tau$. Then,

$$\begin{aligned} \sum_{k=1}^{n+1} \mathbb{E} \left[|X_{n+1,k}^{(1)}|^{2+\tau} \right] &= \sum_{k=1}^{n+1} \mathbb{E} \left[|\gamma_{n+1}^{-1/2} \gamma_k \psi_*(n+1, k+1) e_k 1_{A_{m,k-1}}|^{2+\tau} \right] \\ &\leq \sup_k \mathbb{E} \left[|e_k 1_{A_{m,k-1}}|^{2+\tau} \right] \gamma_{n+1}^{-1-\tau/2} \sum_{k=1}^{n+1} \gamma_k^{2+\tau} |\psi_*(n+1, k+1)|^{2+\tau} . \end{aligned}$$

Under C1b-c, C2b and C4, Lemmas 5.8 and 5.9 imply

$$\limsup_n \gamma_{n+1}^{-(1+\tau)} \sum_{k=1}^{n+1} \gamma_k^{2+\tau} |\psi_*(n+1, k+1)|^{2+\tau} < +\infty$$

since $(2 + \tau)L\gamma_\star > 1 + \tau$, Lemma 5.9 applies even in the case C4b). Hence,

$$\sum_{k=1}^{n+1} \mathbb{E} \left[|X_{n+1,k}^{(1)}|^{2+\tau} \right] = o(\gamma_n^{\tau/2}).$$

Consider now $X_{n+1,k}^{(2)}$. Since there exists a random variable K such that $1_{\mathcal{A}_m} (1 - 1_{\mathcal{A}_{m,k-1}}) 1_{\lim_q \theta_q = \theta_\star} = 0$ for any $k \geq K$, it holds for any $n \geq K$,

$$\begin{aligned} & 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \sum_{k=1}^n \mathbb{E} \left[|X_{n+1,k}^{(2)}|^2 1_{|X_{n+1,k}| \geq \epsilon} | \mathcal{F}_{k-1} \right] \\ &= 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \sum_{k=1}^K \mathbb{E} \left[|X_{n+1,k}|^2 1_{|X_{n+1,k}| \geq \epsilon} | \mathcal{F}_{k-1} \right] (1 - 1_{\mathcal{A}_{m,k-1}}) \\ &\leq 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \gamma_n^{-1} \sum_{k=1}^K \gamma_k^2 |\psi_\star(n+1, k+1)|^2 \mathbb{E} [|e_k|^2 | \mathcal{F}_{k-1}] (1 - 1_{\mathcal{A}_{m,k-1}}). \end{aligned}$$

Under C4, this term is $o_{w.p.1}(1)$. Therefore, the first condition of [15, Corollary 3.1.] is satisfied.

Limiting variance We prove the second condition of [15, Corollary 3.1.]. Set

$$\begin{aligned} V_n^{(1)} &\stackrel{\text{def}}{=} \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 \psi_\star(n, k+1) U_\star \psi_\star(n, k+1)^T 1_{\lim_q \theta_q = \theta_\star}, \\ \bar{V}_n^{(2)} &\stackrel{\text{def}}{=} \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 \psi_\star(n, k+1) \cdots \\ &\quad \times (\mathbb{E}[e_k e_k^T | \mathcal{F}_{k-1}] 1_{\mathcal{A}_{k-1}} - U_\star 1_{\lim_q \theta_q = \theta_\star}) \psi_\star(n, k+1)^T; \end{aligned}$$

We prove that $V_n^{(1)} \xrightarrow{\mathbb{P}} V 1_{\lim_q \theta_q = \theta_\star}$ and $\bar{V}_n^{(2)} \xrightarrow{\mathbb{P}} 0$. It holds on $\{\lim_q \theta_q = \theta_\star\}$,

$$\begin{aligned} V_{n+1}^{(1)} &= \gamma_{n+1} U_\star + \frac{\gamma_n}{\gamma_{n+1}} (\text{Id} + \gamma_{n+1} \nabla h(\theta_\star)) V_n^{(1)} (\text{Id} + \gamma_{n+1} \nabla h(\theta_\star))^T \\ &= V_n^{(1)} + \gamma_n (U_\star + \nabla h(\theta_\star) V_n^{(1)} + V_n^{(1)} \nabla h(\theta_\star)^T) + \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}} V_n^{(1)} \\ &\quad + (\gamma_{n+1} - \gamma_n) U_\star + \gamma_n \gamma_{n+1} \nabla h(\theta_\star) V_n^{(1)} \nabla h(\theta_\star)^T \end{aligned}$$

and by Lemma 5.11, $\lim_n V_n^{(1)} = V 1_{\lim_q \theta_q = \theta_\star}$ almost-surely. Following the same lines as above, it can be proved that $\bar{V}_n^{(2)}$ and $V_n^{(2)}$ given by

$$V_n^{(2)} = 1_{\lim_q \theta_q = \theta_\star} \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 \psi_\star(n, k+1) (\mathbb{E}[e_k e_k^T | \mathcal{F}_{k-1}] - U_\star) \psi_\star(n, k+1)^T$$

have the same limit in probability. By C2c, we write $V_n^{(2)} = \left(V_n^{(2,a)} + V_n^{(2,b)} \right) 1_{\lim_q \theta_q = \theta_\star}$ with

$$\begin{aligned} V_n^{(2,a)} &= \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 \psi_\star(n, k+1) D_{k-1}^{(1)} \psi_\star(n, k+1)^T \\ V_n^{(2,b)} &= \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 \psi_\star(n, k+1) D_{k-1}^{(2)} \psi_\star(n, k+1)^T . \end{aligned}$$

We have $\left| V_n^{(2,a)} \right| \leq \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 |\psi_\star(n, k+1)|^2 |D_{k-1}^{(1)}|$. By Lemma 5.9, there exists a constant C such that on $\{\lim_q \theta_q = \theta_\star\}$

$$\limsup_n \left| V_n^{(2,a)} \right| \leq C \limsup_k \left| D_k^{(1)} \right| ,$$

where we used (15). The rhs tends to zero w.p.1. by C2c. We now consider $V_n^{(2,b)}$. Since $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_\star) = 1$, it is sufficient to prove that for any $m \geq 1$, $V_n^{(2,b)} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} \xrightarrow{\mathbb{P}} 0$ when $n \rightarrow \infty$. Let $m \geq 1$. Set

$$\Xi_n \stackrel{\text{def}}{=} \sum_{j=0}^n D_j^{(2)} 1_{\lim_q \theta_q = \theta_\star} 1_{\mathcal{A}_m} .$$

By the Abel transform, we have

$$\begin{aligned} V_{n+1}^{(2,b)} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} &= \gamma_{n+1} \Xi_n + \gamma_{n+1}^{-1} \sum_{k=0}^{n-1} \{ \gamma_{k+1}^2 \psi_\star(n+1, k+2) \Xi_k \psi_\star(n+1, k+2)^T \\ &\quad - \gamma_{k+2}^2 \psi_\star(n+1, k+3) \Xi_k \psi_\star(n+1, k+3)^T \} \end{aligned}$$

Under C2c, $\gamma_n \Xi_n \xrightarrow{\mathbb{P}} 0$. For the second term, following the same lines as in Delyon [10, Proof of Theorem 24, Chapter 4], it can be proved that the expectation of the second term is upper bounded by

$$C \gamma_{n+1}^{-1} \sum_{k=0}^{n-1} \gamma_{k+1}^2 |\psi_\star(n+1, k+2)|^2 (\gamma_k \mathbb{E} [|\Xi_k|]) .$$

Since $\lim_k \gamma_k \mathbb{E} [|\Xi_k|] = 0$, Lemma 5.9 implies that $V_n^{(2,b)} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \xrightarrow{\mathbb{P}} 0$. This concludes the proof.

Remark 5.3. From the proof above, it can be seen that the assumption on the r.v. $D_n^{(2)}$ can be relaxed in

$$\lim_n \gamma_n \mathbb{E} \left[\left| \sum_{k=1}^n D_k^{(2)} 1_{A_k} 1_{\mathcal{A}_{m,k}} \right| \right] = 0 .$$

Observe indeed that in probability,

$$\lim_n V_n^{(2,b)} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_*} = \lim_n \gamma_n^{-1} \sum_{k=1}^n \gamma_k^2 \psi_*(n, k+1) D_{k-1}^{(2)} \psi_*(n, k+1)^T 1_{\mathcal{A}_{m,k-1}} 1_{A_{k-1}} .$$

5.5 Proof of Proposition 3.1

The proof is prefaced with a preliminary lemma.

Lemma 5.4. Let $\{\gamma_n, n \geq 1\}$ is a (deterministic) positive sequence satisfying $C4a$ and A be a (deterministic) $d \times d$ Hurwitz matrix. Let $\{x_n, n \geq 0\}$ be a sequence of \mathbb{R}^d -valued r.v. satisfying

$$x_{n+1} = x_n + \gamma_{n+1} A x_n + \gamma_{n+1} \zeta_{n+1}^{(1)} + \gamma_{n+1} \zeta_{n+1}^{(2)}, \quad n \geq 0 ,$$

where

$$\sum_{k=1}^n \gamma_k \left(\prod_{j=k+1}^{n+1} (\text{Id} + \gamma_j A) \right) \zeta_k^{(1)} 1_{\lim_q x_q = 0} = \sqrt{\gamma_n} O_{w.p.1}(1) O_{L^2}(1) ,$$

$$|\zeta_n^{(2)}| 1_{\lim_q x_q = 0} = |x_n|^2 O_{w.p.1}(1) .$$

Then

$$\gamma_n^{-1} |x_n|^2 1_{\lim_q x_q = 0} = O_{w.p.1}(1) O_{L^1}(1) .$$

Proof. The proof is adapted from Delyon [10, Theorems 20 and 23]. For $n \geq 0$, set $x_n 1_{\lim_q x_q = 0} = y_n + z_n$ where

$$y_{n+1} = (\text{Id} + \gamma_{n+1} A) y_n + \gamma_{n+1} \zeta_{n+1}^{(1)} 1_{\lim_q x_q = 0}, \quad n \geq 0, \quad (31)$$

and $y_0 = 0$. The first step of the proof is to show

$$y_n = \sqrt{\gamma_n} O_{w.p.1}(1) O_{L^2}(1), \quad z_n = \gamma_n O_{w.p.1}(1) O_{L^1}(1). \quad (32)$$

Then, upon noting that $(y+z)^2 \leq y^2 + 2(y+z)z$, we write

$$|x_n|^2 1_{\lim_q x_q = 0} \leq |y_n|^2 + 2|x_n||z_n| 1_{\lim_q x_q = 0} \leq \gamma_n O_{L^1}(1) + 2\gamma_n O_{w.p.1}(1) O_{L^1}(1)$$

since $|x_n|_{1_{\lim_q x_q=0}} = O_{w.p.1}(1)$. This concludes the proof of the Lemma. We turn to the proof of (32). By iterating (31), we have

$$y_{n+1} = \sum_{k=1}^{n+1} \gamma_k \left\{ \prod_{j=k+1}^{n+1} (\text{Id} + \gamma_j A) \right\} \zeta_k^{(1)}_{1_{\lim_q x_q=0}}.$$

Lemmas 5.8 and 5.9 imply that $y_n = \sqrt{\gamma_n} O_{w.p.1}(1) O_{L^2}(1)$. It holds

$$\begin{aligned} z_{n+1} &= x_{n+1} 1_{\lim_q x_q=0} - y_{n+1} \\ &= (\text{Id} + \gamma_{n+1} A) (x_n 1_{\lim_q x_q=0} - y_n) + \gamma_{n+1} \zeta_{n+1}^{(2)} 1_{\lim_q x_q=0} \\ &= (\text{Id} + \gamma_{n+1} A) z_n + \gamma_{n+1} \zeta_{n+1}^{(2)} 1_{\lim_q x_q=0}. \end{aligned}$$

Under the stated assumptions, Lemmas 5.8 and 5.9 imply that $z_n = o_{w.p.1}(1)$. We thus also have $y_n = x_n 1_{\lim_q x_q=0} - z_n = o_{w.p.1}(1)$. In addition,

$$|z_{n+1}| \leq |\text{Id} + \gamma_{n+1} A| |z_n| + \gamma_{n+1} |\zeta_{n+1}^{(2)}|_{1_{\lim_q x_q=0}},$$

and since A is a Hurwitz matrix, there exists a constant $L' > 0$ such that $|\text{Id} + \gamma_{n+1} A| \leq \exp(-L' \gamma_{n+1})$ (see Lemma 5.8). Hence,

$$\begin{aligned} |z_{n+1}| &\leq \exp(-L' \gamma_{n+1}) |z_n| + O_{w.p.1}(1) \gamma_{n+1} (|y_n|^2 + |z_n|^2) \\ &\leq \exp(-L' \gamma_{n+1}) \{1 + O_{w.p.1}(1) \exp(L' \gamma_{n+1}) \gamma_{n+1} |z_n|\} |z_n| + O_{w.p.1}(1) \gamma_{n+1} |y_n|^2. \end{aligned}$$

Let $\delta \in (0, L')$. Since $z_n = o_{w.p.1}(1)$, there exists a r.v. K which is finite w.p.1. such that for any $k \geq K$, $|O_{w.p.1}(1) \exp(L' \gamma_{k+1}) z_k| \leq \delta$. Therefore, upon noting that for any $x \geq 0$, $1 + x \leq \exp(x)$, for any $n \geq K$,

$$\begin{aligned} |z_{n+1}| &\leq \exp(-(L' - \delta) \gamma_{n+1}) |z_n| + O_{w.p.1}(1) \gamma_{n+1} |y_n|^2 \\ &\leq \exp\left(- (L' - \delta) \sum_{k=K+1}^{n+1} \gamma_k\right) |z_K| \\ &\quad + O_{w.p.1}(1) \sum_{k=K+1}^{n+1} \gamma_k \exp\left(- (L' - \delta) \sum_{j=k+1}^{n+1} \gamma_j\right) |y_{k-1}|^2 \\ &\leq O_{w.p.1}(1) \exp\left(- (L' - \delta) \sum_{k=1}^{n+1} \gamma_k\right) \\ &\quad + O_{w.p.1}(1) \sum_{k=1}^{n+1} \gamma_k \exp\left(- (L' - \delta) \sum_{j=k+1}^{n+1} \gamma_j\right) |y_{k-1}|^2 \\ &\quad + O_{w.p.1}(1) \exp\left(- (L' - \delta) \sum_{j=K+1}^{n+1} \gamma_j\right). \end{aligned}$$

Since $y_n = \sqrt{\gamma_n} O_{L^2}(1)$, C4a and Lemma 5.9 imply that $z_n = \gamma_n O_{w.p.1}(1) O_{L^1}(1)$. \square

Proof of Proposition 3.1 By (20)

$$\begin{aligned} \theta_{n+1} - \theta_\star &= \theta_n - \theta_\star + \gamma_{n+1} \nabla h(\theta_\star) (\theta_n - \theta_\star) \\ &\quad + \gamma_{n+1} (e_{n+1} + r_{n+1}) + \gamma_{n+1} (\theta_n - \theta_\star)^T R_\bullet^{(n)} (\theta_n - \theta_\star) \end{aligned}$$

Let $m \geq 1$. We apply Lemma 5.4 with $x_n \leftarrow (\theta_n - \theta_\star) 1_{\mathcal{A}_m}$, $A \leftarrow \nabla h(\theta_\star)$, $\zeta_{n+1}^{(1)} = (e_{n+1} + r_{n+1}) 1_{\mathcal{A}_m}$ and $\zeta_{n+1}^{(2)} = (\theta_n - \theta_\star)^T R_\bullet^{(n)} (\theta_n - \theta_\star) 1_{\mathcal{A}_m}$. Under C1c, A is a Hurwitz matrix and $|\zeta_{n+1}^{(2)}| 1_{\text{lim}_q \theta_q = \theta_\star} = O_{w.p.1}(1) |x_n|^2$.

We write $\zeta_{n+1}^{(1)} = (e_{n+1} 1_{\mathcal{A}_{m,n}} + e_{n+1} (1 - 1_{\mathcal{A}_{m,n}}) + r_{n+1}) 1_{\mathcal{A}_m}$. Under C4, AVER1a-b, Lemmas 5.8 and 5.9 imply

$$\sum_{k=1}^n \gamma_k \psi_\star(n+1, k+1) e_k 1_{\mathcal{A}_{m,k-1}} = \sqrt{\gamma_n} O_{L^2}(1).$$

Upon noting that $1_{\mathcal{A}_m} (1 - 1_{\mathcal{A}_{m,k}}) = 0$ for all $k \geq K$ where K is a r.v. finite w.p.1.

$$\begin{aligned} &\left(\sum_{k=1}^n \gamma_k \psi_\star(n+1, k+1) e_k (1 - 1_{\mathcal{A}_{m,k-1}}) \right) 1_{\mathcal{A}_m} \\ &= \left(\sum_{k=1}^K \gamma_k \psi_\star(n+1, k+1) e_k (1 - 1_{\mathcal{A}_{m,k-1}}) \right) 1_{\mathcal{A}_m}. \end{aligned}$$

Therefore, by Lemma 5.9, this second term is $\sqrt{\gamma_n} O_{w.p.1}(1)$. Finally, Lemma 5.9 and AVER2a-b imply that the last term is $\sqrt{\gamma_n} O_{w.p.1}(1) O_{L^2}(1)$ (the proof is on the same lines as the proof of Proposition 5.2 and details are omitted).

5.6 Proof of Theorem 3.2

The proof is adapted from the proof of Delyon [10, Theorem 26]. Under C1c, $\nabla h(\theta_\star)$ is invertible. By (4) and Lemma 5.12 applied with $x_k \leftarrow \theta_k - \theta_\star$ and $A \leftarrow \nabla h(\theta_\star)$, we have

$$\sqrt{n} (\bar{\theta}_n - \theta_\star) = -\nabla h(\theta_\star)^{-1} \frac{\sqrt{n}}{n+1} \sum_{k=0}^n e_{k+1} + \sqrt{n} Z_n$$

where

$$\begin{aligned} \nabla h(\theta_\star) Z_n &\stackrel{\text{def}}{=} -\frac{1}{n+1} \sum_{k=0}^n r_{k+1} - \frac{1}{n+1} \sum_{k=0}^n (h(\theta_k) - \nabla h(\theta_\star)(\theta_k - \theta_\star)) \\ &\quad + \frac{1}{n+1} \left(\frac{\theta_{n+1} - \theta_\star}{\gamma_{n+1}} - \frac{\theta_0 - \theta_\star}{\gamma_1} \right) + \frac{1}{n+1} \sum_{k=1}^n \left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k+1}} \right) (\theta_k - \theta_\star). \end{aligned}$$

We prove that $\sqrt{n} Z_n 1_{\lim_q \theta_q = \theta_\star} \xrightarrow{\mathbb{P}} 0$; combined with AVER1c, this will conclude the proof. Since $\lim_m \mathbb{P}(\mathcal{A}_m | \lim_q \theta_q = \theta_\star) = 1$, it is sufficient to prove that for any $m \geq 1$, $\sqrt{n} Z_n 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \xrightarrow{\mathbb{P}} 0$. Let $m \geq 1$. By AVER2c, it holds $n^{-1/2} \sum_{k=0}^n r_{k+1} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \xrightarrow{\mathbb{P}} 0$. By (20),

$$\frac{1}{n+1} \sum_{k=0}^n (h(\theta_k) - \nabla h(\theta_\star)(\theta_k - \theta_\star)) = \frac{1}{n+1} \sum_{k=0}^n (\theta_k - \theta_\star)^T R_\bullet^{(k)} (\theta_k - \theta_\star),$$

and by C1b, $R_\bullet^{(k)} 1_{\lim_q \theta_q = \theta_\star} = O_{w.p.1}(1)$. Therefore, by Proposition 3.1,

$$\frac{\sqrt{n}}{n+1} \sum_{k=0}^n (h(\theta_k) - \nabla h(\theta_\star)(\theta_k - \theta_\star)) 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} = \left(\frac{\sqrt{n}}{n+1} \sum_{k=0}^n \gamma_k W_k \overline{W}_k \right),$$

where $W_k = O_{w.p.1}(1)$ and $\overline{W}_k = O_{L^1}(1)$. AVER3 implies that this term tends to zero in probability. Proposition 3.1 and AVER3 imply that

$$1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \frac{\sqrt{n}}{n+1} \left(\frac{\theta_{n+1} - \theta_\star}{\gamma_{n+1}} - \frac{\theta_0 - \theta_\star}{\gamma_1} \right) = \frac{O_{L^1}(1) O_{w.p.1}(1)}{\sqrt{(n+1)\gamma_{n+1}}} + O_{w.p.1}(1) \xrightarrow{\mathbb{P}} 0.$$

Finally, Proposition 3.1 and AVER3 also imply that

$$\begin{aligned} 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_\star} \frac{\sqrt{n}}{n+1} \sum_{k=1}^n \left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k+1}} \right) (\theta_k - \theta_\star) \\ = \left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \left| \frac{1}{\gamma_k} - \frac{1}{\gamma_{k+1}} \right| \gamma_k^{1/2} W_k \sqrt{\overline{W}_k} \right) \end{aligned}$$

where $W_k = O_{w.p.1}(1)$ and $\overline{W}_k = O_{L^1}(1)$. This term tends to zero in probability.

Lemma 5.5. *C2 and $\lim_n n\gamma_n > 0$ imply AVER1.*

Proof. C2 implies trivially AVER1a-b. We only have to check AVER1c, or equivalently, prove that for any $m \geq 1$,

$$\lim_n \mathbb{E} \left[\exp \left(it^T \mathcal{E}_{n+1} 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \right) \right] = \mathbb{E} \left[\exp \left(it^T U_* t 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \right) \right].$$

Write $\mathcal{E}_{n+1} 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} = T_{1,n} + T_{2,n}$ with $T_{1,n} = (n+1)^{-1/2} \sum_{k=0}^n e_{k+1} 1_{\mathcal{A}_{m,k}} 1_{A_k}$. By (15) and C2b, $T_{2,n} = o_{w.p.1.}(1)$. Observe that $\mathbb{E} [e_{k+1} 1_{\mathcal{A}_{m,k}} 1_{A_k} | \mathcal{F}_k] = 0$ so that the convergence in distribution of $T_{1,n}$ will be established by applying results on martingale-arrays: we check the assumptions of Hall and Heyde [15, Corollary 3.1.]. By C2b, it is easily checked that for any $\epsilon > 0$, there exists a constant C such that for any n ,

$$\mathbb{E} \left[\frac{1}{n} \sum_{k=0}^n \mathbb{E} \left[|e_{k+1}|^2 1_{|e_{k+1}| \geq \epsilon \sqrt{n}} | \mathcal{F}_k \right] 1_{\mathcal{A}_{m,k}} \right] \leq \frac{C}{n^{\tau/2}}.$$

Hence, $n^{-1} \sum_{k=0}^n \mathbb{E} \left[|e_{k+1}|^2 1_{|e_{k+1}| \geq \epsilon \sqrt{n}} | \mathcal{F}_k \right] 1_{\mathcal{A}_{m,k}} 1_{A_k} \xrightarrow{\mathbb{P}} 0$. We now prove that

$$\frac{1}{n+1} \sum_{k=0}^n \mathbb{E} [e_{k+1} e_{k+1}^T | \mathcal{F}_k] 1_{\mathcal{A}_{m,k}} 1_{A_k} \xrightarrow{\mathbb{P}} U_* 1_{\mathcal{A}_m} 1_{\lim_q \theta_q = \theta_*}. \quad (33)$$

As above, we claim that this is equivalent to the proof that for any $m \geq 1$,

$$1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \frac{1}{n+1} \sum_{k=0}^n \left(\mathbb{E} [e_{k+1} e_{k+1}^T | \mathcal{F}_k] - U_* \right) \xrightarrow{\mathbb{P}} 0.$$

C2c and the Cesaro lemma imply that w.p.1, on the set $\mathcal{A}_m \cap \{\lim_q \theta_q = \theta_*\}$, $(n+1)^{-1} \sum_{k=0}^n D_k^{(1)} \xrightarrow{\text{a.s.}} 0$. Finally, under C2c,

$$\frac{1}{n+1} \mathbb{E} \left[\left| \sum_{k=0}^n D_k^{(2)} 1_{\lim_q \theta_q = \theta_*} 1_{\mathcal{A}_m} \right| \right] = \frac{o(1)}{n\gamma_n}$$

and the rhs tends to zero since $\lim_n n\gamma_n > 0$. This concludes the proof of (33) and the proof of the Lemma. \square

5.7 Technical lemmas

Lemma 5.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a measured space, where μ is a bounded positive measure. Let \mathcal{G} be an algebra generating \mathcal{A} . Then for all $B \in \mathcal{A}$ and $\epsilon > 0$, we can find $A \in \mathcal{G}$ such that $\mu(A \Delta B) < \epsilon$.*

Proof. Let $\mathcal{S} \stackrel{\text{def}}{=} \{A \subset \Omega, \forall \epsilon > 0, \exists A' \in \mathcal{G}, \mu(A \Delta A') \leq \epsilon\}$. We prove that \mathcal{S} is a σ -algebra; since it contains \mathcal{G} by definition, this yields the result.

$\Omega \in \mathcal{S}$ since $\Omega \in \mathcal{G}$. Let $A \in \mathcal{S}$: we prove that $A^c \stackrel{\text{def}}{=} \Omega \setminus A \in \mathcal{S}$. Let $\epsilon > 0$; there exists $A' \in \mathcal{G}$ such that $\mu(A \Delta A') \leq \epsilon$. Since $A \Delta B = A^c \Delta B^c$, it holds

$$\mu(A^c \Delta (A')^c) = \mu(A \Delta A') \leq \epsilon;$$

$(A')^c \in \mathcal{G}$ since \mathcal{G} is an algebra, thus showing that $A^c \in \mathcal{S}$.

Finally, we prove that \mathcal{S} is stable by countable union. We first prove it is stable by finite union, or equivalently by union of two elements. Let A_1, A_2 be elements of \mathcal{S} and fix $\epsilon > 0$. There exists $A'_k \in \mathcal{G}$ such that $\mu(A_k \Delta A'_k) \leq \epsilon/2$. Upon noting that

$$(A_1 \cup A_2) \Delta (A'_1 \cup A'_2) \subset (A_1 \Delta A'_1) \cup (A_2 \Delta A'_2) \quad (34)$$

it holds

$$\mu((A_1 \cup A_2) \Delta (A'_1 \cup A'_2)) \leq \epsilon.$$

This concludes the proof since $A'_1 \cup A'_2 \in \mathcal{G}$. Let us consider the countable case. Let $(A_k, k \geq 1)$ be a sequence of \mathcal{S} and fix $\epsilon > 0$; since \mathcal{S} is stable under complement and finite union, we can assume without loss of generality that the sets A_k are pairwise disjoint. For any k , there exists $A'_k \in \mathcal{G}$ such that $\mu(A_k \Delta A'_k) \leq \epsilon 2^{-k}$. Since $(A_k, k \geq 0)$ are pairwise disjoint

$$\mu\left(\bigcup_k A_k\right) = \sum_{k \geq 1} \mu(A_k);$$

since μ is finite, there exists K_ϵ such that $\mu(\sum_{k > K_\epsilon} A_k) \leq \epsilon/2$. Using again (34) it holds

$$\begin{aligned} \mu\left(\left(\bigcup_k A_k\right) \Delta \left(\bigcup_{k \leq K_\epsilon} A'_k\right)\right) &\leq \mu\left(\left(\bigcup_{k \leq K_\epsilon} A_k\right) \Delta \left(\bigcup_{k \leq K_\epsilon} A'_k\right)\right) + \mu\left(\bigcup_{k > K_\epsilon} A_k\right) \\ &\leq \sum_{k=1}^{K_\epsilon} \mu(A_k \Delta A'_k) + \epsilon/2 \\ &\leq \epsilon. \end{aligned}$$

Since $\bigcup_{k \leq K_\epsilon} A'_k \in \mathcal{G}$, this concludes the proof of the sub-additivity. □

Lemma 5.7. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \{\mathcal{F}_n, n \geq 0\})$ be a filtered probability space and set $\mathcal{F}_\infty = \sigma(\mathcal{F}_n, n \geq 1)$. Let $B \in \mathcal{F}_\infty$. There exists a \mathcal{F}_n -adapted sequence $\{A_n, n \geq 0\}$ such that $\lim_n 1_{A_n} = 1_B$ \mathbb{P} -a.s.*

Proof. For any $\epsilon > 0$, there exist $m \geq 1$ and $\tilde{A} \in \mathcal{F}_m$ such that $\mathbb{E} [|1_{\tilde{A}} - 1_B|] \leq \epsilon$ (see Lemma 5.6). Therefore, for any $n \geq 1$, there exist sets $\tilde{A}_n \in \mathcal{F}_{m_n}$ such that $\mathbb{E} [|1_{\tilde{A}_n} - 1_B|] \leq 1/n$. This implies almost-sure convergence of a subsequence $\{\tilde{A}_{\phi_n}, n \geq 0\}$ to 1_B , with $\tilde{A}_{\phi_n} \in \mathcal{F}_{m_{\phi_n}}$. Note that we can assume without loss of generality that the sequence $\{m_{\phi_n}, n \geq 1\}$ is non decreasing. For any $k \in [m_{\phi_n}, m_{\phi_{n+1}}[$, set $A_k = \tilde{A}_{\phi_n}$. Then, $A_k \in \mathcal{F}_{m_{\phi_n}} \subseteq \mathcal{F}_k$ and

$$\lim_k 1_{A_k} = \lim_n 1_{\tilde{A}_{\phi_n}} = 1_B .$$

□

Lemma 5.8. *Let $|\cdot|$ be any matrix norm. Let $\{A_k, k \geq 0\}$ be a sequence of square matrix such that $\lim_k |A_k - A| = 0$ where A is a Hurwitz matrix. Denote by $-L, L > 0$, the largest real part of its eigenvalues. Let $\{\gamma_k, k \geq 0\}$ be a positive sequence such that $\lim_k \gamma_k = 0$. For any $0 < L' < L$, there exists a positive constant C such that for any $k \leq n$*

$$|(\text{Id} + \gamma_n A_n) \cdots (\text{Id} + \gamma_{k+1} A_{k+1})(\text{Id} + \gamma_k A_k)| \leq C \exp(-L' \sum_{j=k}^n \gamma_j) .$$

Proof. Let $\lambda_i, i \leq d$ be the eigenvalues of A . By using the Jordan decomposition, we write $A = SJS^{-1}$ where S is a non-singular matrix, and J is a Jordan matrix (as defined by Horn and Johnson [17, Definition 3.1.1] - note that the diagonal entries of J are λ_i).

For any $t > 0$, denote by D_t the diagonal matrix with diagonal entries (t, t^2, \dots, t^d) and set

$$A = (SD_t) (D_t^{-1} J D_t) (SD_t)^{-1} = (SD_t) (\Lambda + R_t) (SD_t)^{-1}$$

with $\Lambda \stackrel{\text{def}}{=} \text{diag}(\lambda_i)$, upon noting that

$$D_t^{-1} J D_t = \begin{bmatrix} \lambda_1 & tu_1 & 0 & \cdot & 0 \\ 0 & \lambda_2 & tu_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \lambda_{d-1} & tu_{d-1} \\ 0 & \cdot & \cdot & \cdot & \lambda_d \end{bmatrix} .$$

Note also that $|R_t| \rightarrow 0$ as $t \rightarrow 0$. We write

$$\begin{aligned} (SD_t)^{-1} (I + \gamma_\ell A_\ell) (SD_t) &= (SD_t)^{-1} (I + \gamma_\ell A) (SD_t) + \gamma_\ell (SD_t)^{-1} (A_\ell - A) (SD_t) \\ &= I + \gamma_\ell D_t^{-1} J D_t + \gamma_\ell (SD_t)^{-1} (A_\ell - A) (SD_t) \\ &= I + \gamma_\ell \Lambda + \gamma_\ell R_t + \gamma_\ell (SD_t)^{-1} (A_\ell - A) (SD_t) . \end{aligned}$$

Therefore,

$$\left| (SD_t)^{-1} (I + \gamma_\ell A_\ell) (SD_t) \right| \leq |I + \gamma_\ell \Lambda| + \gamma_\ell |R_t| + \gamma_\ell |A_\ell - A| |SD_t| |(SD_t)^{-1}| .$$

Let $0 < L < L'' < L$. There exists t_0 such that for any $t \in (0, t_0)$, $|R_t| \leq (L'' - L')/2$; and there exists K such that for any $\ell \geq K$ and any $t \leq t_0$, $|I + \gamma_\ell \Lambda| \leq 1 - \gamma_\ell L''$ and $|A_\ell - A| |SD_t| |(SD_t)^{-1}| \leq (L'' - L')/2$. Therefore, for any $\ell \geq K$ and any $t \in (0, t_0)$

$$\left| (SD_t)^{-1} (I + \gamma_\ell A_\ell) (SD_t) \right| \leq 1 - \gamma_\ell L' .$$

Now we write for $K \leq k < n$ and $t \leq t_0$,

$$|(I + \gamma_n A_n) \cdots (I + \gamma_k A_k)| \leq |SD_t| |(SD_t)^{-1}| \prod_{\ell=k}^n (1 - \gamma_\ell L')$$

which concludes the proof. \square

Lemma 5.9. *Let γ_k be a positive sequence such that $\lim_k \gamma_k = 0$ and $\sum_k \gamma_k = \infty$. Let $\{e_k, k \geq 0\}$ be a non-negative sequence. Then*

$$\limsup_n \gamma_n^{-p} \sum_{k=1}^n \gamma_k^{p+1} e_k \exp(-b \sum_{j=k+1}^n \gamma_j) \leq \frac{1}{C(b, p)} \limsup_n e_n ,$$

(i) with $C(b, p) = b$, for any $b > 0, p \geq 0$ if $\log(\gamma_{k-1}/\gamma_k) = o(\gamma_k)$.

(ii) with $C(b, p) = b - p/\gamma_*$, for any $b\gamma_* > p \geq 0$ if there exists $\gamma_* > 0$ such that $\log(\gamma_{k-1}/\gamma_k) \sim \gamma_k/\gamma_*$.

By convention, $\sum_{j=n+1}^n \gamma_j = 0$.

Proof. The proof is from Delyon [10, Theorem 19, Chapter 4]. Let $\{x_n, n \geq 0\}$ be defined by $x_n = \exp(-b\gamma_n)x_{n-1} + \gamma_n^{p+1}e_n$ where $x_0 = 0$. Then by a trivial recursion, it holds

$$x_n = \sum_{k=1}^n \gamma_k^{p+1} e_k \exp(-b \sum_{j=k+1}^n \gamma_j) .$$

Set $u_n \stackrel{\text{def}}{=} \gamma_n^{-p} x_n$. Then

$$\begin{aligned} u_n &= \left(\frac{\gamma_{n-1}}{\gamma_n} \right)^p \exp(-b\gamma_n) u_{n-1} + \gamma_n e_n \\ &= \exp(p \log(\gamma_{n-1}/\gamma_n) - b\gamma_n) u_{n-1} + \gamma_n e_n \\ &= (1 - b_n \gamma_n) u_{n-1} + b_n \gamma_n (b_n^{-1} e_n) , \end{aligned}$$

where $b_n \gamma_n \stackrel{\text{def}}{=} 1 - \exp(p \log(\gamma_{n-1}/\gamma_n) - b \gamma_n)$. Observe that $\gamma_n b_n \sim 1 - \exp(-b \gamma_n)$ in case (i) and $\gamma_n b_n \sim 1 - \exp(-(b - p/\gamma_\star) \gamma_n)$ in case (ii). Therefore, $\lim_n b_n = b$ (resp. $b - p/\gamma_\star$) in case (i) (resp. (ii)).

Let $v \geq \limsup_n b_n^{-1} e_n$. We have

$$u_n - v = (1 - b_n \gamma_n)(u_{n-1} - v) + b_n \gamma_n (b_n^{-1} e_n - v)$$

and upon noting that $(a + b)_+ \leq a_+ + b_+$, it holds

$$(u_n - v)_+ \leq (1 - b_n \gamma_n)(u_{n-1} - v)_+ + b_n \gamma_n (b_n^{-1} e_n - v)_+ \leq (1 - b_n \gamma_n)(u_{n-1} - v)_+ .$$

Since $\lim_n \gamma_n b_n = 0$ and $\sum_n b_n \gamma_n = +\infty$, $\lim_n (u_n - v)_+ = 0$ thus implying that $\limsup_n u_n \leq v$. This holds for any $v \geq \limsup_n b_n^{-1} e_n$ thus concluding the proof. \square

Lemma 5.10. *For any matrices A, B, C*

$$|ABA^T - CBC^T| = |(A - C)BA^T - CB(C - A)^T| \leq |A - C| |B| (|A| + |C|) .$$

Lemma 5.11. *Let U_\star be a positive definite matrix.*

(a) *Assume C1b-c and C4a. Consider the equation*

$$v_{n+1} = v_n + \gamma_n f(v_n) + \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}} v_n + (\gamma_{n+1} - \gamma_n) U_\star + \gamma_n \gamma_{n+1} \nabla h(\theta_\star) v_n \nabla h(\theta_\star)^T ,$$

where $f(v) \stackrel{\text{def}}{=} U_\star + \nabla h(\theta_\star) v + v \nabla h(\theta_\star)^T$. Then there exists an unique positive definite matrix V such that $f(V) = 0$ and $\lim_n v_n = V$.

(b) *Assume C1b-c and C4b. Consider the equation*

$$v_{n+1} = v_n + \gamma_n f(v_n) + (\gamma_{n+1} - \gamma_n) U_\star + \gamma_n \gamma_{n+1} \nabla h(\theta_\star) v_n \nabla h(\theta_\star)^T ,$$

where $f(v) \stackrel{\text{def}}{=} U_\star + \nabla h(\theta_\star) v + v \nabla h(\theta_\star)^T + \gamma_\star^{-1} v$. Then there exists an unique positive definite matrix V such that $f(V) = 0$ and $\lim_n v_n = V$.

Proof. (a) Let V such that $f(V) = 0$. We have

$$\begin{aligned} v_{n+1} - V &= v_n - V + \gamma_n (H(v_n - V) + (v_n - V)H^T) \\ &\quad + \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}} (v_n - V) + \gamma_n \gamma_{n+1} \nabla h(\theta_\star) (v_n - V) \nabla h(\theta_\star)^T \\ &\quad + \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}} V + \gamma_n \gamma_{n+1} \nabla h(\theta_\star) V \nabla h(\theta_\star)^T + (\gamma_{n+1} - \gamma_n) U_\star . \end{aligned}$$

Under C4a, $|\gamma_n/\gamma_{n+1} - 1| = o(\gamma_n)$ and $\gamma_n - \gamma_{n+1} = o(\gamma_n^2)$. Then, denoting by \bar{v}_n the vectorialized form of the matrix $v_n - V$, this yields

$$\bar{v}_{n+1} = (\text{Id} + \gamma_n A_n) \bar{v}_n + B_n$$

where $\{A_n, n \geq 0\}$ is a sequence of Hurwitz matrix that converges to a Hurwitz matrix A , and $B_n = o(\gamma_n)$. Then, there exists $L' > 0$ such that

$$|v_{n+1} - V| \leq (1 - \gamma_n L') |v_n - V| + \gamma_n \epsilon_n ,$$

where $\epsilon_n = o(1)$. As in the proof of Lemma 5.9, it can be proved that $\limsup_n |v_{n+1} - V| \leq \limsup_n \epsilon_n = 0$.

(b) Under C4b, $\frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}} = \gamma_n/\gamma_\star + o(\gamma_n)$ and $\gamma_{n+1} - \gamma_n = O(\gamma_n^2)$. As in the previous case, we write

$$v_{n+1} - V = v_n - V + \gamma_n \tilde{H}(v_n - V) + \gamma_n (v_n - V) H^T + o(\gamma_n)$$

where $\tilde{H} \stackrel{\text{def}}{=} \nabla h(\theta_\star) + (2\gamma_\star)^{-1} \text{Id}$. Under the assumptions on γ_\star , \tilde{H} is a Hurwitz matrix. As in the proof of Lemma 5.9, it can be proved that $\limsup_n |v_{n+1} - V| \leq o(1) = 0$. \square

Lemma 5.12. *Define the sequence $\{x_n, n \geq 0\}$ by*

$$x_{n+1} = x_n + \gamma_{n+1} A x_n + \gamma_{n+1} \zeta_{n+1} , \quad x_0 \in \mathbb{R}^d ,$$

where $\{\gamma_n, n \geq 1\}$ is a positive sequence, $\{\zeta_n, n \geq 1\}$ is a \mathbb{R}^d -valued sequence and A is a $d \times d$ matrix. Then

$$A \sum_{k=0}^n x_k = - \sum_{k=0}^n \zeta_{k+1} + \left(\frac{x_{n+1}}{\gamma_{n+1}} - \frac{x_0}{\gamma_1} \right) + \sum_{k=1}^n \left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k+1}} \right) x_k .$$

Proof. By definition of $\{x_n, n \geq 0\}$, for any $n \geq 0$ it holds

$$A x_n = \frac{1}{\gamma_{n+1}} (x_{n+1} - x_n) - \zeta_{n+1} .$$

Therefore,

$$A \sum_{k=0}^n x_k = \sum_{k=0}^n \frac{1}{\gamma_{k+1}} (x_{k+1} - x_k) - \sum_{k=0}^n \zeta_{k+1} .$$

We then conclude by the Abel transform. \square

Lemma 5.13. *Let $\{\gamma_n, n \geq 1\}$ be a positive non-increasing sequence. Then*

$$\lim_n n \gamma_n = +\infty \implies \lim_n \frac{1}{\sqrt{n}} \sum_{k=1}^n \gamma_k^{-1/2} \left| 1 - \frac{\gamma_k}{\gamma_{k+1}} \right| = 0 .$$

Proof. The following proof can be found in the proof of Delyon [10, Theorem 26, Chapter 4]. We have

$$\begin{aligned} \sum_{k=1}^n \gamma_k^{-1/2} \left(\frac{\gamma_k}{\gamma_{k+1}} - 1 \right) &= \sum_{k=1}^n \gamma_k^{1/2} \left(\frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} \right) \\ &= - \sum_{k=2}^{n+1} \gamma_k^{-1} (\sqrt{\gamma_k} - \sqrt{\gamma_{k-1}}) - \frac{1}{\sqrt{\gamma_1}} + \frac{1}{\sqrt{\gamma_{n+1}}} \\ &\leq - \sum_{k=2}^{n+1} \gamma_k^{-1/2} \gamma_{k-1}^{-1/2} (\sqrt{\gamma_k} - \sqrt{\gamma_{k-1}}) + \frac{1}{\sqrt{\gamma_{n+1}}} = \frac{1}{\sqrt{\gamma_n}} + \frac{1}{\sqrt{\gamma_{n+1}}} . \end{aligned}$$

□

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